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A Proof of Completeness for the Eigenfunctions of the Multi-species Liouville Operator and its Green's Function

L. Nocera*

CNR-IPCF, Theoretical Plasma Physics, Via Moruzzi 1, I-56124 Pisa, Italy†

In this report, we determine the spectrum and the eigenfunctions of the free-streaming oscillations of electrons and mobile ions about inhomogeneous, collisionless, electrostatic, space-periodic and non-periodic plasma equilibria. The eigenfunctions are given in the Fourier transformed velocity space. We show that the spectrum has a continuous as well as a discrete part, both extending over the whole real axis. The eigenfunctions of the continuous spectrum pertain to particles which are unrestricted in their motion. Those belonging to the discrete spectrum pertain to particles which are either constrained by boundary conditions or trapped in their equilibrium potential wells. We show that, near the boundaries of these wells, the space distribution of the eigenfunctions is algebraically singular. All of the eigenfunctions have two discrete finite degeneracies and two continuous infinite degeneracies. A further discrete, infinite degeneracy appears for space-periodic equilibria. We prove that the eigenfunctions are mutually orthogonal and that they form a complete set. Examples are presented of the eigenfunctions of both electron and ion oscillations in solitary and in double layer equilibria and, for space-periodic equilibria, their Bloch form is also worked out.

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* nocera@ipcf.cnr.it

† Homepage: http://mizar.pi.ipcf.cnr.it/~tpp/

I. INTRODUCTION

Collisionless plasma equilibrium structures, even in their simplest unmagnetized, electrostatic manifestations, generally involve space distributions of the electric field and velocity distribution functions of charged particles which are inhomogeneous along at least one space coordinate.

A complete understanding of the dynamics of the oscillations within these inhomogeneus structures is based on the knowledge of the oscillations' spectrum and eigenfunctions. This information may be used to determine the ability of the plasma structures to resonantly absorb external radiation, and to damp or de-stabilize the oscillations [1].

Determining these eigenfunctions must take into account contributions coming from kinetic effects, proper of spatially-homogeneous, e.g. Maxwellian, plasma equilibria, and those arising from the inhomogeneous state of the plasma.

A further contribution, pointed out by recent research, is provided by those collisionless plasma structures having discontinuous particle velocity distributions [2]. These equilibria arise when the electric potential within the plasma is not endowed with sufficient spatial symmetry [3] as, e.g., in non monotonic double layers.

For an infinitely extended spatially homogeneous equilibrium based on a Maxwellian electron distribution and on immobile ions, the linear eigenvalue problem for the whole kinetic operator was reduced [4] to the eigenvalue problem of van Kampen [1] for the oscillations of the one-particle electron distribution function governed by the homogeneous Vlasov equation. This proved that the eigenfunctions of such kinetic system belong to a purely continuous infinite real spectrum.

Continuous spectra also arise, irrespective of kinetic effects, e.g., in fluids, precisely because of the above mentioned spatial inhomogeneity of the equilibrium plasma state [5].

One first peculiar outcome of the joint effects of inhomogeneity and kinetics is the appearance of a discrete spectrum. In physical terms, the eigenfunctions of the discrete spectrum describe plasma particles confined in the wells which now appear in the space distribution of their inhomogeneous potential energy. Their corresponding eigenvaluaes are integer multiples of the bouncing frequencies within the wells' boundaries

Thus when the kinetic equilibria are inhomogeneous (as in the present paper), the treatment used in Ref. [4] is inapplicable: it rather leads to a Vlasov equation with spatially inhomogeneous coefficients for the one-particle distributions.

The analyis may then proceed by first determining the eigenfunctions of a part of the Vlasov operator, known as the inhomogeneous free-streaming operator.

The relevance of this method is that the intricate spectral, degeneracy and singular properties, the orthogonality relations and the completeness of the eigenfunctions of the complete inhomogeneous Vlasov operator (which are due to inhomogeneity and which are key issues of the present paper) are already embodied in those of the inhomogeneous free-streaming operator.

Specifically, the remaining part of the Vlasov operator can be treated as a "perturbation" (in the operatorial sense) of the free-streaming operator: this perturbation leaves the essential spectrum and the orthogonality properties of the eigenfunctions unchanged [6].

An example of this approach can be found in Ref. [7], where the concomitant contributions of kinetic effects and of the spatial inhomogeneity of a background magnetic field to the doubly degenerate eigenfunctions of one species (electron) collisionless oscillations were treated by an action-angle, integral-equation approach based on the knowledge of the free-streaming eigenfunctions.

Ref. [8] proposed a normal mode and initial value differential approach and the technique of the Green's function, spectrally built on the free-streaming eigenfunctions, to determine the permittivity to electromagnetic oscillations of a multispecies inhomogeneous plasma.

In Ref. [9], a truncated discrete Bloch electron free-streaming eigenfunction approach was used to find the structure of one species (electron) electrostatic unstable oscillations about a one-dimensional spatially inhomogeneous Bernstein Greene Kruskal spatially-periodic equilibrium neutralized by an immobile ion background.

The works above focussed on the free-streaming eigenfunctions of the electron population which, being the most mobile plasma component, predominantly contributes to the permittivity at high frequencies. Ion eigenfunctions (a further issue of the present paper) may however become important at lower frequencies in the kinetic or in the hybrid plasma regimes.

Also, the equilibrium particle distribution functions had to be well behaved functions of velocity, in order for certain functional scalar products appearing in the plasma dielectric function to be well defined.

Finally, being hindered by their singular (distributional) nature [9], the spatial profile of the free-streaming eigenfunctions could not be developed, nor could the proof of their completeness.

As we shall see, the nature of these so far undeveloped issues much depends on the morphology of the plasma structure hosting the oscillations. Their analysis requires a careful and extensive classification of their spectral, degeneracy, singularity and orthogonality properties.

A valuable help in addressing these issues comes from the representation of the particle distributions in the Fourier transformed velocity space [10–13]. This approach seamlessly provides functional scalar products in the velocity Fourier conjugate space involving well behaved equilibrium distribution functions, irrespective of any possible discontinuity of theirs in the velocity space. Also, it naturally unfolds the above mentioned δ-shaped spatial eigenfunctions' profiles into ordinary functions, thus unveiling their algebraic singularities (a further contribution of our work). Finally, it allows a thorough proof of their completeness, i.e. of the possibility for a generic function to be represented as a superposition of those eigenfunctions.

In our study, practical examples of this analysis will be given for three types of morphologically different, collisionless plasma equilibria of considerable physical relevance. Their equilibrium electric potential has: (*a*) a monotonic spatial behaviour, typical of double layers (Section V); (*b*) a bell-shaped structure, typical of solitary waves and of phase space holes (Section VI); (*c*) a both infinitely and finitely extended periodic structure, typical, e.g., of Bernstein Greene Kruskal wave equilibria (Section VII).

II. NOTATIONS AND BASIC EQUATIONS

Let

$$\hat{\Phi} + \hat{\phi},\ \hat{\Phi} = \min \hat{\Phi} + \Phi_0 \Phi,\ \hat{\phi} = \Phi_0 \tilde{\phi}, \tag{1}$$

$$\Phi_0 = \max \hat{\Phi} - \min \hat{\Phi} \tag{2}$$

be the electric potential (of which Φ is the scaled, normalized steady state equilibrium part and $\tilde{\phi}$ is the normalized perturbation part) and its scale, e the elementary charge, $\alpha = \mathrm{e}$ or $\alpha = \mathrm{i}$ a label denoting the electron and ion quantities,

$$Q_\alpha = Z_\alpha e,\ Z_\alpha = Q_\alpha / |Q_\mathrm{e}|,\ \mu_\alpha = m_\alpha / m_\mathrm{e} \tag{3}$$

the particle charges, charge and mass ratios (of which $Z_\mathrm{e} = -1$, $\mu_\mathrm{e} = 1$),

$$-V_\mathrm{e} = Z_\mathrm{e} \Phi,\ -V_\mathrm{i} = Z_\mathrm{i}(\Phi - 1), \tag{4}$$

the scaled and normalized particle potential energies in the equilibrium potential Φ, n_0 a density scale, and

$$\hat{x} = Lx,\ \hat{y} = Ly,\ \hat{v}_x = v_0 v_x,\ \hat{v}_y = v_0 v_y,\ \hat{t} = \omega_\mathrm{p}^{-1} t, \tag{5}$$

$$L = \surd[e\Phi_0/(4\pi n_0 e^2)],\ v_0 = \surd(e\Phi_0/m_\mathrm{e}),\ \omega_\mathrm{p}^{-1} = L/v_0 \tag{6}$$

the space and velocity coordinates, time and their respective scales.

We direct the coordinate x along the gradient (which we assume to be uni-directional) of the equilibrium potential Φ and particle velocity distribution $\tilde{F}_\alpha$ which, together with the perturbation $\tilde{f}_\alpha$ determine the one particle distribution

$$\hat{f}_\alpha(\hat{x}, \hat{y}, \hat{v}_x, \hat{v}_y, \hat{t}) = \frac{n_0}{v_0 Z_\alpha}[\tilde{F}_\alpha(x, v_x, v_y) + \tilde{f}_\alpha(x, y, v_x, v_y, t)], \tag{7}$$

The equilibrium quantities Φ and $\tilde{F}_\alpha$ are assumed to be known in the domain

$$x_1 < x < x_2, \tag{8}$$

where x_1 and/or x_2 may take infinite values. Last, we introduce the Fourier transforms of the physical quantities

$$F_\alpha(x, q_x, q_y) = \int_{-\infty}^{\infty} \mathrm{d}v_x \int_{-\infty}^{\infty} \mathrm{d}v_y e^{\mathrm{i}(q_x v_x + q_y v_y)} \tilde{F}_\alpha(x, v_x, v_y), \tag{9}$$

$$f_{\alpha k_y \omega}(x, q_x, q_y) = \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}v_x \int_{-\infty}^{\infty} \mathrm{d}v_y \int_{-\infty}^{\infty} \mathrm{d}t \times$$
$$e^{\mathrm{i}(k_y y + q_x v_x + q_y v_y - \omega t)} \tilde{f}_\alpha(x, y, v_x, v_y, t), \tag{10}$$

$$\phi_{k_y \omega}(x) = \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}t e^{\mathrm{i}(k_y y - \omega t)} \tilde{\phi}(x, y, t). \tag{11}$$

In the above notation, the electrostatic oscillations in a collisionless plasma are governed by the linearised electrostatic Maxwell equations and by the velocity Fourier transformed Vlasov equations for electrons and ions:

$$e'_{x k_y \omega} - \mathrm{i} k_y e_{y k_y \omega} = n_{k_y \omega}, \tag{12}$$

$$e'_{y k_y \omega} + \mathrm{i} k_y e_{x k_y \omega} = 0, \tag{13}$$

$$-\omega f_{\alpha k_y \omega} + \left[\frac{\partial^2}{\partial x \partial q_x} - \mathrm{i} k_y \frac{\partial}{\partial q_y} + q_x \frac{V'_\alpha}{\mu_\alpha} \right] f_{\alpha k_y \omega} =$$
$$-\frac{Z_\alpha}{\mu_\alpha} F_\alpha [q_x e_{x k_y \omega} + q_y e_{y k_y \omega}], \tag{14}$$

where a "$'$" denotes differentiation with respect to x,

$$e_{x k_y \omega} = -\phi'_{k_y \omega}, \; e_{x k_y \omega} = \mathrm{i} k_y \phi_{k_y \omega}, \tag{15}$$

$$n_{k_y \omega} = (f_{\mathrm{i} k_y \omega} - f_{\mathrm{e} k_y \omega})|_{q_x = q_y = 0}, \tag{16}$$

are the y- and t-Fourier analyzed perturbations of the x and y components of the electric field and of the plasma density respectively.

