METASTABILITY AND DYNAMICS OF DISCRETE TOPOLOGICAL SINGULARITIES IN TWO DIMENSIONS: A Γ-CONVERGENCE APPROACH

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ABSTRACT. This paper aims at building a variational approach to the dynamics of discrete topological singularities in two dimensions, based on Γ-convergence.

We consider discrete systems, described by scalar functions defined on a square lattice and governed by periodic interaction potentials. Our main motivation comes from XY spin systems, described by the phase parameter, and screw dislocations, described by the displacement function. For these systems, we introduce a discrete notion of vorticity. As the lattice spacing tends to zero we derive the first order Γ-limit of the free energy which is referred to as renormalized energy and describes the interaction of vortices.

As a byproduct of this analysis, we show that such systems exhibit increasingly many metastable configurations of singularities. Therefore, we propose a variational approach to depinning and dynamics of discrete vortices, based on minimizing movements. We show that, letting first the lattice spacing and then the time step of the minimizing movements tend to zero, the vortices move according with the gradient flow of the renormalized energy, as in the continuous Ginzburg-Landau framework.

CONTENTS

1. Introduction

Phase transitions mediated by the formation of topological singularities characterize many physical phenomena such as superconductivity, superfluidity and plasticity. For its central role in Materials Science, this subject has attracted much attention in the last decades ([9], [32], [33], see also [44]), and has brought new interest on fascinating research fields in the mathematical community, as in the theory of harmonic maps on manifolds ([4], [15], [10]). In particular, new variational methods have been developed to describe and predict the relevant phenomena, such as the formation of topological singularities and the corresponding concentration of energy. Two paradigmatic examples of the appearance of topological singularities are given by screw dislocations in crystals and vortices in superconductors. We now introduce two basic discrete models to describe these phenomena.

Given an open set $\Omega \subset \mathbb{R}^2$, consider the square lattice $\varepsilon\mathbb{Z}^2 \cap \Omega$, representing the reference configuration of our physical system. In the case of screw dislocations we consider the elastic energy defined on scalar functions $u : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{R}$ given by

$$SD_\varepsilon(u) := \frac{1}{2} \sum_{i,j\in\varepsilon\mathbb{Z}^2\cap\Omega\ ,\ |i-j|=\varepsilon} \mathrm{dist}^2(u(i) - u(j), \mathbb{Z}). \tag{1.1}$$

Here ε represents the lattice spacing of a cubic lattice casted in a cylindrical crystal, $\varepsilon\mathbb{Z}^2 \cap \Omega$ is a reference planar section of the crystal, and u represents the vertical displacement (scaled by $1/\varepsilon$). The periodicity of the energy is consistent with the fact that plastic deformations, corresponding to integer jumps of u, do not store elastic energy, according with Nabarro Peierls and Frenkel Kontorova theories [26]. Potentials as in (1.1) are commonly used in models for dislocations (see e.g. [18], [28], [23], [37]; see also [8] for more general discrete lattice energies accounting for defects).

A celebrated discrete model which allows to describe the formation of topological singularities, as vortices in superconductors, is the so-called XY spin model. Here, the order parameter is a vectorial spin field $v : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathcal{S}^1$ and the corresponding energy is given by

$$XY_\varepsilon(v) := \frac{1}{2} \sum_{i,j\in\varepsilon\mathbb{Z}^2\cap\Omega\ \ |i-j|=\varepsilon} |v(i) - v(j)|^2.$$

Notice that $XY_\varepsilon(v)$ can be written in terms of a representative of the phase of v, defined as a scalar field u such that $v = e^{2\pi i u}$. In this respect, both models can be regarded as specific examples of scalar systems governed by periodic potentials f acting on first neighbors, whose energy is of the type

$$F_\varepsilon(u) := \sum_{i,j\in\varepsilon\mathbb{Z}^2\cap\Omega\ ,\ |i-j|=\varepsilon} f(u(i) - u(j)).$$

How do dislocations or vortices enter in this description? Loosely speaking, they are defined through a discrete notion of topological degree of the field $v = e^{2\pi i u}$; they are point singularities, and can be identified by the discrete vorticity measure $\mu(u)$. This is a finite sum of Dirac masses centered in the squares of the lattice, and with multiplicities equal to either $+1$ or -1. This notion in the case of dislocations corresponds to the discrete circulation of the plastic strain, and $\mu(u)$ represents the Nye dislocation density.

This paper aims at studying the statics and the dynamics of such topological singularities, by variational principles.

The first step is the asymptotic analysis by Γ-convergence of the discrete energies F_ε, as $\varepsilon \to 0$. This analysis relies on the powerful machinery developed in the recent past for the analysis of Ginzburg-Landau functionals, which can be somehow considered the continuous counterpart of the energies F_ε. We recall that, for a given $\varepsilon > 0$, the Ginzburg-Landau energy $GL_\varepsilon : H^1(\Omega; \mathbb{R}^2) \to \mathbb{R}$ is defined by

$$GL_\varepsilon(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 \, \mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega (1 - |w|^2)^2 \, \mathrm{d}x. \tag{1.2}$$

Starting from the pioneering book [10], the variational analysis as $\varepsilon \to 0$ of GL_ε has been the subject of a vast literature. The analysis in [10] shows that, as ε tends to zero, vortex-like singularities appear by energy minimization (induced for instance by the boundary conditions), and each singularity carries a quantum of energy of order $|\log \varepsilon|$. Removing this leading term from the energy, a finite quantity remains, called renormalized energy, depending on the positions of the singularities. This asymptotic analysis has been also developed through the solid formalism of Γ-convergence ([30], [31], [39], [41], [3]). It turns out that the relevant object to deal with is the distributional Jacobian Jw, which, in the continuous setting, plays the role of the discrete vorticity measure. A remarkable fact is that these results also contain a compactness statement. Indeed, for sequences with bounded energy the vorticity measure is not in general bounded in mass; this is due to the fact that many dipoles are compatible with a logarithmic energy bound. Therefore, the compactness of the vorticity measures fails in the usual sense of weak star convergence. Nevertheless, compactness holds in the *flat topology*, i.e., in the dual of Lipschitz continuous functions with compact support.

Recently, part of this Γ-convergence analysis has been exported to two-dimensional discrete systems. In [36], [1], [2] it has been proved that the functionals $\frac{1}{|\log \varepsilon|} F_\varepsilon$ Γ-converge to $\pi|\mu(\Omega)|$, where μ is the limiting vorticity measure and is given by a finite sum of Dirac masses. This Γ-limit is not affected by the position of the singularities and hence does not account for their interaction, which is an essential ingredient in order to study the dynamics. In this paper, we make a further step in this direction, deriving the renormalized energy for our discrete systems by Γ-convergence, using the notion of Γ-convergence expansion introduced in [7] (see also [14]). Precisely, in Theorem 4.2 we prove that, given $M \in \mathbb{N}$, the functionals $F_\varepsilon(u) - M\pi|\log \varepsilon|$ Γ-converge to $\mathbb{W}(\mu) + M\gamma$, where μ is a sum of M singularities x_i with degrees $d_i = \pm 1$. Here $\mathbb{W}$ is the renormalized energy as in the Ginzburg-Landau setting, defined by

$$\mathbb{W}(\mu) := -\pi \sum_{i \neq j} d_i d_j \log |x_i - x_j| - \pi \sum_i d_i R_0(x_i),$$

where R_0 is a suitable harmonic function (see (4.1)), and γ can be viewed as a core energy, depending on the specific discrete interaction energy (see (4.6)).

An intermediate step to prove Theorem 4.2 is Theorem 3.1 (ii), which establishes a localized lower bound of the energy around the limiting vortices. This result is obtained using a tool introduced by Sandier [38] and Jerrard [30] for the functionals GL_ε, referred to as *ball construction*; it consists in providing suitable pairwise disjoint annuli, where much of the energy is stored, and estimating from below the energy on each of such annuli. In the continuous case, the lower bound on each annulus is the straightforward estimate

$$\frac{1}{2}\int_{B_R\setminus B_r} |\nabla w|^2\,\mathrm{d}x \geq \pi|\deg(w,\partial B_R)|\log\frac{R}{r}, \quad w\in H^1(B_R\setminus B_r;\mathcal{S}^1).$$

In Proposition 3.2 we prove a similar lower bound for F_ε, with R/r replaced by $R/(r+C\varepsilon|\log\varepsilon|)$, the error being due to the discrete structure of our energies. This weaker estimate, inserted in the ball construction machinery, is refined enough to prove the lower bound in Theorem 3.1 (ii).

The second part of the paper is devoted to the analysis of metastable configurations for F_ε and to our variational approach to the dynamics of discrete topological singularities.

We now draw a parallel between the continuous Ginzburg-Landau model and our discrete systems, stressing out the peculiarities of our framework.

In [34], [29], [40], it has been proved that the parabolic flow of GL_ε can be described, as $\varepsilon\to 0$, by the gradient flow of the renormalized energy $\mathbb{W}(\mu)$. Precisely the limiting flow is a measure $\mu(t)=\sum_{i=1}^M d_{i,0}\delta_{x_i(t)}$, where $x(t)=(x_1(t),\dots,x_M(t))$ solves

$$\begin{cases}\dot{x}(t) = -\dfrac{1}{\pi}\nabla W(x(t))\\ x(0)=x_0\,,\end{cases} \tag{1.3}$$

with $W(x(t))=\mathbb{W}(\mu(t))$. The advantage of this description is that the effective dynamics is described by an ODE involving only the positions of the singularities. This result has been derived through a purely variational approach in [40], based on the idea that the gradient flow structure is consistent with Γ-convergence, under some assumptions which imply that the slope of the approximating functionals converges to the slope of their Γ-limit. The gradient flow approach to dynamics used in the Ginzburg-Landau context fails for our discrete systems. In fact, the free energy of discrete systems is often characterized by the presence of many energy barriers, which affect the dynamics and are responsible for pinning effects (for a variational description of pinning effects in discrete systems see [13] and the references therein). As a consequence of our Γ-convergence analysis, we show that F_ε has many local minimizers. Precisely, in Theorem 5.5 and Theorem 5.6 we show that, under suitable assumptions on the potential f, given any configuration of singularities $x\in\Omega^M$, there exists a stable configuration $\tilde{x}$ at a distance of order ε from x. Starting from these configurations, the gradient flow of F_ε is clearly stuck. Moreover, these stable configurations are somehow attractive wells for the dynamics. These results are proven for a general class of energies, including SD_ε, while the case of the XY_ε energy, to our knowledge, is still open. A similar analysis of stable configurations in the triangular lattice has been recently carried on in

[27], combining PDEs techniques with variational arguments, while our approach is purely variational and based on Γ-convergence.

On one hand, our analysis is consistent with the well-known pinning effects due to energy barriers in discrete systems; on the other hand, it is also well understood that dislocations are able to overcome the energetic barriers to minimize their interaction energy (see [17], [23], [28], [37]). The mechanism governing these phenomena is still matter of intense research. Certainly, thermal effects and statistical fluctuations play a fundamental role. Such analysis is beyond the purposes of this paper. Instead, we raise the question whether there is a simple variational mechanism allowing singularities to overcome the barriers, and then which would be the effective dynamics. We face these questions, following the minimizing movements approach à la De Giorgi ([5], [6], [12]). More precisely, we discretize time by introducing a time scale $\tau > 0$, and at each time step we minimize a total energy, which is given by the sum of the free energy plus a dissipation. For any fixed τ, we refer to this process as *discrete gradient flow*. This terminology is due to the fact that, as τ tends to zero, the discrete gradient flow is nothing but the Euler implicit approximation of the continuous gradient flow of F_ε. Therefore, as $\tau \to 0$ it inherits the degeneracy of F_ε, and pinning effects are dominant. The scenario changes completely if instead we keep τ fixed, and send $\varepsilon \to 0$. In this case, it turns out that, during the step by step energy minimization, the singularities are able to overcome the energy barriers, that are of order ε. Finally, sending $\tau \to 0$ the solutions of the discrete gradient flows converge to a solution of (1.3). In our opinion, this purely variational approach based on minimizing movements, mimics in a realistic way more complex mechanisms, providing an efficient and simple view point on the dynamics of discrete topological singularities in two dimensions.

Summarizing, in order to observe an effective dynamics of the vortices we are naturally led to let $\varepsilon \to 0$ for a fixed time step τ, obtaining a discrete gradient flow of the renormalized energy. A technical issue is that the renormalized energy is not bounded from below, and therefore, in the step by step minimization we are led to consider local rather than global minimizers. Precisely, we minimize the energy in a δ neighborhood of the minimizer at the previous step. Without this care, already at the first step we would have the trivial solution $\mu = 0$, corresponding to the fact that dipoles annihilate and the remaining singularities reach the boundary of the domain. Nevertheless, for τ small the minimizers do not touch the constraint, so that they are in fact true local minimizers.

We will adopt the above scheme dealing with two specific choices for the dissipation. On one hand, the canonical choice corresponding to continuous parabolic flows is clearly the L^2 dissipation (see Section 7). On the other hand, once ε is sent to zero, we have a finite dimensional gradient flow of the renormalized energy, for which it is more natural to consider as dissipation the Euclidean distance between the singularities. This, for $\varepsilon > 0$, corresponds to the introduction of a 2-Wasserstein type dissipation, D_2, between the vorticity measures. For two Dirac deltas D_2 is nothing but the square of the Euclidean distance of the masses (see Definition (6.4)). We are then led to consider also the discrete gradient flow with this dissipation (see Section 6). By its very definition D_2 is continuous with respect to the flat norm and this makes the analysis as $\varepsilon \to 0$ rather simple and somehow instructive in order to face the more complex case of L^2 dissipation.

In conclusion, we believe that this paper provides a better understanding of equilibria of discrete systems characterized by energy concentration, and contributes to the debate in the mathematical community over the microscopic mechanisms governing the dynamics of discrete topological singularities, as vortices in XY spin systems and dislocations in crystals. For the latter, richer models could be considered, with more realistic energy densities and dissipations, taking into account the specific material properties and the kinematic constraints of the crystal lattice. Our variational approach, rather than giving a complete analysis of a specific model, aims to be simple and robust, with possible applications to a wide class of discrete systems.

2. The discrete model for topological singularities

In this Section we introduce the discrete formalism used in the analysis of the problem we deal with. We will follow the approach of [8]; specifically, we will use the formalism and the notations in [2] (see also [36]).

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, representing the domain of definition of the relevant fields in the models we deal with.

The discrete lattice. For every $\varepsilon > 0$, we define $\Omega_\varepsilon \subset \Omega$ as follows

$$\Omega_\varepsilon := \bigcup_{i \in \varepsilon\mathbb{Z}^2 :\, i+\varepsilon Q \subset \overline{\Omega}} (i + \varepsilon Q),$$

where $Q = [0,1]^2$ is the unit square. Moreover we set $\Omega^0_\varepsilon := \varepsilon\mathbb{Z}^2 \cap \Omega_\varepsilon$, and $\Omega^1_\varepsilon := \left\{(i,j) \in \Omega^0_\varepsilon \times \Omega^0_\varepsilon :\ |i-j| = \varepsilon,\, i \le j\right\}$ (where $i \le j$ means that $i_l \le j_l$ for $l \in \{1,2\}$). These objects represent the reference lattice and the class of nearest neighbors, respectively. The cells contained in Ω_ε are labeled by the set of indices $\Omega^2_\varepsilon = \left\{i \in \Omega^0_\varepsilon :\ i + \varepsilon Q \subset \Omega_\varepsilon\right\}$. Finally, we define the *discrete boundary* of Ω as

$$\partial_\varepsilon \Omega := \partial\Omega_\varepsilon \cap \varepsilon\mathbb{Z}^2. \tag{2.1}$$

In the following, we will extend the use of these notations to any given open subset A of $\mathbb{R}^2$.

2.1. **Discrete functions and discrete topological singularities.**

Here we introduce the classes of discrete functions on Ω^0_ε, and a notion of discrete topological singularities. To this purpose, we first set

$$\mathcal{AF}_\varepsilon(\Omega) := \left\{u : \Omega^0_\varepsilon \to \mathbb{R}\right\},$$

which represents the class of admissible scalar functions on Ω^0_ε.

Moreover, we introduce the class of admissible fields from Ω^0_ε to the set $\mathcal{S}^1$ of unit vectors in $\mathbb{R}^2$

$$\mathcal{AXY}_\varepsilon(\Omega) := \left\{v : \Omega^0_\varepsilon \to \mathcal{S}^1\right\}, \tag{2.2}$$

Notice that, to any function $u \in \mathcal{AF}_\varepsilon(\Omega)$, we can associate a function $v \in \mathcal{AXY}_\varepsilon(\Omega)$ setting

$$v = v(u) := e^{2\pi i u}.$$

With a little abuse of notation for every $v : \Omega^0_\varepsilon \to \mathbb{R}^2$ we denote

$$\|v\|^2_{L^2} = \sum_{j \in \Omega^0_\varepsilon} \varepsilon^2 |v(j)|^2 . \tag{2.3}$$

Now we can introduce a notion of discrete vorticity corresponding to both scalar and $\mathcal{S}^1$ valued functions. To this purpose, let $P:\mathbb{R}\to\mathbb{Z}$ be defined as follows

$$P(t)=\operatorname{argmin}\{|t-s|:\,s\in\mathbb{Z}\}\,, \tag{2.4}$$

with the convention that, if the argmin is not unique, then we choose the smallest among the two.

Let $u\in\mathcal{AF}_\varepsilon(\Omega)$ be fixed. For every $i\in\Omega^2_\varepsilon$ we introduce the vorticity

$$\begin{aligned}\alpha_u(i):=&P(u(i+\varepsilon e_1)-u(i))+P(u(i+\varepsilon e_1+\varepsilon e_2)-u(i+\varepsilon e_1))\\&-P(u(i+\varepsilon e_1+\varepsilon e_2)-u(i+\varepsilon e_2))-P(u(i+\varepsilon e_2)-u(i)).\end{aligned} \tag{2.5}$$

One can easily see that the vorticity α_u takes values in $\{-1,0,1\}$. Finally, we define the vorticity measure $\mu(u)$ as follows

$$\mu(u):=\sum_{i\in\Omega^2_\varepsilon}\alpha_u(i)\delta_{i+\frac{\varepsilon}{2}(e_1+e_2)}. \tag{2.6}$$

This definition of vorticity extends to $\mathcal{S}^1$ valued fields in the obvious way, by setting $\mu(v)=\mu(u)$ where u is any function in $\mathcal{AF}_\varepsilon(\Omega)$ such that $v(u)=v$.

Let $\mathcal{M}(\Omega)$ be the space of Radon measures in Ω and set

$$\begin{aligned}X&:=\left\{\mu\in\mathcal{M}(\Omega)\,:\;\mu=\sum_{i=1}^N d_i\delta_{x_i}\,,\,N\in\mathbb{N},d_i\in\mathbb{Z}\setminus\{0\}\,,x_i\in\Omega\right\},\\X_\varepsilon&:=\left\{\mu\in X:\mu=\sum_{i\in\Omega^2_\varepsilon}\alpha(i)\delta_{i+\frac{\varepsilon}{2}(e_1+e_2)}\,,\;\alpha(i)\in\{-1,0,1\}\right\}.\end{aligned} \tag{2.7}$$

We will denote by $\mu_n\overset{\text{flat}}{\to}\mu$ the flat convergence of μ_n to μ, i.e., in the dual $W^{-1,1}$ of $W^{1,\infty}_0$.

2.2. **The discrete energy.** Here we introduce a class of energy functionals defined on $\mathcal{AF}_\varepsilon(\Omega)$. We will consider periodic potentials $f:\mathbb{R}\to\mathbb{R}$ which satisfy the following assumptions: For any $a\in\mathbb{R}$

(1) $f(a+z)=f(a)$ for any $z\in\mathbb{Z}$,
(2) $f(a)\geq\frac{1}{2}|e^{2\pi ia}-1|^2=1-\cos 2\pi a$,
(3) $f(a)=2\pi^2(a-z)^2+\mathrm{O}(|a-z|^3)$ for any $z\in\mathbb{Z}$.

For any $u\in\mathcal{AF}_\varepsilon(\Omega)$, we define

$$F_\varepsilon(u):=\sum_{(i,j)\in\Omega^1_\varepsilon}f(u(i)-u(j)). \tag{2.8}$$

As explained in the Introduction, the main motivation for our analysis comes from the study discrete screw dislocations in crystals and XY spin systems. We introduce the basic energies for these two models as in [2].

Regarding the screw dislocations, for any $u:\Omega^0_\varepsilon\to\mathbb{R}$, we define

$$SD_\varepsilon(u):=\frac{1}{2}\sum_{(i,j)\in\Omega^1_\varepsilon}\operatorname{dist}^2(u(i)-u(j),\mathbb{Z}). \tag{2.9}$$

It is easy to see that this potential fits (up to the prefactor $4\pi^2$) with our general assumptions.

As for the XY model, for any $v : \Omega^0_\varepsilon \to \mathcal{S}^1$, we define

$$XY_\varepsilon(v) := \frac{1}{2} \sum_{(i,j)\in\Omega^1_\varepsilon} |v(i) - v(j)|^2. \tag{2.10}$$

Also this potential fits our framework, once we rewrite it in terms of the phase u of v. Indeed, setting $f(a) = 1 - \cos(2\pi a)$, we have

$$XY_\varepsilon(v) = \sum_{(i,j)\in\Omega^1_\varepsilon} f(u(i) - u(j)) \qquad \text{with} \;\; v = e^{2\pi i u}. \tag{2.11}$$

We notice that assumption (2) on F_ε reads as

$$F_\varepsilon(u) \geq XY_\varepsilon(e^{2\pi i u}). \tag{2.12}$$

Let $\{T_i^\pm\}$ be the family of the ε-simplices of $\mathbb{R}^2$ whose vertices are of the form $\{i, i \pm \varepsilon e_1, i \pm \varepsilon e_2\}$, with $i \in \varepsilon\mathbb{Z}^2$. For any $v : \Omega^0_\varepsilon \to \mathcal{S}^1$, we denote by $\tilde v : \Omega_\varepsilon \to \mathbb{R}^2$ the piecewise affine interpolation of v, according with the triangulation $\{T_i^\pm\}$. It is easy to see that, up to boundary terms, $XY_\varepsilon(v)$ corresponds to the Dirichlet energy of $\tilde v$ in Ω_ε; more precisely

$$\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla \tilde v|^2 \, \mathrm{d}x + \frac{1}{2}\int_{B_\varepsilon} |\nabla \tilde v|^2 \, \mathrm{d}x \geq XY_\varepsilon(v) \geq \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla \tilde v|^2 \, \mathrm{d}x, \tag{2.13}$$

where $B_\varepsilon := \{x \in \Omega_\varepsilon : \; \mathrm{dist}(x, \partial\Omega_\varepsilon) \leq \varepsilon\}$.

Remark 2.1. Let $v : \Omega^0_\varepsilon \to \mathcal{S}^1$. One can easily verify that if A is an open subset of Ω and if $|\tilde v| > c > 0$ on ∂A_ε, then

$$\mu(v)(A_\varepsilon) = \deg(\tilde v, \partial A_\varepsilon), \tag{2.14}$$

where, given an open bounded set $V \subset \mathbb{R}^2$ with Lipschitz boundary, the *degree* of a function $w \in H^{\frac{1}{2}}(\partial V; \mathbb{R}^2)$ with $|w| \geq c > 0$, is defined by

$$\deg(w, \partial V) := \frac{1}{2\pi} \int_{\partial V} \left(\frac{w_1}{|w|} \nabla \frac{w_2}{|w|} - \frac{w_2}{|w|} \nabla \frac{w_1}{|w|} \right) \cdot \tau \, \mathrm{d}s \,. \tag{2.15}$$

In [16] it is proved that the quantities above are well defined and that the definition in (2.15) is well posed. Note that $\mu(v)(i + \varepsilon Q) = 0$ whenever $|\tilde v| > 0$ on $i + \varepsilon Q$.

3. Localized lower bounds

In this section we will prove a lower bound for the energies F_ε localized on open subsets $A \subset \Omega$. We will use the standard notation $F_\varepsilon(\cdot, A)$ (and as well $XY_\varepsilon(\cdot, A)$) to denote the functional F_ε defined in (2.8) with Ω replaced by A.

To this purpose, thanks to assumption (2) on the energy density f, it will be enough to prove a lower bound for the XY_ε energy. As a consequence of this lower bound, we obtain a sharp zero-order Γ-convergence result for the functionals F_ε. As explained in the Introduction, the appropriate topology with respect to which compactness results hold true is that induced by the flat norm.

3.1. The zero-order Γ-convergence.

We recall that the space of finite sums of weighted Dirac masses has been denoted in (2.7) by X.

Theorem 3.1. *Let F_ε be defined by* (2.8) *with f satisfying (1)–(3). The following Γ-convergence result holds.*

(i) *(Compactness) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $F_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$ for some positive C. Then, up to a subsequence, $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$, for some $\mu \in X$.*

(ii) *(Localized Γ-liminf inequality) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^M d_i \delta_{x_i}$ with $d_i \in \mathbb{Z} \setminus \{0\}$ and $x_i \in \Omega$. Then, there exists a constant $C \in \mathbb{R}$ such that, for any $i = 1, \ldots, M$ and for every $\sigma < \frac{1}{2} dist(x_i, \partial\Omega \cup \bigcup_{j \neq i} x_j)$, we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, B_\sigma(x_i)) - \pi |d_i| \log \frac{\sigma}{\varepsilon} \geq C. \tag{3.1}$$

In particular

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) - \pi |\mu|(\Omega) \log \frac{\sigma}{\varepsilon} \geq C. \tag{3.2}$$

(iii) *(Γ-limsup inequality) For every $\mu \in X$, there exists a sequence $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$ and*

$$\pi |\mu|(\Omega) \geq \limsup_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|}.$$

The above theorem has been proved in [36] for $F_\varepsilon = SD_\varepsilon$ and in [2] for $F_\varepsilon = XY_\varepsilon$, with (ii) replaced by the standard global Γ-liminf inequality

$$\pi |\mu|(\Omega) \leq \liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|}, \tag{3.3}$$

which is clearly implied by (3.2). We underline that the estimate in (3.1) implies the boundedness of the energy far from the limiting singularities and it will play a central role in the first order Γ-convergence analysis in Subsection 4.2.

By (2.12), the compactness property (i) follows directly from the zero-order Γ-convergence result for the XY_ε energies, while the proof of (ii) requires a specific analysis. For the convenience of the reader we will give a self contained proof of both (i) and (ii) of Theorem 3.1. We will omit the proof of the Γ-lim sup inequality (iii) which is standard and identical to the XY_ε case.

Before giving the proof of Theorem 3.1, we need to revisit a construction referred to as ball construction and introduced in the continuous framework in [38], [30].

