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ISTI Technical Reports

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ISTI-TR-2022/018

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Closure spaces, Topological spaces, Spatial logics, Spatial bisimilarities, Stuttering equivalence

Citation
Ciancia V.; Latella D.; Massink M.; de Vink E.P. *Back-and-Forth in Space: On Logics and Bisimilarity in Closure Spaces. Preliminary Extended Version* ISTI Technical Reports 2022/018. DOI: 10.32079/ISTI-TR-2022/018

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Back-and-Forth in Space: On Logics and Bisimilarity in Closure Spaces Preliminary Extended Version*

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Abstract. We adapt the standard notion of bisimilarity for topological models to closure models and refine it for quasi-discrete closure models. We also define an additional, weaker notion of bisimilarity that is based on paths in space and expresses a form of *conditional* reachability in a way that is reminiscent of *Stuttering Equivalence* on transition systems. For each bisimilarity we provide a characterisation with respect to a suitable spatial logic.

Keywords: Closure Spaces; Topological Spaces; Spatial Logics; Spatial Bisimilarities; Stuttering Equivalence.

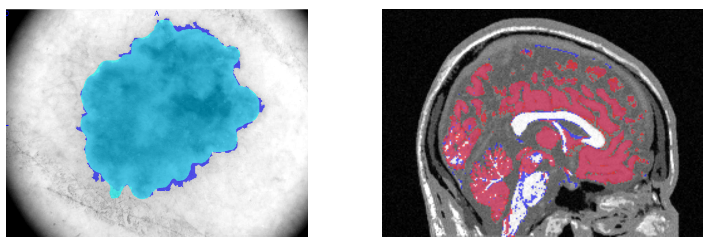
1 Introduction

The use of modal logics for the description of properties of topological spaces—where a point in space satisfies formula $\Diamond \Phi$ whenever it belongs to the *topological closure* of the set $[\![\Phi]\!]$ of the points satisfying formula Φ—has a well established tradition, dating back to the fourties, and has given rise to the research area of *Spatial Logics* (see e.g. [5]). More recently, the class of underlying models of space have been extended to include, for instance, *closure spaces*, a generalisation of topological spaces (see e.g. [20]). The relevant logics have been extended

* Research partially supported by the MIUR Project PRIN 2017FTXR7S IT-MaTTerS". The authors are listed in alphabetical order; they contributed to this work equally.
This is an extended version of the paper "Back-and-Forth in Space: On Logics and Bisimilarity in Closure Spaces", by V. Ciancia, D. Latella, M. Massink, and E-P. de Vink. In: N. Jansen, M. Stoelinga, P. van den Bos (eds), A Journey from Process Algebra via Timed Automata to Model Learning. Lecture Notes in Computer Science, vol 13560. Springer, Cham. pp. 98 -115, Springer, 2022, at: https://link.springer.com/chapter/10.1007/978-3-031-15629-8_6#chapter-info.
This document contains all the detailed proofs of the results presented in the above mentioned paper, that had been omitted in the paper for space reasons.

accordingly. The approach has been enriched with algorithms for spatial (and spatio-temporal) logic *model checking* [14, 13] and associated tools [4, 11, 24, 12, 23], and has been applied in various domains, such as bike-sharing [17], Turing patterns [30], medical image analysis [2, 10, 4, 3]—Figure 1 shows the segmentation of a nevus (Fig. 1a) and a segmentation of a cross-section of brain grey matter (Fig. 1b); the original manual segmentation of both the nevus [29] and the grey matter [1] is shown in blue, while that resulting using spatial model checking is shown in cyan for the nevus and in red for grey matter. As the figures show, the manual segmentation of the nevus and that obtained using spatial model-checking have a very good correspondence; those of the grey matter coincide almost completely, so that very little blue is visible.

Notions of spatial bisimilarity have been proposed as well, and their potential for model minimisation plays an important role in the context of model-checking optimisation. Consequently, a key question, when reasoning about modal logics and their models, is the relationship between logical equivalences and notions of bisimilarity on their models.



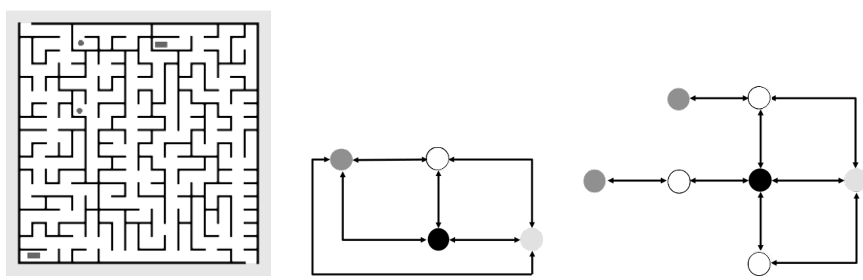
(a) (b)

Fig. 1: Segmentation of (a) nevus and (b) grey matter in the brain.

In this paper we study three different notions of bisimilarity for closure models, i.e. models based on closure spaces. The first one is *closure model bisimilarity* (CM-bisimilarity for short). This bisimilarity is an adaptation for closure models of classical *topo-bisimilarity* for topological models [5]. The former uses the interior operator where topo-bisimilarity uses open sets. Actually, due to monotonicity of the interior operator, CM-bisimilarity is an instantiation to closure models of *monotonic bisimulation* on neighbourhood models [27, 6, 25]. We provide a logical characterisation of CM-bisimilarity, using Infinitary Modal Logic, a modal logic with infinite conjunction [8].