We now cast the electron and ion distributions in the vector valued function

$$|f_{k_y \omega}\rangle = [f_{\mathrm{e} k_y \omega}(x, q_x, q_y), f_{\mathrm{i} k_y \omega}(x, q_x, q_y)]^{\mathrm{T}}, \tag{17}$$

introduce the matrix valued free-streaming operator S_{k_y}

$$S_{\alpha \alpha k_y} = \frac{\partial^2}{\partial x \partial q_x} - \mathrm{i} k_y \frac{\partial}{\partial q_y} + q_x \frac{V'_\alpha}{\mu_\alpha}, \tag{18}$$

$$S_{\alpha \alpha' k_y} = 0 \text{ if } \alpha' \neq \alpha, \tag{19}$$

and rewrite Eqs. (12)-(14) as

$$\nabla^2 e_{x k_y \omega} = n'_{k_y \omega} \tag{20}$$

$$[S_{k_y} |f_{k_y \omega}\rangle]_\alpha - \omega f_{\alpha k_y \omega} = -\frac{Z_\alpha}{\mu_\alpha} F_\alpha [q_x e_{x k_y \omega} + q_y e_{y k_y \omega}]. \tag{21}$$

The vector valued functions

$$|\chi_{\mathrm{e}k_y\sigma}\rangle = [\chi_{\mathrm{e}k_y\sigma}, 0]^{\mathrm{T}}, \ |\chi_{\mathrm{i}k_y\sigma}\rangle = [0, \chi_{\mathrm{i}k_y\sigma}]^{\mathrm{T}} \tag{22}$$

are eigenfunctions of the operator S_{k_y} (Eq. (18)) corresponding to the eigenvalue σ provided

$$S_{\alpha\alpha k_y}\chi_{\alpha k_y\sigma} = \sigma\chi_{\alpha k_y\sigma}. \tag{23}$$

For simplicity, our treatment considers only one ion species. It may be extended to, say, m ion species by allowing the function vectors $|f_{k_y\omega}\rangle$ (Eq. (17)) to have $m+1$ components. In this case, there would be $m+1$ eigenfunction vectors: $|\chi_{\mathrm{e}k_y\sigma}\rangle$, having $m+1$ vanishing components save the first, and $|\chi_{\mathrm{i}_jk_y\sigma}\rangle$, $j = 1\dots m$, having $m+1$ vanishing components, save the $(j+1)$-th.

III. SUPPORT, SINGULARITY, SYMMETRY RELATIONS AND DEGENERACY OF THE EIGENFUNCTIONS

We seek a solution of Eq. (23) in the form $\chi_{\alpha k_y\sigma}(x, q_x, q_y) = e^{-\mathrm{i}f_\alpha(x)+\mathrm{i}q_xu_\alpha(x)+\mathrm{i}q_yc_\alpha}$, where c_α is an arbitrary real quantity. Substitution into Eq. (23) gives $[u_\alpha^2]' = 2V_\alpha/\mu_\alpha$, $f_\alpha' = \mathrm{i}u_\alpha'/u_\alpha + (\sigma - k_yc_\alpha)/u_\alpha$, and

$$\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha} = \frac{C_\alpha}{|u_{\alpha\gamma_\alpha}|}e^{-\mathrm{i}s_\alpha(\sigma-k_yc_\alpha)\xi_{\alpha\gamma_\alpha}+\mathrm{i}s_\alpha q_x|u_{\alpha\gamma_\alpha}|+\mathrm{i}q_yc_\alpha}, \tag{24}$$

$$u_{\alpha\gamma_\alpha}(x) = s_\alpha\sqrt{\{2[\gamma_\alpha + V_\alpha(x)]/\mu_\alpha\}}, \tag{25}$$

$$s_\alpha = \pm, \tag{26}$$

$$\xi_{\alpha\gamma_\alpha}(x) = \int_{x_{0\alpha\gamma_\alpha}}^{x}\frac{\mathrm{d}x'}{|u_{\alpha\gamma_\alpha}(x')|}, \tag{27}$$

where, γ_α is an integration constant and C_α is a normalization constant; in Eq. (27) the arbitrary integration bound $x_{0\alpha\gamma_\alpha}$ will be referred to as the phase terminal: this quantity is independent of σ, but it depends on the particle species α and on γ_α, as specified by the labels; It will be chosen later according to the potential profile Φ.

For $\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}$ in Eq. (24) to be bounded for all values of q, $u_{\alpha\gamma_\alpha}$ must be real and thus $\gamma_\alpha + V_\alpha$ must be non negative. This implies that $\gamma_\alpha \geq -V_\alpha \geq \min(-V_\alpha)$, i.e. (Eq. (4)), $\gamma_\alpha \geq -|Z_\alpha|$. Furthermore, if γ_α is larger than the maximum of the particle equilibrium potential energy, i.e. if $\gamma_\alpha > \max(-V_\alpha) = 0$, then $\gamma_\alpha + V_\alpha > 0$ for all values of x. The corresponding eigenfunctions describe particles which freely move over the whole x-domain. However, if $-|Z_\alpha| \leq \gamma_\alpha < 0$, then the eigenfunctions describe particles which experience reflections at points where $u_\alpha = 0$.

An important property of the eigenfunctions arises when x approaches a root of $u_{\alpha\gamma_\alpha}$ or, equivalently, when γ_α approaches $-V_\alpha(x)$. Basing the phase terminal $x_{0\alpha\gamma_\alpha}$ (Eq. (27)) on that root and defining $d_{\alpha_{\gamma_\alpha}} = \mu_\alpha/|2Z_\alpha\Phi'(x_{0\alpha\gamma_\alpha})|$, Eqs. (25), (27) and (24) approximately give $|u_{\alpha\gamma_\alpha}| \simeq \sqrt{[|x - x_{0\alpha\gamma_\alpha}|/d_{\alpha_{\gamma_\alpha}}]}$, $\xi_{\alpha\gamma_\alpha} \simeq d_{\alpha_{\gamma_\alpha}}|u_{\alpha\gamma_\alpha}|$,

$$\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha} \simeq \frac{C_\alpha}{|u_{\alpha\gamma_\alpha}|}e^{\mathrm{i}s_\alpha[q_x-(\sigma-k_yc_\alpha)d_{\alpha_{\gamma_\alpha}}]|u_{\alpha\gamma_\alpha}|+\mathrm{i}q_yc_\alpha}. \tag{28}$$

We thus see that, as x approaches a reflection point, the eigenfunction diverges, however being integrable there. This behaviour is manifest in the examples considered below.

With the above limitations on the value of γ_α, the phase of the eigenfunctions $\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}$ (Eq. (24)) is real, and the following properties are verified by inspection:

$$\begin{aligned}&\bar{\chi}^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(x,q_x,q_y) = \chi^{-s_\alpha}_{\alpha k_y\sigma(-c_\alpha)\gamma_\alpha}(x,q_x,q_y) = \\ &\chi^{s_\alpha}_{\alpha(-k_y)(-\sigma)c_\alpha\gamma_\alpha}(x,-q_x,-q_y).\end{aligned} \tag{29}$$

Following the above considerations, the species label α (Eq. (22)), the phase sign s_α (Eq. (26)) are discrete degeneracy parameters of the eigenfunctions $\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}$; the real numbers c_α and γ_α are continuous degeneracy parameters: two eigenfunctions having a different value of any of these parameters are solutions of Eq. (23) corresponding to the same eigenvalue σ.

IV. ORTHOGONALITY OF THE EIGENFUNCTIONS

The degenerate eigenfunctions of the free-streaming operator (Eqs. (24)-(27)) will now be shown to be linearly independent and in fact orthogonal. Given the generic vector valued functions

$$|g\rangle = [g_\mathrm{e}(x,q_x,q_y), g_\mathrm{i}(x,q_x,q_y)]^\mathrm{T}, \tag{30}$$

$$|h\rangle = [h_\mathrm{e}(x,q_x,q_y), h_\mathrm{i}(x,q_x,q_y)]^\mathrm{T}, \tag{31}$$

defined in some sub-domain of (x_1, x_2) and for $-\infty < q_x < \infty$, $-\infty < q_y < \infty$, we introduce their functional scalar product

$$\langle g|h\rangle = \int_a^b \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}q_x \int_{-\infty}^{\infty} \mathrm{d}q_y (g(x,q_x,q_y), h(x,q_x,q_y)), \tag{32}$$

where

$$\begin{aligned}&(g(x,q_x,q_y), h(x,q_x,q_y)) = \\ &\Re(g_\mathrm{e}(x,q_x,q_y)\bar{h}_\mathrm{e}(x,q_x,q_y) + g_\mathrm{i}(x,q_x,q_y)\bar{h}_\mathrm{i}(x,q_x,q_y))\end{aligned} \tag{33}$$

is their point-wise scalar product and $\Re$ and the overbar denote the real part and complex conjugation.