3.2. **Lower bound on annuli.** Let $w \in H^1(B_R \setminus B_r; \mathcal{S}^1)$ with $\deg(w, \partial B_R) = d$. By Jensen's inequality, the following lower bound holds

$$\begin{aligned} \frac{1}{2} \int_{B_R \setminus B_r} |\nabla w|^2 \, \mathrm{d}x &\geq \frac{1}{2} \int_r^R \int_{\partial B_\rho} |(w \times \nabla w) \cdot \tau|^2 ds \, d\rho \\ &\geq \int_r^R \frac{1}{\rho} \pi |d|^2 \, \mathrm{d}\rho \geq \pi |d| \log \frac{R}{r}. \end{aligned} \tag{3.4}$$

The latter is a key estimate in the context of continuous Ginzburg-Landau. In the following we will prove an analogous lower bound for the energy $XY_\varepsilon(v, \cdot)$ in an annulus in which the piecewise affine interpolation $\tilde{v}$ satisfies $|\tilde{v}| \geq \frac{1}{2}$. In view of (2.12) such a lower bound will hold also for the energy F_ε.

Proposition 3.2. *Fix $\varepsilon > 0$ and let $\sqrt{2}\varepsilon < r < R - 2\sqrt{2}\varepsilon$. For any function $v : (B_R \setminus B_r) \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}^1$ with $|\tilde{v}| \geq \frac{1}{2}$ in $B_{R - \sqrt{2}\varepsilon} \setminus B_{r + \sqrt{2}\varepsilon}$, it holds*

$$XY_\varepsilon(v, B_R \setminus B_r) \geq \pi |\mu(v)(B_r)| \log \frac{R}{r + \varepsilon\big(\alpha |\mu(v)(B_r)| + \sqrt{2}\big)}, \tag{3.5}$$

where $\alpha > 0$ is a universal constant.

Proof. By (2.13), using Fubini's theorem, we have that

$$XY_\varepsilon(v, B_R \setminus B_r) \geq \frac{1}{2}\int_{r+\sqrt{2}\varepsilon}^{R-\sqrt{2}\varepsilon}\int_{\partial B_\rho} |\nabla \tilde{v}|^2 \,\mathrm{d}s\,\mathrm{d}\rho. \tag{3.6}$$

Fix $r + \sqrt{2}\varepsilon < \rho < R - \sqrt{2}\varepsilon$ and let T be a simplex of the triangulation of the ε-lattice. Set $\gamma_T(\rho) := \partial B_\rho \cap T$, let $\bar{\gamma}_T(\rho)$ be the segment joining the two extreme points of $\gamma_T(\rho)$ and let $\bar{\gamma}(\rho) = \bigcup_T \bar{\gamma}_T(\rho)$; then

$$\begin{aligned}\frac{1}{2}\int_{\partial B_\rho} |\nabla \tilde{v}|^2 \,\mathrm{d}s &= \frac{1}{2}\int_{\cup_T \gamma_T(\rho)} |\nabla \tilde{v}|^2 \,\mathrm{d}s = \frac{1}{2}\sum_T |\nabla \tilde{v}_{|_T}|^2 \mathcal{H}^1(\gamma_T(\rho)) \\ &\geq \frac{1}{2}\sum_T |\nabla \tilde{v}_{|_T}|^2 \mathcal{H}^1(\bar{\gamma}_T(\rho)) = \frac{1}{2}\int_{\bar{\gamma}(\rho)} |\nabla \tilde{v}|^2 \,\mathrm{d}s,\end{aligned} \tag{3.7}$$

where we have used that $\nabla \tilde{v}$ is constant in each simplex T. Set $m(\rho) := \min_{\bar{\gamma}(\rho)} |\tilde{v}|$; using Jensen's inequality and the fact that $\mathcal{H}^1(\bar{\gamma}(\rho)) \leq \mathcal{H}^1(\partial B_\rho)$ we get

$$\begin{aligned}\frac{1}{2}\int_{\bar{\gamma}(\rho)} |\nabla \tilde{v}|^2 \,\mathrm{d}s &\geq \frac{1}{2}\int_{\bar{\gamma}(\rho)} m^2(\rho)\left|\left(\frac{\tilde{v}}{|\tilde{v}|} \times \nabla \frac{\tilde{v}}{|\tilde{v}|}\right) \cdot \tau\right|^2 \,\mathrm{d}s \\ &\geq \frac{1}{2}\frac{m^2(\rho)}{\mathcal{H}^1(\bar{\gamma}(\rho))}\left|\int_{\bar{\gamma}(\rho)} \left(\frac{\tilde{v}}{|\tilde{v}|} \times \nabla \frac{\tilde{v}}{|\tilde{v}|}\right) \cdot \tau \,\mathrm{d}s\right|^2 \\ &\geq \frac{m^2(\rho)}{\rho}\pi|d|^2 \geq \frac{m^2(\rho)}{\rho}\pi|d| \end{aligned} \tag{3.8}$$

where we have set $d := \deg(\tilde{v}, \partial B_\rho) = \mu(v)(B_r)$, which does not depend on ρ since $|\tilde{v}| \geq 1/2$.

Now, let $T(\rho)$ be the simplex in which the minimum $m(\rho)$ is attained and let $T_1(\rho)$, $T_2(\rho)$, $T_3(\rho)$ be the simplices sharing a side with $T(\rho)$. By (3.7)

$$\frac{1}{2}\int_{\partial B_\rho} |\nabla \tilde{v}|^2 \,\mathrm{d}x \geq \frac{1}{2}|\nabla \tilde{v}_{|_{T(\rho)}}|^2 \mathcal{H}^1(\bar{\gamma}_{T(\rho)}(\rho)) + \frac{1}{2}\sum_{j=1}^{3} |\nabla \tilde{v}_{|_{T_j(\rho)}}|^2 \mathcal{H}^1(\bar{\gamma}_{T_j(\rho)}(\rho)).$$

If $\bar{\gamma}_{T(\rho)}(\rho)$ does not lie on any of the sides of $T(\rho)$, using the explicit formula of the affine interpolation $\tilde{v}$ on $T(\rho)$, a simple but somehow lengthy computation shows that

$$|\nabla \tilde{v}_{|_{T(\rho)}}|^2 \mathcal{H}^1(\bar{\gamma}_{T(\rho)}(\rho)) \geq \alpha_1 \frac{1 - m^2(\rho)}{\varepsilon} \tag{3.9}$$

for some universal constant α_1. If $\bar{\gamma}_{T(\rho)}(\rho)$ lies on the side shared by $T(\rho)$ and $T_j(\rho)$ for some j, using that $\rho > \sqrt{2}\varepsilon$, a simple geometric argument yields

$$\mathcal{H}^1(\bar{\gamma}_{T(\rho)}(\rho)) + \mathcal{H}^1(\bar{\gamma}_{T_j(\rho)}(\rho)) \geq \alpha_2 \varepsilon, \tag{3.10}$$

where $\alpha_2 > 0$. By combining (3.9) and (3.10), we get

$$\frac{1}{2}\int_{\partial B_\rho} |\nabla \tilde{v}|^2 \,\mathrm{d}s \geq \tilde{\alpha}\frac{1 - m^2(\rho)}{\varepsilon}, \tag{3.11}$$

where $\tilde{\alpha}$ is the smallest among α_1 and α_2.

In view of (3.7), (3.8) and (3.11), for any $r + \sqrt{2}\varepsilon < \rho < R - \sqrt{2}\varepsilon$ we have

$$\frac{1}{2}\int_{\partial B_\rho} |\nabla \tilde{v}|^2 \,\mathrm{d}s \geq \frac{m^2(\rho)}{\rho}\pi|d| \vee \tilde{\alpha}\frac{1 - m^2(\rho)}{\varepsilon} \geq \frac{\pi|d|\tilde{\alpha}}{\varepsilon\pi|d| + \tilde{\alpha}\rho}.$$

By this last estimate and (3.6) we get

$$XY_\varepsilon(v, B_R \setminus B_r) \geq \pi|\mu(v)(B_r)| \log \frac{\varepsilon(\frac{\pi}{\tilde\alpha}|\mu(v)(B_r)| - \sqrt{2}) + R}{\varepsilon(\frac{\pi}{\tilde\alpha}|\mu(v)(B_r)| + \sqrt{2}) + r}. \tag{3.12}$$

Assuming, without loss of generality, $\tilde\alpha < 1$, we immediately get (3.5) for $\alpha = \frac{\pi}{\tilde\alpha}$. □

3.3. **Ball Construction.** Here we introduce a construction referred to as *ball construction*, introduced in [38], [30]. Let $\mathcal{B} = \{B_{R_1}(x_1), \dots, B_{R_N}(x_N)\}$ be a finite family of pairwise disjoint balls in $\mathbb{R}^2$ and let $\mu = \sum_{i=1}^N d_i \delta_{x_i}$ with $d_i \in \mathbb{Z} \setminus \{0\}$. Let F be a positive superadditive set function on the open subsets of $\mathbb{R}^2$, i.e., such that $F(A \cup B) \geq F(A) + F(B)$, whenever A and B are open and disjoint. We will assume that there exists $c > 0$ such that

$$F(A_{r,R}(x)) \geq \pi|\mu(B_r(x))| \log \frac{R}{c+r}, \tag{3.13}$$

for any annulus $A_{r,R}(x) = B_R(x) \setminus \bar B_r(x)$, with $A_{r,R}(x) \subset \Omega \setminus \bigcup_i B_{R_i}(x_i)$.

The purpose of this construction is to select a family of larger and larger annuli in which the main part of the energy F concentrates. Let t be a parameter which represents an artificial time. For any $t > 0$ we want to construct a finite family of balls $\mathcal{B}(t)$ which satisfies the following properties

(1) $\bigcup_{i=1}^N B_{R_i}(x_i) \subset \bigcup_{B \in \mathcal{B}(t)} B$,
(2) the balls in $\mathcal{B}(t)$ are pairwise disjoint,
(3) $F(B) \geq \pi|\mu(B)| \log(1+t)$ for any $B \in \mathcal{B}(t)$ with $B \subseteq \Omega$,
(4) $\sum_{B \in \mathcal{B}(t)} R(B) \leq (1+t)\sum_i R_i + (1+t)cN(N^2+N+1)$, where $R(B)$ denotes the radius of the ball B.

We construct the family $\mathcal{B}(t)$, closely following the strategy of the ball construction due to Sandier and Jerrard, that we need to slightly revise in order to include our case: The only difference in our discrete setting is the appearance of the error term $c > 0$ in (3.13) and in (4), while in the continuous setting $c = 0$.

The ball construction consists in letting the balls alternatively expand and merge each other as follows. It starts with an expansion phase if $\mathrm{dist}(B_{R_i}(x_i), B_{R_j}(x_j)) > 2c$ for all $i \neq j$, and with a merging phase otherwise. Assume that the first phase is an expansion. It consists in letting the balls expand, without changing theirs centers, in such a way that, at each (artificial) time, the ratio $\theta(t) := \frac{R_i(t)}{c+R_i}$ is independent of i. We will parametrize the time enforcing $\theta(t) = 1 + t$. Note that with this choice $R_i(0) = R_i + c$ so that the balls $\{B_{R_i(0)}(x_i)\}$ are pairwise disjoint. The first expansion phase stops at the first time T_1 when two balls bump into each other. Then the merging phase begins. It consists in identifying a suitable partition $\{S^1_j\}_{j=1,\dots,N_n}$ of the family $\{B_{R_i(T_1)}(x_i)\}$, and, for each subclass S^1_j, in finding a ball $B_{R^1_j}(x^1_j)$ which contains all the balls in S^1_j such that the following properties hold:

i) for every $j \neq k$, $\mathrm{dist}(B_{R^1_j}(x^1_j), B_{R^1_k}(x^1_k)) > 2c$;
ii) $R^1_j - Nc$ is not larger than the sum of all the radii of the balls $B_{R_i(T_1)}(x_i) \in S^1_j$, i.e., contained in $B_{R^1_j}(x^1_j)$.

This construction consists in applying the usual merging procedure described in [38] to the balls in the family $\{B_{R_i(T_1)+c}(x_i)\}$. In such a way one obtains a family of pairwise disjoint balls $\{B_{\tilde R^1_j}(x^1_j)\}$ such that

$$\tilde R^1_j \le \sum_{i: B_{R_i(T_1)}(x_i)\subset B_{\tilde R^1_j}(x^1_j)} (R_i(T_1)+c).$$

The family $\{B_{R^1_j}(x^1_j)\}$ is obtained by setting $R^1_j = \tilde R^1_j - c$.

After the merging, another expansion phase begins, during which we let the balls $\left\{B_{R^1_j}(x^1_j)\right\}$ expand in such a way that, for $t \ge T_1$, for every j we have

$$\frac{R^1_j(t)}{c+R^1_j} = \frac{1+t}{1+T_1}\,. \tag{3.14}$$

Again note that $R^1_j(T_1) = R^1_j + c$. We iterate this process obtaining a set of merging times $\{T_1,\dots,T_n\}$, and a family $\mathcal{B}(t) = \{B_{R^k_j(t)}(x^k_j)\}_j$ for $t \in [T_k, T_{k+1})$, for all $k = 1,\dots,n-1$. Notice that $n \le N$. If the condition $\mathrm{dist}(B_{R_i}(x_i), B_{R_j}(x_j)) > 2c$ for all $i \ne j$, is not satisfied we clearly can start this process with a merging phase (in this case $T_1 = 0$).

By construction, we clearly have (1) and (2). We now prove (4). Set $N(t) = \sharp\{B \in \mathcal{B}(t)\}$ and $I(t) = \{1,\dots,N(t)\}$. Moreover, for every merging time T_k and $1 \le j \le N(T_k)$, set

$$I_j(T_k) := \left\{ i \in I(T_{k-1}) :\, B_{R^{k-1}_i}(x^{k-1}_i) \subset B_{R^k_j}(x^k_j) \right\}.$$

By ii) and (3.14) it follows that for any $1 \le k \le n$

$$\begin{aligned}
\sum_{j=1}^{N(T_k)} (R^k_j - Nc) &\le \sum_{j=1}^{N(T_k)} \sum_{l\in I_j(T_k)} R^{k-1}_l(T_k)\\
&= \sum_{j=1}^{N(T_{k-1})} \left(\frac{1+T_k}{1+T_{k-1}} R^{k-1}_j + \frac{1+T_k}{1+T_{k-1}} c\right)\\
&= \frac{1+T_k}{1+T_{k-1}} \sum_{j=1}^{N(T_{k-1})} R^{k-1}_j + \frac{1+T_k}{1+T_{k-1}} cN(T_{k-1})\\
&\le \frac{1+T_k}{1+T_{k-1}} \sum_{j=1}^{N(T_{k-1})} R^{k-1}_j + (1+T_k)cN.
\end{aligned} \tag{3.15}$$

Let $T_k \le t < T_{k+1}$ for some $1 \le k \le n$; by (3.14) and iterating (3.15) we get

$$\begin{aligned}
\sum_{j=1}^{N(T_k)} R^k_j(t) &= \frac{1+t}{1+T_k} \sum_{j=1}^{N(T_k)} R^k_j + \frac{1+t}{1+T_k} c\,N(T_k)\\
&\le (1+t)\sum_{i=1}^{N} R_i + (1+t)cN(N^2+N+1),
\end{aligned} \tag{3.16}$$

and this concludes the proof of (4).

It remains to prove (3). For $t = 0$ it is trivially satisfied. We will show that it is preserved during the merging and the expansion times. Let T_k be a merging time and assume that (3) holds for all $t < T_k$. Then for every $j \in I(T_k)$

$$\begin{aligned} F(B_{R_j^k}(x_j^k)) &\geq \sum_{l \in I_j(T_k)} F(B_{R_l^{k-1}(T_k)}(x_l^{k-1})) \\ &\geq \pi \log(1+T_k) \sum_{l=1}^{j} |\mu(B_{R_l^{k-1}(T_k)}(x_l^{k-1}))| \\ &\geq \pi \log(1+T_k) |\mu(B_{R_j^k}(x_j^k))|. \end{aligned}$$

Finally, for a given $t \in [T_k, T_{k+1})$ and for any ball $B_{R_i^k(t)}(x_i^k(t)) \in \mathcal{B}(t)$ we have

$$\begin{aligned} F(B_{R_i^k(t)}(x_i^k)) &\geq F(B_{R_i^k(t)}(x_i^k) \setminus \bar{B}_{R_i^k}(x_i^k)) + F(B_{R_i^k}(x_i^k)) \\ \geq \pi |\mu(B_{R_i^k(t)}(x_i^k))| \log \frac{1+t}{1+T_k} &+ \pi |\mu(B_{R_i^k(t)}(x_i^k))| \log(1+T_k) \\ &= \pi |\mu(B_{R_i^k(t)}(x_i^k))| \log(1+t), \end{aligned}$$

where we have used that $\frac{R_i^k(t)}{c+R_i^k} = \frac{1+t}{1+T_k}$.

3.4. **Proof of Theorem 3.1.** First, we give an elementary lower bound of the energy localized on a single square of the lattice.

Proposition 3.3. *There exists a positive constant β such that for any $\varepsilon > 0$, for any function $v \in \mathcal{AXY}_\varepsilon(\Omega)$ and for any $i \in \Omega_\varepsilon^2$ such that the piecewise affine interpolation $\tilde{v}$ of v satisfies $\min_{i+\varepsilon Q} |\tilde{v}| < \frac{1}{2}$, it holds $XY_\varepsilon(v, i+\varepsilon Q) \geq \beta$.*

Proof. Using the very definition of the interpolation $\tilde{v}$, the condition $\min_{i+\varepsilon Q} |\tilde{v}| < \frac{1}{2}$ immediately implies that there are a universal constant $\beta > 0$ and two nearest neighbors j, k in $i + \varepsilon Q$ such that $|v(j) - v(k)| \geq \sqrt{2\beta}$. □

Proof of Theorem 3.1. By (2.12), it is enough to prove (i) and (ii) for $F_\varepsilon = XY_\varepsilon$, using as a variable $v_\varepsilon = e^{2\pi i u_\varepsilon}$. The proof of (iii) is standard and left to the reader. *Proof of (i).* For every $\varepsilon > 0$, set $I_\varepsilon := \left\{ i \in \Omega_\varepsilon^2 : \min_{i+\varepsilon Q} |\tilde{v}_\varepsilon| \leq \frac{1}{2} \right\}$. Notice that, by definition (see (2.6)), $\mu(v_\varepsilon)$ is supported in $I_\varepsilon + \frac{\varepsilon}{2}(e_1 + e_2)$.

Starting from the family of balls $B_{\frac{\sqrt{2}\varepsilon}{2}}(i + \frac{\varepsilon}{2}(e_1 + e_2)))$, and eventually passing through a merging procedure (see Subsection 3.3) we can construct a family of pairwise disjoint balls

$$\mathcal{B}_\varepsilon := \left\{ B_{R_{i,\varepsilon}}(x_{i,\varepsilon}) \right\}_{i=1,\dots,N_\varepsilon},$$

with $\sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq \varepsilon \sharp I_\varepsilon$. Then, by Proposition 3.3 and by the energy bound, we immediately have that $\sharp I_\varepsilon \leq C |\log \varepsilon|$ and hence

$$\sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq C\varepsilon |\log \varepsilon|. \tag{3.17}$$

We define the sequence of measures

$$\mu_\varepsilon := \sum_{i=1}^{N_\varepsilon} \mu(v_\varepsilon)(B_{R_{i,\varepsilon}}(x_{i,\varepsilon})) \delta_{x_{i,\varepsilon}}.$$

Since $|\mu_\varepsilon(B)| \leq \sharp I_\varepsilon$ for each ball $B \in \mathcal{B}_\varepsilon$, by (3.5) we deduce that (3.13) holds with $F(\cdot) = XY_\varepsilon(v_\varepsilon, \cdot \setminus \cup_{B \in \mathcal{B}_\varepsilon} B)$ and $c = \varepsilon(\alpha \sharp I_\varepsilon + 2\sqrt{2})$.

We let the balls in the families $\mathcal{B}_\varepsilon$ grow and merge as described in Subsection 3.3, and let $\mathcal{B}_\varepsilon(t) := \{B_{R_{i,\varepsilon}(t)}(x_{i,\varepsilon}(t))\}$ be the corresponding family of balls at time t. Set moreover $t_\varepsilon := \frac{1}{\sqrt{\varepsilon}} - 1$, $N_\varepsilon(t_\varepsilon) := \sharp\mathcal{B}_\varepsilon(t_\varepsilon)$ and define

$$\nu_\varepsilon := \sum_{\substack{i=1,\dots,N_\varepsilon(t_\varepsilon)\\ B_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon))\subset\Omega}} \mu_\varepsilon(B_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon)))\delta_{x_{i,\varepsilon}(t_\varepsilon)}. \tag{3.18}$$

By (3) in Subsection 3.3, for any $B \in \mathcal{B}_\varepsilon(t_\varepsilon)$, with $B \subseteq \Omega$, we have

$$XY_\varepsilon(v_\varepsilon, B) \geq \pi|\mu_\varepsilon(B)|\log(1+t_\varepsilon) = \pi|\nu_\varepsilon(B)|\tfrac{1}{2}|\log\varepsilon|;$$

by the energy bound, we have immediately that $|\nu_\varepsilon|(\Omega) \leq M$ and hence $\{\nu_\varepsilon\}$ is precompact in the weak* topology. By (4) in Subsection 3.3, it follows that

$$\sum_{j=1}^{N_\varepsilon(t_\varepsilon)} R_j(t_\varepsilon) \leq C\sqrt{\varepsilon}\ (\sharp I_\varepsilon)^4,$$

which, using the definition of the flat norm, implies that $\|\nu_\varepsilon - \mu_\varepsilon\|_{\mathrm{flat}} \to 0$ (see [3] for more details); similarly, using (3.17), one can show that $\|\mu_\varepsilon - \mu(v_\varepsilon)\|_{\mathrm{flat}} \to 0$ as $\varepsilon \to 0$. We conclude that also $\mu(v_\varepsilon)$ is precompact in the flat topology.

Proof of (ii). Fix $i \in \{1,\dots,M\}$. Without loss of generality, and possibly extracting a subsequence, we can assume that

$$\liminf_{\varepsilon\to 0} XY_\varepsilon(v_\varepsilon, B_\sigma(x_i)) - \pi|d_i||\log\varepsilon| = \lim_{\varepsilon\to 0} XY_\varepsilon(v_\varepsilon, B_\sigma(x_i)) - \pi|d_i||\log\varepsilon| < +\infty. \tag{3.19}$$

We consider the restriction $\bar v_\varepsilon \in \mathcal{AXY}_\varepsilon(B_\sigma(x_i))$ of v_ε to $B_\sigma(x_i)$. Notice that $\mathrm{supp}(\mu(\bar v_\varepsilon) - \mu(v_\varepsilon)\llcorner B_\sigma(x_i)) \subseteq B_\sigma(x_i)\setminus B_{\sigma-\varepsilon}(x_i)$. On the other hand, by (3.19) and Proposition 3.3 it follows that

$$|\mu(v_\varepsilon)|(B_\sigma(x_i)\setminus B_{\sigma-\varepsilon}(x_i)) \leq C|\log\varepsilon|. \tag{3.20}$$

Then, using (3.20) one can easily get

$$\|\mu(\bar v_\varepsilon) - \mu(v_\varepsilon)\llcorner B_\sigma(x_i)\|_{\mathrm{flat}} \to 0, \tag{3.21}$$

and hence

$$\|\mu(\bar v_\varepsilon) - d_i\delta_{x_i}\|_{\mathrm{flat}} \to 0. \tag{3.22}$$

We repeat the ball construction procedure used in the proof of (i) with Ω replaced by $B_\sigma(x_i)$, v_ε by $\bar v_\varepsilon$ and I_ε by

$$I_{i,\varepsilon} := \left\{ j \in (B_\sigma(x_i))^2_\varepsilon \,:\, \min_{j+\varepsilon Q} |\tilde v_\varepsilon| \leq \frac{1}{2} \right\}.$$

We denote by $\mathcal{B}_{i,\varepsilon}$ the corresponding family of balls and by $\mathcal{B}_{i,\varepsilon}(t)$ the family of balls constructed at time t.

Fix $0 < \gamma < 1$ such that

$$(1-\gamma)(|d_i|+1) > |d_i|\,. \tag{3.23}$$

Let $t_{\varepsilon,\gamma} = \varepsilon^{\gamma-1} - 1$ and let $\nu_{\varepsilon,\gamma}$ be the measure defined as in (3.18) with Ω replaced by $B_\sigma(x_i)$ and t_ε replaced by $t_{\varepsilon,\gamma}$. As in the previous step, since $\gamma > 0$ we deduce that $\|\nu_{\varepsilon,\gamma} - d_i\delta_{x_i}\|_{\mathrm{flat}} \to 0$; moreover, for any $B \in \mathcal{B}_{i,\varepsilon}(t_{\varepsilon,\gamma})$ we have

$$XY_\varepsilon(v_\varepsilon, B) \geq \pi|\nu_{\varepsilon,\gamma}(B)|(1-\gamma)|\log\varepsilon|. \tag{3.24}$$

Now, if $\liminf_{\varepsilon\to 0}|\nu_{\varepsilon,\gamma}|(B_\sigma(x_i)) > |d_i|$, then, thanks to (3.23), (3.1) holds true. Otherwise we can assume that $|\nu_{\varepsilon,\gamma}|(B_\sigma(x_i)) = |d_i|$ for ε small enough. Then $\nu_{\varepsilon,\gamma}$ is a sum of Dirac masses concentrated on points which converge to x_i, with weights all having the same sign and summing to d_i. Let $C_1 > 0$ be given and set $\bar t_\varepsilon := \frac{\sigma}{C_1(\sharp I_{i,\varepsilon})^4\varepsilon} - 1$. By (3.16), we have that any ball $B \in \mathcal{B}_{i,\varepsilon}(\bar t_\varepsilon)$ satisfies

$$\mathrm{diam}(B) \le \frac{C_2}{C_1}\sigma,$$

where $C_2 > 0$ is a universal constant. We fix $C_1 > 2C_2$ so that $\mathrm{diam}(B) < \frac{\sigma}{2}$. Recall that, for ε small enough, $\mathrm{supp}(\nu_{\varepsilon,\gamma}) \subseteq B_{\sigma/2}(x_i)$; hence if $B \in \mathcal{B}_{i,\varepsilon}(\bar t_\varepsilon)$ with $\mathrm{supp}(\nu_{\varepsilon,\gamma}) \cap B \neq \emptyset$, then $B \subseteq B_\sigma(x_i)$ and one can easily show that

$$\mu(\bar v_\varepsilon)\Big(\bigcup_{\substack{B\in\mathcal{B}_{i,\varepsilon}(\bar t_\varepsilon)\\ B\subset B_\sigma(x_i)}} B\Big) = d_i.$$

We have immediately that

$$XY_\varepsilon(\bar v_\varepsilon, B_\sigma(x_i)\setminus \cup_{B\in\mathcal{B}_{i,\varepsilon}} B) \ge \pi \sum_{\substack{B\in\mathcal{B}_{i,\varepsilon}(\bar t_\varepsilon)\\ B\subset B_\sigma(x_i)}} |\mu(\bar v_\varepsilon)(B)| \log(1+\bar t_\varepsilon) \ge \pi|d_i|\log\frac{\sigma}{C_1(\sharp I_{i,\varepsilon})^4\varepsilon}.$$

On the other hand, by Proposition 3.3 there exists a positive constant β such that

$$XY_\varepsilon(\bar v_\varepsilon, j+\varepsilon Q) \ge \beta \quad \text{for any } j \in I_{i,\varepsilon};$$

therefore, $XY_\varepsilon(\bar v_\varepsilon, \bigcup_{B\in\mathcal{B}_{i,\varepsilon}} B) \ge \beta \sharp I_{i,\varepsilon}$. Finally, we get

$$\begin{aligned} XY_\varepsilon(\bar v_\varepsilon, B_\sigma(x_i)) &\ge XY_\varepsilon(\bar v_\varepsilon, B_\sigma(x_i)\setminus \cup_{B\in\mathcal{B}_{i,\varepsilon}} B) + XY_\varepsilon(\bar v_\varepsilon, \cup_{B\in\mathcal{B}_{i,\varepsilon}} B)\\ &\ge \pi|d_i|\log\frac{\sigma}{\varepsilon} - \log\big(C_1(\sharp I_{i,\varepsilon})^4\big) + \sharp I_{i,\varepsilon}\beta \ge \pi|d_i|\log\frac{\sigma}{\varepsilon} + C \end{aligned}$$

and (3.1) follows sending $\varepsilon \to 0$. □

4. The renormalized energy and the first order Γ-convergence.

In this section we will prove the first order Γ-convergence of F_ε to the renormalized energy, introduced in the continuous framework of Ginzburg-Landau energies in [10]. To this purpose we begin by recalling the many definitions and results of [10] we need.