We show that, for *quasi-discrete* closure models, i.e. closure models where every point has a minimal neighbourhood, CM-bisimilarity gets a considerably simpler definition—based on the the closure operator instead of the interior operator—that is reminiscent of the definition of bisimilarity for transition systems. The advantage of the direct use of the closure operator, which is the foundational operator of closure spaces, is given by its intuitive interpretation in quasi-discrete closure models that makes several proofs simpler. We then present a refinement of CM-bisimilarity, specialised for *quasi-discrete* closure models. In *quasi-discrete* closure spaces, the closure of a set of points—and so also its interior—can be expressed using an underlying binary relation; this gives rise to both a *direct* closure and interior of a set, and a *converse* closure and interior, the latter being obtained using the inverse of the binary relation. This, in turn, induces a refined notion of bisimilarity, *CM-bisimilarity with converse*, CMC-bisimilarity, which is shown to be strictly stronger than CM-bisimilarity. We also present a closure-based definition for CMC-bisimilarity [15]. Interestingly, the latter resembles *Strong Back-and-Forth bisimilarity* proposed by De Nicola, Montanari and Vaandrager in [19].

We extend the Infinitary Modal Logic with the converse of its unary modal operator and show that the resulting logic characterises CMC-bisimilarity. CM-bisimilarity, and CMC-bisimilarity, play an important role as they are the closure model counterpart of classical topo-bisimilarity. On the other hand, they turn out to be too strong, when considering intuitive relations on space, such as scaling or reachability, that may be useful when dealing with models representing images[3]. Consider, for instance, the image of a maze in Figure 2a, where walls are represented in black and the exit area is shown in light grey (the floor is represented in white). A typical question one would ask is whether, starting from a given point (i.e. pixel)—for instance one of those shown in dark grey in the picture—one can reach the exit area, at the border of the image.



(a) (b) (c)

Fig. 2: A maze (a) and its path- and CoPa-minimal models ((b) and (c))

Essentially, we are interested in those paths in the picture, rooted at dark grey points, leading to light grey points *passing only* through white points. In [18] we introduced path-bisimilarity; it requires that, in order for two points to be equivalent, for every path rooted in one point there must be a path rooted in the other point and the end-points of the two paths must be bisimilar. Path-bisimilarity is too weak; nothing whatsoever is required about the internal structure of the relevant paths. For instance, Figure 2b shows the minimal model for the image

[3] Images can be modeled as quasi-discrete closure spaces where the underlying relation is a pixel/voxel adjacency relation; see [2, 10, 4, 3] for details.

of the maze shown in Figure 2a according to path-bisimilarity. We see that all dark grey points are equivalent and so are all white points. In other words, we are unable to distinguish those dark grey (white) points from which one can reach an exit from those from which one cannot. So, we look for *reachability* of bisimilar points by means of paths over the underlying space. Such reachability is not unconditional; we want the relevant paths to share some common structure. For that purpose, we resort to a notion of "compatibility" between relevant paths that essentially requires each of them to be composed by a sequence of non-empty "zones", with the total number of zones in each of the paths being the same, while the length of each zone being arbitrary; each element of one path in a given zone is required to be related by the bisimulation to all the elements in the corresponding zone in the other path. This idea of compatibility gives rise to the third notion of bisimulation we present in this paper, namely *Compatible Path bisimulation*, CoPa-bisimulation. We show that, for quasi-discrete closure models, CoPa-bisimulation is strictly weaker than CMC-bisimilarity[4]. Figure 2c shows the minimal model for the image of the maze shown in Figure 2 according to CoPa-bisimilarity. We see that, in this model, dark grey points from which one can reach light grey ones passing only by white points are distinguished from those from which one cannot. Similarly, white points through which an exit can be reached from a dark grey point are distinguished both from those that can't be reached from dark grey points and from those through which no light grey point can be reached.

We provide a logical characterisation of CoPa-bisimularity too. The notion of CoPa-bisimulation is reminiscent of that of the *Equivalence with respect to Stuttering* for transition systems [9, 22], although in a different context and with different definitions as well as different underlying notions. The latter, in fact, is defined via a convergent sequence of relations and makes use of a different notion of path than the one of CS used in this paper. Finally, stuttering equivalence is focussed on CTL/CTL*, which implies a flow of time with single past (i.e. trees), which is not the case for structures representing space.

The paper is organised as follows: after having settled the context and offered some preliminary notions and definitions in Section 2, in Section 3 we present CM-bisimilarity. Section 4 deals with CMC-bisimularity. Section 5 addresses CoPa-bisimilarity. We conclude the paper with Section 6. All detailed proofs can be found in the Appendix.

2 Preliminaries

In this paper, given a set X, $\mathcal{P}(X)$ denotes the powerset of X; for $Y \subseteq X$ we use $\overline{Y}$ to denote $X \setminus Y$, i.e. the complement of Y. For a function $f : X \to Y$ and $A \subseteq X$, we let $f(A)$ be defined as $\{f(a) \,|\, a \in A\}$. We briefly recall several definitions and results on closure spaces, most of which are taken from [20].