In Eq. (32), the x-bounds of the scalar product a and b are to be chosen according to the shape of the potential Φ. The interval $a < x < b$, where $|g\rangle$ and $|h\rangle$ are defined, may be smaller than the entire domain $x_1 < x < x_2$ (Eq. (8)) where the equilibrium quantities are defined.

Next, we observe that, because of the definition of the vectors $|\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}\rangle$ (Eq. (22)) and of their point-wise scalar product (Eq. (33)), the quantity $\langle\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}|\chi^{s'_{\alpha'}}_{\alpha' k_y \sigma' c'_{\alpha'} \gamma'_{\alpha'}}\rangle$ vanishes if $\alpha' \neq \alpha$. Also, in taking the q_x-integration in the scalar product of two eigenfunctions of the same species, the quantity

$$\int_{-\infty}^{\infty} \mathrm{d}q_x e^{\mathrm{i}q_x[s_\alpha|u_{\alpha\gamma_\alpha}(x)|-s'_\alpha|u_{\alpha\gamma'_\alpha}(x)|]} \tag{34}$$

vanishes if $s_\alpha|u_{\alpha\gamma_\alpha}(x)| \neq s'_\alpha|u_{\alpha\gamma'_\alpha}(x)|$, which certainly occurs when the signs s_α and s'_α (Eq. (26)) are different.

The above considerations are summarized in the relation

$$\langle\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}|\chi^{s'_{\alpha'}}_{\alpha' k_y \sigma' c'_{\alpha'} \gamma'_{\alpha'}}\rangle = 0 \text{ if } \alpha' \neq \alpha \text{ or if } s_\alpha \neq s'_\alpha. \tag{35}$$

On the other hand, when $\alpha' = \alpha$ and $s'_\alpha = s_\alpha$, by carrying out the q_x and q_y integrations, the scalar product (Eq. (32)) of the eigenfunctions (Eq. (24)) gives

$$\begin{aligned}
&\langle\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}|\chi^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha}\rangle = \\
&4\pi^2|C_\alpha|^2\Re\int_a^b \mathrm{d}x \frac{e^{-\mathrm{i}s_\alpha[\sigma\xi_{\alpha\gamma_\alpha}(x)-\sigma'\xi_{\alpha\gamma'_\alpha}(x)]}}{|u_{\alpha\gamma_\alpha}(x)||u_{\alpha\gamma'_\alpha}(x)|} \times \\
&\delta(c_\alpha - c'_\alpha)\delta(|u_{\alpha\gamma_\alpha}(x)| - |u_{\alpha\gamma'_\alpha}(x)|).
\end{aligned} \tag{36}$$

But since, for a given x, $|u_{\alpha\gamma'_\alpha}(x)|$ is a monotonic function of γ'_α (Eq. (25)) and thus there is only one root, $\gamma'_\alpha = \gamma_\alpha$, of $U_\alpha(\gamma'_\alpha) = |u_{\alpha\gamma_\alpha}| - |u_{\alpha\gamma'_\alpha}| = 0$, we may use the identity $\delta(U_\alpha(\gamma'_\alpha)) = [|\mathrm{d}U_\alpha/\mathrm{d}\gamma'_\alpha|_{\gamma'_\alpha=\gamma_\alpha}]^{-1}\delta(\gamma'_\alpha - \gamma_\alpha)$ and, using also Eq. (25), we rearrange Eq. (36) as

$$\begin{aligned}
&\langle\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}|\chi^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha}\rangle = 4\pi^2\mu_\alpha|C_\alpha|^2 \times \\
&\Re\int_a^b \mathrm{d}x \frac{e^{-\mathrm{i}s_\alpha(\sigma-\sigma')\xi_{\alpha\gamma_\alpha}(x)}}{|u_{\alpha\gamma_\alpha}(x)|}\delta(c_\alpha - c'_\alpha)\delta(\gamma_\alpha - \gamma'_\alpha).
\end{aligned} \tag{37}$$

Now, being the integral of a positive quantity, $\xi_{\alpha\gamma_\alpha}$ (Eq. (27)) is a monotonic function of x: thus we may uniquely make the substitutions

$$t = \xi_{\alpha\gamma_\alpha}(x) = \int_{x_{0\alpha\gamma_\alpha}}^{x} \frac{\mathrm{d}x'}{|u_{\alpha\gamma_\alpha}(x')|}, \tag{38}$$

$$t_{a\alpha\gamma_\alpha} = \xi_{\alpha\gamma_\alpha}(a),\ t_{b\alpha\gamma_\alpha} = \xi_{\alpha\gamma_\alpha}(b), \tag{39}$$

and write Eq. (37) as

$$\begin{aligned}
&\langle\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}|\chi^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha}\rangle = 4\pi^2\mu_\alpha|C_\alpha|^2 \times \\
&\Re\int_{t_{a\alpha\gamma_\alpha}}^{t_{b\alpha\gamma_\alpha}} \mathrm{d}t e^{-\mathrm{i}s_\alpha(\sigma-\sigma')t}\delta(c_\alpha - c'_\alpha)\delta(\gamma_\alpha - \gamma'_\alpha).
\end{aligned} \tag{40}$$

In the following sections, the eigenfunctions will be made orthonormal by an appropriate choice of the constant C_α, according to the values of the integration bounds $t_{a\alpha\gamma_\alpha}$ and $t_{b\alpha\gamma_\alpha}$ arising for several equilibrium potential profiles of practical relevance.

V. EIGENFUNCTIONS OF PARTICLES IN A DOUBLE LAYER

In a double layer, extending for $x_1 = -\infty < x < \infty = x_2$, we assume that the equilibrium potential monotonucally increases, so that, when $\gamma_\alpha < 0$, there may be at most one reflection point at which $\gamma_\alpha + V_\alpha = 0$ and the eigenfunctions are defined in the whole semi-infinite interval lying on one side of that point: thus, one of the scalar product's bounds (Eq. (32)) will be placed at that point and the other lies at infinity. It is also convenient to base the phase terminal $x_{0\alpha\gamma_\alpha}$ in Eq. (27) at the reflection point so that

$$\text{electrons} : a = x_{0\mathrm{e}\gamma_\mathrm{e}} = a_{\mathrm{e}\gamma_\mathrm{e}} < x < \infty = b, \tag{41}$$

$$\text{ions} : a = -\infty < x < a_{\mathrm{i}\gamma_\mathrm{i}} = x_{0\mathrm{i}\gamma_\mathrm{i}} = b. \tag{42}$$

Inserting these values into Eqs. (39) and (40), we find

$$t_{a\mathrm{e}\gamma_\mathrm{e}} = 0,\ t_{b\mathrm{e}\gamma_\mathrm{e}} = \infty,\ t_{a\mathrm{i}\gamma_\mathrm{i}} = -\infty,\ t_{b\mathrm{i}\gamma_\mathrm{i}} = 0, \tag{43}$$

$$\langle \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha} | \chi^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha} \rangle =$$

$$\delta(\sigma - \sigma')\delta(c_\alpha - c'_\alpha)\delta(\gamma_\alpha - \gamma'_\alpha). \tag{44}$$

Here, orthonormality was ensured by choosing, in Eq. (40),

$$C_\alpha = e^{\mathrm{i}\zeta}/\sqrt{(4\pi^3\mu_\alpha)}, \tag{45}$$

where ζ is an arbitrary phase. There are no limitations on the real eigenvalues and the spectrum is continuous.

If $\gamma_\alpha \geq 0$, the phase terminals and the scalar product bounds will be based at their respective limit values attained as $\gamma_\alpha \to 0^-$: this "continuity rule" ensures the continuity of the eigenfunctions' phase and scalar product as γ_α goes through zero. For the adopted infinitely extended, increasing potential, the reflection points shift to infinity as $\gamma_\alpha \to 0^-$: $a_{\mathrm{e}\gamma_\mathrm{e}} \to -\infty$, $a_{\mathrm{i}\gamma_\mathrm{i}} \to \infty$. Inserting these values into Eqs. (41) and (42), and these latter into Eqs. (39) and (40), we find the same integration bounds $t_{a\alpha\gamma_\alpha}$ and $t_{b\alpha\gamma_\alpha}$ (Eq. (43)), orthogonality relation (Eq. (44)), normalization constant (Eq. (45)) and continuous nature of the spectrum as for the reflected particle eigenfunctions.

The eigenfunctions' reflection points and phases in the double layer equilibrium potential

$$\Phi(x) = [1 + \tanh(Kx)]/2 \tag{46}$$

are

$$a_{\alpha\gamma_\alpha} = \operatorname{arctanh}([|Z_\alpha| + 2\min(\gamma_\alpha, 0)]/Z_\alpha)/K, \tag{47}$$

and

$$\xi_{\alpha\gamma_\alpha} = \frac{1}{2K}\left[I^+_{\alpha\gamma_\alpha}(s) + I^-_{\alpha\gamma_\alpha}(s)\right]^{s=x}_{s=x_{0\alpha\gamma_\alpha}}, \tag{48}$$

$$I^\pm_{\alpha\gamma_\alpha}(s) = \frac{1}{u_{\alpha\gamma_\alpha}(\pm\infty)} \ln\frac{|u_{\alpha\gamma_\alpha}(s)| \mp u_{\alpha\gamma_\alpha}(\pm\infty)}{|u_{\alpha\gamma_\alpha}(s)| \pm u_{\alpha\gamma_\alpha}(\pm\infty)}, \tag{49}$$

$$u_{\alpha\gamma_\alpha}(\pm\infty) = \sqrt{[(\gamma_\alpha + |Z_\alpha|) + (\gamma_\alpha \mp Z_\alpha)]}/\sqrt{\mu_\alpha}, \tag{50}$$

where $x_{0\alpha\gamma_\alpha}$ are the phase terminals (Eqs. (41) and (42)) and $u_{\alpha\gamma_\alpha}$ was given in Eq. (25).

The corresponding eigenfunctions are presented in Fig. 1. In this figure, the ion eigenfunction has eigenvalue $1/\sqrt{\mu_{\rm i}}$ smaller than that of the electron eigenfunction. This choice avoids the fine scale oscillations of the ion eigenfunction, whose phase $\xi_{{\rm i}\gamma_{\rm i}}$ is proportional to $\sqrt{\mu_{\rm i}} \gg 1$ (Eqs. (24) and (27)).