4.1. Revisiting the analysis of Bethuel-Brezis-Hélein.

Fix $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $d_i \in \{-1,+1\}$ and $x_i \in \Omega$. In order to define the renormalized energy, consider the following problem

$$\begin{cases} \Delta\Phi = 2\pi\mu & \text{in } \Omega\\ \Phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $R_0(x) = \Phi(x) - \sum_{i=1}^M d_i \log|x-x_i|$. Notice that R_0 is harmonic in Ω and $R_0(x) = -\sum_{i=1}^M d_i\log|x-x_i|$ for any $x\in\partial\Omega$. The *renormalized energy* corresponding to the configuration μ is then defined by

$$\mathbb{W}(\mu) := -\pi\sum_{i\neq j} d_i d_j \log|x_i-x_j| - \pi\sum_i d_i R_0(x_i). \tag{4.1}$$

Let $\sigma > 0$ be such that the balls $B_\sigma(x_i)$ are pairwise disjoint and contained in Ω and set $\Omega^\sigma := \Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i)$. A straightforward computation shows that

$$\mathbb{W}(\mu) = \lim_{\sigma\to 0} \frac{1}{2}\int_{\Omega^\sigma} |\nabla\Phi|^2 \,\mathrm{d}x - M\pi|\log\sigma| \,. \tag{4.2}$$

In this respect the renormalized energy represents the finite energy induced by μ once the leading logarithmic term has been removed.

It is convenient to consider (as done in [10]) suitable cell problems and auxiliary minimum problems. Set

$$m(\sigma,\mu) := \min_{w\in H^1(\Omega^\sigma;\mathcal{S}^1)} \left\{ \frac{1}{2}\int_{\Omega^\sigma} |\nabla w|^2 \,\mathrm{d}x \,:\, \deg(w,\partial B_\sigma(x_i)) = d_i \right\},$$

$$\begin{aligned} \tilde{m}(\sigma,\mu) := \min_{w\in H^1(\Omega^\sigma;\mathcal{S}^1)} & \left\{ \frac{1}{2}\int_{\Omega^\sigma} |\nabla w|^2 \,\mathrm{d}x \,: \right. \\ & \left. w(\cdot) = \frac{\alpha_i}{\sigma^{d_i}}(\cdot - x_i)^{d_i} \text{on } \partial B_\sigma(x_i),\, |\alpha_i| = 1 \right\}. \end{aligned} \tag{4.3}$$

For any $x \in \mathbb{R}^2 \setminus \{0\}$, we define $\theta(x)$ as the polar coordinate $\arctan x_2/x_1$, also referred to as the lifting of the function $\frac{x}{|x|}$. Given $\varepsilon > 0$ we introduce a discrete minimization problem in the ball B_σ

$$\gamma(\varepsilon,\sigma) := \min_{u\in\mathcal{AF}_\varepsilon(B_\sigma)} \left\{ F_\varepsilon(u,B_\sigma) \,:\, 2\pi u(\cdot) = \theta(\cdot) \text{ on } \partial_\varepsilon B_\sigma \right\}, \tag{4.4}$$

where the discrete boundary ∂_ε is defined in (2.1).

Theorem 4.1. *It holds*

$$\lim_{\sigma\to 0} m(\sigma,\mu) - \pi|\mu|(\Omega)|\log\sigma| = \lim_{\sigma\to 0} \tilde{m}(\sigma,\mu) - \pi|\mu|(\Omega)|\log\sigma| = \mathbb{W}(\mu). \tag{4.5}$$

Moreover, for any fixed $\sigma > 0$, the following limit exits finite

$$\lim_{\varepsilon\to 0}(\gamma(\varepsilon,\sigma) - \pi|\log\frac{\varepsilon}{\sigma}|) =: \gamma \in \mathbb{R}. \tag{4.6}$$

The proof of (4.5) is contained in [10], whereas the statement in (4.6) is a discrete version of Lemma III.1 in [10] and can be proved similarly. We give the details of the proof of (4.6) for the convenience of the reader.

Proof of (4.6). First, by scaling, it is easy to see that $\gamma(\varepsilon,\sigma) = I(\frac{\varepsilon}{\sigma})$ where $I(t)$ is defined by

$$I(t) := \min\left\{ F_1(\theta, B_{\frac{1}{t}}) \,|\, 2\pi u = \theta \text{ on } \partial_1 B_{\frac{1}{t}} \right\}.$$

We aim to prove that

$$0 < t_1 \le t_2 \Rightarrow I(t_1) \le \pi\log\frac{t_2}{t_1} + I(t_2) + O(t_2). \tag{4.7}$$

Notice that by (4.7) it easily follows that $\lim_{t\to 0^+}(I(t) - \pi|\log t|)$ exists and is not $+\infty$. Moreover, by Theorem 3.1, there exists a universal constant C such that

$$I(t) \ge \pi|\log t| + C \qquad \forall t \in (0,1].$$

We conclude that $\lim_{t\to 0^+}(I(t) - \pi|\log t|)$ is not $-\infty$.

In order to complete the proof we have to show that (4.7) holds. To this end, let θ be the lifting of the function $\frac{x}{|x|}$. Since $|\nabla\theta(x)| \le c/r$ for every $x \in A_{r,R} = B_R \setminus B_r$, by standard interpolation estimates (see for instance [20]) and using assumption (3) on f, we have that, as $r < R \to \infty$,

$$F_1(\theta/2\pi, A_{r,R}) \leq \pi \log \frac{R}{r} + O(1/r). \tag{4.8}$$

Let u_2 be a minimizer for $I(t_2)$ and for any $i \in \mathbb{Z}^2$ define

$$u_1(i) := \begin{cases} u_2(i) & \text{if } |i| \leq \frac{1}{t_2} \\ \frac{\theta(i)}{2\pi} & \text{if } \frac{1}{t_2} \leq |i| \leq \frac{1}{t_1}, \end{cases}$$

By (4.8) we have

$$\begin{aligned} I(1/R) \leq & \sum_{\substack{(i,j)\in(B_r)_1^1 \\ i,j\in(B_r)_1}} f(u_1(i) - u_1(j)) + \sum_{\substack{(i,j)\in(A_{r-\sqrt{2},R})_1^1 \\ i,j\in(A_{r-\sqrt{2},R})_1}} f(u_1(i) - u_1(j)) \\ & \leq I(1/r) + \pi \log \frac{R}{r} + O(1/r), \end{aligned}$$

which yields (4.7) for $r = \frac{1}{t_2}$ and $R = \frac{1}{t_1}$.

□

4.2. **The main Γ-convergence result.** We are now in a position to state the first-order Γ-convergence theorem for the functionals F_ε.

Theorem 4.2. *The following* Γ*-convergence result holds.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be a sequence satisfying $F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \leq C$. Then, up to a subsequence, $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \leq M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for any i.*

(ii) *(Γ-lim inf inequality) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$, with $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i. Then,*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \geq \mathbb{W}(\mu) + M\gamma. \tag{4.9}$$

(iii) *(Γ-lim sup inequality) Given $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i, there exists $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ with $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ such that*

$$F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \to \mathbb{W}(\mu) + M\gamma.$$

In our analysis it will be convenient to introduce the energy functionals F_ε in term of the variable μ, i.e., by minimizing F_ε with respect to all $u \in \mathcal{AF}_\varepsilon(\Omega)$ with $\mu(u) = \mu$. Precisely, let $\mathcal{F}_\varepsilon : X \to [0, +\infty]$ be defined by

$$\mathcal{F}_\varepsilon(\mu) := \inf\{F_\varepsilon(u) \ : \ u \in \mathcal{AF}_\varepsilon(\Omega), \mu(u) = \mu\}. \tag{4.10}$$

Theorem 4.2 can be rewritten in terms of $\mathcal{F}_\varepsilon$ as follows.

Theorem 4.3. *The following* Γ*-convergence result holds.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{\mu_\varepsilon\} \subset X$ be a sequence satisfying $\mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log\varepsilon| \leq C$. Then, up to a subsequence, $\mu_\varepsilon \overset{\text{flat}}{\to} \mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \leq M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for every i.*

(ii) *(Γ-$\liminf$ inequality) Let $\{\mu_\varepsilon\} \subset X$ be such that $\mu_\varepsilon \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i. Then,*

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log\varepsilon| \geq \mathbb{W}(\mu) + M\gamma. \tag{4.11}$$

(iii) *(Γ-$\limsup$ inequality) Given $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i, there exists $\{\mu_\varepsilon\} \subset X$ with $\mu_\varepsilon \stackrel{\text{flat}}{\to} \mu$ such that*

$$\mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log\varepsilon| \to \mathbb{W}(\mu) + M\gamma. \tag{4.12}$$

4.3. **The proof of Theorem 4.2.** Recalling that $F_\varepsilon(u) \geq XY_\varepsilon(e^{2\pi i u})$, the proof of the compactness property (i) will be done for $F_\varepsilon = XY_\varepsilon$, and will be deduced by Theorem 3.1. On the other hand, the constant γ in the definition of the Γ-limit depends on the details of the discrete energy F_ε, and its derivation requires a specific proof.

Proof of (i): Compactness. The fact that, up to a subsequence, $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $\sum_{i=1}^N |d_i| \leq M$ is a direct consequence of the zero order Γ-convergence result stated in Theorem 3.1 (i). Assume now $\sum_{i=1}^N |d_i| = M$ and let us prove that $|d_i| = 1$. Let $0 < \sigma_1 < \sigma_2$ be such that $B_{\sigma_2}(x_i)$ are pairwise disjoint and contained in Ω and let ε be small enough so that $B_{\sigma_2}(x_i)$ are contained in Ω_ε. For any $0 < r < R$ and $x \in \mathbb{R}^2$, set $A_{r,R}(x) := B_R(x) \setminus B_r(x)$. Since $F_\varepsilon(u_\varepsilon) \geq XY_\varepsilon(e^{2\pi i u_\varepsilon})$,

$$F_\varepsilon(u_\varepsilon) \geq \sum_{i=1}^N XY_\varepsilon(e^{2\pi i u_\varepsilon}, B_{\sigma_1}(x_i)) + \sum_{i=1}^N XY_\varepsilon(e^{2\pi i u_\varepsilon}, A_{\sigma_1,\sigma_2}(x_i)). \tag{4.13}$$

To ease notation we set $v_\varepsilon = e^{2\pi i u_\varepsilon}$ and we indicate with $\tilde v_\varepsilon$ the piecewise affine interpolation of v_ε. Moreover let t be a positive number and let ε be small enough so that $t > \sqrt{2}\varepsilon$. Then, by (3.1) and (2.13), we get

$$F_\varepsilon(u_\varepsilon) \geq \pi\sum_{i=1}^N |d_i|\log\frac{\sigma_1}{\varepsilon} + \frac{1}{2}\sum_{i=1}^N \int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} |\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x + C. \tag{4.14}$$

By the energy bound, we deduce that $\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} |\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x \leq C$ and hence, up to a subsequence, $\tilde v_\varepsilon \rightharpoonup v_i$ in $H^1(A_{\sigma_1+t,\sigma_2-t}(x_i);\mathbb{R}^2)$ for some field v_i. Moreover, since

$$\frac{1}{\varepsilon^2}\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} (1-|\tilde v_\varepsilon|^2)^2\,\mathrm{d}x \leq CXY_\varepsilon(v_\varepsilon) \leq C\log\frac{1}{\varepsilon},$$

(see Lemma 2 in [1] for more details), we deduce that $|v_i| = 1$ a.e.

Furthermore, by standard Fubini's arguments, for a.e. $\sigma_1 + t < \sigma < \sigma_2 - t$, up to a subsequence the trace of $\tilde v_\varepsilon$ is bounded in $H^1(\partial B_\sigma(x_i);\mathbb{R}^2)$, and hence it converges uniformly to the trace of v_i. By the very definition of degree it follows that $\deg(v_i, \partial B_\sigma(x_i)) = d_i$.

Hence, by (3.4), for every i we have

$$\frac{1}{2}\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} |\nabla v_i|^2\,\mathrm{d}x \geq |d_i|^2\pi\log\frac{\sigma_2-t}{\sigma_1+t}. \tag{4.15}$$

By (4.14) and (4.15), we conclude that for ε sufficiently small

$$F_\varepsilon(u_\varepsilon) \geq \pi \sum_{i=1}^{N} \left(|d_i| \log \frac{\sigma_1}{\varepsilon} + |d_i|^2 \log \frac{\sigma_2 - t}{\sigma_1 + t} \right) + C$$

$$= M\pi |\log \varepsilon| + \pi \sum_{i=1}^{N} (|d_i|^2 - |d_i|) \log \frac{1}{\sigma_1} + \pi \sum_{i=1}^{N} |d_i|^2 \log \frac{\sigma_1(\sigma_2 - t)}{\sigma_1 + t} + C.$$

The energy bound yields that the sum of the last two terms is bounded; letting $t \to 0$ and $\sigma_1 \to 0$, we conclude $|d_i| = 1$.

Proof of (ii): Γ-liminf inequality. Fix $r > 0$ so that the balls $B_r(x_i)$ are pairwise disjoint and compactly contained in Ω. Let moreover $\{\Omega^h\}$ be an increasing sequence of open smooth sets compactly contained in Ω such that $\cup_{h\in\mathbb{N}}\Omega^h = \Omega$. Without loss of generality we can assume that $F_\varepsilon(u_\varepsilon) \leq M\pi|\log\varepsilon| + C$, which together with Theorem 3.1 yields

$$F_\varepsilon(u_\varepsilon, \Omega \setminus \bigcup_{i=1}^{M} B_r(x_i)) \leq C. \tag{4.16}$$

We set $v_\varepsilon := e^{2\pi i u_\varepsilon}$ and we denote by $\tilde{v}_\varepsilon$ the piecewise affine interpolation of v_ε. For every $r > 0$, by (4.16) and by (2.12) we deduce $XY_\varepsilon(v_\varepsilon \setminus \bigcup_{i=1}^{N} B_r(x_i)) \leq C$. Fix $h \in \mathbb{N}$ and let ε be small enough so that $\Omega^h \subset \Omega_\varepsilon$. Then,

$$\frac{1}{2} \int_{\Omega^h \setminus \bigcup_{i=1}^{N} B_{2r}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 \, \mathrm{d}x \leq C;$$

therefore, by a diagonalization argument, there exists a unitary field v with $v \in H^1(\Omega \setminus \cup_{i=1}^{M} B_\rho(x_i); \mathcal{S}^1)$ for any $\rho > 0$ and a subsequence $\{\tilde{v}_\varepsilon\}$ such that $\tilde{v}_\varepsilon \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega \setminus \cup_{i=1}^{M}\{x_i\}; \mathbb{R}^2)$.

Let $\sigma > 0$ be such that $B_\sigma(x_i)$ are pairwise disjoint and contained in Ω^h. Recalling the definition of $A_{r,R}$ in the proof of (i), we set $A_{r,R} := A_{r,R}(0)$. Let $t \leq \sigma$, and consider the minimization problem

$$\min_{w \in H^1(A_{t/2,t}; \mathcal{S}^1)} \left\{ \frac{1}{2} \int_{A_{t/2,t}} |\nabla w|^2 \, \mathrm{d}x : \deg(w, \partial B_{\frac{t}{2}}) = 1 \right\}.$$

It is easy to see that the minimum is $\pi \log 2$ and that the set of minimizers is given by (the restriction at $A_{t/2,t}$ of) the rotations of $\frac{x}{|x|}$. Let $\mathcal{K}$ be the set of such functions. To ease the notations in the rest of the proof, it is convenient to introduce a complex notation for $\mathcal{K}$: Identifying $\mathbb{R}^2$ with $\mathbb{C}$ and setting $g(z) := \frac{z}{|z|}$ (with $z \in \mathbb{C}$), we have that

$$\mathcal{K} = \{\alpha \, g(z) \, : \, \alpha \in \mathbb{C}, |\alpha| = 1\}. \tag{4.17}$$

Set

$$\mathrm{d}_t(w, \mathcal{K}) := \min \left\{ \|\nabla w - \nabla v\|_{L^2(A_{t/2,t}; \mathbb{R}^2)} \, : \, v \in \mathcal{K} \right\}. \tag{4.18}$$

It is easy to see that for any given $\delta > 0$ there exists a positive $\omega(\delta)$ (independent of t) such that if $\mathrm{d}_t(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K}) \geq \delta$, then

$$\liminf_{\varepsilon \to 0} \frac{1}{2} \int_{A_{\frac{t}{2}+\sqrt{2}\varepsilon, t-\sqrt{2}\varepsilon}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 \, \mathrm{d}x \geq \pi \log 2 + \omega(\delta). \tag{4.19}$$

By a scaling argument we can assume $t = 1$. Then, arguing by contradiction, if there exists a subsequence $\{\tilde v_\varepsilon\}$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_{A_{\frac12+\sqrt2\varepsilon,1-\sqrt2\varepsilon}(x_i)}|\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x=\pi\log 2,$$

then, by the lower semicontinuity of the L^2 norm, we get

$$\pi\log 2\le\frac12\int_{A_{1/2,1}(x_i)}|\nabla v|^2\,\mathrm{d}x\le\lim_{\varepsilon\to 0}\frac12\int_{A_{\frac12+\sqrt2\varepsilon,1-\sqrt2\varepsilon}(x_i)}|\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x=\pi\log 2. \tag{4.20}$$

It follows that $v(\cdot+x_i)\in\mathcal{K}$, and that $\tilde v_\varepsilon\to v$ strongly in $H^1(A_{1/2,1}(x_i);\mathbb{R}^2)$, which yields the contradiction $\mathrm{dist}(v(\cdot+x_i),\mathcal{K})\ge\delta$.

Let $L\in\mathbb{N}$ be such that $L\,\omega(\delta)\ge\mathbb{W}(\mu)+M(\gamma-\pi\log\sigma-C)$ where C is the constant in (3.1). For $l=1,\dots,L$, set $C_l(x_i):=B_{2^{1-l}\sigma}(x_i)\setminus B_{2^{-l}\sigma}(x_i)$.

We distinguish among two cases.

First case: for ε small enough and for every fixed $1\le l\le L$, there exists at least one i such that $\mathrm{d}_{2^{1-l}\sigma}(\tilde v_\varepsilon(\cdot+x_i),\mathcal{K})\ge\delta$. Then, by (3.1), (4.19) and the lower semicontinuity of the L^2 norm, we conclude

$$\begin{aligned}F_\varepsilon(u_\varepsilon,\Omega^h)&\ge\sum_{i=1}^M XY_\varepsilon(v_\varepsilon,B_{2^{-L}\sigma}(x_i))+\sum_{l=1}^L\sum_{i=1}^M XY_\varepsilon(v_\varepsilon,C_l(x_i))\\&\ge M(\pi\log\frac{\sigma}{2^L}+\pi|\log\varepsilon|+C)+L(M\pi\log 2+\omega(\delta))+\mathrm{o}(\varepsilon)\\&\ge M\pi|\log\varepsilon|+M\gamma+\mathbb{W}(\mu)+\mathrm{o}(\varepsilon).\end{aligned}$$

Second case: Up to a subsequence, there exists $1\le\bar l\le L$ such that for every i we have $\mathrm{d}_{\bar\sigma}(\tilde v_\varepsilon(\cdot+x_i),\mathcal{K})\le\delta$, where $\bar\sigma:=2^{1-\bar l}\sigma$. Let $\alpha_{\varepsilon,i}$ be the unitary vector such that $\|\tilde v_\varepsilon-\alpha_{\varepsilon,i}\frac{x-x_i}{|x-x_i|}\|_{H^1(C_{\bar l}(x_i);\mathbb{R}^2)}=\mathrm{d}_{\bar\sigma}(\tilde v_\varepsilon(\cdot+x_i),\mathcal{K})$.

One can construct a function $\bar u_\varepsilon\in\mathcal{AF}_\varepsilon(\Omega)$ such that

(i) $\bar u_\varepsilon=u_\varepsilon$ on $\partial_\varepsilon(\mathbb{R}^2\setminus B_{2^{-\bar l}\sigma}(x_i))$;
(ii) $e^{2\pi i\bar u_\varepsilon}=\alpha_\varepsilon e^{i\theta}$ on $\partial_\varepsilon B_{2^{1-\bar l}\sigma}(x_i)$
(iii) $F_\varepsilon(u_\varepsilon,B_{\bar\sigma}(x_i))\ge F_\varepsilon(\bar u_\varepsilon,B_{\bar\sigma}(x_i))+r(\varepsilon,\delta)$ with $\lim_{\delta\to0}\lim_{\varepsilon\to0}r(\varepsilon,\delta)=0$.

The proof of (i)-(iii) is quite technical, and consists in adapting standard cut-off arguments to our discrete setting. For the reader convenience we skip the details of the proof, and assuming (i)-(iii) we conclude the proof of the lower bound.

By Theorem (4.1), we have that

$$\begin{aligned}F_\varepsilon(u_\varepsilon)&\ge XY_\varepsilon(v_\varepsilon,\Omega^h\setminus\bigcup_{i=1}^M B_{\bar\sigma}(x_i))+\sum_{i=1}^M F_\varepsilon(u_\varepsilon,B_{\bar\sigma}(x_i))\\&\ge\frac12\int_{\Omega^h\setminus\bigcup_{i=1}^M B_{\bar\sigma}(x_i)}|\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x+\sum_{i=1}^M F_\varepsilon(\bar u_\varepsilon,B_{\bar\sigma}(x_i))+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)\\&\ge\frac12\int_{\Omega^h\setminus\bigcup_{i=1}^M B_{\bar\sigma}(x_i)}|\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x+M(\gamma-\pi\log\frac{\varepsilon}{\bar\sigma})+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)\\&\ge\frac12\int_{\Omega\setminus\bigcup_{i=1}^M B_{\bar\sigma}(x_i)}|\nabla v|^2\,\mathrm{d}x+M(\gamma-\pi\log\frac{\varepsilon}{\bar\sigma})+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)+\mathrm{o}(1/h)\\&\ge M\pi|\log\varepsilon|+M\gamma+\mathbb{W}(\mu)+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)+\mathrm{o}(\bar\sigma)+\mathrm{o}(1/h).\end{aligned}$$

The proof follows sending $\varepsilon \to 0$, $\delta \to 0$, $\sigma \to 0$ and $h \to \infty$.

Proof of (iii): Γ-limsup inequality. This proof is standard in the continuous case, and we only sketch its discrete counterpart. Let w_σ be a function that agrees with a minimizer of (4.3) in $\Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i) =: \Omega^\sigma$. Then, $w_\sigma = \alpha_i \frac{x-x_i}{\sigma}$ on $\partial B_\sigma(x_i)$ for some $|\alpha_i| = 1$.

For every $\rho > 0$ we can always find a function $w_{\sigma,\rho} \in C^\infty(\overline{\Omega^\sigma}; \mathcal{S}^1)$ such that $w_{\sigma,\rho} = \alpha_i \frac{x-x_i}{\sigma}$ on $\partial B_\sigma(x_i)$, and

$$\frac{1}{2}\int_{\Omega^\sigma} |\nabla w_{\sigma,\rho}|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega^\sigma} |\nabla w_\sigma|^2 \,\mathrm{d}x \le \rho.$$

Moreover, for every i let $w_i \in \mathcal{AXY}_\varepsilon(B_\sigma(x_i))$ be a function which agrees with $\alpha_i \frac{x-x_i}{|x-x_i|}$ on $\partial_\varepsilon B_\sigma(x_i)$ and such that its phase minimizes problem (4.4). If necessary, we extend w_i to $(\overline{B_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2) \setminus (B_\sigma(x_i))^0_\varepsilon$ to be equal to $\alpha_i \frac{x-x_i}{|x-x_i|}$. Finally, define the function $w_{\varepsilon,\sigma,\rho} \in \mathcal{AXY}_\varepsilon(\Omega)$ which coincides $w_{\sigma,\rho}$ on $\Omega^\sigma \cap \varepsilon\mathbb{Z}^2$ and with w_i on $\overline{B_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2$. Then, in view of assumption (3) on f, a straightforward computation shows that any phase $u_{\varepsilon,\sigma,\rho}$ of $w_{\varepsilon,\sigma,\rho}$ is a recovery sequence, i.e.,

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_{\varepsilon,\sigma,\rho}) - M\pi|\log\varepsilon| = M\gamma + \mathbb{W}(\mu) + o(\rho,\sigma),$$

with $\lim_{\sigma\to 0}\lim_{\rho\to 0} o(\rho,\sigma) = 0$.

4.4. **Γ-convergence analysis in the L^2 topology.** Here we prove a Γ-convergence result for $F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon|$, where M is fixed positive integer, with respect to the flat convergence of $\mu(u_\varepsilon)$ and the L^2-convergence of $\tilde v_\varepsilon$, where $\tilde v_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^2$ is the piecewise affine interpolation of $e^{2\pi i u_\varepsilon}$.