Definition 1 (Closure Space – CS). *A* closure space*, CS for short, is a pair* $(X, \mathcal{C})$ *where* X *is a non-empty set (of* points*) and* $\mathcal{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is a*

[4] CoPa-bisimilarity is stronger than path-bisimilarity (see [18] for details).

function satisfying the following axioms: (i) $\mathcal{C}(\emptyset) = \emptyset$*; (ii)* $A \subseteq \mathcal{C}(A)$ *for all* $A \subseteq X$*; and (iii)* $\mathcal{C}(A_1 \cup A_2) = \mathcal{C}(A_1) \cup \mathcal{C}(A_2)$ *for all* $A_1, A_2 \subseteq X$*.* •

It is worth pointing out that topological spaces coincide with the sub-class of CSs that satisfy the *idempotence* axiom $\mathcal{C}(\mathcal{C}(A) = \mathcal{C}(A)$. The *interior* operator is the dual of closure: $\mathcal{I}(A) = \overline{\mathcal{C}(\overline{A})}$. It holds that $\mathcal{I}(X) = X$, $\mathcal{I}(A) \subseteq A$, and $\mathcal{I}(A_1 \cap A_2) = \mathcal{I}(A_1) \cap \mathcal{I}(A_2)$. A *neighbourhood* of a point $x \in X$ is any set $A \subseteq X$ such that $x \in \mathcal{I}(A)$. A minimal neighbourhood of a point x is a neighbourhood A of x such that $A \subseteq A'$ for every other neighbourhood A' of x. We recall that the closure operator, and consequently the interior operator, is monotonic: if $A_1 \subseteq A_2$ then $\mathcal{C}(A_1) \subseteq \mathcal{C}(A_2)$ and $\mathcal{I}(A_1) \subseteq \mathcal{I}(A_2)$.
We have occasion to use the following property of closure spaces[5]:

Lemma 1. *Let* $(X, \mathcal{C})$ *be a CS. For* $x \in X$*,* $A \subseteq X$*, it holds that* $x \in \mathcal{C}(A)$ *iff* $U \cap A \neq \emptyset$ *for each neighbourhood* U *of* x*.* □

Definition 2 (Quasi-discrete CS – QdCS). *A* quasi-discrete closure space *is a CS* $(X, \mathcal{C})$ *such that any of the two following equivalent conditions holds: (i) each* $x \in X$ *has a minimal neighbourhood; or (ii) for each* $A \subseteq X$ *it holds that* $\mathcal{C}(A) = \bigcup_{x \in A} \mathcal{C}(\{x\})$*.* •

Given a relation $R \subseteq X \times X$, define the function $\mathcal{C}_R : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows: for all $A \subseteq X$, $\mathcal{C}_R(A) = A \cup \{x \in X \mid a \in A \text{ exists s.t. } (a, x) \in R\}$. It is easy to see that, for any R, $\mathcal{C}_R$ satisfies all the axioms of Definition 1 and so $(X, \mathcal{C}_R)$ is a CS. The following theorem is a standard result in the theory of CSs [20]:

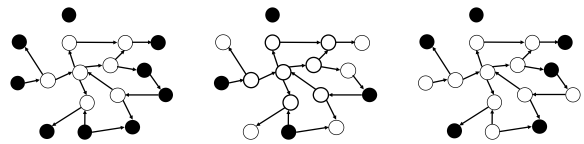
Theorem 1. *A CS* $(X, \mathcal{C})$ *is quasi-discrete if and only if there is a relation* $R \subseteq X \times X$ *such that* $\mathcal{C} = \mathcal{C}_R$*.* □

In the sequel, whenever a CS $(X, \mathcal{C})$ is quasi-discrete, we use $\vec{\mathcal{C}}$ to denote $\mathcal{C}_R$, and, consequently, $(X, \vec{\mathcal{C}})$ to denote the closure space, abstracting from the specification of R, when the latter is not necessary. Moreover, we let $\overleftarrow{\mathcal{C}}$ denote $\mathcal{C}_{R^{-1}}$. Finally, we use the simplified notation $\vec{\mathcal{C}}(x)$ for $\vec{\mathcal{C}}(\{x\})$ and similarly for $\overleftarrow{\mathcal{C}}(x)$. An example of the difference between $\vec{\mathcal{C}}$ and $\overleftarrow{\mathcal{C}}$ is shown in Figure 3.

Regarding the interior operator $\mathcal{I}$, the notations $\vec{\mathcal{I}}$ and $\overleftarrow{\mathcal{I}}$ are defined in the obvious way: $\vec{\mathcal{I}} A = \overline{\vec{\mathcal{C}}(\overline{A})}$ and $\overleftarrow{\mathcal{I}} A = \overline{\overleftarrow{\mathcal{C}}(\overline{A})}$.
In the context of the present paper, *paths* over closure spaces play an important role. Therefore, we give a formal definition of paths based on continuous functions below.

Definition 3 (Continuous function). *Function* $f : X_1 \to X_2$ *is a* continuous *function from* $(X_1, \mathcal{C}_1)$ *to* $(X_2, \mathcal{C}_2)$ *if and only if for all sets* $A \subseteq X_1$ *we have* $f(\mathcal{C}_1(A)) \subseteq \mathcal{C}_2(f(A))$*.* •

[5] See also [32] Corollary 14.B.7.



(a) (b) (c)

Fig. 3: In white: (a) a set of points A, (b) $\vec{\mathcal{C}}(A)$, and (c) $\overleftarrow{\mathcal{C}}(A)$.