VI. EIGENFUNCTIONS OF PARTICLES IN A PHASE SPACE HOLE

In a phase space hole, extending for $x_1 = -\infty < x < \infty = x_2$, we position the single extremum of the equilibrium potential at $x = 0$. For for $\gamma_\alpha < 0$, reflection and trapping points at which $\gamma_\alpha + V_\alpha = 0$ now occur in pair: $a_{\alpha\gamma_\alpha}$ and $b_{\alpha\gamma_\alpha}$.

In an electron hole, for which that extremum is a maximum, and for $\gamma_\alpha < 0$, ions are reflected at these points and electrons are trapped between them. In an ion hole, the potential has a single minimum, ions are trapped and electrons are reflected: a convenient choice of the phase terminals $x_{0\alpha\gamma_\alpha}$ Eq. (27) and of the scalar product bounds a and b (Eq. (32)) is

$$x < 0 : a = -\infty < x < a_{\alpha\gamma_\alpha} = x_{0\alpha\gamma_\alpha} = b, \tag{51}$$

$$x > 0 : a = x_{0\alpha\gamma_\alpha} = b_{\alpha\gamma_\alpha} < x < \infty = b \tag{52}$$

for the reflected particles and

$$a = x_{0\alpha\gamma_\alpha} = a_{\alpha\gamma_\alpha} < x < b_{\alpha\gamma_\alpha} = b \tag{53}$$

for the trapped ones. Inserting these values in Eq. (39) gives

$$x < 0 : t_{a\alpha\gamma_\alpha} = -\infty,\ t_{b\alpha\gamma_\alpha} = 0, \tag{54}$$

$$x > 0 : t_{a\alpha\gamma_\alpha} = 0,\ t_{b\alpha\gamma_\alpha} = \infty \tag{55}$$

for the reflected particles and

$$t_{a\alpha\gamma_\alpha} = 0,\ t_{b\alpha\gamma_\alpha} = \int_{a_{\alpha\gamma_\alpha}}^{b_{\alpha\gamma_\alpha}} \frac{{\rm d}x'}{|u_{\alpha\gamma_\alpha}(x')|} \tag{56}$$

for the trapped ones. Inserting Eqs. (54) and (55) into Eq. (40) gives, for the reflected particle eigenfunctions, the same orthogonality relation, normalization constant and continuous nature of the spectrum as for the reflected particle eigenfunctions of the double layer (Eqs. (43)-(45)).

On the other hand, inserting Eq. (56) into Eq. (40), we see that, if the trapped particle eigenfunctions are to be orthogonal, then, necessarily,

$$\sigma = m\pi/t_{\alpha\gamma_\alpha},\ \sigma' = m'\pi/t_{\alpha\gamma_\alpha}, \tag{57}$$

$$t_{\alpha\gamma_\alpha} = t_{b\alpha\gamma_\alpha} - t_{a\alpha\gamma_\alpha}, \tag{58}$$

where the mode numbers m, m' are integers. For both species of trapped particles, Eq. (40) now reduces to

$$\begin{aligned}&\langle \chi^{s_\alpha}_{\alpha k_y \sigma c_e \gamma_\alpha} | \chi^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha} \rangle = \\ &\delta(c_\alpha - c'_\alpha)\delta(\gamma_\alpha - \gamma'_\alpha)[t_{\alpha\gamma_\alpha}\delta_{(\sigma t_{\alpha\gamma_\alpha}/\pi)(\sigma' t_{\alpha\gamma_\alpha}/\pi)}].\end{aligned} \tag{59}$$

Here, orthonormality was ensured by choosing C_α as in Eq. (45) and, given the integers m and m', $\delta_{mm'}$ is Kronecker's symbol: notice that $\sigma t_{\alpha\gamma_\alpha}/\pi$ and $\sigma' t_{\alpha\gamma_\alpha}/\pi$ are integers (Eq. (57)).

When $\gamma_\alpha \geq 0$, we set the values of a, b and $x_{0\alpha\gamma_\alpha}$ to their respective limit values attained as $\gamma_\alpha \to 0^-$, according to the continuity rule (Section V). For the adopted single-extremum, infinitely extended potential, the reflection points of the reflected particles approach the position of that extremum as $\gamma_\alpha \to 0^-$: $a_{\alpha\gamma_\alpha} \to b_{\alpha\gamma_\alpha} \to 0$; the trapping points of the trapped particles shift to infinity: $a_{\alpha\gamma_\alpha} \to -\infty$, $b_{\alpha\gamma_\alpha} \to \infty$.

Inserting these values into Eqs. (51)-(53), and these latter into Eqs. (39) and (40), we find, for the eigenfunctions of free particles of both species, the same integration bounds $t_{0\alpha\gamma_\alpha}$ and $t_{1\alpha\gamma_\alpha}$, orthogonality relation, normalization constant and continuous nature of the spectrum as for the free particles in a double layer (Eqs. (43)-(45)).

The eigenfunctions' reflection points and phases in the electron phase space hole equilibrium potential

$$\Phi(x) = \mathrm{sech}^2(Kx), \tag{60}$$

are

$$b_{\alpha\gamma_\alpha} = -a_{\alpha\gamma_\alpha} = \mathrm{arcsinh}(|u_{\alpha\gamma_\alpha}(0)/u_{\alpha\gamma_\alpha}(\infty)|)/K, \tag{61}$$

$$\xi_{\alpha\gamma_\alpha} = \frac{1}{K u_{\alpha\gamma_\alpha}(\infty)}[I_{\alpha\gamma_\alpha}(s)]^{s=Kx}_{s=Kx_{0\alpha\gamma_\alpha}}, \tag{62}$$

$$I_{\alpha\gamma_\alpha}(s) = \ln(\sinh s + \surd\{\sinh^2(s) + [u_{\alpha\gamma_\alpha}(0)/u_{\alpha\gamma_\alpha}(\infty)]^2\}) \tag{63}$$

where $x_{0\alpha\gamma_\alpha}$ are the phase terminals (Eqs. (41) and (42)) and $u_{\alpha\gamma_\alpha}$ was defined in Eq. (25). The corresponding eigenfunctions are presented in Fig. 2.

VII. EIGENFUNCTIONS OF PARTICLES IN A PERIODIC POTENTIAL AND THEIR BLOCH STRUCTURE

The periodic equilibrium potential extends to the domain of Eq. (8)

$$x_1 \leq x < x_2,\ x_2 = x_1 + N\lambda, \tag{64}$$

where λ is its period and $N \geq 1$ is an integer. Although the potential is unknown outside this domain, we assume that the profile $\Phi(x)$ which models it satisfies the periodic boundary conditions

$$\Phi(x+\lambda) = \Phi(x) \text{ for } x_1 \leq x < x_1 + N\lambda, \tag{65}$$

$$\Phi(x) = \Phi(x - N\lambda) \text{ for } x \geq x_1 + N\lambda, \tag{66}$$

as usually done for numerical simulations over a finite, periodic numerical mesh.

It is convenient to refer the electron (respectively ion) eigenfunctions to a coordinate scale in which the origin is translated to the position of the potential's first minimum (respectively maximum): both scales extend from 0 to $N\lambda$. These translations multiply the eigenfunctions by a phase factor (Eqs. (24)-(27)) which may be absorbed into the normalization constant of Eq. (45), without affecting their eigenvalue and degeneracy parameters. In the following, without changing their names, we tacitly refer to these eigenfunctions and to these new scales.

Given $\gamma_\alpha < 0$, we define $a_{\alpha\gamma_\alpha 0}$ as the root of $\gamma_\alpha + V_\alpha = 0$ closest to the origin. We write the pairs of contiguous roots $a_{\alpha\gamma_\alpha}, b_{\alpha\gamma_\alpha} > a_{\alpha\gamma_\alpha}$ of $\gamma_\alpha + V_\alpha = 0$ — which we now label by an integer ν_α — as

$$a_{\alpha\gamma_\alpha\nu_\alpha} = a_{\alpha\gamma_\alpha 0} + \nu_\alpha\lambda,\ b_{\alpha\gamma_\alpha\nu_\alpha} = b_{\alpha\gamma_\alpha 0} + \nu_\alpha\lambda, \tag{67}$$
$$\text{for } \nu_\alpha = 0, \ldots N-1.$$

We also extend, again without changing their names, the eigenfunctions $\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}$ (Eq. (24)) — which we also label by the integer ν_α — from the potential well $I_{\alpha\gamma_\alpha\nu_\alpha}$, where they are defined, to the whole coordinate domain by prescribing that

$$\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha}(x, q_x, q_y) = 0 \text{ if } x \notin I_{\alpha\gamma_\alpha\nu_\alpha}, \tag{68}$$

$$I_{\alpha\gamma_\alpha\nu_\alpha} = \{x | a_{\alpha\gamma_\alpha\nu_\alpha} < x < b_{\alpha\gamma_\alpha\nu_\alpha}\}, \tag{69}$$
$$\text{for } \nu_\alpha = 0, \ldots N-1.$$

Given the set of N these extended "single-well" eigenfunctions $\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha}$, $\nu_\alpha = 0 \ldots N-1$ and $t_{\alpha\gamma_\alpha}$ (Eqs. (56) and (57)), which, owing to the periodicity of the potential, may be calculated here between any of the trapping point pairs $(a_{\alpha\gamma_\alpha\nu_\alpha}, b_{\alpha\gamma_\alpha\nu_\alpha})$, we construct, by linear superposition, the function

$$\eta^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y) = \frac{1}{\sqrt{N}} \times \\ \sum_{\nu_\alpha=0}^{N-1} e^{-2is_\alpha \nu_\alpha \sigma t_{\alpha\gamma_\alpha}/N} \times \\ \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha}(x, q_x, q_y). \tag{70}$$

Since each of the eigenfunctions $\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha}$ has the same eigenvalue σ and the same degeneracy parameters c_α, γ_α, s_α, so does $\eta^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}$, which thus is a legitimate "N-well" eigenfunction of the free-streaming operator.