To this purpose, for $N \in \mathbb{N}$ let us first introduce the set

$$\mathcal{D}_N := \{v \in L^2(\Omega;\mathcal{S}^1) : Jv = \pi\sum_{i=1}^N d_i\delta_{x_i} \text{ with } |d_i| = 1,\, x_i \in \Omega,\; v \in H^1_{\mathrm{loc}}(\Omega \setminus \mathrm{supp}(Jv);\mathcal{S}^1)\}. \tag{4.21}$$

Notice that, if $v \in \mathcal{D}_M$, then the function

$$\frac{1}{2}\int_{\Omega\setminus\bigcup_{i=1}^M B_\sigma(x_i)} |\nabla v|^2 \,\mathrm{d}x - M\pi|\log\sigma|,$$

is monotonically decreasing with respect to σ. Therefore, it is well defined the functional $\mathcal{W} : L^2(\Omega;\mathcal{S}^1) \to \bar{\mathbb{R}}$ given by

$$\mathcal{W}(v) = \begin{cases} \lim_{\sigma\to 0}\frac{1}{2}\int_{\Omega\setminus\bigcup_{i=1}^M B_\sigma(x_i)} |\nabla v|^2 \,\mathrm{d}x - M\pi|\log\sigma| & \text{if } v \in \mathcal{D}_M; \\ -\infty & \text{if } v \in \mathcal{D}_N \text{ for some } N < M; \\ +\infty & \text{otherwise} \end{cases} \tag{4.22}$$

Notice that, by (4.5) we have that, for every $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$

$$\mathbb{W}(\mu) = \min_{\substack{v \in H^1_{\mathrm{loc}}(\Omega\setminus\mathrm{supp}(\mu);\mathcal{S}^1) \\ Jv = \mu}} \mathcal{W}(v). \tag{4.23}$$

Remark 4.4. We can rewrite $\mathcal{W}(v)$ as follows

$$\mathcal{W}(v) = \frac{1}{2}\int_{\Omega\setminus\cup_i B_\rho(x_i)} |\nabla v|^2 \,\mathrm{d}x + M\pi\log\rho + \sum_{i=1}^{M}\sum_{j=0}^{+\infty}\left(\frac{1}{2}\int_{C_{i,j}} |\nabla v|^2\,\mathrm{d}x - \pi\log 2\right),$$

where $C_{i,j}$ denotes the annulus $B_{2^{-j}\rho}(x_i)\setminus B_{2^{-(j+1)}\rho}(x_i)$. In particular, for the lower bound (3.4) we deduce that

$$\sup_{i,j}\frac{1}{2}\int_{C_{i,j}}|\nabla v|^2\,\mathrm{d}x \le \pi\log 2 + \mathcal{W}(v) - M\pi\log\rho. \tag{4.24}$$

Theorem 4.5. *Let $M\in\mathbb{N}$ be fixed. The following Γ-convergence result holds.*

(i) (Compactness) Let $\{u_\varepsilon\}\subset\mathcal{AF}_\varepsilon(\Omega)$ be such that $F_\varepsilon(u_\varepsilon)\le M\pi|\log\varepsilon| + C$. Then, up to a subsequence, $\mu(u_\varepsilon)\stackrel{\text{flat}}{\to}\mu=\sum_{i=1}^N d_i\delta_{x_i}$ with $d_i\in\mathbb{Z}\setminus\{0\}$, $x_i\in\Omega$ and $\sum_{i=1}^N|d_i|\le M$. Moreover, if $\sum_{i=1}^N|d_i| = M$, then $|d_i|=1$ and up to a further subsequence $\tilde v_\varepsilon\rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega\setminus\mathrm{supp}(\mu);\mathbb{R}^2)$ for some $v\in\mathcal{D}_M$.

(ii) (Γ-liminf inequality) Let $v\in\mathcal{D}_M$ and let $\{u_\varepsilon\}\subset\mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon)\stackrel{\text{flat}}{\to} Jv$ and $\tilde v_\varepsilon\to v$ in $L^2(\Omega;\mathbb{R}^2)$. Then,

$$\liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \ge \mathcal{W}(v) + M\gamma. \tag{4.25}$$

(iii) (Γ-limsup inequality) Given $v\in\mathcal{D}_M$, there exists $\{u_\varepsilon\}\subset\mathcal{AF}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon)\stackrel{\text{flat}}{\to} Jv$, $\tilde v_\varepsilon\rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega\setminus\mathrm{supp}(Jv);\mathbb{R}^2)$ and

$$\lim_{\varepsilon\to0} F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| = \mathcal{W}(v) + M\gamma. \tag{4.26}$$

Proof. Proof of (i). The compactness properties concerning the sequence $\{\mu(u_\varepsilon)\}$ are given in Theorem 4.2 (i) while the weak convergence up to a subsequence of $\{\tilde v_\varepsilon\}$ to a unitary field v such that $v\in\mathcal{D}_M$ has been shown in the first lines of the proof of Theorem 4.2 (ii).
Proof of (ii). The proof of Γ-liminf inequality follows strictly the one of Theorem 4.2 (ii) and we leave it to the reader.
Proof of (iii). Let $Jv = \pi\sum_{i=1}^M d_i\delta_{x_i}$, with $x_i\in\Omega$, $|d_i|=1$. Fix $\sigma>0$ and $\Omega^\sigma := \Omega\setminus\cup_{i=1}^M B_\sigma(x_i)$. Without loss of generality we can assume that $\mathcal{W}(v)<+\infty$ and hence for some fixed constant $C>0$ and for every σ

$$\frac{1}{2}\int_{\Omega^\sigma}|\nabla v|^2\,\mathrm{d}x \le M\pi|\log\sigma| + C.$$

Now, fix $\sigma>0$, and let $C_{i,j}$ denote the annulus $B_{2^{-j}\sigma}(x_i)\setminus B_{2^{-(j+1)}\sigma}(x_i)$. By Remark 4.4, it follows that for every $i=1,\dots,M$

$$\lim_{j\to\infty}\frac{1}{2}\int_{C_{i,j}}|\nabla v|^2\,\mathrm{d}x = \pi\log 2. \tag{4.27}$$

z Recall that $\pi\log 2$ is the minimal possible energy in each annulus, and that the class of minimizers is given by the set $\mathcal{K}$ defined in (4.17). Using standard scaling arguments and (4.27), one can show (see (4.20)) that for any $j\in\mathbb{N}$ there exists a unitary vector $\alpha_{i,j}$ such that

$$\frac{1}{2}\int_{C_{i,j}}\left|\nabla\left(v-\alpha_{i,j}\frac{x-x_i}{|x-x_i|}\right)\right|^2\,\mathrm{d}x = r(i,j), \tag{4.28}$$

with $\lim_{j\to\infty} r(i,j) = 0$. Moreover, we can find a function $w_j \in C^\infty(\overline{\Omega^{2^{-j}\sigma}};\mathcal{S}^1)$ such that

$$\frac{1}{2}\int_{\Omega^{2^{-j}\sigma}} |\nabla w_j - \nabla v|^2 \,\mathrm{d}x \le \frac{1}{j}. \tag{4.29}$$

Let $\varphi \in C^1([\frac{1}{2},1];[0,1])$ be such that $\varphi(\frac{1}{2}) = 1$ and $\varphi(1) = 0$, and let define the function $v_{i,j}$ in $C_{i,j}$, with

$$v_{i,j}(x) := \varphi(2^j\sigma^{-1}|x-x_i|)\alpha_{i,j}\frac{x-x_i}{|x-x_i|} + (1-\varphi(2^j\sigma^{-1}|x-x_i|))w_j(x).$$

Then define the function v_j as follows

$$v_j = \begin{cases} w_j & \text{in } \Omega^{2^{-j}\sigma} \\ v_{i,j} & \text{in } C_{i,j}\,. \end{cases} \tag{4.30}$$

Finally for every i we denote by $\bar{v}^\varepsilon_{i,j} \in \mathcal{AXY}_\varepsilon(B_{2^{-j-1}\sigma}(x_i))$ a function which agrees with $\alpha_{i,j}\frac{x-x_i}{|x-x_i|}$ on $\partial_\varepsilon B_{2^{-j-1}\sigma}(x_i)$ and such that its phase (up to an additive constant) minimizes problem (4.4). If necessary, we extend $\bar{v}_{i,j}$ to $(\overline{B_{2^{-j-1}\sigma}(x_i)} \cap \varepsilon\mathbb{Z}^2) \setminus (B_{2^{-j-1}\sigma}(x_i))^0_\varepsilon$ to be equal to $\alpha_{i,j}\frac{x-x_i}{|x-x_i|}$. Finally, consider the field the $v_{\varepsilon,j}$ which coincides with v_j on the nodes of $\Omega^{2^{-j-1}\sigma}$ and with $\bar{v}^\varepsilon_{i,j}$ on $\overline{B_{2^{-j}\sigma}(x_i)} \cap \varepsilon\mathbb{Z}^2$. In view of assumption (3) on f, a straightforward computation shows that any phase $u_{\varepsilon,j}$ of $v_{\varepsilon,j}$ satisfies

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_{\varepsilon,j}) - M\pi|\log\varepsilon| = M\gamma + \frac{1}{2}\int_{\Omega^{2^{-j}\sigma}} |\nabla v|^2 \,\mathrm{d}x - M\pi|\log(2^{-j}\sigma)| + \mathrm{o}(j),$$

with $\lim_{j\to\infty} \mathrm{o}(j) = 0$. A standard diagonal argument yields that there exists $j(\varepsilon) \to 0$ such that $u_{\varepsilon,j(\varepsilon)}$ is a recovery sequence in the sense of (4.26). □

5. Analysis of local minimizers

In this section we will prove the existence of many local minimizers for a large class of interaction potentials. We will assume some further hypotheses for the energy density f in addition to (1), (2) and (3):

(4) $f \in C^0([-\frac{1}{2},\frac{1}{2}]) \cap C^2((-\frac{1}{2},\frac{1}{2}))$;
(5) There exists a neighborhood I of $\frac{1}{2}$ such that for every $x \in I$ we have $C_1(\frac{1}{2}-x)^2 < f(\frac{1}{2}) - f(x)$ for some $C_1 > 0$ and $\sup_{t\in(-\frac{1}{2},\frac{1}{2})} f''(t) < \frac{1}{9}C_1$;
(6) f is increasing in $[0,\frac{1}{2}]$ and even.

Notice that these conditions are satisfied by the energy density of the screw dislocations functionals, $f(a) = \mathrm{dist}^2(a,\mathbb{Z})$, while they are not satisfied by the energy density of the XY model.

5.1. **Antipodal configurations and energy barriers.** When a discrete singularity of $\mu(v)$ moves to a neighboring cell, then v has to pass through an antipodal configuration $v(i) = -v(j)$ (i.e., such that the corresponding phase u satisfies $\mathrm{dist}(u(i)-u(j),\mathbb{Z}) = \frac{1}{2}$). We will show that such configurations are energy barriers.

Lemma 5.1. *There exist $\alpha > 0$ and $E > 0$ such that the following holds: Let $u \in \mathcal{AF}_\varepsilon(\Omega)$ such that $dist(u(i)-u(j),\mathbb{Z}) > \frac{1}{2}-\alpha$ for some $(i,j) \in \Omega^1_\varepsilon$. Then there exists a function w, with $w = u$ in $\Omega^0_\varepsilon \setminus \{i\}$ such that $F_\varepsilon(w) \le F_\varepsilon(u) - E$.*

Proof. As a consequence of assumption (5), it is easy to see that there exist $\gamma > 0$ and a positive constant C_2 such that

$$f(\tfrac{1}{2}) - f(\gamma) - f(\tfrac{1}{2} - \gamma) > C_2\,. \tag{5.1}$$

First, we prove the statement assuming $f \in C^2(\mathbb{R})$. In this case, assumption (5) implies that $f'(\frac{1}{2}) = 0$ and $|f''(\frac{1}{2})| > C_1$.

Without loss of generality we can assume that $u(i) = 0$. For sake of notation we set

$$E^i(u) = \sum_{|l-i|=\varepsilon} f(u(l))\,. \tag{5.2}$$

We will assume that $i \notin \partial_\varepsilon\Omega$, so that i has exactly four nearest neighbors, denoted by j, k_1, k_2 and k_3. The case $i \in \partial_\varepsilon\Omega$ is fully analogous (some explicit computations are indeed shorter), and left to the reader. By assumption

$$E^i(u) \geq f(\tfrac{1}{2} + \alpha) + \sum_{l=1}^{3} f(u(k_l))\,. \tag{5.3}$$

We will distinguish two cases.

Case 1: There exists at least a nearest neighbor, say k_1, such that $\operatorname{dist}(u(k_1), \mathbb{Z}) \geq \frac{1}{2} - \alpha$. In this case we have that

$$E^i(u) \geq 2f(\tfrac{1}{2} + \alpha) + f(u(k_2)) + f(u(k_3)). \tag{5.4}$$

Now there are two possibilities. In fact we may have either that $\operatorname{dist}(u(k_2), \mathbb{Z}) \vee \operatorname{dist}(u(k_3), \mathbb{Z}) < 3\alpha$, or that $\operatorname{dist}(u(k_2), \mathbb{Z}) \vee \operatorname{dist}(u(k_3), \mathbb{Z}) \geq 3\alpha$.

In the first case, set $w(i) = \gamma$ with γ as in (5.1). Then, by continuity we have

$$E^i(w) = 2f(\tfrac{1}{2} - \gamma) + 2f(\gamma) + \mathrm{o}(1),$$

where $\mathrm{o}(1) \to 0$ as $\alpha \to 0$. From (5.4) we have $E^i(u) \geq 2f(\frac{1}{2} + \alpha)$, which together with (5.1) yields

$$E^i(u) - E^i(w) \geq 2(f(\tfrac{1}{2} + \alpha) - f(\tfrac{1}{2})) + C_2 + \mathrm{o}(1) = C_2 + \mathrm{o}(1) \tag{5.5}$$

as $\alpha \to 0$. Suppose now that $\operatorname{dist}(u(k_2), \mathbb{Z}) \vee \operatorname{dist}(u(k_3), \mathbb{Z}) \geq 3\alpha$. Then we define $w(i) = \frac{1}{2}$ and we get

$$E^i(w) \leq 2f(\alpha) + f(\tfrac{1}{2}) + f(\tfrac{1}{2} + 3\alpha)\,.$$

Moreover, thanks to assumption (6) of f we have $E^i(u) \geq 2f(\frac{1}{2} + \alpha) + f(3\alpha)$. We conclude that

$$E^i(u) - E^i(w) \geq \frac{7}{2}\alpha^2(f''(0) - f''(\tfrac{1}{2})) \geq \frac{7}{2}\alpha^2 C_1\,. \tag{5.6}$$

Case 2: For every i it holds $\mathrm{dist}(u(k_i),\mathbb{Z}) < \frac{1}{2}-\alpha$. Set $w(i)=\eta$ with $|\eta|=3\alpha$ and $\eta\sum_{l=1}^{3} f'(u(k_l))\geq 0$. Then

(5.7)
$$\begin{aligned}E^i(u)-E^i(w) &\geq f(\tfrac{1}{2}+\alpha)-f(\tfrac{1}{2}+\alpha-|\eta|)+\sum_{l=1}^{3} f(u(k_l))-f(u(k_l)-\eta)\\ &= \frac{1}{2}|f''(\tfrac{1}{2})||\eta|(|\eta|-2\alpha)+\eta\sum_{l=1}^{3} f'(u(k_l))-\frac{1}{2}\eta^2\sum_{l=1}^{3} f''(u(k_l))+o(\eta^2)\\ \geq \frac{1}{2}|f''(\tfrac{1}{2})|3\alpha^2-\frac{9}{2}\alpha^2\sum_{l=1}^{3} f''(u(k_l))+o(\alpha^2) &\geq \frac{3}{2}(C_1-9\sup_t f''(t))\alpha^2+o(\alpha^2).\end{aligned}$$

The combination of Step 1 and Step 2 concludes the proof in the case of $f\in C^2(\mathbb{R})$, by choosing α small enough and $E=(7C_1\wedge 3(C_1-9\sup_t f''(t)))\alpha^2/2$. The general case can be recovered by approximating f in a neighborhood of $\frac{1}{2}$ with C^2 functions still satisfying assumptions (4)-(6). □

Note that in the case of $f(a)=\mathrm{dist}^2(a,\mathbb{Z})$ the proof of the above Lemma can be obtained by a direct computation without the regularization.

Remark 5.2. Note that the function w constructed in Lemma 5.1 has a discrete vorticity that can be different from the one of u only in the four ε-squares sharing i as a vertex, and hence $\|\mu(u)-\mu(w)\|_{\mathrm{flat}}\leq 2\varepsilon$.

Definition 5.3. *Let $\alpha>0$. We say that a function $u\in\mathcal{AF}_\varepsilon(\Omega)$ satisfies the α-cone condition if*

$$\mathit{dist}(u(i)-u(j),\mathbb{Z})\leq\frac{1}{2}-\alpha\qquad \textit{for every}\ \ (i,j)\in\Omega^1_\varepsilon\,.$$

Remark 5.4. Note that if $u\in\mathcal{AF}_\varepsilon(\Omega)$ satisfies the α-cone condition for some $\alpha>0$, then for every $w\in\mathcal{AF}_\varepsilon(\Omega)$ such that $\sum_{i\in\Omega^0_\varepsilon}|w(i)-u(i)|^2<\frac{\alpha^2}{16}$ we have $\mu(w)=\mu(u)$. In other words, the vorticity measure $\mu(u)$ is stable with respect to small variations of u.

5.2. **Metastable configurations and pinning.** As a consequence of Lemma 5.1 we prove the existence of a minimizer for the energy F_ε, under assumptions (1)-(6) with singularities close to prescribed positions.

Theorem 5.5. *Given $\mu_0=\sum_{i=1}^{M} d_i\delta_{x_i}$ with $x_i\in\Omega$ and $d_i\in\{1,-1\}$ for $i=1,\dots,M$, there exists a constant $K\in\mathbb{N}$ such that, for ε small enough, there exists $k_\varepsilon\in\{1,\dots,K\}$ such that the following minimum problem is well-posed*

(5.8)
$$\min\{F_\varepsilon(u)\,:\,\|\mu(u)-\mu_0\|_{\mathrm{flat}}\leq k_\varepsilon\varepsilon\}.$$

Moreover, let α be given by Lemma 5.1; any minimizer u_ε of the problem in (5.8) *satisfies the α-cone condition and it is a local minimizer for F_ε.*

Proof. For any $k\in\mathbb{N}\cup\{0\}$, we set

(5.9)
$$I^k_\varepsilon:=\inf\{F_\varepsilon(u)\,:\,\|\mu(u)-\mu_0\|_{\mathrm{flat}}\leq(M+2k)\varepsilon\}\,.$$

By constructing explicit competitors one can show that

(5.10)
$$I^0_\varepsilon\leq M\pi|\log\varepsilon|+C.$$

Then, we consider a minimizing sequence $\{u_\varepsilon^{k,n}\}$ for I_ε^k. It is not restrictive to assume that $0 \le u_\varepsilon^{k,n}(i) \le 1$ for any $i \in \Omega_\varepsilon^0$; therefore, up to a subsequence, $u_\varepsilon^{k,n} \to u_\varepsilon^k$ as $n \to \infty$ for some $u_\varepsilon^k \in \mathcal{AF}_\varepsilon(\Omega)$. Note that if u_ε^k satisfies the α-cone condition, then it is a minimizer for I_ε^k.

Set $\bar{k} := \lceil \frac{C-\mathbb{W}(\mu_0)-M\gamma}{E} \rceil + 1$ and assume by contradiction that there exists a subsequence, still labeled with ε, such that for every $k \in \{0, 1, \dots, \bar{k}\}$, there exists a bond $(i_\varepsilon, j_\varepsilon) \in \Omega_\varepsilon^1$, with $\operatorname{dist}(u_\varepsilon^k(i_\varepsilon) - u_\varepsilon^k(j_\varepsilon), \mathbb{Z}) > \frac{1}{2} - \alpha$. Thus, for n large enough, we have

$$\operatorname{dist}(u_\varepsilon^{k,n}(i_\varepsilon) - u_\varepsilon^{k,n}(j_\varepsilon), \mathbb{Z}) > \frac{1}{2} - \alpha.$$

By Lemma 5.1, there exists a function $w_\varepsilon^{k,n} \in \mathcal{AF}_\varepsilon(\Omega)$ such that $w_\varepsilon^{k,n} \equiv u_\varepsilon^{k,n}$ in $\Omega_\varepsilon^0 \setminus \{i\}$ and $F_\varepsilon(w_\varepsilon^{k,n}) \le F_\varepsilon(u_\varepsilon^{k,n}) - E$ for some $E > 0$. By construction (see Remark 5.2) we have that $\|\mu(w_\varepsilon^{k,n}) - \mu(u_\varepsilon^{k,n})\|_{\text{flat}} \le 2\varepsilon$. It follows that

$$I_\varepsilon^{k+1} \le F_\varepsilon(w_\varepsilon^{k,n}) \le I_\varepsilon^k - E.$$

By an easy induction argument on k and by (5.10), we have immediately that

$$I_\varepsilon^k \le I_\varepsilon^0 - kE \le M\pi|\log\varepsilon| + C - kE. \tag{5.11}$$

By the lower bound (4.9) in Theorem 4.2, (5.11), and the definition of $\bar{k}$ we get

$$\mathbb{W}(\mu_0) + M\gamma \le \liminf_{\varepsilon\to 0} I_\varepsilon^{\bar{k}} - M\pi|\log\varepsilon| \le C - \bar{k}E \le \mathbb{W}(\mu_0) + M\gamma - E,$$

and so the contradiction. Then the statement holds true for $K = M + 2\bar{k}$.

□

Let $\varepsilon > 0$ and let $u_\varepsilon^0 \in \mathcal{AF}_\varepsilon(\Omega)$. We say that $u_\varepsilon = u_\varepsilon(t)$ is a solution of the gradient flow of F_ε from u_ε^0 if u_ε satisfies

$$\begin{cases} \frac{1}{|\log\varepsilon|}\dot{u}_\varepsilon = -\nabla F_\varepsilon(u_\varepsilon) & \text{in } (0,+\infty)\times\Omega_\varepsilon^0 \\ u_\varepsilon(0) = u_\varepsilon^0 & \text{in } \Omega_\varepsilon^0. \end{cases}$$

Clearly $u_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$, and we will write $u_\varepsilon(t,i)$ in place of $u_\varepsilon(t)(i)$.

Theorem 5.6. *Let $\mu_0 = \sum_{i=1}^M d_i\delta_{x_i}$ with $x_i \in \Omega$ and $d_i \in \{1,-1\}$ for $i = 1,\dots,M$. Let $\{u_\varepsilon^0\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon^0) - M\pi|\log\varepsilon| = \mathbb{W}(\mu_0) + M\gamma. \tag{5.12}$$

Let α be given by Lemma 5.1. Then, for ε small enough, the following facts hold:

- (i) *u_ε^0 satisfy the α-cone condition.*
- (ii) *The solution $u_\varepsilon(t)$ of the gradient flow of F_ε from u_ε^0 satisfies $\mu(u_\varepsilon(t)) = \mu(u_\varepsilon^0)$ for every $t > 0$.*
- (iii) *There exists $\bar{u}_\varepsilon^0$ such that $\bar{u}_\varepsilon^0 \in \operatorname{argmin}\{F_\varepsilon(u) : \mu(u) = \mu(u_\varepsilon^0)\}$. Moreover $\bar{u}_\varepsilon^0$ satisfies the α-cone condition and it is a local minimizer for F_ε.*

Proof. Proof of (i). Assume, by contradiction, that there exists a sequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k}^0$ does not satisfy the α-cone condition, namely for every $k \in \mathbb{N}$ there exists a bond $(i_k, j_k) \in \Omega_{\varepsilon_k}^1$ with

$$\operatorname{dist}(u_{\varepsilon_k}^0(i_k) - u_{\varepsilon_k}^0(j_k), \mathbb{Z}) > \frac{1}{2} - \alpha.$$

By Lemma 5.1, for any k there exists a function $w_{\varepsilon_k} \in \mathcal{AF}_{\varepsilon_k}(\Omega)$ such that $w_{\varepsilon_k} \equiv u^0_{\varepsilon_k}$ in $\Omega^0_{\varepsilon_k} \setminus \{i_k\}$ and

$$F_{\varepsilon_k}(w_{\varepsilon_k}) \le F_\varepsilon(u^0_{\varepsilon_k}) - E \le F_{\varepsilon_k}(u^0_{\varepsilon_k}) - E. \tag{5.13}$$

Moreover, by construction (see Remark 5.2) we have that $\|\mu(w_{\varepsilon_k}) - \mu(u^0_{\varepsilon_k})\|_{\mathrm{flat}} \le 2\varepsilon_k$ and so $\mu(w_{\varepsilon_k}) \overset{\mathrm{flat}}{\to} \mu_0$. By the lower bound (4.9) in Theorem 4.2, we get

$$\begin{aligned} \mathbb{W}(\mu_0) + M\gamma &\le \liminf_{\varepsilon_k \to 0} F_{\varepsilon_k}(w_{\varepsilon_k}) - M\pi|\log\varepsilon_k| \\ &\le \lim_{\varepsilon_k \to 0} F_{\varepsilon_k}(u^0_{\varepsilon_k}) - M\pi|\log\varepsilon_k| - E = \mathbb{W}(\mu_0) + M\gamma - E, \end{aligned} \tag{5.14}$$

and so the contradiction.

Proof of (ii). Assume, by contradiction, that there exists a sequence $\varepsilon_k \to 0$ such that the solutions $u_{\varepsilon_k}(t)$ of the gradient flows of F_{ε_k} from $u^0_{\varepsilon_k}$ do not satisfy (ii). Let t_k be the first time (in fact, the infimum) for which $\mu(u_{\varepsilon_k}(t_k)) \ne \mu(u^0_{\varepsilon_k})$; then, there exists $(i_k, j_k) \in \Omega^1_{\varepsilon_k}$ such that $\mathrm{dist}(u_{\varepsilon_k}(t_k, i_k) - u_{\varepsilon_k}(t_k, j_k), \mathbb{Z}) > \frac{1}{2} - \alpha$. By Lemma 5.1 there exists $w_{\varepsilon_k}(t_k) \in \mathcal{AF}_{\varepsilon_k}(\Omega)$ such that $w_{\varepsilon_k}(t_k) \equiv u_{\varepsilon_k}(t_k)$ in $\Omega^0_{\varepsilon_k} \setminus \{i_k\}$ and $F_{\varepsilon_k}(w_{\varepsilon_k}(t_k)) \le F_{\varepsilon_k}(u_{\varepsilon_k}(t_k)) - E$, for some positive constant E independent of k. Moreover, by (5.2), we have that

$$\|\mu(u^0_{\varepsilon_k}) - \mu(w_{\varepsilon_k}(t_k))\|_{\mathrm{flat}} = \|\mu(u_{\varepsilon_k}(t_k)) - \mu(w_{\varepsilon_k}(t_k))\|_{\mathrm{flat}} \le 2\varepsilon_k\,.$$

Therefore, by the lower bound (4.9) in Theorem 4.2, arguing as in (5.14), we get a contradiction.