Definition 4 (Index space). *An* index space *is a connected*[6] *CS* $(I, \mathcal{C})$ *equipped with a total order* $\leqslant\, \subseteq I \times I$ *with a bottom element* 0. *We often write* $\iota_1 < \iota_2$ *whenever* $\iota_1 \leqslant \iota_2$ *and* $\iota_1 \neq \iota_2$, (ι_1, ι_2) *for* $\{\iota \mid \iota_1 < \iota < \iota_2\}$, $[\iota_1, \iota_2)$ *for* $\{\iota \mid \iota_1 \leq \iota < \iota_2\}$, *and* $(\iota_1, \iota_2]$ *for* $\{\iota \mid \iota_1 < \iota \leq \iota_2\}$. •

Definition 5 (Path). *A* path *in CS* $(X, \mathcal{C})$ *is a continuous function from an index space* $\mathcal{J} = (I, \mathcal{C}^{\mathcal{J}})$ *to* $(X, \mathcal{C})$. *A path* π *is* bounded *if there exists* $\ell \in I$ *such that* $\pi(\iota) = \pi(\ell)$ *for all* ι *such that* $\ell \leqslant \iota$; *we call the minimal such* ℓ *the* length *of* π, *written* $\texttt{len}(\pi)$. •

Particularly relevant in the present paper are *quasi-discrete* paths, i.e. paths having $(\mathbb{N}, \mathcal{C}_{\texttt{succ}})$ as index space, where $\mathbb{N}$ is the set of natural numbers and $\texttt{succ}$ is the *successor* relation $\texttt{succ} = \{(m, n) \mid n = m + 1\}$.

The following lemmas state some useful properties of closure and interior operators as well as of paths.

Lemma 2. *For all QdCSs* $(X, \vec{\mathcal{C}}), A, A_1, A_2 \subseteq X, x_1, x_2 \in X$, *and* $\pi : \mathbb{N} \to X$ *the following holds:*

1. $\overleftarrow{\mathcal{C}}(A) = A \cup \{x \in X \mid$ *there exists* $a \in A$ *such that* $(x, a) \in R\}$;
2. $x_1 \in \overleftarrow{\mathcal{C}}(\{x_2\})$ *if and only if* $x_2 \in \vec{\mathcal{C}}(\{x_1\})$;
3. $\overleftarrow{\mathcal{C}}(A) = \{x \mid x \in X$ *and exists* $a \in A$ *such that* $a \in \vec{\mathcal{C}}(\{x\})\}$;
4. *if* $A_1 \subseteq A_2$, *then* $\overleftarrow{\mathcal{C}}(A_1) \subseteq \overleftarrow{\mathcal{C}}(A_2)$ *and* $\overleftarrow{\mathcal{I}}(A_1) \subseteq \overleftarrow{\mathcal{I}}(A_2)$;
5. π *is a path over* X *if and only if for all* $j \neq 0$ *the following holds:* $\pi(j) \in \vec{\mathcal{C}}(\pi(j-1))$ *and* $\pi(j-1) \in \overleftarrow{\mathcal{C}}(\pi(j))$. □

Lemma 3. *Let* $(X, \vec{\mathcal{C}})$ *be a QdCS. Then* $\vec{\mathcal{C}}(x) \subseteq A$ *iff* $x \in \overleftarrow{\mathcal{I}}(A)$ *and* $\overleftarrow{\mathcal{C}}(x) \subseteq A$ *iff* $x \in \vec{\mathcal{I}}(A)$, *for all* $x \in X$ *and* $A \subseteq X$. □

In the sequel we will assume a set $\texttt{AP}$ of *atomic proposition letters* is given and we introduce the notion of closure *model*.

Definition 6 (Closure model – CM). *A* closure model*, CM for short, is a tuple* $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$, *with* $(X, \mathcal{C})$ *a CS, and* $\mathcal{V} : \texttt{AP} \to \mathcal{P}(X)$ *the (atomic proposition) valuation function, assigning to each* $p \in \texttt{AP}$ *the set of points where* p *holds.* •

[6] Given CS $(X, \mathcal{C})$, $A \subseteq X$ is *connected* if it is *not* the union of two non-empty separated sets. Two subsets $A_1, A_2 \subseteq X$ are called *separated* if $A_1 \cap \mathcal{C}(A_2) = \emptyset = \mathcal{C}(A_1) \cap A_2$. CS $(X, \mathcal{C})$ is *connected* if X is connected.

All the definitions given above for CSs apply to CMs as well; thus, a *quasi-discrete closure model* (QdCM for short) is a CM $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$ where $(X, \vec{\mathcal{C}})$ is a QdCS. For a closure model $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$ we often write $x \in \mathcal{M}$ when $x \in X$. Similarly, we speak of paths in $\mathcal{M}$ meaning paths in $(X, \mathcal{C})$. For $x \in \mathcal{M}$, we let $\mathtt{BPaths^F}_{\mathcal{J},\mathcal{M}}(x)$ denote the set of all *bounded* paths π in $\mathcal{M}$ with indices in $\mathcal{J}$, such that $\pi(0) = x$ (paths *rooted* in x); similarly $\mathtt{BPaths^T}_{\mathcal{J},\mathcal{M}}(x)$ denotes the set of all *bounded* paths π in $\mathcal{M}$ with indices in $\mathcal{J}$, such that $\pi(\mathtt{len}(\pi)) = x$ (paths *ending* in x). We refrain from writing the subscripts $_{\mathcal{J},\mathcal{M}}$ when not necessary.

In the sequel, for a logic $\mathcal{L}$, a formula $\Phi \in \mathcal{L}$, and a model $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$ we let $[\![\Phi]\!]^{\mathcal{M}}_{\mathcal{L}}$ denote the set $\{x \in X \mid \mathcal{M}, x \models_{\mathcal{L}} \Phi\}$ of all the points in $\mathcal{M}$ that satisfy Φ, where $\models_{\mathcal{L}}$ is the satisfaction relation for $\mathcal{L}$. For the sake of readability, we refrain from writing the subscript $_{\mathcal{L}}$ when this does not cause confusion.