We now prove that both the single- and the N-well eigenfunctions are orthonormal. For the former ones, a convenient choice of the phase terminal in Eq. (27) and of the scalar product bounds in Eq. (32) is

$$a = x_{0\alpha\gamma_\alpha} = a_{\alpha\gamma_\alpha\nu_\alpha},\ b = b_{\alpha\gamma_\alpha\nu_\alpha}. \tag{71}$$

The calculation of their scalar product involving the q-integration part (Eq. (34)) still holds and so does Eq. (35). Also, two eigenfunctions having a different ν_α are non zero only over disjoint intervals (Eq. (68)) and thus that scalar product vanishes if $\alpha \neq \alpha'$, or $s_\alpha \neq s'_\alpha$, or $\nu_\alpha \neq \nu'_\alpha$. On the other hand, when $\alpha = \alpha'$, $s_\alpha = s'_\alpha$ and $\nu_\alpha = \nu'_\alpha$, the proof proceeds, in each potential well, exactly as for the eigenfunctions of the particles moving in the single potential well of a phase space hole (Section VI). This gives the eigenfunctions' normalization constant as in Eq. (57) and the orthonormality condition

$$\langle \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha} | \chi^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha \nu_\alpha} \rangle = \\ \delta(c_\alpha - c'_\alpha)\delta(\gamma_\alpha - \gamma'_\alpha)[t_{\alpha\gamma_\alpha}\delta_{(\sigma t_{\alpha\gamma_\alpha}/\pi)(\sigma' t_{\alpha\gamma_\alpha}/\pi)}]. \tag{72}$$

Concerning the normalization of the N-well eigenfunctions, we choose the phase terminal in Eq. (27) and the scalar product bounds in Eq. (32) as

$$a = x_{0\alpha\gamma_\alpha} = a_{\alpha\gamma_\alpha 0},\ b = b_{\alpha\gamma_\alpha(N-1)} \tag{73}$$

and we observe that, if two of those eigenfunctions have a different α or s_α, so do the single-well eigenfunctions by which they are constructed, whence the orthogonality of both of them. On the other hand, if they have the same α and s_α, then, from Eq. (70),

$$\langle \eta^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha} | \eta^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha} \rangle = \\ \frac{1}{N} \sum_{\nu_\alpha=0}^{N-1} \sum_{\nu'_\alpha=0}^{N-1} e^{2is_\alpha[\nu_\alpha \sigma t_{\alpha\gamma_\alpha} - \nu'_\alpha \sigma' t_{\alpha\gamma'_\alpha}]/N} \times \\ \langle \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha}, \chi^{s_\alpha}_{\alpha k_y \sigma' c'_\alpha \gamma'_\alpha \nu'_\alpha} \rangle = \\ \delta(c_\alpha - c'_\alpha)\delta(\gamma_\alpha - \gamma'_\alpha)[t_{\alpha\gamma_\alpha}\delta_{(\sigma t_{\alpha\gamma_\alpha}/\pi)(\sigma' t_{\alpha\gamma_\alpha}/\pi)}], \tag{74}$$

where we took into account Eq. (72) and that two single-well eigenfunctions having a different ν_α are orthogonal. In Eqs. (72) and (74), orthonormality was ensured by choosing, in Eq. (40), the normalization constant C_α as in Eq. (45), and $t_{\alpha\gamma_\alpha}$ (Eqs. (56) and (57)) is calculated between any of the trapping point pairs $(a_{\alpha\gamma_\alpha\nu_\alpha}, b_{\alpha\gamma_\alpha\nu_\alpha})$.

It is instructive to analyze the behaviour of the N-well eigenfunctions (Eq. (70)) under discrete modular translations $x \mapsto [x+\lambda]\mathrm{mod}(N\lambda)$, λ being the period of the background equilibrium potential (Eqs. (65) and (66)). We first notice the recurrence relation

$$\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha ([\nu_\alpha+1]\mathrm{mod}\, N)}([x+\lambda]\mathrm{mod}(N\lambda), q_x, q_y) = \\ \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha}(x, q_x, q_y) \tag{75}$$

of the extended single-well eigenfunctions (Eq. (68)) calculated according to Eqs. (24)-(27) and using the phase terminals of Eq. (73).

Now, if x belongs to any of the $\nu_\alpha = 0 \ldots N-1$ potential wells, then $[x+\lambda]\mathrm{mod}(N\lambda)$ lies in the $([\nu_\alpha+1]\mathrm{mod}\, N)$-th potential well. According to its definition (Eq. (70)), the sole contributions to the N-well eigenfunction at those two coordinates are $e^{-2\mathrm{i}s_\alpha \nu_\alpha \sigma t_{\alpha\gamma_\alpha}/N}\ \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha \nu_\alpha}(x, q_x, q_y)$ and $e^{-2\mathrm{i}s_\alpha ([\nu_\alpha+1]\mathrm{mod}\, N)\sigma t_{\alpha\gamma_\alpha}/N}\ \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha ([\nu_\alpha+1]\mathrm{mod}\, N)}([x+\lambda]\mathrm{mod}(N\lambda), q_x, q_y)$ respectively and, because of Eq. (75) and of the integer value of $\sigma t_{\alpha\gamma_\alpha}/\pi$ (Eq. (57)), they are related by

$$\eta^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}([x+\lambda]\mathrm{mod}(N\lambda), q_x, q_y) = \\ e^{-2\mathrm{i}s_\alpha \sigma t_{\alpha\gamma_\alpha}/N}\eta^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y). \tag{76}$$

In Eq. (70), we thus constructed an eigenfunction of the free-streaming operator having eigenvalue σ which is also an eigenfunction under discrete modular translations $x \mapsto [x+\lambda]\mathrm{mod}(N\lambda)$, with respect to which its eigenvalue is $e^{-2\mathrm{i}s_\alpha \sigma t_{\alpha\gamma_\alpha}/N}$. This is sufficient to write the eigenfunction in Bloch form [14]:

$$\eta^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y) = \\ e^{-2\mathrm{i}s_\alpha \sigma t_{\alpha\gamma_\alpha} x/(N\lambda)} w^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y), \tag{77}$$

where, as it can be directly verified, $w^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}$ is a suitable modular-periodic function of x having the same period λ of the equilibrium potential: $w^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}([x+\lambda]\mathrm{mod}(N\lambda), q_x, q_y) = w^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y)$.

The N-well eigenfunctions $\eta^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}$, constructed for $\gamma_\alpha < 0$, are immediately extended to values of $\gamma_\alpha > 0$, by adopting, for the quantities $a_{\alpha\gamma_\alpha\nu_\alpha}$ and $b_{\alpha\gamma_\alpha\nu_\alpha}$, their limit values as $\gamma_\alpha \to 0^-$, according to the continuity rule of Section V. In this limit, each of the potential wells $I_{\alpha\gamma_\alpha\nu_\alpha}$ (Eq. (69)) invades the entire ν_α-th period of the equilibrium potential: the domain of the N-well eigenfunction thus invades the whole coordinate domain. This is shown by Eq. (67) and by the relations

$$\lim_{\gamma_\alpha=0^-}(a_{\alpha\gamma_\alpha 0}, b_{\alpha\gamma_\alpha 0}, b_{\alpha\gamma_\alpha(N-1)}) = (0, \lambda, N\lambda). \tag{78}$$

The free N-well eigenfunctions inherit the Bloch structure of Eq. (77) and the normalization of Eq. (74) if $\alpha = \alpha'$ and $s_\alpha = s'_\alpha$, and they are otherwise orthogonal.

The eigenfunctions's phases for particles in the electron phase-space hole equilibrium potential

$$\Phi(x) = [1 - \cos(Kx)]/2 \tag{79}$$

are

$$\xi_{\alpha\gamma_\alpha} = \frac{\sqrt{\mu_\alpha}}{K}\frac{\sqrt{2}}{\sqrt{|Z_\alpha|}}F\left(\arcsin(\sin(Kx/2)/\kappa), \kappa\right), \tag{80}$$

$$\kappa = \sqrt{[(\gamma_\alpha + |Z_\alpha|)/|Z_\alpha|]}, \tag{81}$$

where F is the elliptic integral of first kind and κ is its modulus. The particles' reflection points are given by Eq. (67), in which

$$a_{\alpha\gamma_\alpha 0} = \arccos(1 + 2\gamma_\alpha/|Z_\alpha|)/K, \tag{82}$$

$$b_{\alpha\gamma_\alpha 0} = 2\pi/K - a_{\alpha\gamma_\alpha 0}. \tag{83}$$

The corresponding single-well eigenfunctions are presented in Fig. 3.

VIII. GREEN'S FUNCTION OF THE FREE-STREAMING OPERATOR

Given the diagonal nature of the matrix valued free-streaming operator S_{k_y} (Eq. (18)), its matrix valued Green function $G_{k_y}(x, q_x, q_y; s, p_x, p_y)$ solves

$$S_{\alpha\alpha k_y} G_{\alpha\alpha' k_y} = \delta_{\alpha\alpha'}\delta(x - s)\delta(q_x - p_x)\delta(q_y - p_y). \tag{84}$$

We see that the Green's function admits off-diagonal elements if $S_{\alpha\alpha k_y}$ has non trivial eigenfunctions corresponding to a zero eigenvalue. In fact, these eigenfunctions do exist and they are recovered from the eigenfunctions given in Eq. (24) by setting $\sigma = 0$. Correspondingly, in Eq. (97), the contributions to $f_\alpha(x, q_x, q_y)$ coming from the off-diagonal elements $G_{\alpha\alpha' k_y}$ belong to the null space of the free-streaming operator and they are to be accepted or rejected according to whether they meet the physical boundary and initial conditions at hand.

We now prove that the Green's function may be given by a spectral representation. We first treat the case of a purely continuous spectrum and we analyze the contributions of a discrete spectrum at the end of this section. Given the coordinate x, we introduce the quantity

$$\varsigma_\alpha = 1 \text{ if } V'_\alpha(x) \geq 0, \tag{85}$$

$$\varsigma_\alpha = -1 \text{ if } V'_\alpha(x) < 0 \tag{86}$$

and we write the Green's function according to the sign of ς_α:

$$G^{\varsigma_\alpha}_{\alpha\alpha' k_y}(x, q_x, q_y; s, p_x, p_y) = \delta_{\alpha\alpha'} \times \\ \left\{ \sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(s)}^{\infty} \mathrm{d}\gamma_\alpha \int_{-\infty}^{\infty} \frac{A^{s_\alpha}_\alpha \mathrm{d}\sigma}{\sigma + \mathrm{i}s_\alpha\varsigma_\alpha 0^+} \times \right. \\ \chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(x, q_x, q_y)\bar{\chi}^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(s, p_x, p_y) + \\ \sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(x)}^{\infty} \mathrm{d}\gamma_\alpha \int_{-\infty}^{\infty} \frac{B^{s_\alpha}_\alpha \mathrm{d}\sigma}{\sigma + \mathrm{i}s_\alpha\varsigma_\alpha 0^+} \times \\ \left. \bar{\chi}^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(x, q_x, q_y)\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(s, p_x, p_y) \right\}. \tag{87}$$

Here, 0^+ stands for an arbitrarily small positive quantity and $A^{s_\alpha}_\alpha$ and $B^{s_\alpha}_\alpha$ are arbitrary constants.