Proof of (iii). Let $\{u^n_\varepsilon\}$ be a minimizing sequence for the minimum problem in (iii). We can always assume that $0 \le u^n_\varepsilon(i) \le 1$ for any $i \in \Omega^0_\varepsilon$; therefore, up to a subsequence, $u^n_\varepsilon \to \bar{u}^0_\varepsilon$ as $n \to \infty$ for some $\bar{u}^0_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$. To prove that $\bar{u}^0_\varepsilon$ (for ε small enough) is a minimizer, it is enough to show that $\mu(\bar{u}^0_\varepsilon) = \mu(u_\varepsilon)$; this follows once we have proved that $\bar{u}^0_\varepsilon$ satisfies the α-cone condition (see Remark 5.4). Assume by contradiction that there exists a sequence $\varepsilon_k \to 0$ such that $\mathrm{dist}(\bar{u}^0_\varepsilon(i_k) - \bar{u}^0_\varepsilon(j_k), \mathbb{Z}) > \frac{1}{2} - \alpha$ for some bond $(i_k, j_k) \in \Omega^1_{\varepsilon_k}$. Then, for n large enough, we have

$$F_{\varepsilon_k}(u^n_{\varepsilon_k}) \le F_{\varepsilon_k}(u^0_{\varepsilon_k}) + \varepsilon_k, \qquad \mathrm{dist}(u^n_{\varepsilon_k}(i) - u^n_{\varepsilon_k}(j), \mathbb{Z}) > \frac{1}{2} - \alpha\,. \tag{5.15}$$

Let $\bar{n}$ be such that (5.15) holds. By Lemma 5.1, there exists a function $w_{\varepsilon_k} \in \mathcal{AF}_{\varepsilon_k}(\Omega)$ such that $w_{\varepsilon_k} \equiv u^{\bar{n}}_{\varepsilon_k}$ in $\Omega^0_{\varepsilon_k} \setminus \{i\}$ and

$$F_{\varepsilon_k}(w_{\varepsilon_k}) \le F_\varepsilon(u^{\bar{n}}_{\varepsilon_k}) - E \le F_{\varepsilon_k}(u^0_{\varepsilon_k}) - E + \varepsilon_k. \tag{5.16}$$

By construction (see Remark 5.2), we have that $\|\mu(w_{\varepsilon_k}) - \mu(u^0_{\varepsilon_k})\|_{\mathrm{flat}} = \|\mu(w_{\varepsilon_k}) - \mu(u^{\bar{n}}_{\varepsilon_k})\|_{\mathrm{flat}} \le 2\varepsilon_k$. Therefore, by the lower bound (4.9) in Theorem 4.2, arguing as in (5.14), we get a contradiction.

Finally, by the α-cone condition and Remark 5.4, we have immediately that $F_\varepsilon(\bar{u}^0_\varepsilon) \le F_\varepsilon(w)$ for any function $w \in \mathcal{AF}_\varepsilon(\Omega)$ with $\|w - u\|_{L^2} \le \frac{\alpha}{4}$, and hence $\bar{u}^0_\varepsilon$ is a local minimizer of F_ε .

□

Remark 5.7. By Theorem 5.6 it easily follows that there exists $t_n \to \infty$ such that $u^\infty_\varepsilon := \lim_{t_n \to \infty} u_\varepsilon(t_n)$ is a critical point of F_ε.

6. Discrete gradient flow of $\mathcal{F}_\varepsilon$ with flat dissipation

In Section 5 we have seen that the energy $\mathcal{F}_\varepsilon$ has many local minimizers. In particular, Theorem 5.5 shows that the length-scale of metastable configurations of singularities is of order ε. In this section we introduce and analyze an effective dynamics of vortices, which overcome the pinning effect due to the presence of these local minima. This is done considering a discrete in time gradient flow, following the minimizing movements method. It turns out that, for ε smaller than the time step τ, the vortices overcome the energetic barriers and the dynamics is described (as $\varepsilon, \tau \to 0$) by the gradient flow of the renormalized energy (see Definition 6.3). This process requires the introduction of a suitable dissipation.

In this section we consider a dissipation which is continuous with respect to the flat norm. To this purpose, we notice that, identifying each $\mu = \sum_{i=1}^N d_i \delta_{x_i}$ with a 0-current, it can be shown that

$$\|\mu\|_{\text{flat}} = \min\{|S|\,,\, S \text{ 1-current}, \partial S \llcorner \Omega = \mu\} \tag{6.1}$$

(see [22, Section 4.1.12]). Moreover, it is an established result in the optimal transport theory (see for instance [45, Theorem 5.30]) that the minimization in (6.1) can be restricted to the family

$$\mathcal{S}(\mu) := \Bigg\{ S = \sum_{l=1}^{L} m_l [p_l, q_l] \,:\, L \in \mathbb{N}, m_l \in \mathbb{Z}, \,,\; p_l\,, q_l \in \text{supp}(\mu) \cup \partial\Omega\,,$$
$$\partial S \llcorner \Omega = \sum_{l=1}^{L} m_l (\delta_{q_l} - \delta_{p_l}) \llcorner \Omega = \mu \Bigg\}\,,$$

where $m[p,q]$ denotes the 1-rectifiable current supported on the oriented segment of vertices p and q, and with multiplicity m (for a self-contained proof of this fact we refer also to [35, Proposition 4.4]). Notice that, given $S \in \mathcal{S}(\mu)$, $|S| = \sum_{l=1}^{L} |m_l||q_l - p_l|$.

We define our dissipation in two steps.

First assume that $\nu_1 = \sum_{i=1}^{N_1} d_i^1 \delta_{x_i^1}$ and $\nu_2 = \sum_{j=1}^{N_2} d_j^2 \delta_{x_j^2}$ with $d_i^1, d_j^2 \in \mathbb{N}$ for every $i = 1, \dots, N_1$ and $j = 1, \dots, N_2$ and set

$$\tilde{D}_2(\nu_1, \nu_2) := \min \Bigg\{ \sum_{l=1}^{L} |q_l - p_l|^2 \,:\, L \in \mathbb{N},\, q_l \in \text{supp}(\nu_1) \cup \partial\Omega,\, p_l \in \text{supp}(\nu_2) \cup \partial\Omega,$$
$$\sum_{l=1}^{L} \delta_{q_l} \llcorner \Omega = \nu_1, \sum_{l=1}^{L} \delta_{p_l} \llcorner \Omega = \nu_2 \Bigg\}.$$

It is easy to see that $\tilde{D}_2^{\frac{1}{2}}$ is a distance. Actually, $\|\nu_1 - \nu_2\|_{\text{flat}}$ and $D_2(\nu_1, \nu_2)$ can be rewritten as

$$\|\nu_1 - \nu_2\|_{\text{flat}} = \min_{\lambda} \int_{\bar{\Omega} \times \bar{\Omega}} |x - y| d\lambda(x, y)\,,$$
$$\tilde{D}_2(\nu_1, \nu_2) = \min_{\lambda} \int_{\bar{\Omega} \times \bar{\Omega}} |x - y|^2 d\lambda(x, y)\,,$$

where the minimum is taken over all measures λ which are sums of Dirac deltas in $\bar{\Omega} \times \bar{\Omega}$ with integer coefficients, and have marginals restricted to Ω given by ν_1 and ν_2. This clarifies the connection of the flat distance and of our dissipation with the

Wasserstein distances W_1 and W_2, defined on pairs of probability measures in $\mathbb{R}^2$, respectively (see for instance [45]).

From the very definition of $\tilde{D}_2$ one can easily check that

$$\tilde{D}_2(\nu_1+\rho_1,\nu_2+\rho_2)\le \tilde{D}_2(\nu_1,\nu_2)+\tilde{D}_2(\rho_1,\rho_2) \tag{6.2}$$

for any ρ_1 and ρ_2 sums of positive Dirac masses, and

$$\tilde{D}_2(\nu_1,\nu_2)\le \operatorname{diam}(\Omega)\|\nu_1-\nu_2\|_{\mathrm{flat}}\,. \tag{6.3}$$

For the general case of $\mu_1=\sum_{i=1}^{N_1} d_i^1\delta_{x_i^1}$ and $\mu_2=\sum_{i=1}^{N_2} d_i^2\delta_{x_i^2}$ with $d_i^1,d_i^2\in\mathbb{Z}$ we set

$$D_2(\mu_1,\mu_2):=\tilde{D}_2(\mu_1^++\mu_2^-,\mu_2^++\mu_1^-), \tag{6.4}$$

where μ_j^+ and μ_j^- are the positive and the negative part of μ_j. As a consequence of (6.2) and (6.3) we have that D_2 is continuous with respect to the flat norm.

We are now in a position to introduce the discrete gradient flow of F_ε with respect to the dissipation D_2.

Definition 6.1. *Fix $\delta>0$ and let $\varepsilon,\tau>0$. Given $\mu_{\varepsilon,0}\in X_\varepsilon$, we say that $\{\mu_{\varepsilon,k}^\tau\}$, with $k\in\mathbb{N}\cup\{0\}$, is a solution of the flat discrete gradient flow of $\mathcal{F}_\varepsilon$ from $\mu_{\varepsilon,0}$ if $\mu_{\varepsilon,0}^\tau=\mu_{\varepsilon,0}$, and for any $k\in\mathbb{N}$, $\mu_{\varepsilon,k}^\tau$ satisfies*

$$\mu_{\varepsilon,k}^\tau\in\operatorname{argmin}\left\{\mathcal{F}_\varepsilon(\mu)+\frac{\pi \mathrm{D}_2(\mu,\mu_{\varepsilon,k-1}^\tau)}{2\tau}:\mu\in X_\varepsilon,\ \|\mu-\mu_{\varepsilon,k-1}^\tau\|_{\mathrm{flat}}\le\delta\right\}. \tag{6.5}$$

Notice that the existence of a minimizer is obvious, since μ lies in X_ε which is a finite set.

We want to analyze the limit as $\varepsilon\to0$ of the flat discrete gradient flow. To this purpose, let $\mu_0:=\sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$ with $|d_{i,0}|=1$, and let $\mu_{\varepsilon,0}\in X_\varepsilon$ be such that

$$\mu_{\varepsilon,0}\overset{\mathrm{flat}}{\to}\mu_0,\qquad \lim_{\varepsilon\to0}\frac{\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})}{|\log\varepsilon|}=\pi|\mu_0|(\Omega).$$

In Theorem 6.7 we will show that, as $\varepsilon\to0$, the sequence $\mu_{\varepsilon,k}^\tau$ converges to some $\mu_k^\tau\in X$, whose singularities have the same degrees of those of the initial datum. Therefore, it is convenient to regard the renormalized energy as a function only of the positions of M singularities. To this end we introduce the following notation

$$W(x):=\mathbb{W}(\mu)\qquad\text{where }\mu=\sum_{i=1}^M d_{i,0}\delta_{x_i}\quad\text{ and }x=(x_1,\dots,x_M)\in\Omega^M.$$

The right notion for the limit as $\varepsilon\to0$ of flat discrete gradient flows of $\mathcal{F}_\varepsilon$ is given by the following definition of discrete gradient flow of the renormalized energy.

Definition 6.2. *Let $\delta>0$, $K\in\mathbb{N}\cup\{0\}$, and $\tau>0$. Fix $x_0\in\Omega^M$. We say that $\{x_k^\tau\}$ with $k=0,1,\dots,K$, is a solution of the discrete gradient flow of W from x_0 if $x_0^\tau=x_0$ and, for any $k=1,\dots,K$, $x_k^\tau\in\Omega^M$ satisfies*

$$x_k^\tau\in\operatorname{argmin}\left\{W(x)+\frac{\pi|x_k^\tau-x_{k-1}^\tau|^2}{2\tau}:\ x\in\Omega^M,\ \sum_{i=1}^M|x_i-x_{i,k-1}^\tau|\le\delta\right\}, \tag{6.6}$$

where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^k$ for any $k\in\mathbb{N}$.

In Theorem 6.6 we show that, as $\tau \to 0$, this discrete in time evolution converges, until a maximal time $\tilde{T}_\delta$, to the gradient flow of the renormalized energy given by the following definition.

Definition 6.3. *Let $M \in \mathbb{N}$ and $x_0 \in \Omega^M$. The gradient flow of the renormalized energy from x_0 is given by*

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{\pi}\nabla W(x(t)) \\ x(0) = x_0. \end{cases} \tag{6.7}$$

We denote by T^* the maximal time of existence of the solution, and we notice that until the time T^* the solution is unique, and that T^* is the minimal critical time among the first collision time and the exit time from Ω.

As $\delta \to 0$, $\tilde{T}_\delta$ converges to the critical time T^*. Notice that the renormalized energy is not bounded from below and it blows up to $-\infty$ whenever one of these critical events occur. This justifies the introduction of the parameter δ, in order to explore local minima. Nevertheless, the solutions of flat discrete gradient flows defined above do not touch the constraint and hence, they satisfy the corresponding unconstrained Euler-Lagrange equations.

6.1. **Flat discrete gradient flow of W.** Fix initial conditions

$$x_0 = (x_{1,0}, \dots, x_{M,0}) \in \Omega^M, \qquad d_{1,0}, \dots, d_{M,0} \in \{-1, 1\},$$

and fix $\delta > 0$ such that

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,0}, x_{j,0}), \mathrm{dist}(x_{i,0}, \partial\Omega)\} - 2\delta =: c_\delta > 0. \tag{6.8}$$

Definition 6.4. *We say that a solution of the discrete gradient flow $\{x_k^\tau\}$ of W from x_0 is maximal if the minimum problem in* (6.6) *does not admit a solution for $k = K + 1$.*

Let $\{x_k^\tau\}$ be a maximal solution of the flat discrete gradient flow of W from x_0, according with Definitions 6.2, 6.4; we set

$$k_\delta^\tau = k_\delta^\tau(\{x_k^\tau\}) := \min\{k \in \{1, \dots, K\} \ : \ \min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,k}^\tau, x_{j,k}^\tau), \mathrm{dist}(x_{i,k}^\tau, \partial\Omega)\} \le 2\delta\,\}. \tag{6.9}$$

We notice that, since $|x_{k_\delta^\tau}^\tau - x_{k_\delta^\tau - 1}^\tau| \le \delta$ and

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,k_\delta^\tau-1}^\tau, x_{j,k_\delta^\tau-1}^\tau), \mathrm{dist}(x_{i,k_\delta^\tau-1}^\tau, \partial\Omega)\} > 2\delta,$$

then

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,k_\delta^\tau}^\tau, x_{j,k_\delta^\tau}^\tau), \mathrm{dist}(x_{i,k_\delta^\tau}^\tau, \partial\Omega)\} > \delta,$$

i.e., $k_\delta^\tau < K$. It follows that, for any $k = 0, 1, \dots, k_\delta^\tau$, we have

$$x_k^\tau \in K_\delta, \tag{6.10}$$

where K_δ is the compact set given by

$$K_\delta := \left\{x \in \Omega^M \ : \ \min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_i, x_j), \mathrm{dist}(x_i, \partial\Omega)\} \ge \delta\right\}. \tag{6.11}$$

Notice that W is smooth on K_δ. In particular, we can set

$$C_\delta := \max_{x\in K_\delta}(W(x_0) - W(x)). \tag{6.12}$$

Proposition 6.5. *For τ small enough the following holds. For every $k = 1, \dots, k^\tau_\delta$, we have that $\sum_{i=1}^M |x^\tau_{i,k} - x^\tau_{i,k-1}| < \delta$ and*

$$\partial_{x_i} W(x^\tau_k) + \pi \frac{x^\tau_{i,k} - x^\tau_{i,k-1}}{\tau} = 0 \qquad \text{for } i = 1, \dots, M. \tag{6.13}$$

In particular, for every $k = 1, \dots, k^\tau_\delta$

$$|x^\tau_k - x^\tau_{k-1}| \le \max_{x\in K_\delta} |\nabla W(x)|\tau. \tag{6.14}$$

Proof. Since the energy W is clearly decreasing in k, for every $k = 1, \dots, k^\tau_\delta$ we have

$$\frac{|x^\tau_k - x^\tau_{k-1}|^2}{2\tau} \le \frac{1}{\pi}(W(x^\tau_{k-1}) - W(x^\tau_k)) \le W(x_0) - W(x^\tau_k) \le C_\delta. \tag{6.15}$$

It follows that for τ small enough $\sum_{i=1}^M |x^\tau_{i,k} - x^\tau_{i,k-1}| < \delta$. Therefore, the minimality of x^τ_k clearly implies (6.13), as well as (6.14). □

Let $x(t)$ be the solution of the gradient flow of W with initial datum x_0 (see (6.7)) and let T^* be its maximal existence time. We set

$$T_\delta := \inf\left\{ t \in [0, T^*] : \ \min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_i(t), x_j(t)), \mathrm{dist}(x_i(t), \partial\Omega)\} \le 2\delta \right\}. \tag{6.16}$$

Notice that by definition we have

$$\lim_{\delta\to 0} T_\delta = T^*. \tag{6.17}$$

For $0 \le t \le k^\tau_\delta \tau$, we denote by $x^\tau(t) = (x^\tau_1(t), \dots, x^\tau_M(t))$ the piecewise affine in time interpolation of $\{x^\tau_k\}$.

Theorem 6.6. *Let $\{x^\tau_k\}_{\tau>0}$ be a family of maximal solutions of the flat discrete gradient flow of W from x_0. Then,*

$$\tilde T_\delta := \liminf_{\tau\to 0} k^\tau_\delta \tau \ge T_\delta, \tag{6.18}$$

where k^τ_δ is defined in (6.9) and T_δ is defined in (6.16).

Moreover, for every $0 < T < \tilde T_\delta$, $x^\tau \to x$ uniformly on $[0, T]$. Finally, $\tilde T_\delta \to T^*$ as $\delta \to 0$.

Proof. By the very definition of k^τ_δ, it is easy to prove that

$$|x^\tau_{k^\tau_\delta} - x^\tau_0| > c_\delta\,,$$

where c_δ is defined in (6.8). Moreover, by (6.14), for τ small enough we get

$$|x^\tau_{k^\tau_\delta} - x^\tau_0| \le \sum_{k=1}^{k^\tau_\delta} |x^\tau_k - x^\tau_{k-1}| \le \max_{x\in K_\delta} |\nabla W(x)| k^\tau_\delta \tau,$$

and hence

$$k^\tau_\delta \tau \ge \frac{c_\delta}{\max_{x\in K_\delta} |\nabla W(x)|} > 0.$$

From (6.14) it is easy to see that x^τ are equibounded and equicontinuous in $[0, \tau k^\tau_\delta]$, and hence by Ascoli Arzelà Theorem, they uniformly converge, up to a subsequence, to a function x on $[0,T]$, for every $T<\tilde T_\delta$. Let $t\in(0,\tilde T_\delta)$ and let $h>0$, by (6.13) we get

$$x^\tau(\tau\lfloor(t+h)/\tau\rfloor)-x^\tau(\tau\lfloor t/\tau\rfloor)=\sum_{k=\lfloor t/\tau\rfloor}^{\lfloor(t+h)/\tau\rfloor-1}x^\tau_{k+1}-x^\tau_k=-\frac{\tau}{\pi}\sum_{k=\lfloor t/\tau\rfloor}^{\lfloor(t+h)/\tau\rfloor-1}\nabla W(x^\tau_k).$$

Taking the limit as $\tau\to0$, and then $h\to0$, we obtain that the limit x is the unique solution of (6.7).

Moreover, it is easy to see that $x^\tau(\tau k^\tau_\delta)\to x(\tilde T_\delta)$ as $\tau\to0$ and hence by the very definition of k^τ_δ, it is immediate to see that (6.18) holds true and $\tilde T_\delta<T^*$. Since $T_\delta\to T^*$ (see (6.17)) we conclude that $\tilde T_\delta\to T^*$ as $\delta\to0$.

□

6.2. **Flat discrete gradient flow of $\mathcal{F}_\varepsilon$.** We are now in a position to state and prove the convergence of the discrete gradient flows as $\varepsilon\to0$.

Theorem 6.7. *Let $\mu_0:=\sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$ with $|d_{i,0}|=1$. Let $\mu_{\varepsilon,0}\in X_\varepsilon$ be such that*

$$\mu_{\varepsilon,0}\overset{\text{flat}}{\to}\mu_0,\qquad \lim_{\varepsilon\to0}\frac{\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})}{|\log\varepsilon|}=\pi|\mu_0|(\Omega).$$

Let $\delta>0$ be fixed such that $\min\left\{\frac12 dist_{i\neq j}(x_{i,0},x_{j,0}), dist(x_{i,0},\partial\Omega)\right\}>2\delta$. Given $\tau>0$, let $\mu^\tau_{\varepsilon,k}$ be a solution of the flat discrete gradient flow of $\mathcal{F}_\varepsilon$ from $\mu_{\varepsilon,0}$.

Then, up to a subsequence, for any $k\in\mathbb{N}$ we have $\mu^\tau_{\varepsilon,k}\overset{\text{flat}}{\to}\mu^\tau_k$, for some $\mu^\tau_k\in X$ with $|\mu^\tau_k|(\Omega)\le M$.

Moreover there exists a maximal solution of the discrete gradient flow, $x^\tau_k=(x^\tau_{1,k},\dots,x^\tau_{M,k})$, of W from $x_0=(x_{1,0},\dots,x_{M,0})$, according with Definition 6.2, such that

$$\mu^\tau_k=\sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,k}}\qquad \text{for every } k=1,\dots,k^\tau_\delta,$$

where k^τ_δ is defined in (6.9).

Proof. Since $\mathcal{F}_\varepsilon(\mu^\tau_{\varepsilon,k})$ is not increasing in k, we have

$$\mathcal{F}_\varepsilon(\mu^\tau_{\varepsilon,k})\le\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})\le M\pi|\log\varepsilon|+\mathrm{o}(|\log\varepsilon|).$$

By Theorem 4.3(i), we have that, up to a subsequence, $\mu^\tau_{\varepsilon,k}\overset{\text{flat}}{\to}\mu^\tau_k\in X$, with $|\mu^\tau_k|(\Omega)\le M$ and $\|\mu^\tau_k-\mu^\tau_{k-1}\|_{\text{flat}}\le\delta$. Let $\tilde k^\tau_\delta$ be defined by

$$\begin{aligned}\tilde k^\tau_\delta:=\sup\{k\in\mathbb{N}:\ &\mu^\tau_l=\sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,l}},\\ &\min\{\frac12\mathrm{dist}_{i\neq j}(x^\tau_{i,l},x^\tau_{j,l}),\mathrm{dist}(x^\tau_{i,l},\partial\Omega)\}>2\delta,\ l=0,\dots,k\}.\end{aligned}\tag{6.19}$$

Since $|\mu^\tau_{\tilde k^\tau_\delta+1}|(\Omega)\le M$ and $\|\mu^\tau_{\tilde k^\tau_\delta+1}-\mu^\tau_{\tilde k^\tau_\delta}\|_{\text{flat}}\le\delta$, we deduce that $\mu_{\tilde k^\tau_\delta+1}=\sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,\tilde k^\tau_\delta+1}}$, while

$$\min\{\frac12\mathrm{dist}_{i\neq j}(x^\tau_{i,\tilde k^\tau_\delta+1},x^\tau_{j,\tilde k^\tau_\delta+1}),\mathrm{dist}(x^\tau_{i,\tilde k^\tau_\delta+1},\partial\Omega)\}\le2\delta\,.\tag{6.20}$$

Moreover, since $\|\mu^\tau_k-\mu^\tau_{k-1}\|_{\text{flat}}\le\delta$, it is easy to see that at each step $k=1,\dots,\tilde k^\tau_\delta+1$ and for every singularity $x^\tau_{i,k-1}$ of μ^τ_{k-1}, there is exactly one singularity of μ_k^τ at distance at most δ from $x_{i,k-1}^\tau$; we relabel it $x_{i,k}^\tau$. Therefore, by definition of D_2 , we have that for $k = 1, \dots, \tilde{k}_\delta^\tau + 1$

$$D_2(\mu_k^\tau, \mu_{k-1}^\tau) = |x_k^\tau - x_{k-1}^\tau|^2. \tag{6.21}$$

We now show that for $k = 0, 1, \dots, \tilde{k}_\delta^\tau + 1$, x_k^τ satisfies (6.6). For any measure $\mu = \sum_{i=1}^M d_{i,0}\delta_{y_i}$ with $\|\mu - \mu_{k-1}^\tau\|_{\mathrm{flat}} \le \delta$, by Theorem 4.3 (iii) there exists a recovery sequence $\{\mu_\varepsilon\}$ such that $\mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log\varepsilon| \to \mathbb{W}(\mu) + M\gamma$ as $\varepsilon \to 0$. By a standard density argument we can assume that $\|\mu_\varepsilon - \mu_{\varepsilon,k-1}^\tau\|_{\mathrm{flat}} \le \delta$. Therefore by (ii) of Theorem 4.3, using the fact that $\mu_{\varepsilon,k}^\tau$ satisfies (6.5) and the continuity of D_2 with respect to the flat norm, we get

$$\begin{aligned}
\mathbb{W}(\mu_k^\tau) + M\gamma \quad &+ \quad \frac{\pi D_2(\mu_k^\tau, \mu_{k-1}^\tau)}{2\tau} \\
&\le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mu_{\varepsilon,k}^\tau) - \pi M|\log\varepsilon| + \frac{\pi D_2(\mu_{\varepsilon,k}^\tau, \mu_{\varepsilon,k-1}^\tau)}{2\tau} \\
&\le \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon) - \pi M|\log\varepsilon| + \frac{\pi D_2(\mu_\varepsilon, \mu_{\varepsilon,k-1}^\tau)}{2\tau} \\
&= \mathbb{W}(\mu) + M\gamma + \frac{\pi D_2(\mu, \mu_{k-1}^\tau)}{2\tau},
\end{aligned}$$

i.e., μ_k^τ satisfies

$$\mu_k^\tau \in \operatorname{argmin}\left\{ \mathbb{W}(\mu) + \frac{\pi D_2(\mu, \mu_{k-1}^\tau)}{2\tau} \; : \; \mu = \sum_{i=1}^M d_{i,0}\delta_{x_i},\ \left\|\mu, \mu_{k-1}^\tau\right\|_{\mathrm{flat}} \le \delta \right\}.$$

By (6.21) we have that x_k^τ is a solution of the discrete gradient flow of W from $x_0 = (x_{1,0}, \dots, x_{M,0})$ and by (6.20) that $\tilde{k}_\delta^\tau + 1 = k_\delta^\tau$.

□

7. Discrete gradient flow of F_ε with L^2 dissipation

In this section we introduce and analyze the discrete gradient flow of F_ε with L^2 dissipation (for the L^2 norm, we will use the notation introduced in (2.3)).

Definition 7.1. *Fix $\delta > 0$ and let $\varepsilon, \tau > 0$. Given $u_{\varepsilon,0} \in \mathcal{AF}_\varepsilon(\Omega)$, we say that $\{u_{\varepsilon,k}^\tau\}$, with $k \in \mathbb{N} \cup \{0\}$, is a solution of the L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$ if $u_{\varepsilon,0}^\tau = u_{\varepsilon,0}$, and for any $k \in \mathbb{N}$, $u_{\varepsilon,k}^\tau$ satisfies*

$$\begin{aligned}
u_{\varepsilon,k}^\tau \in \operatorname{argmin}\Bigg\{ F_\varepsilon(u) + \frac{\|e^{2\pi i u} - e^{2\pi i u_{\varepsilon,k-1}^\tau}\|_{L^2}^2}{2\tau|\log\tau|} \; &: \; u \in \mathcal{AF}_\varepsilon(\Omega), \\
&\|\mu(u) - \mu(u_{\varepsilon,k-1}^\tau)\|_{\mathrm{flat}} \le \delta \Big\}.
\end{aligned} \tag{7.1}$$

The constraint $\|\mu(u) - \mu(u_{\varepsilon,k-1}^\tau)\|_{\mathrm{flat}} \le \delta$ is not closed in the L^2 topology. Nevertheless, in Subsection 7.2 we prove an existence result for such a discrete gradient flow.