3 Bisimilarity for Closure Models

In this section, we introduce the first notion of bisimilarity that we consider, namely CM-bisimilarity, for which we also provide a logical characterisation.

3.1 CM-bisimilarity

Definition 7. *Given a CM $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$, a symmetric relation $B \subseteq X \times X$ is a* CM-bisimulation *for $\mathcal{M}$ if, whenever $(x_1, x_2) \in B$, the following holds:*

1. *for all $p \in \mathtt{AP}$ we have $x_1 \in \mathcal{V}(p)$ if and only if $x_2 \in \mathcal{V}(p)$;*
2. *for all $S_1 \subseteq X$ such that $x_1 \in \mathcal{I}(S_1)$ exists $S_2 \subseteq X$ such that $x_2 \in \mathcal{I}(S_2)$ and for all $s_2 \in S_2$ exists $s_1 \in S_1$ such that $(s_1, s_2) \in B$.*

Two points $x_1, x_2 \in X$ are called CM-bisimilar in $\mathcal{M}$ if $(x_1, x_2) \in B$ for some CM-bisimulation B for $\mathcal{M}$. Notation, $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CM}} x_2$. •

The above notion is the natural adaptation for CMs of the notion of topo-bisimilation for topological models [5]. In such models the underlying set is equiped with a topology, i.e. a special case of a CS. For a topological model $\mathcal{M} = (X, \tau, \mathcal{V})$ with τ a topology on X the requirements for a relation $B \subseteq X \times X$ to be a topo-bisimulation are similar to those in Definition 7; see [5] for details.

3.2 Logical characterisation of CM-bisimilarity

Next, we show that CM-bisimilarity is characterised by an infinitary version of Modal Logic, IML for short, where the classical modal operator $\Diamond$ is interpreted as closure and is denoted by $\mathcal{N}$—for "near". We first recall the definition of IML [15], i.e. Modal Logic with infinite conjunction.

Definition 8. *The abstract language of* IML *is defined as follows:*

$$\Phi ::= p \mid \neg\Phi \mid \bigwedge_{i \in I} \Phi_i \mid \mathcal{N}\Phi$$

where p ranges over `AP` *and I ranges over a collection of index sets.*

The satisfaction relation for all CMs $\mathcal{M}$, points $x \in \mathcal{M}$, and `IML` *formulas Φ is recursively defined on the structure of Φ as follows:*

$$\begin{array}{ll} \mathcal{M}, x \models_{\texttt{IML}} p & \Leftrightarrow x \in \mathcal{V}(p); \\ \mathcal{M}, x \models_{\texttt{IML}} \neg \Phi & \Leftrightarrow \mathcal{M}, x \models_{\texttt{IML}} \Phi \textit{ does not hold}; \\ \mathcal{M}, x \models_{\texttt{IML}} \bigwedge_{i \in I} \Phi_i & \Leftrightarrow \mathcal{M}, x \models_{\texttt{IML}} \Phi_i \textit{ for all } i \in I; \\ \mathcal{M}, x \models_{\texttt{IML}} \mathcal{N}\Phi & \Leftrightarrow x \in \mathcal{C}([\![\Phi]\!]^{\mathcal{M}}). \end{array}$$ •

Below we define `IML`-equivalence, i.e. the equivalence induced by `IML`.

Definition 9. *Given CM $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$, the equivalence relation $\simeq^{\mathcal{M}}_{\texttt{IML}} \subseteq X \times X$ is defined as: $x_1 \simeq^{\mathcal{M}}_{\texttt{IML}} x_2$ if and only if for all* `IML` *formulas Φ the following holds: $\mathcal{M}, x_1 \models_{\texttt{IML}} \Phi$ if and only if $\mathcal{M}, x_2 \models_{\texttt{IML}} \Phi$.* •

It holds that `IML`-equivalence $\simeq^{\mathcal{M}}_{\texttt{IML}}$ includes CM-bisimilarity.

Lemma 4. *For all points x_1, x_2 in a CM $\mathcal{M}$, if $x_1 \rightleftharpoons^{\mathcal{M}}_{\texttt{CM}} x_2$ then $x_1 \simeq^{\mathcal{M}}_{\texttt{IML}} x_2$.* □

The converse of the lemma follows from Lemma 5 below.

Lemma 5. *For a CM $\mathcal{M}$, it holds that $\simeq^{\mathcal{M}}_{\texttt{IML}}$ is a CM-bisimulation for $\mathcal{M}$.* □

From this lemma we immediately obtain that $x_1 \simeq^{\mathcal{M}}_{\texttt{IML}} x_2$ implies $x_1 \rightleftharpoons^{\mathcal{M}}_{\texttt{CM}} x_2$, for all points x_1, x_2 in a CM $\mathcal{M}$. Summarizing, we get the following result.

Theorem 2. *For every CM $\mathcal{M}$ it holds that* `IML`*-equivalence $\simeq^{\mathcal{M}}_{\texttt{IML}}$ coincides with CM-bisimilarity $\rightleftharpoons^{\mathcal{M}}_{\texttt{CM}}$.* □

4 CMC-bisimilarity for QdCMs

Definition 7 defines CM-bisimilarity in terms of the interior operator $\mathcal{I}$; however, conceptually it is striking that CM-bisimilarity is defined in terms of interior rather than in terms of closure. In the case of QdCMs, an alternative formulation, exploiting the symmetric nature of the operators in such spaces, can be given as we will see below.