To understand the structure of the Green's function, we observe that, in the first term in the braces on the rigth hand side of Eq. (87), the γ_α-integration starts at $-V_\alpha(s)$ and thus $\gamma_\alpha + V_\alpha(s) \geq 0$: this ensures that the phase of $\bar{\chi}^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(s, p_x, p_y)$ is purely imaginary (Eqs. (24)-(27)) and thus that this eigenfunction does not diverge exponentially $|p_x| \to \infty$.

Likewise, by writing $\gamma_\alpha + V_\alpha(x) = \gamma_\alpha + V_\alpha(s) + [V_\alpha(x) - V_\alpha(s)] \geq [V_\alpha(x) - V_\alpha(s)]$, we see that, for the phase of $\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(x, q_x, q_y)$ to be also purely imaginary, $[V_\alpha(x) - V_\alpha(s)]$ must be non negative: taking $\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}$ from Eq. (24), we see that this is ensured, in the first term in the braces on the right hand side of Eq. (87), precisely by the factor $\int_{-\infty}^{\infty} \mathrm{d}\sigma/(\sigma + \mathrm{i}s_\alpha\varsigma_\alpha 0^+)$ $e^{-\mathrm{i}s_\alpha\sigma[\xi_{\alpha\gamma_\alpha}(x) - \xi_{\alpha\gamma_\alpha}(s)]}$: indeed, taking into account that, being the integral of a positive quantity, $\xi_{\alpha\gamma_\alpha}$ is an increasing function (Eq. (27)), that factor gives $-\mathrm{i}\pi s_\alpha\varsigma_\alpha\theta(\varsigma_\alpha[\xi_{\alpha\gamma_\alpha}(x) - \xi_{\alpha\gamma_\alpha}(s)]) = -\mathrm{i}\pi s_\alpha\varsigma_\alpha\theta(\varsigma_\alpha[x - s]) = -\mathrm{i}\pi s_\alpha\varsigma_\alpha\theta(V_\alpha(x) - V_\alpha(s))$, where θ denotes the step function

$$\theta(x) = 1 \text{ if } x \geq 0, \tag{88}$$

$$\theta(x) = -1 \text{ if } x < 0. \tag{89}$$

Similar arguments holds for the second term in the braces on the right hand side of Eq. (87), where the γ_α-integration starts at $-V_\alpha(x)$, and where the the σ-integration results in the factor $-2\mathrm{i}\pi s_\alpha\varsigma_\alpha\theta(V(s) - V(x))$.

Last, we point out that, through the identity $1/(\sigma \pm \mathrm{i}0^+) = \mathrm{P}(1/\sigma) \mp \mathrm{i}\pi\delta(\sigma)$, the σ-integrations in Eq. (87) give terms containing the null space eigenfunction $\chi^{s_\alpha}_{\alpha k_y 0 c_\alpha\gamma_\alpha}$ corresponding to the eigenvalue $\sigma = 0$. In this way, an arbitrary function belonging to the null space of $S_{\alpha\alpha k_y}$ is built in Eq. (87).

We now prove that $G_{\alpha\alpha k_y}$ as given by Eq. (87) solves Eq. (84). By definition, $\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}$ is an eigenfunction of $S_{\alpha\alpha k_y}$ with eigenvalue σ: substituting its expression from Eq. (24) into Eq. (87), and operating by $S_{\alpha\alpha k_y}$ on the first term in the braces on the right hand side Eq. (87) gives

$$
\begin{aligned}
&\delta_{\alpha\alpha'} \sum_{s_\alpha=\pm} \int_{-V_\alpha(s)}^{\infty} \mathrm{d}\gamma_\alpha |C_\alpha|^2 \frac{e^{\mathrm{i}s_\alpha[q_x|u_{\alpha\gamma_\alpha}(x)|-p_x|u_{\alpha\gamma_\alpha}(s)|]}}{|u_{\alpha\gamma_\alpha}(x)||u_{\alpha\gamma_\alpha}(s)|} \times \\
&\int_{-\infty}^{\infty} \mathrm{d}c_\alpha e^{\mathrm{i}c_\alpha\{(q_y-p_y)+s_\alpha k_y[\xi_{\alpha\gamma_\alpha}(x)-\xi_{\alpha\gamma_\alpha}(s)]\}} \times \\
&\int_{-\infty}^{\infty} \frac{A^{s_\alpha}_\alpha \sigma \mathrm{d}\sigma}{\sigma + \mathrm{i}s_\alpha\varsigma_\alpha 0^+} e^{-\mathrm{i}s_\alpha\sigma[\xi_{\alpha\gamma_\alpha}(x)-\xi_{\alpha\gamma_\alpha}(s)]}.
\end{aligned}
\tag{90}
$$

Here, we carry out first the σ-integration by means of the identity $\int_{-\infty}^{\infty} \sigma \mathrm{d}\sigma/(\sigma + \mathrm{i}s_\alpha\varsigma_\alpha 0^+)$ $e^{-\mathrm{i}s_\alpha\sigma[\xi_{\alpha\gamma_\alpha}(x)-\xi_{\alpha\gamma_\alpha}(s)]} = 2\pi\delta(\xi_{\alpha\gamma_\alpha}(x) - \xi_{\alpha\gamma_\alpha}(s)) = 2\pi|u_{\alpha\gamma_\alpha}(x)|\delta(x - s)$ and then we perform the straightforward c_α-integration to get

$$
\begin{aligned}
&4\pi^2 \delta_{\alpha\alpha'} \delta(x - s)\delta(q_y - p_y) \sum_{s_\alpha=\pm} A^{s_\alpha}_\alpha \times \\
&\int_{-V_\alpha(s)}^{\infty} \mathrm{d}\gamma_\alpha |C_\alpha|^2 \frac{e^{\mathrm{i}s_\alpha|u_{\alpha\gamma_\alpha}(s)|(q_x-p_x)}}{|u_{\alpha\gamma_\alpha}(s)|}.
\end{aligned}
\tag{91}
$$

Finally, changing the integration variable from γ_α to $u = |u_{\alpha\gamma_\alpha}(s)|$ (Eq. (25)), taking the normalization constant C_α from Eq. (45) and using the identity $\int_0^\infty \mathrm{d}u e^{\pm \mathrm{i}ut} = \pi\delta(t) \pm \mathrm{i}\mathrm{P}/(2t)$, reduces Eq. (91) to

$$
\begin{aligned}
&\delta_{\alpha\alpha'} \delta(x - s)\delta(q_x - p_x)\delta(q_y - p_y) \times \\
&\left[(A^+_\alpha + A^-_\alpha)\delta(q_x - p_x) - (A^+_\alpha - A^-_\alpha) \frac{1}{2\mathrm{i}\pi} \frac{\mathrm{P}}{q_x - p_x} \right].
\end{aligned}
\tag{92}
$$

To operate by $S_{\alpha\alpha k_y}$ on the second term in the braces on the right hand side of Eq. (87), we split the γ_α-integration there in a part running from $-V_\alpha(x)$ to 0 and in a part running from 0 to ∞. In the latter part, the operator $\partial^2/\partial x \partial q_x$ obviously commutes with the γ_α-integration and so does the whole free-streaming operator $S_{\alpha\alpha k_y}$ (Eq. (18)). In the former part, we first regularize the fractional γ_α-integral by adding and subtracting, in the numerator of its integrand, the value of that numerator at $\gamma_\alpha = -V_\alpha(x)$: that value turns out to be independent of q_x (Eq. (28)) and thus, again, the operator $\partial^2/\partial x \partial q_x$ commutes with the γ_α-integration.

In this way, a contribution analogous to Eq. (92) (with x, q_x, q_y interchanged respectively with s, p_x, p_y and $B^{s_\alpha}_\alpha$ replacing $A^{s_\alpha}_\alpha$) results from the action of $S_{\alpha\alpha k_y}$ on the second term in the braces on the right hand side of Eq. (87) and its action on the whole right hand side of that equation thus gives the sought Eq. (84), provided we choose

$$A^+_\alpha + A^-_\alpha + B^+_\alpha + B^-_\alpha = 1, \tag{93}$$

$$A^+_\alpha - A^-_\alpha - (B^+_\alpha - B^-_\alpha) = 0. \tag{94}$$

When, for $\gamma_\alpha < 0$, the spectrum contains a discrete part (Eq. (57)), each of the γ_α-integrals on the right hand side of Eq. (87) will be split in a term running from 0 to infinity and in a term running from $-V_\alpha$ to 0. The contribution of the former term to the Green's function only involves the continuous part of the spectrum and it can be treated as above. In the contribution of the latter term, the σ-integration in Eq. (87) are replaced by a series over the mode number of the eigenfunctions.

The proof that this latter contribution complies with Eq. (84) closely follows the one used for the continuous spectrum and it needs not be given here. That proof basically relied on the spectral representations of the Heaviside and Dirac delta functions in terms of certain singular Fourier integrals over a continuous variable. In the case of a discrete spectrum, those spectral representations are given in terms infinite trigonometric series over a discrete variable [6].