In the parabolic flow of Ginzburg-Landau functionals it is well known that, as $\varepsilon \to 0$, the dynamics becomes slower and slower, and in order to capture a non trivial dynamics it is needed to scale the time by $|\log\varepsilon|$ (see for instance [40]). In our discrete in time evolution, with $\tau \gg \varepsilon$, it turns out that the natural scaling involves the time step τ instead of the length scale ε. Such a time-scaling is plugged into the discrete dynamics through the $1/|\log\tau|$ pre-factor in front of the L^2 dissipation. A heuristic argument to justify this pre-factor is that it is the correct scaling for the canonical vortex $x/|x|$. Indeed, given $V \in \mathbb{R}^2$ representing the vortex velocity, a direct computation shows that

$$\lim_{\tau\to 0} \frac{1}{\tau|\log\tau|} \left\| \frac{x}{|x|} - \frac{x-\tau V}{|x-\tau V|} \right\|_2^2 = \pi |V|^2 \,. \tag{7.2}$$

As in Section 6, we want to consider the limit as $\varepsilon \to 0$ of such a discrete gradient flow. To this purpose, we will exploit the Γ-convergence analysis developed in Section 4.4. The limit dynamics will be described by a discrete gradient flow (that we shall define in the following) of the functional $\mathcal{W}$ (defined in (4.22)).

Let $v_0 \in \mathcal{D}_M$ (see (4.21)) be an initial condition with $\mathcal{W}(v_0) < +\infty$, and let $u_{\varepsilon,0}$ be a recovery sequence for v_0 in the sense of (4.26). We will show that the solutions $u^\tau_{\varepsilon,k}$ of the L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$ converge (according with the topology of our Γ-convergence analysis in Subsection 4.4) to some limit v^τ_k. Moreover, at each time step k, $v^\tau_k \in \mathcal{D}_M$, the Γ-limit $\mathcal{W}$ is finite, and the degrees of the singularities coincide with the degrees $d_{i,0}$ of the initial datum. Finally, $\{v^\tau_k\}$ is a solution of the L^2 discrete gradient flow according with the following definition.

Definition 7.2. *Let $\delta, \tau > 0$ and $K \in \mathbb{N}$. We say that $\{v^\tau_k\}$, with $k = 0, 1, \dots, K$, is a solution of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0 if $v^\tau_0 = v_0$ and, for any $k = 1, \dots, K$, v^τ_k satisfies*

$$\begin{aligned} v^\tau_k \in \operatorname{argmin}\Big\{ \mathcal{W}(v) + \frac{\|v - v^\tau_{k-1}\|^2_{L^2}}{2\tau|\log\tau|} \ : \ Jv = \sum_{i=1}^M d_{i,0}\delta_{y_{i,k}},\, y_{i,k} \in \Omega, \\ v \in H^1_{\mathrm{loc}}(\Omega \setminus \cup_{i=1}^M \{y_{i,k}\}; \mathcal{S}^1),\ \|Jv - Jv^\tau_{k-1}\|_{\mathrm{flat}} \le \delta \Big\}. \end{aligned} \tag{7.3}$$

As in Section 6, we first do the asymptotic analysis as $\tau \to 0$. This step is much more delicate than in the case of flat dissipation. Indeed, it is at this stage that we adopt the abstract method [40], and exploit it in the context of minimizing movements instead of gradient flows. This method relies on the proof of two energetic inequalities; the first relates the slope of the approximating functionals with the slope of the renormalized energy; the second one relates the scaled L^2 norm underlying the parabolic flow of GL_ε with the Euclidean norm of the time derivative of the limit singularities. In our discrete in time framework, we adapt the arguments in [40] by replacing derivatives by finite differences. The explicit computation in (7.2) has not an easy counterpart for general solutions v^τ_k, and (7.2) has to be replaced by more sophisticated estimates (see (7.18) and (7.57)). This point is indeed quite technical, and makes use of a lot of analysis developed in [40], [41].

7.1. **L^2 discrete gradient flow of $\mathcal{W}$.** Let $v_0 \in \mathcal{D}_M$ with $Jv_0 = \sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$, and fix $\delta > 0$ such that (6.8) holds true.

Definition 7.3. *We say that a solution of the L^2 discrete gradient flow $\{v^\tau_k\}$ of $\mathcal{W}$ from v_0 is maximal if the minimum problem in* (7.3) *does not admit a solution for $k = K + 1$.*

Let $\{v^\tau_k\}$ be a maximal solution of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0, let $Jv^\tau_k := \sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,k}}$, and let k^τ_δ be defined as in (6.9).

Remark 7.4. Since for any $i = 1, \dots, M$, we have that $|d_{i,0}| = 1$ and thanks to the constraint $\|Jv^\tau_k - Jv^\tau_{k-1}\|_{\mathrm{flat}} \le \delta$, we get that at each step $k = 1, \dots, k^\tau_\delta$ and for each singularity $x^\tau_{i,k-1}$ of Jv^τ_{k-1}, there is exactly one singularity of Jv^τ_k whose distance from $x^\tau_{i,k-1}$ is less than δ. We label this singularity $x^\tau_{i,k}$.

The above remark guarantees that the following definition is well posed.

Definition 7.5. *We set $x^\tau_k := (x^\tau_{1,k}, \dots, x^\tau_{M,k})$, where $x^\tau_{i,k}$ are labeled according with Remark 7.4. Moreover, we define $x^\tau(t) := (x^\tau_1(t), \dots, x^\tau_M(t))$ as the piecewise affine in time interpolation of $\{x^\tau_k\}$.*

As in Section 6 we have that $x^\tau_k \in K_\delta$, where K_δ is defined in (6.11). Moreover, the energy $\mathcal{W}$ is clearly decreasing in k. Since, for every $k = 1, \dots, k^\tau_\delta$ we have

$$\frac{\|v^\tau_k - v^\tau_{k-1}\|^2_{L^2}}{2\tau|\log\tau|} \le \mathcal{W}(v^\tau_{k-1}) - \mathcal{W}(v^\tau_k),$$

then

$$\begin{aligned}(7.4)\qquad \sum_{k=1}^{k^\tau_\delta} \frac{\|v^\tau_k - v^\tau_{k-1}\|^2_{L^2}}{2\tau|\log\tau|} &\le \mathcal{W}(v_0) - \mathcal{W}(v^\tau_{k^\tau_\delta}) \\ &\le \mathcal{W}(v_0) - W(x^\tau_{k^\tau_\delta}) \le \mathcal{W}(v_0) - W(x_0) + C_\delta\,,\end{aligned}$$

where C_δ is defined in (6.12).

Proposition 7.6. *For every $k = 0, 1, \dots, k^\tau_\delta$ we have that $\|Jv^\tau_k - Jv^\tau_{k-1}\|_{\mathrm{flat}} < C\sqrt{\tau|\log\tau|}$, where $C > 0$ depends only on δ (and on the initial condition v_0).*

Proof. Fix $1 \le k \le k^\tau_\delta$ and $1 \le i \le M$. Set $\rho^\tau_{i,k} := \frac{1}{4}\mathrm{dist}(x^\tau_{i,k}, x^\tau_{i,k-1})$. Note that

$$(7.5)\qquad \deg(v^\tau_k, \partial B_{\rho^\tau_{i,k}}(x^\tau_{i,k})) \neq 0 = \deg(v^\tau_{k-1}, \partial B_{\rho^\tau_{i,k}}(x^\tau_{i,k}))\,.$$

Moreover, since $\mathcal{W}(v^\tau_k) \le \mathcal{W}(v_0)$, from (4.24) we have that

$$(7.6)\qquad \int_{B_{2\rho^\tau_{i,k}}(x^\tau_{i,k}) \setminus B_{\rho^\tau_{i,k}}(x^\tau_{i,k})} (|\nabla v^\tau_k|^2 + |\nabla v^\tau_{k-1}|^2)\,\mathrm{d}x \le 2\mathcal{W}(v_0) + C\,.$$

As a consequence of (7.5) and (7.6), we have that

$$(7.7)\qquad (\mathrm{dist}(x^\tau_{i,k}, x^\tau_{i,k-1}))^2 \le C \int_{B_{2\rho^\tau_{i,k}}(x^\tau_{i,k}) \setminus B_{\rho^\tau_{i,k}}(x^\tau_{i,k})} |v^\tau_k - v^\tau_{k-1}|^2\,\mathrm{d}x\,.$$

Indeed, if by contradiction (7.7) does not hold, by a scaling argument we could find two sequences $\{w^n_1\}$ and $\{w^n_2\}$ of functions in $H^1(B_2 \setminus B_1; \mathcal{S}^1)$ such that

$$\int_{B_2\setminus B_1} (|\nabla w^n_1|^2 + |\nabla w^n_2|^2)\,\mathrm{d}x \le 2\mathcal{W}(v_0) + C, \qquad \int_{B_2\setminus B_1} |w^n_1 - w^n_2|^2\,\mathrm{d}x \to 0\,,$$

and such that $\deg(w^n_1, \partial B_\rho) \neq \deg(w^n_2, \partial B_\rho)$ for almost every $\rho \in [1, 2]$. This is impossible in view of the stability of the degree with respect to uniform convergence for continuous maps from $\mathcal{S}^1$ to $\mathcal{S}^1$.

Now, from (7.4) we have that

$$\int_{B_{2\rho^\tau_{i,k}}(x^\tau_{i,k}) \setminus B_{\rho^\tau_{i,k}}(x^\tau_{i,k})} |v^\tau_k - v^\tau_{k-1}|^2\,\mathrm{d}x \le C\tau|\log\tau|,$$

which together with (7.7) yields

$$(7.8)\qquad \|Jv^\tau_k - Jv^\tau_{k-1}\|_{\mathrm{flat}} \le C\sqrt{\tau|\log\tau|}\,.$$

□

For every $k = 0, 1, \dots, k_\delta^\tau$ we set

$$D_k^\tau := \mathcal{W}(v_k^\tau) - W(x_k^\tau). \tag{7.9}$$

Moreover, set $\tilde{T}_\delta := \liminf_{\tau\to 0} k_\delta^\tau \tau$, and define for any $t \in [0, \tilde{T}_\delta)$, the *energy excess*

$$D(t) = \limsup_{\tau\to 0} D^\tau_{\lfloor t/\tau \rfloor} \geq 0. \tag{7.10}$$

Since $\mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0)$, by (6.10) we have

$$D_k^\tau = \mathcal{W}(v_k^\tau) - W(x_k^\tau) \leq \mathcal{W}(v_0) - W(x_k^\tau) \leq D(0) + C_\delta\,, \tag{7.11}$$

where C_δ is defined in (6.12). From now on we will say that an initial condition v_0 is *well prepared* if $W(x_0) = \mathcal{W}(v_0)$, i.e., $D(0) = 0$.

We are in a position to state the main theorem of this section.

Theorem 7.7. *Let v_0 be a well prepared initial condition. Let $\{v_k^\tau\}_{\tau>0}$ be a family of maximal solutions of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0. Then,*

$$\tilde{T}_\delta := \liminf_{\tau\to 0} k_\delta^\tau \tau \geq T_\delta, \tag{7.12}$$

where k_δ^τ is defined in (6.9) *and T_δ is defined in* (6.16).

Moreover, for every $0 < T < \tilde{T}_\delta$, $x^\tau \to x$ uniformly on $[0, T]$, where x^τ is defined in Definition 7.5, and x is the solution of the gradient flow of W from x_0 according with Definition 6.3. Finally, $D(t) = 0$ for every $0 \leq t < \tilde{T}_\delta$ and $\tilde{T}_\delta \to T^*$ as $\delta \to 0$.

Remark 7.8. As a consequence of the uniform convergence of x^τ and the estimate (7.8), one can prove that the 1-current associated to the polygonal x^τ (with the natural orientation and multiplicity given by the integers $d_{i,0}$), converges to the current associated to the limit x in the flat norm.

The proof of Theorem 7.7 is postponed at the end of the section, and will be obtained as a consequence of Theorem 7.9 below, which can be regarded as the discrete in time counterpart of Theorem 1.4 in [41].

Theorem 7.9. *Let v_0 be a well prepared initial datum, i.e., with $W(x_0) = \mathcal{W}(v_0)$. Let $\{v_k^\tau\}_{\tau>0}$ be solutions of the L^2 discrete gradient flow for $\mathcal{W}$ from v_0, let $T > 0$ be such that $k_\delta^\tau \geq \lfloor T/\tau \rfloor$ for every τ, and assume that $x^\tau \to x$ uniformly in $[0, T]$ for some $x(t) \in H^1([0, T]; \Omega^M)$. Moreover, assume that (i) and (ii) below are satisfied:*

(i) *(*Lower bound*) For any $s \in [0, T]$*

$$\liminf_{\tau\to 0} \frac{\tau}{|\log \tau|} \sum_{k=1}^{\lfloor \frac{s}{\tau} \rfloor} \left\| \frac{v_k^\tau - v_{k-1}^\tau}{\tau} \right\|^2_{L^2} \geq \pi \int_0^s |\dot{x}(t)|^2 \, \mathrm{d}t \ .$$

(ii) *(*Construction*) For any $k = 0, 1, \dots, \lfloor T/\tau \rfloor - 1$, there exists a field $w_{k+1}^\tau \in H^1_{\mathrm{loc}}(\Omega \setminus \cup_{i=1}^M \{x_{i,k}^\tau - \frac{\tau}{\pi}\partial_{x_i} W(x_k^\tau)\}; \mathcal{S}^1)$ and a constant $M_\delta > 0$ such that*

$$\mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) \geq \frac{\tau}{\pi} |\nabla W(x_k^\tau)|^2 - \tau M_\delta D_k^\tau + \mathrm{o}(\tau),$$
$$\frac{1}{|\log \tau|} \left\| \frac{w_{k+1}^\tau - v_k^\tau}{\tau} \right\|^2_{L^2} \leq \frac{1}{\pi} |\nabla W(x_k^\tau)|^2 + \mathrm{o}(1).$$

Then, $D(t) = 0$ for every $t \in [0, T]$, and $x(t)$ is a solution of the gradient flow (6.7) *of W from x_0 on $[0, T]$.*

Proof. By (ii) and by the minimality of v^τ_{k+1}, we have

$$\mathcal{W}(v^\tau_k) - \mathcal{W}(v^\tau_{k+1}) = \mathcal{W}(v^\tau_k) - \mathcal{W}(w^\tau_{k+1}) + \mathcal{W}(w^\tau_{k+1}) - \mathcal{W}(v^\tau_{k+1})$$
$$\geq \frac{\tau}{\pi}|\nabla W(x^\tau_k)|^2 - \frac{\tau}{2|\log\tau|}\left\|\frac{w^\tau_{k+1}-v^\tau_k}{\tau}\right\|^2_{L^2} + \frac{\tau}{2|\log\tau|}\left\|\frac{v^\tau_{k+1}-v^\tau_k}{\tau}\right\|^2_{L^2} - \tau M_\delta D^\tau_k + \mathrm{o}(\tau)$$
$$\geq \frac{\tau}{2\pi}|\nabla W(x^\tau_k)|^2 + \frac{\tau}{2|\log\tau|}\left\|\frac{v^\tau_{k+1}-v^\tau_k}{\tau}\right\|^2_{L^2} - \tau M_\delta D^\tau_k + \mathrm{o}(\tau).$$

Now, let $s \in [0,T]$. Summing over $k = 0, 1, \dots, \lfloor s/\tau\rfloor - 1$, we have

$$\mathcal{W}(v^\tau_0) - \mathcal{W}(v^\tau_{\lfloor s/\tau\rfloor}) \geq \frac{1}{2\pi}\int_0^{\tau\lfloor s/\tau\rfloor-\tau} |\nabla W(x^\tau_{\lfloor t/\tau\rfloor})|^2 \,\mathrm{d}t$$
$$+\frac{\tau}{2|\log\tau|}\sum_{k=0}^{\lfloor s/\tau\rfloor-1}\left\|\frac{v^\tau_{k+1}-v^\tau_k}{\tau}\right\|^2_{L^2} - M_\delta\int_0^{\tau\lfloor s/\tau\rfloor-\tau} D^\tau_{\lfloor t/\tau\rfloor}\,\mathrm{d}t + \mathrm{o}(1).$$

By the uniform convergence of x^τ to x in $[0,T]$ and the fact that $x \in H^1$, we have that also $x^\tau_{\lfloor\cdot/\tau\rfloor} \to x$ uniformly in $[0,T]$. Hence, passing to the lim inf as $\tau \to 0$, using (i) and (7.11), we get

$$\liminf_{\tau\to0}(\mathcal{W}(v^\tau_0) - \mathcal{W}(v^\tau_{\lfloor s/\tau\rfloor})) \geq \frac{1}{2}\int_0^s \frac{1}{\pi}|\nabla W(x(t))|^2 + \pi|\dot{x}(t)|^2\,\mathrm{d}t - M_\delta\int_0^s D(t)\,\mathrm{d}t, \tag{7.13}$$

where $D(t)$ is defined in (7.10).

Since $\mathcal{W}(v^\tau_0) = \mathcal{W}(v_0) = W(x_0) = W(x(0))$, we have immediately that

$$\liminf_{\tau\to0}(\mathcal{W}(v^\tau_0) - \mathcal{W}(v^\tau_{\lfloor s/\tau\rfloor})) = W(x(0)) - W(x(s)) - D(s). \tag{7.14}$$

Combining this with (7.13) yields

$$W(x(0)) - W(x(s)) - D(s) \geq \frac{1}{2}\int_0^s \frac{1}{\pi}|\nabla W(x(t))|^2 + \pi|\dot{x}(t)|^2\,\mathrm{d}t - M_\delta\int_0^s D(t)\,\mathrm{d}t. \tag{7.15}$$

Since

$$W(x(0)) - W(x(s)) = \int_0^s \langle -\nabla W(x(t)), \dot{x}(t)\rangle\,\mathrm{d}t \leq \frac{1}{2}\int_0^s \frac{1}{\pi}|\nabla W(x(t))|^2 + \pi|\dot{x}(t)|^2\,\mathrm{d}t, \tag{7.16}$$

then,

$$D(s) \leq M_\delta\int_0^s D(t)\,\mathrm{d}t.$$

Since $D(0) = 0$ by assumption, from Gronwall's lemma we find that $D(s) = 0$ for all $s \in [0,T]$.

Using that $D(s) = 0$, by (7.15) and (7.16) we obtain

$$\int_0^s |\frac{1}{\sqrt{\pi}}\nabla W(x(t)) + \sqrt{\pi}\,\dot{x}(t)|^2\,\mathrm{d}t \leq 0,$$

and hence $\dot{x}(t) = -\frac{1}{\pi}\nabla W(x(t))$ a.e. in $[0,T]$. □

The following propositions are devoted to show that the hypothesis of Theorem 7.9 are satisfied by the L^2 discrete gradient flow defined in Definition 7.2.

Proposition 7.10. *Let $\{v_k^\tau\}_{\tau>0}$ be a family of maximal solutions of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0, let k_δ^τ be as in (6.9), and let x^τ be defined as in Definition 7.5. Then*

$$\tilde{T}_\delta = \liminf_{\tau\to 0} k_\delta^\tau \tau \geq \pi \frac{c_\delta^2}{C_\delta}, \tag{7.17}$$

where C_δ and c_δ are defined in (6.12) and (6.8) respectively.

Moreover, there exists a map $x \in H^1([0,\tilde{T}_\delta];\Omega^M)$ such that, up to a subsequence, $x^\tau \to x$ uniformly on $[0,T]$ for every $0 < T < \tilde{T}_\delta$ and

$$\liminf_{\tau\to 0} \frac{\tau}{|\log\tau|} \sum_{k=1}^{\lfloor \frac{T}{\tau} \rfloor} \left\| \frac{v_k^\tau - v_{k-1}^\tau}{\tau} \right\|_{L^2}^2 \geq \pi \int_0^T |\dot{x}(t)|^2 \,\mathrm{d}t. \tag{7.18}$$

Proof. The starting point of the proof consists in applying Theorem A.1 to piecewise affine interpolations in time of suitable regularizations of v_k^τ. Clearly, the Ginzburg-Landau energy of v_k^τ is not bounded. By the very definition of $\mathcal{W}$, we have

$$\frac{1}{2}\int_{\Omega\setminus\cup_i B_\tau(x_{i,k}^\tau)} |\nabla v_k^\tau|^2 \,\mathrm{d}x - M\pi|\log\tau| \leq \mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0).$$

Moreover, the Dirichlet energy stored in $B_\tau(x_{i,k}^\tau)\setminus B_{\tau/2}(x_{i,k}^\tau)$ is bounded. Therefore, by standard cut off arguments, we can easily construct fields $\hat{v}_k^\tau$ which coincide with v_k^τ in $\Omega\setminus\cup_i B_\tau(x_{i,k}^\tau)$, are equal to zero in $B_{\tau/2}(x_{i,k}^\tau)$ and satisfy

$$\frac{1}{2}\int_\Omega |\nabla \hat{v}_k^\tau|^2 \,\mathrm{d}x \leq M\pi|\log\tau| + C. \tag{7.19}$$

Then, we consider the piecewise affine in time interpolation $\hat{v}^\tau : [0,+\infty)\times\Omega \to \mathbb{R}^2$ of $\hat{v}_k^\tau$ defined by

$$\hat{v}^\tau(t,x) := \begin{cases} (1 - \frac{t-k\tau}{\tau})\hat{v}_k^\tau(x) + \frac{t-k\tau}{\tau}\hat{v}_{k+1}^\tau(x) & \text{if } k\tau \leq t \leq (k+1)\tau \leq k_\delta^\tau\tau, \\ \hat{v}_{k_\delta^\tau}^\tau(x) & \text{if } t > k_\delta^\tau\tau. \end{cases}$$

For every fixed $t > 0$, we denote by $\hat{\mu}^\tau(t)$ the (space) Jacobian of $\hat{v}^\tau$.
We will prove the theorem in several steps.

Step 1. There exists a map $x \in C^{0,\frac{1}{2}}([0,+\infty);\Omega^M)$ such that up to a subsequence, for every $T > 0$ we have

$$\hat{\mu}^\tau(t) \overset{\text{flat}}{\to} \mu(t) := \pi\sum_{i=1}^M d_{i,0}\delta_{x_i(t)} \qquad \textit{for every } t \in [0,T]\,. \tag{7.20}$$

Fix $T > 0$. By the convexity of the Dirichlet energy and by (7.19), it follows that for any $t \in [0,T]$

$$\frac{1}{2}\int_\Omega |\nabla \hat{v}^\tau(t,x)|^2 \,\mathrm{d}x \leq M\pi|\log\tau| + C. \tag{7.21}$$

Moreover, by the definition of $\hat{v}_k^\tau$, it follows that for any $k = 0,\dots,k_\delta^\tau - 1$

$$\left\|\hat{v}_{k+1}^\tau - \hat{v}_k^\tau\right\|_{L^2}^2 \leq \left\|v_{k+1}^\tau - v_k^\tau\right\|_{L^2}^2 + C\tau^2;$$

therefore, by (7.4), we get

$$\int_{[0,T]\times\Omega} |\partial_t \hat{v}^\tau|^2 \,\mathrm{d}t\,\mathrm{d}x = \sum_{k=0}^{k_\delta^\tau - 1} \tau \left\| \frac{\hat{v}_{k+1}^\tau - \hat{v}_k^\tau}{\tau} \right\|_{L^2}^2 \le C|\log\tau|\,.$$

It is easy to see that for every $t \in [k\tau, (k+1)\tau]$

$$\frac{1}{\tau}\int_\Omega (1 - |\hat{v}^\tau(t,x)|^2)^2 \,\mathrm{d}x \le \frac{C}{\tau}\left\|\hat{v}_{k+1}^\tau - \hat{v}_k^\tau\right\|_{L^2}^2 \le C|\log\tau|. \tag{7.22}$$

In conclusion, for every $t \in [0,T]$ we have

$$\frac{1}{2}\int_\Omega |\nabla \hat{v}^\tau|^2 + \frac{1}{\tau}(1 - |\hat{v}^\tau|^2)^2 \,\mathrm{d}x \le C|\log\tau|$$

$$\int_{[0,T]\times\Omega} |\partial_t \hat{v}^\tau|^2 \,\mathrm{d}t\,\mathrm{d}x \le C|\log\tau|.$$

By Theorem A.1 applied with $\varepsilon = \sqrt{\tau}$ and recalling that $\mu(0) = \mu_0 = \sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$, we deduce that

$$\mu(t) = \pi\sum_{i=1}^{M(t)} d_{i,0}\delta_{x_i(t)}, \qquad \text{for all } t \in [0,T]$$

for some $x_i(t) \in C^{0,\frac{1}{2}}([0,T_i);\Omega)$ with $T_i \le T$. Here T_i represents the first time when $x_i(t)$ reaches $\partial\Omega$. Finally, by construction, $x_i(t)$ are defined on $[0,T]$, distinct, and contained in Ω. The conclusion follows by a standard diagonalization argument.

Step 2. Set

$$\hat{T}_\delta := \inf\left\{ t \in [0,+\infty) : \ \min\{\frac{1}{2}dist_{i\neq j}(x_i(t),x_j(t)), dist(x_i(t),\partial\Omega)\} \le 2\delta \right\}.$$

Then, $\tilde{T}_\delta \ge \hat{T}_\delta > 0$.

Since $x \in C^{0,\frac{1}{2}}$ and $x(0) = x_0$ satisfies

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,0},x_{j,0}), \mathrm{dist}(x_{i,0},\partial\Omega)\} > 2\delta,$$

we have $\hat{T}_\delta > 0$. Fixed $t > \tilde{T}_\delta$, by construction and Step 1 we have that

$$\hat{\mu}^\tau(t) = \hat{\mu}^\tau_{k_\delta^\tau \tau} \overset{\text{flat}}{\to} \pi\sum_{i=1}^M d_{i,0}\delta_{x_i(\tilde{T}_\delta)}. \tag{7.23}$$

Set $\mu^\tau(t) := \pi\sum_{i=1}^M d_{i,0}\delta_{x_i^\tau(t)}$ for $t \le k_\delta^\tau\tau$, where x_i^τ are defined in Definition 7.5. Let $0 \le k \le k_\delta^\tau$. Since $\mathrm{supp}(\hat{\mu}^\tau(k\tau))$, $\mathrm{supp}(\mu^\tau(k\tau)) \subseteq \cup_i B_\tau(x_{i,k}^\tau)$ and $\hat{\mu}^\tau(k\tau)(B_\tau(x_{i,k}^\tau)) = \mu^\tau(k\tau)(B_\tau(x_{i,k}^\tau))$, for any $\varphi \in C_c^{0,1}(\Omega)$ we have

$$\begin{aligned}\langle \hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau), \varphi\rangle &= \sum_{i=1}^M \langle \hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau), \varphi - \bar{\varphi}_i\rangle \\ &\le (|\hat{\mu}^\tau(k\tau)|(\Omega) + |\mu^\tau(k\tau)|(\Omega))\,\tau\|\nabla\varphi\|_{L^\infty}\,,\end{aligned}$$

where $\bar{\varphi}_i$ denotes the average of φ on $B_\tau(x_{i,k}^\tau)$. Since, by Remark 4.4, we have

$$|\hat{\mu}^\tau(k\tau)|(\Omega) \le C\sum_{i=1}^M \int_{B_\tau(x_{i,k}^\tau)\setminus B_{\frac{\tau}{2}}(x_{i,k}^\tau)} |\nabla \hat{v}_k^\tau|^2 \,\mathrm{d}x \le C\,,$$

we deduce that

$$\max_{k=0,1,\dots,k_\delta^\tau} \|\hat\mu^\tau(k\tau)-\mu^\tau(k\tau)\|_{\mathrm{flat}} \le C\tau\,. \tag{7.24}$$

This fact together with (7.23) yields

$$\sum_{i=1}^{M_0} d_{i,0}\delta_{x^\tau_{i,k_\delta^\tau}} \overset{\mathrm{flat}}{\to} \sum_{i=1}^{M} d_{i,0}\delta_{x_i(\tilde T_\delta)}.$$

Therefore, by the very definition of k_δ^τ, we have that for every $t>\tilde T_\delta$

$$\min\{\frac12 \mathrm{dist}_{i\neq j}(x_i(t),x_j(t)),\mathrm{dist}(x_i(t),\partial\Omega)\}\le 2\delta.$$

By continuity, the previous inequality holds also for $t=\tilde T_\delta$, so that we conclude that $\tilde T_\delta\ge \hat T_\delta>0$.