Definition 10. *Given a QdCM $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$, a symmetric relation $B \subseteq X \times X$ is a* closure-based *CM-bisimulation for $\mathcal{M}$ if, whenever $(x_1, x_2) \in B$, the following holds:*

1. *for all $p \in$* `AP` *we have $x_1 \in \mathcal{V}(p)$ if and only if $x_2 \in \mathcal{V}(p)$;*
2. *for all x_1' such that $x_1 \in \vec{\mathcal{C}}(x_1')$ exists x_2' with $x_2 \in \vec{\mathcal{C}}(x_2')$ and $(x_1', x_2') \in B$.* •

The above definition is justified by the next lemma.

Lemma 6. *Let $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$ be a QdCM and $B \subseteq X \times X$ a relation. It holds that B is a CM-bisimulation iff B is a closure-based CM-bisimulation.* □

As noted above, when dealing with QdCMs, we can exploit the symmetric nature of the operators in such spaces. Recall in fact that, whenever $\mathcal{M}$ is quasi-discrete, there are actually two interior functions, namely $\vec{\mathcal{I}}(S)$ and $\overleftarrow{\mathcal{I}}(S)$. It is then natural to exploit both functions for the definition of a notion of CM-bisimilarity specifically designed for QdCMs, namely CMC-bisimilarity, presented below.

4.1 CMC-bisimilarity for QdCMs

Definition 11. *Given QdCM $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$, a symmetric relation $B \subseteq X \times X$ is a* CMC-bisimulation *for $\mathcal{M}$ if, whenever $(x_1, x_2) \in B$, the following holds:*

1. *for all $p \in \mathtt{AP}$ we have $x_1 \in \mathcal{V}(p)$ if and only if $x_2 \in \mathcal{V}(p)$;*
2. *for all $S_1 \subseteq X$ such that $x_1 \in \vec{\mathcal{I}}(S_1)$ exists $S_2 \subseteq X$ such that $x_2 \in \vec{\mathcal{I}}(S_2)$ and for all $s_2 \in S_2$, exists $s_1 \in S_1$ with $(s_1, s_2) \in B$;*
3. *for all $S_1 \subseteq X$ such that $x_1 \in \overleftarrow{\mathcal{I}}(S_1)$ exists $S_2 \subseteq X$ such that $x_2 \in \overleftarrow{\mathcal{I}}(S_2)$ and for all $s_2 \in S_2$, exists $s_1 \in S_1$ with $(s_1, s_2) \in B$.*

Two points $x_1, x_2 \in X$ are called CMC-bisimilar in $\mathcal{M}$, if $(x_1, x_2) \in B$ for some CMC-bisimulation B for $\mathcal{M}$. Notation, $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CMC}} x_2$. •

For a QdCM $\mathcal{M}$, as for CM-bisimilarity, we have that CMC-bisimilarity $\rightleftharpoons_{\mathtt{CMC}}$ on $\mathcal{M}$ is a CMC-bisimulation itself, viz. the largest CMC-bisimulation for $\mathcal{M}$, thus including each CMC-bisimulation for $\mathcal{M}$. Also for CMC-bisimilarity, a formulation in terms of closures is possible.

Definition 12. *Given a QdCM $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$, a symmetric relation $B \subseteq X \times X$ is a* closure-based *CMC-bisimulation for $\mathcal{M}$ if, whenever $(x_1, x_2) \in B$, the following holds:*

1. *for all $p \in \mathtt{AP}$ we have $x_1 \in \mathcal{V}(p)$ in and only if $x_2 \in \mathcal{V}(p)$;*
2. *for all $x_1' \in \vec{\mathcal{C}}(x_1)$ exists $x_2' \in \vec{\mathcal{C}}(x_2)$ such that $(x_1', x_2') \in B$;*
3. *for all $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$ exists $x_2' \in \overleftarrow{\mathcal{C}}(x_2)$ such that $(x_1', x_2') \in B$.* •

Remark 1. Note the correspondence of criterium (3) of Definition 12 and criterium (2) of Definition 10. Recall that in the context of QdCMs we have that $x_1 \in \mathcal{C}(x_1')$ if and only if $x_1 \in \vec{\mathcal{C}}(x_1')$ if and only if $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$—see Lemma 2(2).

The above definition was proposed originally in [15], in a slightly different form, and resembles (strong) Back-and-Forth bisimulation of [19], in particular for the presence of condition (3). Should we have deleted that condition, thus making our definition more similar to classical strong bisimulation for transition systems, we would have to consider points v_{12} and v_{22} of Figure 4a bisimilar where $X = \{v_{11}, v_{12}, v_{21}, v_{22}\}$, $\vec{\mathcal{C}}(v_{11}) = \{v_{11}, v_{12}\}$, $\vec{\mathcal{C}}(v_{12}) = \{v_{12}\}$, $\vec{\mathcal{C}}(v_{21}) = \{v_{21}, v_{22}\}$, $\vec{\mathcal{C}}(v_{22}) = \{v_{22}\}$, $\mathcal{V}(w) = \{v_{11}\}, \mathcal{V}(b) = \{v_{21}\}$, and $\mathcal{V}(g) = \{v_{12}, v_{22}\}$, for the atomic propositions g, b, and w. We instead want to consider them as not being bisimilar because they are in the closure of points that are *not* bisimilar, namely v_{11} and v_{21}. For instance, v_{21} might represent a poisoned physical location (whereas v_{11} is not poisoned) and so v_{22} should not be considered equivalent to v_{12} because the former can be reached (by poison aerosol) from the poisoned location while the latter cannot.

The next lemma shows the interchangability of Definitions 11 and 12.