IX. RESOLVENT OF THE FREE-STREAMING OPERATOR

The procedure developed in Section VIII may be employed to find resolvent of the free-streaming operator, i.e. the matrix valued Green's function $G_{k_y\omega}\left(x, q_x, q_y; s, p_x, p_y\right)$ of the operator $S_{k_y} - \omega$. This operator obviously has the same eigenfunctions of S_{k_y}, but corresponding to the eigenvalues $\sigma - \omega$. Therefore, denoting by $A_\alpha^{s_\alpha}$ and $B_\alpha^{s_\alpha}$ constants obeying Eqs. (93) and (94), we may write $G_{\alpha\alpha' k_y\omega}$ as in Eq. (87)

$$
\begin{aligned}
&G^{\varsigma_\alpha}_{\alpha\alpha' k_y\omega}(x, q_x, q_y; s, p_x, p_y) = \delta_{\alpha\alpha'} \times \\
&\left\{ \sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(s)}^{\infty} \mathrm{d}\gamma_\alpha \int_{-\infty}^{\infty} \frac{A_\alpha^{s_\alpha}\mathrm{d}\sigma}{(\sigma-\omega) + \mathrm{i}s_\alpha\varsigma_\alpha 0^+} \times \right. \\
&\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(x, q_x, q_y)\bar{\chi}^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(s, p_x, p_y) + \\
&\sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(x)}^{\infty} \mathrm{d}\gamma_\alpha \int_{-\infty}^{\infty} \frac{B_\alpha^{s_\alpha}\mathrm{d}\sigma}{(\sigma-\omega) + \mathrm{i}s_\alpha\varsigma_\alpha 0^+} \times \\
&\left. \bar{\chi}^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(x, q_x, q_y)\chi^{s_\alpha}_{\alpha k_y\sigma c_\alpha\gamma_\alpha}(s, p_x, p_y) \right\}.
\end{aligned}
\tag{95}
$$

Now consider the inhomogeneous problem

$$
(S_{k_y} - \omega)|g_\omega\rangle = |h\rangle \tag{96}
$$

for the vector valued function $|g_\omega\rangle$ over the domain $a < x < b$, where a and b are the scalar product bounds introduced in Eq. (33) and given in Sections V-VII for several shapes of the equilibrium electric potential. In terms of the resolvent, the solution of Eq. (96) is

$$
\begin{aligned}
g_{\alpha\omega}(x, q_x, q_y) = \int_a^b \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}p_x \int_{-\infty}^{\infty} \mathrm{d}p_y \times \\
G_{\alpha\alpha k_y\omega}(x, q_x, q_y; s, p_x, p_y) h_\alpha(s, p_x, p_y).
\end{aligned}
\tag{97}
$$

The contribution to Eq. (97) of the second term in the braces on the right hand side of Eq. (95) can afford a straightforward interchange of the order of the s- and γ_α-integration. The contribution of the first term also relies on such interchange, which however needs to be performed separately, by Fubini's rule, in each sub-domain of $a < x < b$ where $V_\alpha(s)$ is monotonic: when this is done, the lower bound for the γ_α-integratione becomes $-V_\alpha(x)$. Taking into account the symmetry relations of the eigenfunctions (Eq. (29)), the two contributions add to

$$\begin{aligned}&g_{\alpha\omega}(x, q_x, q_y) = \\ &\sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(x)}^{\infty} \mathrm{d}\gamma_\alpha \times \int_{-\infty}^{\infty} \frac{\mathrm{d}\sigma}{(\sigma-\omega)+\mathrm{i}s_\alpha\varsigma_\alpha 0^+} \times \\ &(A_\alpha^{s_\alpha} + B_\alpha^{-s_\alpha}) H^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha} \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y),\end{aligned} \tag{98}$$

where

$$\begin{aligned}H^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha} = &\int_a^b \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}p_x \int_{-\infty}^{\infty} \mathrm{d}p_y \times \\ &\bar{\chi}^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(s, p_x, p_y) h_\alpha(s, p_x, p_y).\end{aligned} \tag{99}$$

X. COMPLETENESS OF THE EIGENFUNCTIONS OF THE FREE-STREAMING OPERATOR

The resolvent of the free-streaming operator found in Section IX will now be used to prove the completeness of its eigenfunctions. To do so we adapt a technique developed Ref. [15] based on the solutions of the initial value problem

$$(\mathrm{i}\partial/\partial t + S_{k_y})|f\rangle = 0, \tag{100}$$

$$f_\alpha(x, q_x, q_y, 0) = -\mathrm{i}h_\alpha(x, q_x, q_y), \tag{101}$$

$$|h_\alpha(x, q_x, q_y)| < \infty. \tag{102}$$

In terms of the unilateral Fourier transform

$$g_{\alpha\omega}(x, q_x, q_y) = \int_0^\infty \mathrm{d}t e^{-\mathrm{i}\omega t} f_\alpha(x, q_x, q_y, t), \tag{103}$$

Eqs. (100) and (101) reduce to an inhomogeneous problem for $g_{\alpha\omega}$ of the type of Eq. (96) and its solution is given in Eq. (98).

This solution is now used to calculate, by the initial value theorem,

$$f_\alpha(x, q_x, q_y, 0^+) = \lim_{\omega \to -\mathrm{i}\infty} \mathrm{i}\omega g_{\alpha\omega}(x, q_x, q_y), \tag{104}$$

where ω approaches infinity along the negative imaginary axis. Taking into account that, by Eqs. (101) and (102), f_α is continuous and finite at $t = 0$ and that its value there is $-\mathrm{i}h_\alpha$, Eqs. (98) and (104) give

$$h_\alpha(x, q_x, q_y) = \sum_{s_\alpha=\pm} \int_{-\infty}^{\infty} \mathrm{d}c_\alpha \int_{-V_\alpha(x)}^{\infty} \mathrm{d}\gamma_\alpha \times \int_{-\infty}^{\infty} \mathrm{d}\sigma \times (A_\alpha^{s_\alpha} + B_\alpha^{-s_\alpha}) H^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha} \chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}(x, q_x, q_y). \tag{105}$$

Eq. (105) provides a means to express an arbitrary initial condition h_α as a superposition of the eigenfunctions $\chi^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}$ of the free-streaming operator. The superposition coefficient $H^{s_\alpha}_{\alpha k_y \sigma c_\alpha \gamma_\alpha}$ is given in terms of h_α in Eq. (99). The superposition is taken over the whole spectrum of the operator and for all of the allowed values of the degeneracy paramentes of the eigenfunctions. We conclude that those eigenfunctions are complete.

XI. SUMMARY AND DISCUSSION

In this work, we determined the spectrum, singularities, degeneracies and orthogonality relations of the eigenfunctions of free-streaming electrostatic oscillations of electrons and mobile ions in several, physically relevant, inhomogeneous collisionless plasma equilibria.

We found that the eigenfunctions of particles which move over an infinite or semi-infinite spatial coordinate domain (Sections V and VI) belong to an infinite real continuous spectrum. The eigenfunctions of particles which are trapped in the troughs of their respective equilibrium potential energy wells (Section VI and VII) belong to a infinite real discrete spectrum.

We analyzed in detail the degeneracy structure of the eigenfunctions. All of them have two finite discrete degeneracies and two infinite, continuous degeneracies. A further infinite discrete degeneracy arises for periodic equilibria. We showed that all types of eigenfunctions form orthonormal sets and we determined the values of the corresponding eigenfunction normalization constants.

For periodic equilibria, we also analyzed the properties of the eigenfunctions under discrete modular space coordinate translations: we found that the eigenfunctions of both the continuous and discrete spectrum are also eigenfunctions under these translations. Their Bloch form was worked out accordingly.

Besides spatially-periodic equilibria, eigenfunctions were also given for two other types of physically relevant equilibria in which the electric potential is bell-shaped (as in a solitary wave or in a phase-space hole) or monotonic (as in a double layer).

A valuable contribution of our work is the proof of completeness of the eigenfunctions of the free-streaming operator. This rather elaborated proof required a judicious generalization to our three-variable, vector-valued, degenerate and singular eigenfunctions of the matrix-valued, partial differential free-streaming operator of a technique developed in Ref. [15] for the single-variable, scalar-valued, non-degenerate continuous eigenfunctions of a scalar-valued ordinary differential operator.

One further methodological contribution of our treatment is its development in the Fourier transformed velocity space where the eigenfunctions are ordinary functions, rather than distributions. Without any further complication, this treatment can be applied also when the equilibrium background is based on distribution functions which are discontinuous in velocity, but which are well behaved in such space.

The velocity Fourier representation allows focus on the space coordinate dependence of the eigenfunctions. This unveiled that, near the walls of the potential wells, the eigenfunctions of all trapped species pertaining to all kinds of equilibria have an algebraic singularity.

Singularity — partly mitigated by its integrability — should be taken into account in the numerical reproduction of the eigenfunctions by those algorithms working in the Fourier transformed velocity space. The simple analytical expressions of the singular eigenfunctions afforded by our treatment should help investigations in this direction.

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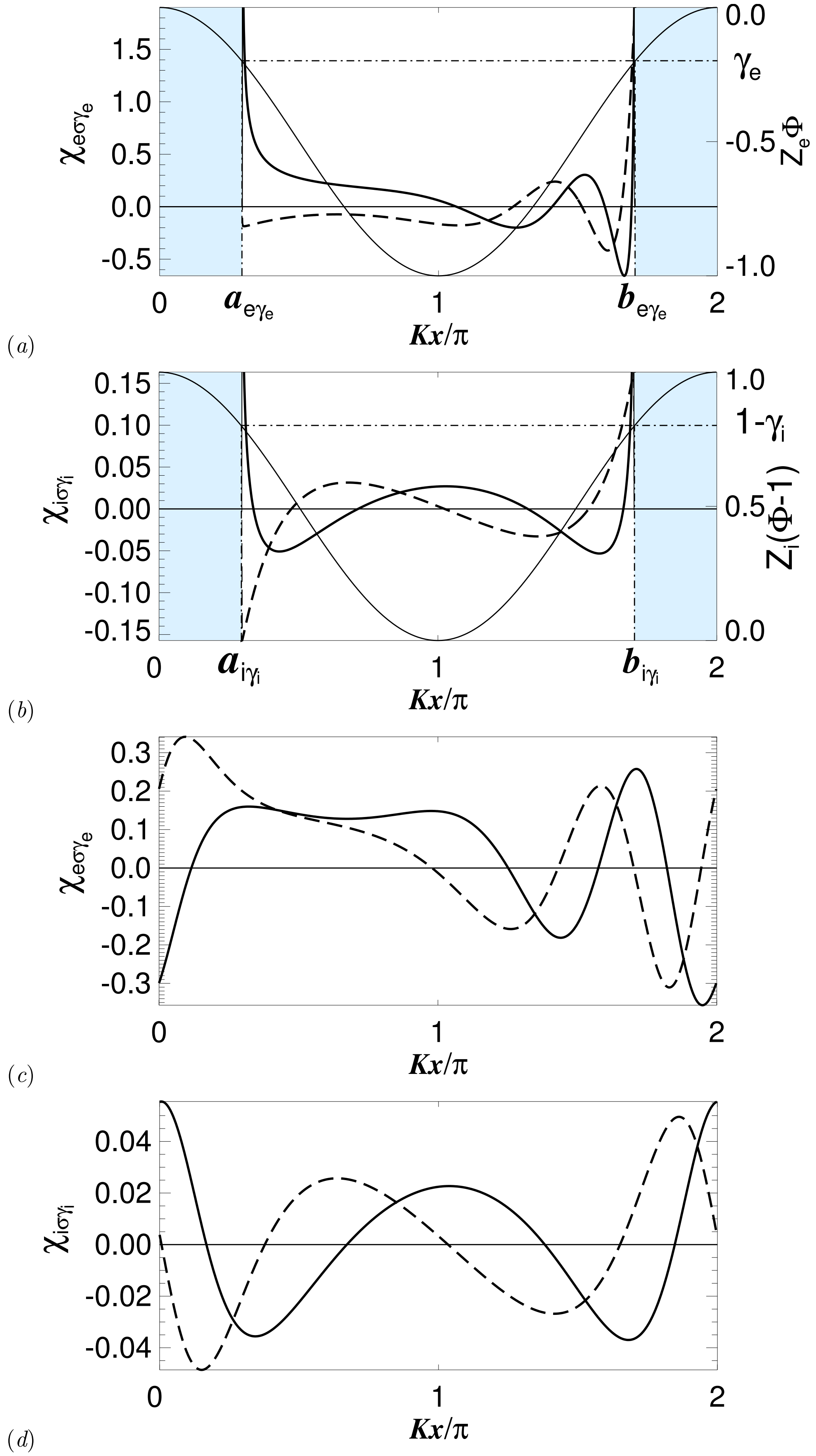