Step 3. $x^\tau\to x$ uniformly on the compact subsets of $[0,\tilde T_\delta)$.

Let us show that

$$\max_{k=0,1,\dots,k_\delta^\tau} \|\hat\mu^\tau(k\tau)-\mu(k\tau)\|_{\mathrm{flat}} =: \|\hat\mu^\tau(\bar k^\tau\tau)-\mu(\bar k^\tau\tau)\|_{\mathrm{flat}}\to 0\,. \tag{7.25}$$

Up to a subsequence we can assume that $\bar k^\tau\tau$ converges to some $t_0\in[0,\tilde T_\delta]$. The fields

$$\tilde v^\tau(t,x):=\begin{cases}\hat v^\tau(t,x) & \text{if } t\le \bar k^\tau\tau\\ \hat v^\tau(\bar k^\tau\tau,x) & \text{if } t>\bar k^\tau\tau\end{cases} \tag{7.26}$$

satisfy the assumptions of Theorem A.1, applied with $\varepsilon=\sqrt\tau$; therefore, denoting by $\tilde\mu^\tau(t)$ the (space) Jacobian of $\tilde v^\tau$, we have that, up to a subsequence,

$$\tilde\mu^\tau(t)\overset{\mathrm{flat}}{\to}\tilde\mu(t):=\begin{cases}\mu(t) & \text{if } t\le t_0\\ \mu(t_0) & \text{if } t>t_0\,,\end{cases} \tag{7.27}$$

where the structure of $\tilde\mu$ is a consequence of the continuity guaranteed by Theorem A.1. From (7.27) one can easily prove that $\hat\mu^\tau(\bar k^\tau\tau)-\mu(t_0)$ converges to zero in the flat norm and hence we get (7.25). Combining (7.24) with (7.25) we also deduce that

$$\max_{k=0,1,\dots,k_\delta^\tau} \|\mu^\tau(k\tau)-\mu(k\tau)\|_{\mathrm{flat}}\to 0. \tag{7.28}$$

Moreover, by the construction of μ^τ and (7.6), we have that

$$\max_{t\in[0,k_\delta^\tau\tau]} \|\mu^\tau(t)-\mu^\tau(\lfloor t/\tau\rfloor\tau)\|_{\mathrm{flat}}\to 0. \tag{7.29}$$

Using (7.29), (7.28) and that $\max_{t\in[0,k_\delta^\tau\tau]}\|\mu(\lfloor t/\tau\rfloor\tau)-\mu(t)\|_{\mathrm{flat}}\to 0$, by the triangular inequality we conclude that

$$\max_{t\in[0,k_\delta^\tau\tau]}|x^\tau(t)-x(t)|=\max_{t\in[0,k_\delta^\tau\tau]}\|\mu^\tau(t)-\mu(t)\|_{\mathrm{flat}}\to 0.$$

Step 4. The function x belongs to $H^1([0,\tilde T_\delta];\Omega^M)$, and, for any $T\in[0,\tilde T_\delta]$, (7.18) holds true. In particular, $\tilde T_\delta\ge \pi c_\delta^2/C_\delta$.

The proof of this step is obtained as a consequence of Proposition A.3 applied to the fields $\hat v^\tau$, with $\varepsilon = \tau$ and $\tilde T = \tilde T_\delta$. By (7.4) and recalling (7.22), it easily follows that

$$\frac{1}{\tau^2}\int_0^{\tilde T_\delta}\int_\Omega (1-|\hat v^\tau|^2)^2 \,\mathrm{d}x\,\mathrm{d}t \le C\sum_{k=1}^{k_\delta^\tau}\tau\left\|\frac{\hat v_{k+1}^\tau-\hat v_k^\tau}{\tau}\right\|_{L^2}^2 \le C|\log\tau|;$$

and hence (A.5) holds with $\varepsilon = \tau$ and $w_\varepsilon = v^\tau$. This fact together with (7.20) and (7.21) guarantees that the hypothesis of Proposition A.5 are satisfied. Therefore, we deduce that (7.18) holds true with v_k^τ replaced by $\hat v_k^\tau$. Since $\|\hat v_k^\tau - v_k^\tau\|_{L^2} = \mathrm{O}(\tau^2)$, we deduce (7.18).

Finally, by (7.4) and recalling (6.12), we have

$$\begin{aligned}(7.30)\qquad \pi\int_0^T |\dot x(t)|^2\,\mathrm{d}t &\le \liminf_{\tau\to 0}\frac{\tau}{|\log\tau|}\sum_{k=1}^{\lfloor\frac{T}{\tau}\rfloor}\left\|\frac{v_k^\tau - v_{k-1}^\tau}{\tau}\right\|_{L^2}^2\\ &\le \liminf_{\tau\to0}(\mathcal{W}(v_0)-\mathcal{W}(v^\tau_{\lfloor\frac{T}{\tau}\rfloor})) \le \liminf_{\tau\to 0}(W(x_0)-W(x^\tau_{\lfloor\frac{T}{\tau}\rfloor}))\le C_\delta.\end{aligned}$$

By Hölder inequality, and recalling (6.8), we conclude

$$(7.31)\qquad c_\delta \le |x(\tilde T_\delta)-x(0)| \le \int_0^{\tilde T_\delta}|\dot x|\,\mathrm{d}t \le \|\dot x\|_{L^2([0,\tilde T_\delta];\mathbb{R}^{2M})}\sqrt{\tilde T_\delta}.$$

By (7.30) and (7.31) we immediately get (7.17) □

Since we have proved assumption (i) in Theorem 7.10, it remains to prove only assumption (ii). To this aim, at each time step $k = 0,1,\dots,k_\delta^\tau$, we construct, a field w_{k+1}^τ whose vortices are obtained translating $x_{i,k}^\tau$ in the direction of the renormalized energy $\nabla W(x_k^\tau)$. The variation of the energy $\mathcal{W}$ associated to the fields v_k^τ and w_{k+1}^τ is proportional to the distance among the vortices of the two functions (i.e. $|\nabla W(x_k^\tau)|$) up to an error given by the energy excess D_k^τ defined in (7.11).

Proposition 7.11. *For any $k = 0,1,\dots,k_\delta^\tau - 1$, there exists a field $w_{k+1}^\tau \in H^1_{\mathrm{loc}}(\Omega\setminus \bigcup_{i=1}^M\left\{x_{i,k}^\tau - \frac{\tau}{\pi}\partial_{x_i}W(x_k^\tau)\right\};\mathcal{S}^1)$ such that*

$$(7.32)\qquad \mathcal{W}(v_k^\tau)-\mathcal{W}(w_{k+1}^\tau) \ge \frac{\tau}{\pi}|\nabla W(x_k^\tau)|^2 - M_\delta\tau D_k^\tau + \mathrm{o}(\tau)$$

$$(7.33)\qquad \frac{\left\|w_{k+1}^\tau - v_k^\tau\right\|_{L^2}^2}{\tau^2|\log\tau|} \le \frac{1}{\pi}|\nabla W(x_k^\tau)|^2 + \mathrm{o}(1),$$

where M_δ is a positive constant depending only on δ.

Proof. Fix $k \in \{0,1,\dots,k_\delta^\tau - 1\}$; to ease notations we set

$$(7.34)\qquad V_i = (V_{i1},V_{i2}) := -\frac{1}{\pi}\partial_{x_i}W(x_k^\tau),\qquad V := (V_1,\dots,V_M).$$

With a little abuse of notations, from now on we will set $x_i := x_{i,k}^\tau$ and $y_i = x_i + \tau V_i$ for every $i = 1,\dots,M$. By (6.9), the balls $B_{\delta/2}(x_i)$ are pairwise disjoint and contained in Ω.

In order to construct the field w_{k+1}^τ, we wish to “push” the vortices x_i along the direction V_i. For every $i = 1,\dots,M$, we can find smooth, compactly supported vector fields in Ω, X_{i1} and X_{i2} such that

$$X_{i1}(x) = (1,0) \quad X_{i2}(x) = (0,1) \qquad \text{for } x \in B_{\delta/2}(x_i),$$
$$X_{i1}(x) = X_{i2}(x) = (0,0) \qquad \text{for } x \in B_{\delta/2}(x_j), j \neq i$$

and such that $\|\nabla X_{ij}\|_{L^\infty} \leq \frac{2}{\delta}$ for every i,j. Then, define $X_V = \sum_{i=1}^M \sum_{j=1,2} V_{ij} X_{ij}$. Since W is smooth in K_δ (see (6.11)), there exists a constant M_δ depending only on δ such that

$$\|\det \nabla X_V\|_{L^\infty} \leq \frac{1}{2} M_\delta \,. \tag{7.35}$$

For any $t \in [0,\tau]$, we define $\chi_t(x) := x + tX_V(x)$ for every $x \in \Omega$; notice that $\chi_t(x) = x + tV_i$ for $x \in B_{\delta/2}(x_i)$. For any $t \in [0,\tau]$ let Φ^t be the solution of

$$\begin{cases} \Delta\Phi^t = 2\pi \sum_{i=1}^M d_{i,0}\delta_{x_i+tV_i} & \text{in } \Omega \\ \Phi^t = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$R^t(x) := \Phi^t(x) - \sum_{i=1}^M d_{i,0} \log|x - x_i - tV_i|. \tag{7.36}$$

By definition R^t are smooth harmonic functions in Ω; we denote by $\tilde{R}^t$ its harmonic conjugates with zero average in Ω. Moreover, we denote by θ_i^t the polar coordinates centered at $x_i + tV_i$ and set $\tilde{\Phi}^t := \sum_{i=1}^M d_{i,0}\theta_i^t + \tilde{R}^t$. Notice that $\nabla\tilde{\Phi}^t$ is nothing but the $\pi/2$ rotation of $\nabla\Phi^t$. We define

$$\psi^t(\cdot) = \tilde{\Phi}^t(\chi_t(\cdot)) - \tilde{\Phi}^0(\cdot). \tag{7.37}$$

Notice that ψ^t is a smooth function in Ω, the singularities at x_i canceling out, and that it is smooth in space-time. In particular, using (6.10) one can show that, for τ small enough, there exists a constant C depending only on δ such that

$$\sup_{t\in[0,\tau]} \left(\|\nabla\psi^t\|_{L^\infty(\Omega)} + \|\frac{\mathrm{d}}{\mathrm{d}t}\psi^t\|_{L^\infty(\Omega)} \right) \leq C \,. \tag{7.38}$$

For any $0 < \sigma < \delta$, we define $\Omega_\sigma^t := \Omega \setminus \cup_{i=1}^M B_\sigma(x_i + tV_i)$. By definition of $\tilde{\Phi}^t$, the renormalized energy associated to the configuration $x_k^\tau + tV$ is given by

$$W(x_k^\tau + tV) = \lim_{\sigma\to 0} \frac{1}{2} \int_{\Omega_\sigma^t} |\nabla\tilde{\Phi}^t|^2 - M\pi|\log\sigma|. \tag{7.39}$$

Since $v_k^\tau \in H^1(\Omega_\sigma^0; \mathcal{S}^1)$, there exist a family $\{L_i\}_{i=1,\dots,M}$ of cuts of the domain Ω (L_i is a segment from x_i to $\partial\Omega$) and a function $\varphi^0 \in H^1(\Omega_\sigma^0 \setminus \cup_{i=1}^M \{L_i\}; \mathbb{R})$ such that $v_k^\tau = e^{i\varphi^0}$.

Recalling (7.37), we introduce the field w_{k+1}^τ defined by the following identity (notice that χ_τ is invertible for τ small enough)

$$w_{k+1}^\tau(\chi_\tau(x)) := v_k^\tau(x) e^{i\psi^\tau(x)} = e^{i(\varphi^0(x) + \psi^\tau(x))}. \tag{7.40}$$

By definition, $w_{k+1}^\tau \in H^1(\Omega_\sigma^\tau; \mathcal{S}^1)$ and $Jw_{k+1}^\tau = \sum_{i=1}^M d_{i,0}\delta_{y_i}$.

We notice that if $\varphi^0 = \tilde{\Phi}^0$, then by (7.39) we get

$$\begin{aligned} \mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) &= \lim_{\sigma\to 0} \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla v_k^\tau|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla w_{k+1}^\tau|^2 \,\mathrm{d}y \qquad (7.41)\\ &= \lim_{\sigma\to 0} \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla \tilde{\Phi}^0|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla \tilde{\Phi}^\tau|^2 \,\mathrm{d}y \\ &= W(x_k^\tau) - W(x_k^\tau + \tau V). \qquad (7.42) \end{aligned}$$

Recalling (6.10) and (6.11), by Taylor expansion we conclude

$$W(x_k^\tau) - W(x_k^\tau - \frac{\tau}{\pi}\nabla W(x_k^\tau)) = \frac{\tau}{\pi}|\nabla W(x_k^\tau)|^2 + \mathrm{O}(\tau^2). \qquad (7.43)$$

We show now that w_{k+1}^τ satisfies (7.32) even when v_k^τ is not optimal in energy. To this purpose, we show that the difference $\mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau)$ can be bounded from below by the variation of the renormalized energy up to an error given by the defect D_k^τ defined in (7.9). More precisely, set

$$D_{\sigma,k}^\tau := \frac{1}{2}\int_{\Omega_\sigma^0} (|\nabla\varphi^0|^2 - |\nabla\tilde{\Phi}^0|^2)\,\mathrm{d}x, \qquad (7.44)$$

so that $D_k^\tau = \lim_{\sigma\to 0} D_{\sigma,k}^\tau$. We want to prove that, for $0 < \sigma \ll \tau$,

$$\begin{aligned} \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla v_k^\tau|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla w_{k+1}^\tau|^2 \,\mathrm{d}y \geq \frac{1}{2}\int_{\Omega_\sigma} |\nabla\tilde{\Phi}^0|^2 \,\mathrm{d}x \\ - \frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla\tilde{\Phi}^\tau|^2 \,\mathrm{d}y - M_\delta \tau D_{\sigma,k}^\tau + \mathrm{O}(\sqrt{\sigma|\log\sigma|}). \end{aligned} \qquad (7.45)$$

Notice that, taking the limit as $\sigma \to 0$ in (7.45), we get

$$\mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) \geq W(x_k^\tau) - W(x_k^\tau - \tau\nabla W(x_k^\tau)) - M_\delta \tau D_k^\tau,$$

which, in view of (7.43), concludes the proof of (7.32).

We now prove (7.45). By the change of variable $y = \chi_\tau(x)$ and by definition of w_{k+1}^τ in (7.40), we get

$$\begin{aligned} \frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla w_{k+1}^\tau|^2 \,\mathrm{d}y = \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla w_{k+1}^\tau(\chi_\tau)|^2 |J\chi_\tau| \,\mathrm{d}x \\ = \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla\varphi^0 + \nabla\psi^\tau|^2 |J\chi_\tau| \,\mathrm{d}x \end{aligned} \qquad (7.46)$$

We claim that the following two estimates hold:

$$\frac{1}{2}\int_{\Omega_\sigma^0} |\nabla\varphi^0|^2 |J\chi_\tau| \,\mathrm{d}x \leq \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla\tilde{\Phi}^0|^2 |J\chi_\tau| \,\mathrm{d}x + (1 + M_\delta\tau) D_{\sigma,k}^\tau, \qquad (7.47)$$

$$\int_{\Omega_\sigma^0} \langle\nabla\psi^\tau, \nabla\varphi^0\rangle |J\chi_\tau| \,\mathrm{d}x = \int_{\Omega_\sigma^0} \langle\nabla\psi^\tau, \nabla\tilde{\Phi}^0\rangle |J\chi_\tau| \,\mathrm{d}x + \mathrm{O}(\sqrt{\sigma|\log\sigma|}). \qquad (7.48)$$

By (7.47) and (7.48), we conclude the proof of (7.45) as follows: Using (7.37) and the change of variables $y = \chi_\tau(x)$, by (7.46) we get

$$\begin{aligned}(7.49)\quad &\frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla w_{k+1}^\tau|^2 \,\mathrm{d}y \\ &\leq \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla\tilde{\Phi}^0 + \nabla\psi^\tau|^2 |J\chi_\tau| \,\mathrm{d}x + (1+M_\delta\tau)D_{\sigma,k}^\tau + \mathrm{O}(\sqrt{\sigma|\log\sigma|}) \\ &= \frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla\tilde{\Phi}^\tau|^2 \,\mathrm{d}y + (1+M_\delta\tau)D_{\sigma,k}^\tau + \mathrm{O}(\sqrt{\sigma|\log\sigma|})\,.\end{aligned}$$

By (7.44) and straightforward algebraic manipulations we obtain (7.45).

Now, we will prove the claims (7.47) and (7.48). Claim (7.47) follows by

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\sigma^0} (|\nabla\varphi^0|^2 - |\nabla\tilde{\Phi}^0|^2)|J\chi_\tau| \,\mathrm{d}x &\leq \|J\chi_\tau\|_{L^\infty} D_{\sigma,k}^\tau \\ &\leq (1+\frac{1}{2}M_\delta\tau + O(\tau^2))D_{\sigma,k}^\tau \leq (1+M_\delta\tau)D_{\sigma,k}^\tau.\end{aligned}$$

We pass to the proof of (7.48). We have

$$\begin{aligned}(7.50)\quad &\int_{\Omega_\sigma^0} \langle\nabla\psi^\tau, \nabla\varphi^0\rangle |J\chi_\tau| \,\mathrm{d}x \\ &= \int_{\Omega_\sigma^0} \langle\nabla\psi^\tau, \nabla\tilde{\Phi}^0\rangle |J\chi_\tau| \,\mathrm{d}x + \int_{\Omega_\sigma^0} \langle\nabla\psi^\tau, \nabla\varphi^0 - \nabla\tilde{\Phi}^0\rangle |J\chi_\tau| \,\mathrm{d}x.\end{aligned}$$

Using again that $\|J\chi_\tau\|_{L^\infty} \leq 1 + M_\delta\tau$ and Hölder inequality, we get

$$\begin{aligned}(7.51)\quad &\int_{\Omega_\sigma^0} \langle\nabla\psi^\tau, \nabla\varphi^0 - \nabla\tilde{\Phi}^0\rangle |J\chi_\tau| \,\mathrm{d}x \\ &\leq (1+M_\delta\tau)\left(\int_{\Omega_\sigma^0} |\nabla\psi^\tau|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\Omega_\sigma^0} |\nabla\varphi^0 - \nabla\tilde{\Phi}^0|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

Moreover, since $\mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0)$, we have

$$\int_{\Omega_\sigma^0} |\nabla\varphi^0 - \nabla\tilde{\Phi}^0|^2 \,\mathrm{d}x \leq 2\int_{\Omega_\sigma^0} (|\nabla\varphi^0|^2 + |\nabla\tilde{\Phi}^0|^2) \,\mathrm{d}x \leq 4\mathcal{W}(v_0) + 4M\pi|\log\sigma| + o_\sigma(1).$$

By (7.37), since X_V has compact support in Ω we have

$$\frac{\partial\psi^t}{\partial\nu} = \frac{\partial\Phi^t}{\partial\nu^\perp} - \frac{\partial\Phi}{\partial\nu^\perp} = 0 \qquad \text{on } \partial\Omega\,.$$

Therefore, in view of (7.38),

$$\int_{\Omega_\sigma^0} |\nabla\psi^\tau|^2 \,\mathrm{d}x = \int_{\partial\Omega} \psi^\tau \frac{\partial\psi^\tau}{\partial\nu} \,\mathrm{d}s - \sum_{i=1}^M \int_{B_\sigma(x_i)} |\nabla\psi^\tau|^2 \,\mathrm{d}s \leq C\sigma^2.$$

Combining the above estimates with (7.50) and (7.51) we get (7.48).

To complete the proof it remains to show that (7.33) holds. By definition of $w^\tau_{k+1}(x)$ (see (7.40)), we have immediately

$$\begin{aligned}\left\|v^\tau_k - w^\tau_{k+1}\right\|^2_{L^2} &= \int_\Omega \left|v^\tau_k - v^\tau_k(\chi_\tau^{-1})e^{i\psi^\tau(\chi_\tau^{-1})}\right|^2 \,\mathrm{d}y \\
&= \int_\Omega \left|v^\tau_k - v^\tau_k(\chi_\tau^{-1})\right|^2 \,\mathrm{d}y \qquad (7.52)\\
&+ \int_\Omega \left|v^\tau_k(\chi_\tau^{-1}) - v^\tau_k(\chi_\tau^{-1})e^{i\psi^\tau(\chi_\tau^{-1})}\right|^2 \,\mathrm{d}y \qquad (7.53)\\
&+ 2\int_\Omega \langle v^\tau_k - v^\tau_k(\chi_\tau^{-1}), v^\tau_k(\chi_\tau^{-1}) - v^\tau_k(\chi_\tau^{-1})e^{i\psi^\tau(\chi_\tau^{-1})}\rangle \,\mathrm{d}y. \qquad (7.54)\end{aligned}$$

In order to prove (7.33) it is enough to show that

$$\int_\Omega \left|v^\tau_k - v^\tau_k(\chi_\tau^{-1})\right|^2 \,\mathrm{d}y \le \pi\tau^2|\log\tau|\;|V|^2 + \mathrm{o}(\tau^2|\log\tau|)\ , \qquad (7.55)$$

$$\int_\Omega \left|v^\tau_k(\chi_\tau^{-1}) - v^\tau_k(\chi_\tau^{-1})e^{i\psi^\tau(\chi_\tau^{-1})}\right|^2 \,\mathrm{d}y \le C\tau^2; \qquad (7.56)$$

indeed, once we got (7.55) and (7.56), by Hölder inequality, we have immediately that the integral in (7.54) is $\mathrm{O}(\tau^2\sqrt{|\log\tau|})$.

First, we prove (7.56). By the change of variable $y = \chi_\tau(x)$ and the fact that $\psi^0 = 0$, in view of (7.38), we obtain

$$\begin{aligned}\int_\Omega \left|v^\tau_k(\chi_\tau^{-1}) - v^\tau_k(\chi_\tau^{-1})e^{i\psi^\tau(\chi_\tau^{-1})}\right|^2 \,\mathrm{d}y &= \int_\Omega \left|1 - e^{i\psi^\tau}\right|^2 |J\chi_\tau| \,\mathrm{d}x \\
&\le (1 + M_\delta\tau)\|\frac{\mathrm{d}}{\mathrm{d}t}\psi^t\|^2_{L^\infty(\Omega)}\tau^2|\Omega| \le C\tau^2\,.\end{aligned}$$

Finally, to complete the proof of the Theorem it remains to show that (7.55) holds. By Hölder inequality, we have

$$\begin{aligned}&\int_{\Omega\setminus\cup_i B_{\delta/2}(y_i)} |v^\tau_k - v^\tau_k(\chi_\tau^{-1})|^2 \,\mathrm{d}y \\
&\le \int_{\Omega\setminus\cup_i B_{\delta/2}(y_i)} \tau\int_0^\tau |\nabla v^\tau_k(\chi_t^{-1})|^2\|\frac{\mathrm{d}}{\mathrm{d}t}\chi_t^{-1}\|^2_{L^\infty} \,\mathrm{d}y\,\mathrm{d}t \\
&\le C(1+M_\delta\tau)\tau^2(\mathcal{W}(v^\tau_k) + M\pi|\log\frac{\delta}{2}|) \le C\tau^2\,,\end{aligned}$$

where C depends only on δ and we have used that $\mathcal{W}(v^\tau_k) \le \mathcal{W}(v_0)$.

In order to complete the proof of(7.55), it is enough to show that

$$\int_{B_{\delta/2}(y_i)} |v^\tau_k - v^\tau_k(\chi_\tau^{-1})|^2 \,\mathrm{d}y \le \pi\tau^2|\log\tau||V_i|^2 + \mathrm{o}(\tau^2|\log\tau|). \qquad (7.57)$$

Let $N > 0$ be given; then, for any $i = 1,\dots,M$,

$$\int_{B_{\delta/2}(y_i)} |v^\tau_k(\chi_\tau^{-1}) - v^\tau_k|^2 \,\mathrm{d}y \le \int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)} |v^\tau_k(\chi_\tau^{-1}) - v^\tau_k|^2 \,\mathrm{d}y + 4N^2\tau^2\pi. \qquad (7.58)$$

Without loss of generality we can assume $d_{i,0} = \deg(v^\tau_k, \partial B_{\delta/2}(x_i)) = 1$.

We first show the estimate (7.57) in the case $v^\tau_k = \frac{x-x_i}{|x-x_i|}$ in $B_{\delta/2}(x_i)$. Let (r,θ) be the polar coordinates with respect to y_i; denoting by $\alpha = \alpha(r,\theta)$ the angle between the vectors $\frac{y-y_i}{|y-y_i|}$ and $v_k^\tau(y)=\frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|}$, we have

(7.59)
$$\int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)}\left|\frac{y-y_i}{|y-y_i|}-\frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|}\right|^2\,\mathrm{d}y=\int_{N\tau}^{\delta/2}r\,\mathrm{d}r\int_0^{2\pi}4\sin^2\frac{\alpha}{2}\,\mathrm{d}\theta.$$

Using elementary geometry identities and Taylor expansion, for $N\tau\le r\le\delta/2$ we get

$$\sin\alpha=\frac{\tau|V_i|\sin\theta}{r}\frac{1}{\sqrt{1+\frac{\tau^2|V_i|^2}{r^2}-2\frac{\tau|V_i|\cos\theta}{r}}}=\frac{\tau|V_i|\sin\theta}{r}(1+\mathrm{O}(1/N)),$$

so that $\sin^2\frac{\alpha}{2}=\frac{\tau^2|V_i|^2\sin^2\theta}{4r^2}+\mathrm{O}(1/N)$. Therefore, by (7.59) we get

$$\begin{aligned}(7.60)\quad &\int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)}\left|\frac{y-y_i}{|y-y_i|}-\frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|}\right|^2\,\mathrm{d}y\\ &=\tau^2|V_i|^2\int_{N\tau}^{\delta/2}\frac{1}{r}\,\mathrm{d}r\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta+\mathrm{O}(1/N)\\ &=\pi\tau^2|\log\tau||V_i|^2+\pi\tau^2\log\frac{\delta}{2N}|V_i|^2+\mathrm{O}(1/N)).\end{aligned}$$

Then, (7.57) follows (in the case $v_k^\tau=\frac{x-x_i}{|x-x_i|}$) by choosing $N=|\log\tau|$.