Lemma 7. *Let $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$ be a QdCM and $B \subseteq X \times X$ a relation. It holds that B is a CMC-bisimulation if and only if B is a closure-based CMC-bisimulation.* □

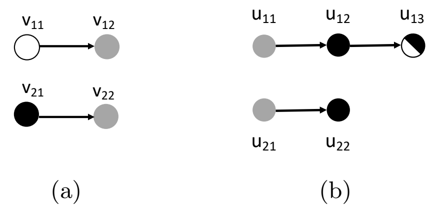


Fig. 4: v_{12} and v_{22} are not bisimilar (a); $u_{11} \rightleftharpoons_{\mathtt{CM}} u_{21}$ but $u_{11} \not\rightleftharpoons_{\mathtt{CMC}} u_{21}$ (b).

The following proposition follows directly from the relevant definitions, keeping in mind that for QdCSs the interior operator $\mathcal{I}$ coincides with the operator $\vec{\mathcal{I}}$.

Proposition 1. *For x_1, x_2 in QdCM $\mathcal{M}$, if $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CMC}} x_2$, then $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CM}} x_2$.* □

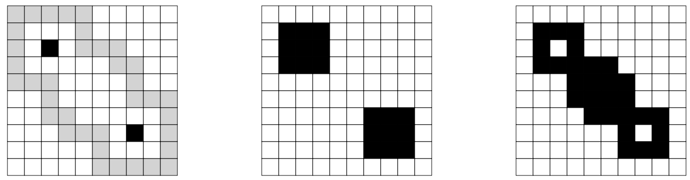
As can be expected, the converse of the proposition does *not* hold. A counter example to Proposition 1 is shown in Figure 4b. Here, $X = \{u_{11}, u_{12}, u_{13}, u_{21}, u_{22}\}$, $\mathcal{C}(u_{11}) = \{u_{11}, u_{12}\}$, $\mathcal{C}(u_{12}) = \{u_{12}, u_{13}\}$, $\mathcal{C}(u_{13}) = \{u_{13}\}$, $\mathcal{C}(u_{21}) = \{u_{21}, u_{22}\}$, $\mathcal{C}(u_{22}) = \{u_{22}\}$, and $\mathcal{V}(g) = \{u_{11}, u_{21}\}$, $\mathcal{V}(b) = \{u_{12}, u_{13}, u_{22}\}$, and $\mathcal{V}(w) = \{u_{13}\}$, for the atomic propositions g, b, and w.

It is easy to see, using Definition 10, that the symmetric closure of relation $B = \{(u_{11}, u_{21}), (u_{12}, u_{22})\}$ is a CM-bisimulation. Thus, we have $u_{11} \rightleftharpoons_{\mathtt{CM}} u_{21}$. Note, the checking of the various requirements does not involve the point u_{13} at all. However, there is no CMC-bisimulation containing the pair (u_{11}, u_{21}). In fact, any such relation would have to satisfy condition (2) of Definition 12. Since $u_{12} \in \vec{\mathcal{C}}(u_{11})$ we would have $(u_{12}, u_{21}) \in B$ or $(u_{12}, u_{22}) \in B$. Since $u_{13} \in \vec{\mathcal{C}}(u_{12})$, similarly, we would have that $(u_{13}, u_{21}) \in B$ or $(u_{13}, u_{22}) \in B$, because $\vec{\mathcal{C}}(u_{21}) = \{u_{21}, u_{22}\}$ and $\vec{\mathcal{C}}(u_{22}) = \{u_{22}\}$. However, $u_{13} \in \mathcal{V}(w)$ and neither $u_{21} \in \mathcal{V}(w)$, nor $u_{22} \in \mathcal{V}(w)$, violating requirement (1) of Definition 12, if $(u_{13}, u_{21}) \in B$ or $(u_{13}, u_{22}) \in B$.

4.2 Logical characterisation of CMC-bisimilarity

In order to provide a logical characterisation of CMC-bisimilarity, we extend `IML` with a "converse" of its modal operator. The result is the *Infinitary Modal Logic with Converse* (`IMLC`), a logic with the two modalities $\vec{\mathcal{N}}$ and $\overleftarrow{\mathcal{N}}$ expressing *proximity*. For example, with reference to the QdCM shown in Figure 5a—where points and atomic propositions are shown as grey-scale coloured squares and the underlying relation is orthodiagonal adjacency[7]—Figure 5b shows in black the points satisfying $\vec{\mathcal{N}}$`black` in the model shown in Figure 5a.

[7] In orthodiagonal adjacency, two squares are related if they share a face or a vertex.



(a) (b) (c)

Fig. 5: A model (a). In black the points satisfying $\vec{\mathcal{N}}$`black` (b), and those satisfying $\vec{\zeta}$`black`[`white`] (c)

Definition 13. *The abstract language of* `IML` *is defined as follows:*

$$\Phi ::= p \mid \neg\Phi \mid \bigwedge_{i\in I} \Phi_i \mid \vec{\mathcal{N}}\,\Phi \mid \overleftarrow{\mathcal{N}}\,\Phi.$$

where p ranges over `AP` *and I ranges over a collection of index sets.*

The satisfaction relation for all QdCMs $\mathcal{M}$, points $x \in \mathcal{M}$, and `IMLC` *formulas Φ is defined recursively on the structure of Φ as follows:*

$$\begin{array}{ll}
\mathcal{M}, x \models_{\texttt{IMLC}} p & \Leftrightarrow x \in \mathcal{V}(p);\\
\mathcal{M}, x \models_{\texttt{IMLC}} \neg\,\Phi & \Leftrightarrow \mathcal{M}, x \models_{\texttt{IMLC}} \Phi \textit{ does not hold};\\
\mathcal{M}, x \models_{\texttt{IMLC}} \bigwedge_{i\in I} \Phi_i & \Leftrightarrow \mathcal{M}, x \models_{\texttt{IMLC}} \Phi_i \textit{ for all } i \in I;\\
\mathcal{M}, x \models_{\texttt{IMLC}} \vec{\mathcal{N}}\,\Phi & \Leftrightarrow x \in \vec{\mathcal{C}}([\![\Phi]\!]^{\mathcal{M}});\\
\mathcal{M}, x \models_{\texttt{IMLC}} \overleftarrow{\mathcal{N}}\,\Phi & \Leftrightarrow x \in \overleftarrow{\mathcal{C}}([\![\Phi]\!]^{\mathcal{M}}).
\end{array}$$