Figure 3. Same as in Fig. 1 for particles moving over a single period of the finitely extended periodic potential $\Phi(x) = [1 - \cos(Kx)]/2$. In panels (a) and (b), the mode number for the trapped particle eigenfunctions is $m = 1$ for both species. Parameters are as in Fig. 1, but $\gamma_\alpha = -0.2$ in panels (a) and

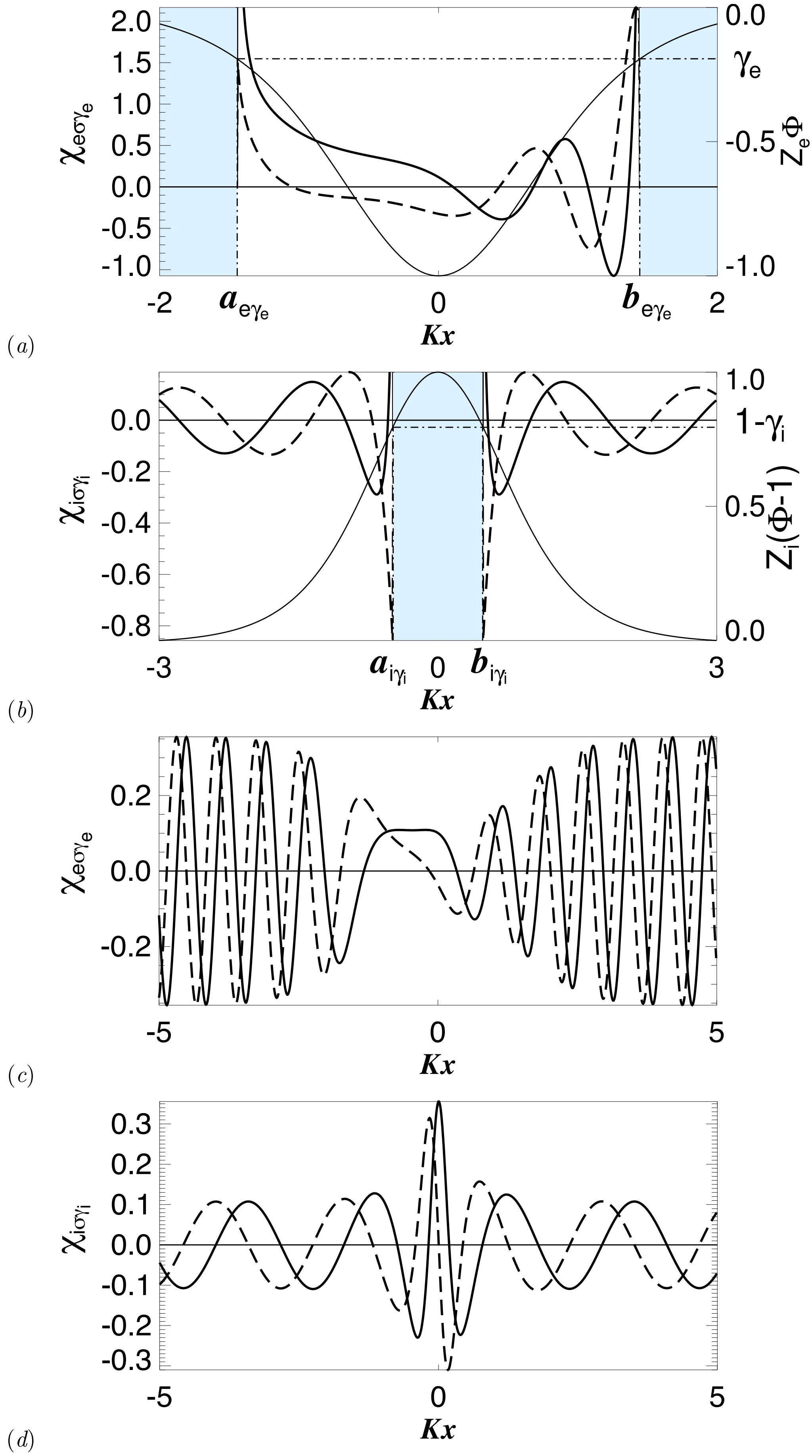

Figure 2. Same as for Fig. 1, but for particles moving in the electron phase space hole potential $\Phi(x) = \text{sech}^2(Kx)$. In panel (a) the mode number for the trapped electron eigenfunction is $m = 1$. Parameters as in Fig. 1, but $\gamma_\alpha = -0.2$ in panels (a) and (b).

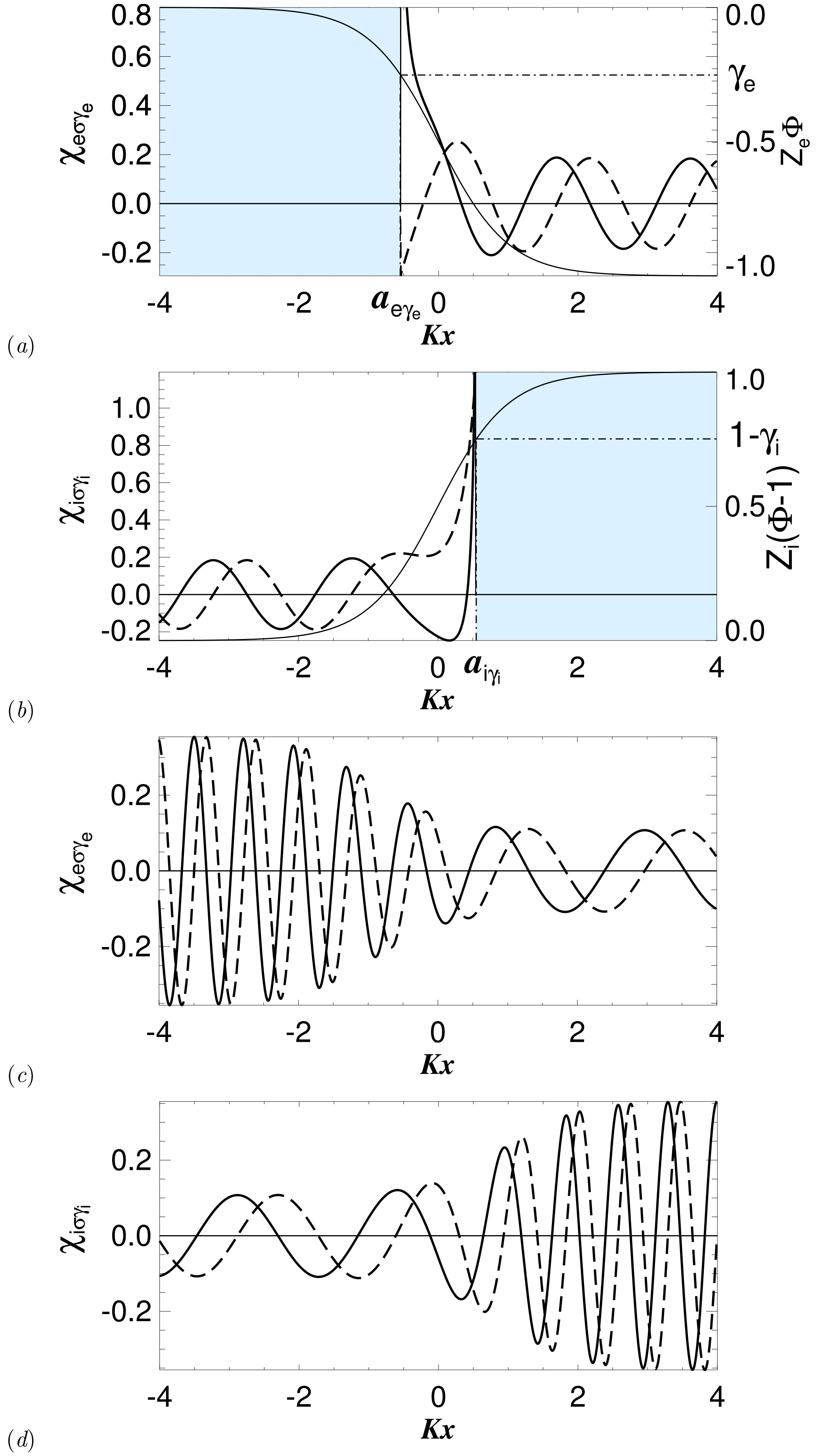

Figure 1. The real (solid bold lines) and imaginary (dashed bold lines) parts of the eigenfunctions $\chi^{s_\mathrm{e}}_{\mathrm{e}k_y\sigma c_\mathrm{e}\gamma_\mathrm{e}}$ and $\chi^{s_\mathrm{i}}_{\mathrm{i}k_y\sigma c_\mathrm{i}\gamma_\mathrm{i}}$ vs. coordinate x for reflected electrons (panel (a)), reflected ions (panel (b)), free electrons (panel (c)) and free ions (panel (d)) moving in a double layer potential $\Phi(x) = [1 +$