Now, we prove (7.57) in the general case, i.e., without assuming $v_k^\tau=\frac{x-x_i}{|x-x_i|}$. Set $L:=\lfloor\log_2\frac{\delta}{2N\tau}\rfloor$ and let θ_i be the angle in polar coordinates with center in y_i, i.e., the phase of the function $\frac{y-y_i}{|y-y_i|}$. For every $l=1,\dots,L$, we set

$$C_l(y_i):=B_{2^{-l}\delta}(y_i)\setminus B_{2^{-l-1}\delta}(y_i),\quad \tilde C_l(y_i):=B_{2^{-l+1}\delta}(y_i)\setminus B_{2^{-l-2}\delta}(y_i).$$

Set $\tilde\varphi^0_{i,l}=\frac{1}{|\tilde C_l(y_i)|}\int_{\tilde C_l(y_i)}\varphi^0(x)\,\mathrm{d}y$ and notice that the average of θ_i is equal to π. We have

$$\begin{aligned}&\int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)}|v_k^\tau(\chi_\tau^{-1})-v_k^\tau|^2\,\mathrm{d}y=\sum_{l=1}^L\int_{C_l(y_i)}|v_k^\tau(\chi_\tau^{-1})-v_k^\tau|^2\,\mathrm{d}y\\ &=\sum_{l=1}^L\int_{C_l(y_i)}|e^{i(\varphi^0(\chi_\tau^{-1})-\tilde\varphi^0_{i,l}+\pi)}-e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi)}|^2\,\mathrm{d}y\\ &=\sum_{l=1}^L\int_{C_l(y_i)}|e^{i\theta_i(\chi_\tau^{-1})}-e^{i\theta_i}|^2\,\mathrm{d}y\\ &+\int_{C_l(y_i)}|e^{i(\varphi^0(\chi_\tau^{-1})-\tilde\varphi^0_{i,l}+\pi)}-e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi)}-(e^{i\theta_i(\chi_\tau^{-1})}-e^{i\theta_i})|^2\,\mathrm{d}y\\ &+2\int_{C_l(y_i)}\langle e^{i\theta_i(\chi_\tau^{-1})}-e^{i\theta_i},e^{i(\varphi^0(\chi_\tau^{-1})-\tilde\varphi^0_{i,l}+\pi)}-e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi)}-(e^{i\theta_i(\chi_\tau^{-1})}-e^{i\theta_i})\rangle\,\mathrm{d}y\,.\end{aligned}$$

Estimating the last term of the right hand side of the above formula by Hölder's inequality and recalling (7.60), in order to prove (7.57) it is enough to show the following estimate

$$\sum_{l=1}^L\int_{C_l(y_i)}\left|e^{i(\varphi^0(\chi_\tau^{-1})-\tilde\varphi^0_{i,l}+\pi)}-e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi)}-(e^{i\theta_i(\chi_\tau^{-1})}-e^{i\theta_i})\right|^2\,\mathrm{d}y\;\le\;C\tau^2\,.$$

By definition of χ_τ, for any $y \in C_l(y_i)$, $\chi_\tau^{-1}(y) = y - \tau V_i$ and then

$$\begin{aligned} e^{i(\varphi^0(\chi_\tau^{-1}(y))-\tilde\varphi^0_{i,l}+\pi)} - e^{i(\varphi^0(y)-\tilde\varphi^0_{i,l}+\pi)} &= -\int_0^\tau \nabla e^{i(\varphi^0(y-tV_i)-\tilde\varphi^0_{i,l}+\pi)} \cdot V_i \,\mathrm{d}t \ , \\ e^{i\theta_i(\chi_\tau^{-1}(y))} - e^{i\theta_i(y)} &= -\int_0^\tau \nabla e^{i\theta_i(y-tV_i)} \cdot V_i \,\mathrm{d}t \ ; \end{aligned}$$

then, by Jensen and Cauchy inequalities,

$$\begin{aligned} &\int_{C_l(y_i)} \left| e^{i(\varphi^0(\chi_\tau^{-1})-\tilde\varphi^0_{i,l}+\pi)} - e^{i(\varphi_k^\tau-\tilde\varphi^0_{i,l}+\pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i}) \right|^2 \,\mathrm{d}y \\ &= \int_{C_l(y_i)} \left| \int_0^\tau (\nabla e^{i\theta_i(y-tV_i)} - \nabla e^{i(\varphi^0(y-tV_i)-\tilde\varphi^0_{i,l}+\pi)}) \cdot V_i \,\mathrm{d}t \right|^2 \,\mathrm{d}y \\ &\leq \tau|V_i|^2 \int_{C_l(y_i)} \mathrm{d}y \int_0^\tau \left| \nabla e^{i\theta_i(y-tV_i)} - \nabla e^{i(\varphi^0(y-tV_i)-\tilde\varphi^0_{i,l}+\pi)} \right|^2 \,\mathrm{d}t \\ &\leq \tau^2|V_i|^2 \int_{\tilde C_l(y_i)} \left| \nabla e^{i\theta_i} - \nabla e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi)} \right|^2 \,\mathrm{d}y. \end{aligned} \tag{7.61}$$

Furthermore

$$\begin{aligned} &\sum_{l=1}^L \int_{\tilde C_l(y_i)} |\nabla e^{i\theta_i} - \nabla e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi)}|^2 \,\mathrm{d}y \\ &\leq 2\sum_{l=1}^L \int_{\tilde C_l(y_i)} |\nabla e^{i\theta_i}|^2 |1 - e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi-\theta_i)}|^2 \,\mathrm{d}y + \int_{\tilde C_l(y_i)} |\nabla(\theta_i - \varphi^0)|^2 \,\mathrm{d}y \\ &\leq 2\sum_{l=1}^L \int_{\tilde C_l(y_i)} 2^{2l+4}\delta^{-2} |e^{i(\theta_i-\varphi^0)} - e^{i(-\tilde\varphi^0_{i,l}+\pi)}|^2 \,\mathrm{d}y + \int_{\tilde C_l(y_i)} |\nabla(\theta_i - \varphi^0)|^2 \,\mathrm{d}y, \end{aligned} \tag{7.62}$$

where the last inequality follows from the fact that $|\nabla e^{i\theta_i}(y)|^2 = \frac{1}{|y-y_i|^2}$ and that $2^{-l-2}\delta \leq |y - y_i| \leq 2^{-l+1}\delta$ for $y \in \tilde C_l(x_i)$.

Finally, by Poincaré inequality, it follows that

$$\begin{aligned} &\int_{\tilde C_l(y_i)} |e^{i(\theta_i-\varphi^0)} - e^{i(-\tilde\varphi^0_{i,l}+\pi)}|^2 \,\mathrm{d}x \\ &\leq \int_{\tilde C_l(x_i)} |\theta_i - \varphi^0 - (\pi - \tilde\varphi^0_{i,l})|^2 \,\mathrm{d}x \leq C 2^{-2l}\delta^2 \int_{\tilde C_l(y_i)} |\nabla(\theta^i - \varphi^0)|^2 \,\mathrm{d}y \ , \end{aligned} \tag{7.63}$$

where C is a positive constant. By the minimality of θ_i, we have

$$\int_{\tilde C_l(y_i)} |\nabla(\varphi^0 - \theta_i)|^2 \,\mathrm{d}y = \int_{\tilde C_l(y_i)} |\nabla\varphi^0|^2 - \int_{\tilde C_l(y_i)} |\nabla\theta_i|^2 \,\mathrm{d}y \ .$$

By (7.62) and Remark 4.4, we obtain

$$\begin{aligned}\sum_{l=1}^{L}\int_{\tilde{C}_l(y_i)}&|\nabla e^{i(\varphi^0-\tilde{\varphi}^\tau_{i,l}+\pi)}-\nabla e^{i\theta_i}|^2\,\mathrm{d}y\\ &\le C\sum_{l=1}^{L}\int_{\tilde{C}_l(y_i)}(|\nabla\varphi^0|^2-|\nabla\theta_i|^2)\,\mathrm{d}y\\ =C\sum_{l=1}^{L}&\left(\int_{\tilde{C}_l(y_i)}|\nabla\varphi^0|^2\,\mathrm{d}x-6\pi\log 2\right)\le C\,.\end{aligned}$$

This together with (7.61) concludes the proof. □

We are now in a position to prove Theorem 7.7.

Proof of Theorem 7.7. By Theorems 7.10, 7.11, we can apply Theorem 7.9 for any $T<\pi\frac{c_\delta^2}{C_\delta}$, and in view of (7.17) we obtain that

1) x^τ converges to the solution x of (6.7), uniformly on $[0,T]$;
2) $D(T)=0$.

Let $T^{max}\le\liminf_{\tau\to0}k_\delta^\tau\tau$ be the maximal time such that 1) and 2) hold true on $[0,T)$ for every $T<T^{max}$. Recalling (see (6.17)) that $T_\delta\to T^*$ as $\delta\to0$, it remains only to prove that $T^{max}\ge T_\delta$. This follows by a standard continuation argument: Assume by contradiction that $T^{max}<T_\delta$, and let $T<T^{max}$. Then we have

$$\min_{t\in[0,T^{max}]}\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_i(t),x_j(t)),\mathrm{dist}(x_i(t),\partial\Omega)\}-2\delta=c'_\delta>0.$$

Consider now $x^\tau_{\lfloor T/\tau\rfloor}$, $v^\tau_{\lfloor T/\tau\rfloor}$ as the initial condition of a new L^2 discrete gradient flow. Notice that, in view of 2), these initial conditions are well prepared; the fact that the initial time is not zero is not relevant, since all the equations are autonomous. Moreover, even if the initial conditions depend on τ, they converge as $\tau\to0$. Therefore, Theorems 7.10, 7.11, and Theorem 7.9 still hold true with the obvious modifications, and we easily deduce that 1) and 2) holds true as long as $0\le t-T\le(c'_\delta)^2/C_\delta$. This time interval in which we can extend the solution is independent of $T<T^{max}$, which contradicts the maximality of T^{max}. □

7.2. **L^2 discrete gradient flow of F_ε.** We conclude this section by analyzing the existence of the L^2 discrete gradient flow of F_ε and studying its asymptotic behaviour as $\varepsilon\to0$. The existence will be obtained for ε small enough by making use of the auxiliary problem studied in the previous section. To this aim it is convenient to introduce a relaxed version of such discrete evolution.

Definition 7.12. *Fix $\delta>0$ and let $\varepsilon,\tau>0$. Given $u_{\varepsilon,0}\in\mathcal{AF}_\varepsilon(\Omega)$, we say that $\{\bar{u}^\tau_{\varepsilon,k}:k\in\mathbb{N}\}$, is a solution of the relaxed L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$ if $\bar{u}^\tau_{\varepsilon,0}=u_{\varepsilon,0}$ and, for any $k\in\mathbb{N}$, there exists a sequence $\{u^\tau_{\varepsilon,k,n}\}_n$ such that*

$$\begin{aligned}\lim_{n\to\infty}\|e^{2\pi iu^\tau_{\varepsilon,k,n}}-e^{2\pi i\bar{u}^\tau_{\varepsilon,k}}\|_{L^2}=0,\\ \|\mu(u^\tau_{\varepsilon,k,n})-\mu(\bar{u}^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}}\le\delta\quad\textit{for every } n\in\mathbb{N},\\ \lim_{n\to\infty}F_\varepsilon(u^\tau_{\varepsilon,k,n})+\frac{\|e^{2\pi iu^\tau_{\varepsilon,k,n}}-e^{2\pi i\bar{u}^\tau_{\varepsilon,k-1}}\|^2_{L^2}}{2\tau|\log\tau|}=I^\tau_{\varepsilon,k}\,,\end{aligned}\tag{7.64}$$

where

$$I^\tau_{\varepsilon,k} = \inf_{u\in\mathcal{AF}_\varepsilon(\Omega)} \left\{ F_\varepsilon(u) + \frac{\|e^{2\pi i u} - e^{2\pi i \bar u^\tau_{\varepsilon,k-1}}\|^2_{L^2}}{2\tau|\log\tau|} \ : \ \|\mu(u) - \mu(\bar u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}} \le \delta \right\}.$$

The existence of such relaxed discrete gradient flow is obvious. To show that it is actually a strong L^2 discrete gradient flow it is enough to show that $\|\mu(\bar u^\tau_{\varepsilon,k}) - \mu(\bar u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}} \le \delta$. A key argument is given by the following estimate that one can easly check by contradiction

(7.65)
$$\limsup_{n\to+\infty} \|\mu(u^\tau_{\varepsilon,k,n}) - \mu(\bar u^\tau_{\varepsilon,k})\|_{\mathrm{flat}} \le C\varepsilon\sharp\{(i,j)\in\Omega^1_\varepsilon : \mathrm{dist}(\bar u^\tau_{\varepsilon,k}(i) - \bar u^\tau_{\varepsilon,k}(j),\mathbb{Z}) = \tfrac12\}$$

Theorem 7.13. *Let v_0 be such that $\mathcal{W}(v_0) < +\infty$ and let $Jv_0 = \sum_{i=1}^M d_{i,0}\delta_{x_{i,0}} =: \mu_0$ with $|d_{i,0}| = 1$. Let $u_{\varepsilon,0}\in\mathcal{AF}_\varepsilon(\Omega)$ such that*

$$\mu(u_{\varepsilon,0}) \stackrel{\mathrm{flat}}{\to} \mu_0, \qquad F_\varepsilon(u_{\varepsilon,0}) \le \pi|\mu_0|(\Omega)\log\varepsilon + C.$$

Let $\delta > 0$ be fixed such that $\min\left\{\frac12 dist_{i\neq j}(x_{i,0},x_{j,0}), dist(x_{i,0},\partial\Omega)\right\} > 2\delta$. Given $\tau > 0$, let $\bar u^\tau_{\varepsilon,k}$ be a solution of the relaxed L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$.

Then, up to a subsequence, for any $k\in\mathbb{N}$ we have $\mu(\bar u^\tau_{\varepsilon,k}) \stackrel{\mathrm{flat}}{\to} \mu^\tau_k$, for some $\mu^\tau_k\in X$ with $|\mu^\tau_k|(\Omega)\le M$ and there exists a maximal solution of the L^2 discrete gradient flow, v^τ_k, of $\mathcal{W}$ from v_0, according with Definition 7.2, such that

$$\mu^\tau_k = Jv^\tau_k = \sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,k}}\,, \ \text{for every } k = 1,\dots,k^\tau_\delta\,, \tag{7.66}$$

with k^τ_δ as defined in (6.9).

Moreover denoting by $\tilde v^\tau_{\varepsilon,k}$ the piecewise affine interpolation of $e^{2\pi i\bar u^\tau_{\varepsilon,k}}$, we have

$$\tilde v^\tau_{\varepsilon,k} \rightharpoonup v^\tau_k \quad \text{in } H^1_{\mathrm{loc}}(\Omega\setminus\cup_{i=1}^M\{x^\tau_{i,k}\};\mathbb{R}^2)\,, \ \text{for every } k = 1,\dots,k^\tau_\delta\,. \tag{7.67}$$

Finally for τ and ε small enough such $\bar u^\tau_{\varepsilon,k}$ is indeed a minimizer of problem (7.1) *and hence it is a solution of the (strong) L^2 discrete gradient flow.*

Proof. The proof of this result uses the first order Γ-convergence result (Theorem 4.5) and follows closely the proof of the analogous statement in Section 6 (see Theorem 6.7). Indeed, by the definition of the relaxed L^2 discrete gradient flow we have that for any $k\in\mathbb{N}$

$$F_\varepsilon(\bar u^\tau_{\varepsilon,k}) + \frac{\left\|e^{2\pi i\bar u^\tau_{\varepsilon,k}} - e^{2\pi i\bar u^\tau_{\varepsilon,k-1}}\right\|^2_{L^2}}{2\pi\tau|\log\tau|} \le F_\varepsilon(\bar u^\tau_{\varepsilon,k-1}).$$

By induction on k, one can show that

$$F_\varepsilon(\bar u^\tau_{\varepsilon,k}) \le F_\varepsilon(u_{\varepsilon,0},\Omega) \le M\pi|\log\varepsilon| + C\,.$$

This estimate together with (7.65) implies that $\|\mu(\bar u^\tau_{\varepsilon,k}) - \mu(\bar u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}} \le \delta + C\varepsilon|\log\varepsilon|$. Then using the Compactness result stated in Theorem 4.2(i), and arguing as in the proof of Theorem 6.7 we deduce (7.66) and (7.67).

In order to show that, for ε small enough, $\bar u^\tau_{\varepsilon,k}$ is a solution of the L^2 discrete gradient flow according with Definition 7.1, it is enough to recall that thanks to Proposition 7.6 we have that $\|\mu^\tau_k - \mu^\tau_{k-1}\|_{\mathrm{flat}} \le C\sqrt{\tau|\log\tau|}$. Then the conclusion follows by the convergence in the flat norm of $\mu(\bar u^\tau_{\varepsilon,k})$ to μ^τ_k. □

8. Conclusions

We have obtained an asymptotic expansion by Γ-convergence for a large class of discrete energies accounting for defects (including the elastic energy in crystals with screw dislocations and the energy of XY spin systems with vortices). Based on this analysis, we have been able to show existence of metastable configurations, and we have introduced a discrete in time variational dynamics, which allows to overcome the energy barriers and mimics the effect of more complex mechanisms, as thermal effects. We have described the dynamics up to the first collision time; it would be interesting to model the collision of discrete vortices, and study the dynamics after the critical time as in the Ginzburg-Landau setting (see [11], [42], [43]). In all the paper we have focused on Neumann boundary conditions, but our analysis could be extended to the case of Dirichlet boundary conditions.

In the proof of our results we have made use of a new variational principle that allows to deduce the presence of local minimizers for Γ-converging sequences, also in the absence of local minimizers in their limit. This has been possible for a large class of interaction potentials, which includes the case of screw dislocations but not the XY model, for which this fact is still unclear.

In the discrete dynamics we have analyzed two different dissipations. This is motivated also by applications. Indeed, the L^2 dissipation is a standard choice for parabolic flows and measures the variations in the spin variable. While, the dissipation D_2 is a natural choice in the study of screw dislocation dynamics, and can be viewed as a measure of the number of energy barriers to be overcome in order to move a dislocation. We note that, in the case of dislocations, one could also consider suitable variants of the D_2 dissipation accounting for the glide directions of the crystal. This would lead to a different effective dynamics. We also believe that our approach could be generalized to anisotropic energies and to more general lattice structures. It is still open the case of edge dislocations, for which a complete Γ-expansion of the energy is not yet available ([21], [19]).

Having proved a pinning phenomenon, it remains open to characterize a critical ε-τ regime for the motion of dislocations, and an effective depinning threshold in this regime. This is a relevant issue and it might be worth facing it by using our variational approach.

The effective dynamics of our discrete systems agrees with the asymptotic parabolic flow of the Ginzburg-Landau functionals. In the latter, the time scaling needed to get a non-trivial effective dynamics depends on the space parameter ε. It is worth noticing that, in our discrete in time gradient flow with L^2 dissipation, the time scaling is expressed only in terms of the time step τ. In this respect, an analysis of critical ε-τ regimes would make an interesting bridge between these two approaches.

Appendix A. Product-Estimate

In this section we collect some results in [39] that are used in the proofs of Section 7.

We first introduce some notation. Let A be an open bounded subset of $\mathbb{R}^3$. Given $w = (w_1, w_2) \in H^1(A; \mathbb{R}^2)$, its Jacobian Jw can be regarded as a 2-form in

$\mathbb{R}^3$ given by

$$Jw = \mathrm{d}w_1 \wedge \mathrm{d}w_2 = \sum_{j<k} (\partial_j w_1 \partial_k w_2 - \partial_j w_2 \partial_k w_1)\, \mathrm{d}x_j \wedge \mathrm{d}x_k. \tag{A.1}$$

Thus Jw acts on vector fields $X, Y \in C(A; \mathbb{R}^3)$ with the standard rule that

$$\mathrm{d}x_j \wedge \mathrm{d}x_k (X, Y) = \frac{1}{2} (X_j Y_k - X_k Y_j)\,.$$

The Jacobian Jw can be also identified with a 1-dimensional current $\star Jw$ which acts on 1 forms $\omega = \omega_1\, \mathrm{d}x_1 + \omega_2\, \mathrm{d}x_2 + \omega_3\, \mathrm{d}x_3$ as

$$\langle \star Jw, \omega \rangle = \int_A Jw \wedge \omega\,.$$

In particular, for any $X, Y \in C(A; \mathbb{R}^3)$

$$\langle \star Jw, X \wedge Y \rangle = \int_A Jw(X, Y)\, \mathrm{d}x,$$

where, with a little abuse of notation, we identify 1-forms with vector fields.

Let Ω be an open bounded subset of $\mathbb{R}^2$ and $T > 0$. For a given $w \in H^1([0,T] \times \Omega; \mathbb{R}^2)$, we denote by μ, V^1, V^2 the L^1 functions such that

$$Jw = \mu\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 + V^1\, \mathrm{d}x_1 \wedge \mathrm{d}t + V^2\, \mathrm{d}x_2 \wedge \mathrm{d}t\,. \tag{A.2}$$

The theorem below collects the results of Theorem 1 and Theorem 3 in [39]. We remind that the definition of the functionals GL_ε is given in (1.2).

Theorem A.1. *Let $w_\varepsilon \in H^1([0,T] \times \Omega; \mathbb{R}^2)$ be such that*

$$\int_0^T GL_\varepsilon(w_\varepsilon(t, \cdot))\, \mathrm{d}t + \int_0^T \int_\Omega |\partial_t w_\varepsilon(t,x)|^2\, \mathrm{d}x\, \mathrm{d}t \le C |\log \varepsilon|\,. \tag{A.3}$$

Then, there exists a rectifiable integer 1-current J such that, up to a subsequence,

$$\frac{\star J u_\varepsilon}{\pi} \to J \quad \textit{in } (C_c^{0,\gamma}([0,T] \times \Omega; \mathbb{R}^3))',\ \forall \gamma \in (0,1].$$

Moreover, for any $X, Y \in C_c^0([0,T] \times \Omega; \mathbb{R}^3)$

$$\liminf_{\varepsilon \to 0} \frac{1}{\pi |\log \varepsilon|} \left(\int_{[0,T]\times\Omega} |X \cdot \nabla w_\varepsilon|^2\, \mathrm{d}x\, \mathrm{d}t \int_{[0,T]\times\Omega} |Y \cdot \nabla w_\varepsilon|^2\, \mathrm{d}x\, \mathrm{d}t \right)^{\frac{1}{2}} \ge |\langle J, X \wedge Y \rangle|\,.$$

If in addition we assume that

$$\sup_{t \in [0,T]} GL_\varepsilon(w_\varepsilon(t, \cdot)) \le C |\log \varepsilon|, \tag{A.4}$$

then, J can be written as in (A.2) *with $\mu \in C^{0,\frac{1}{2}}([0,T]; (C_c^{0,\gamma}(\Omega))')$ for every $\gamma \in (0,1]$ and V^1, $V^2 \in L^2([0,T]; \mathcal{M}(\Omega))$.*

Finally, up to a subsequence,

$$\mu_\varepsilon(t) \overset{\text{flat}}{\to} \mu(t) \qquad \textit{for all } t \in [0,T].$$

We now state a variant of Corollary 4 in [39] which is a direct consequence of Theorem A.1.

Corollary A.2. *Let* $0 \le t_1 < t_2$ *and let* $w_\varepsilon \in H^1([t_1,t_2]\times\Omega;\mathbb{R}^2)$ *be such that* (A.3) *holds true with* $[0,T]$ *replaced by* $[t_1,t_2]$*, and such that for all* $t \in [t_1,t_2]$

$$\frac{1}{2}\int_\Omega |\nabla w_\varepsilon(t,x)|^2 \,\mathrm{d}x \le M\pi|\log\varepsilon| + C$$

for some $M\in\mathbb{N}$ *and* $C\in\mathbb{R}$*. Assume moreover that*

$$\mu_\varepsilon(t) \overset{\text{flat}}{\to} \mu(t) := \pi\sum_{i=1}^M d_i\delta_{x_i(t)},$$

with $|d_i| = 1$ *and* $x_i(t) \in C([t_1,t_2];\Omega)$ *for every* i *with* $x_i(t) \neq x_j(t)$ *for* $i \neq j$*.*

Then, for any $X, Y \in C^0_c(\Omega;\mathbb{R}^3)$

$$\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\int_{[t_1,t_2]\times\Omega}\langle X\cdot\nabla w_\varepsilon, Y\cdot\nabla w_\varepsilon\rangle \,\mathrm{d}x\,\mathrm{d}t = \pi\int_{t_1}^{t_2}\sum_{i=1}^M\langle X(x_i(t)), Y(x_i(t))\rangle \,\mathrm{d}t.$$

Here we state a result analogous to Corollary 7 in [39].

Proposition A.3. *Let* $\tilde T > 0$ *and let* $w_\varepsilon \in H^1([0,\tilde T]\times\Omega;\mathbb{R}^2)$ *be such that* (A.3) *holds true, and such that for all* $t\in[0,\tilde T]$

$$\frac{1}{2}\int_\Omega |\nabla w_\varepsilon(t,x)|^2 \,\mathrm{d}x \le M\pi|\log\varepsilon| + C$$

for some $M\in\mathbb{N}$ *and* $C\in\mathbb{R}$*. Assume moreover that*

$$\mu_\varepsilon(t) \overset{\text{flat}}{\to} \mu(t) := \pi\sum_{i=1}^M d_i\delta_{x_i(t)},$$

with $|d_i| = 1$ *and* $x_i(t)\in C([0,\tilde T];\Omega)$ *for any* i *with* $x_i(t)\neq x_j(t)$ *for* $i\neq j$*. Then*

$$\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\int_{[0,\tilde T]\times\Omega}|\partial_t w_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \ge \pi\sum_{i=1}^M\int_0^{\tilde T}|\dot x_i|^2 \,\mathrm{d}t. \tag{A.5}$$

Proof. The proof of this result coincides with the one of Corollary 7 in [39], the only difference being that [39] assumes that for every $t\in[0,\hat T]$

$$\frac{1}{\varepsilon^2}\int_\Omega(1-|w_\varepsilon(x,t)|^2)^2 \,\mathrm{d}x \le C|\log\varepsilon|.$$

Here this assumption is replaced by (A.3), which is enough to apply Corollary A.2. Once the statement of Corollary A.2 holds true, the rest of the proof follows exactly as in in [39]. □

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