•

`IMLC`-equivalence is defined in the usual way:

Definition 14. *Given QdCM $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$, the equivalence relation $\simeq^{\mathcal{M}}_{\texttt{IMLC}} \subseteq X \times X$ is defined as: $x_1 \simeq^{\mathcal{M}}_{\texttt{IMLC}} x_2$ if and only if for all* `IMLC` *formulas Φ the following holds: $\mathcal{M}, x_1 \models_{\texttt{IMLC}} \Phi$ if and only if $\mathcal{M}, x_2 \models_{\texttt{IMLC}} \Phi$.* •

Next we derive two lemmas which are used to prove that CMC-bisimilarity and `IMLC`-equivalence coincide.

Lemma 8. *For x_1, x_2 in QdCM $\mathcal{M}$, if $x_1 \rightleftharpoons^{\mathcal{M}}_{\texttt{CMC}} x_2$ then $x_1 \simeq^{\mathcal{M}}_{\texttt{IMLC}} x_2$.* □

For what concerns the other direction, i.e. going from `IMLC`-equivalence to CMC-bisimilarity, we have the following result.

Lemma 9. *For a QdCM $\mathcal{M}$, $\simeq^{\mathcal{M}}_{\texttt{IMLC}}$ is a CMC-bisimulation for $\mathcal{M}$.* □

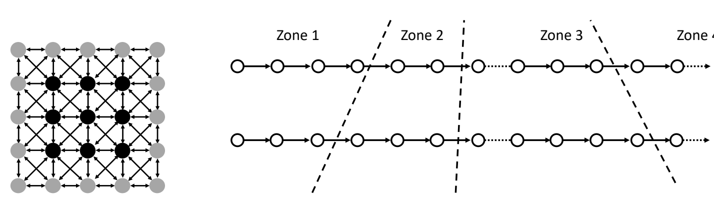
With the two lemmas above in place, we can establish the correspondence of CMC-bisimilarity and `IMLC`-equivalence.

Theorem 3. *For a QdCM $\mathcal{M}$ it holds that $\simeq^{\mathcal{M}}_{\texttt{IMLC}}$ coincides with $\rightleftharpoons^{\mathcal{M}}_{\texttt{CMC}}$.* □

Remark 2. In previous work of Ciancia et al., versions of the *Spatial Logic for Closure Spaces*, `SLCS`, have been defined that are based on the *surrounded* operator $\mathcal{S}$ and/or the *reachability* operator ρ (see e.g. [18, 15, 4, 14]). A point x satisfies $\Phi_1\,\mathcal{S}\,\Phi_2$ if it lays in an area whose points satisfy Φ_1, and that is delimited (i.e., surrounded) by points that satisfy Φ_2; x satisfies $\rho\,\Phi_1[\Phi_2]$ if there is a path rooted in x that *can reach* a point satisfying Φ_1 and whose *internal* points—if any—satisfy Φ_2. In [4], it has been shown that $\mathcal{S}$ can be derived from the logical operator ρ; more specifically, $\Phi_1\,\mathcal{S}\,\Phi_2$ is equivalent to $\Phi_1 \wedge \neg\rho(\neg(\Phi_1 \vee \Phi_2))[\neg\Phi_2]$. Furthermore, for QdCM, ρ gives rise to two symmetric operators, namely $\vec{\rho}$—coinciding with ρ—and $\overleftarrow{\rho}$—meaning that x *can be reached* from a point satisfying Φ_1, via a path whose internal points satisfy Φ_2. It is easy to see that, for such spaces, $\vec{\mathcal{N}}\,\Phi$ ($\overleftarrow{\mathcal{N}}\,\Phi$) is equivalent to $\overleftarrow{\rho}\,\Phi[\texttt{false}]$ ($\vec{\rho}\,\Phi[\texttt{false}]$) and that $\vec{\rho}\,\Phi_1[\Phi_2]$ ($\overleftarrow{\rho}\,\Phi_1[\Phi_2]$) is equivalent to a suitable combination of (possibly infinite) disjunctions and nested $\overleftarrow{\mathcal{N}}$ ($\vec{\mathcal{N}}$); the interested reader is referred to [16]. Thus, on QdCMs, `IMLC` and `ISLCS`—the infinitary version of `SLCS` [18]—share the same expressive power.

5 CoPa-Bisimilarity for QdCM

CM-bisimilarity, and its refinement CMC-bisimilarity, are a fundamental starting point for the study of spatial bisimulations due to their strong links to topo-bisimulation. On the other hand, they are rather fine-grained relations for reasoning about general properties of space. For instance, with reference to the model of Figure 6a, where all black points satisfy only atomic proposition b while the grey ones satisfy only g, the point at the center of the model is *not* CMC-bisimilar to any other black point. This is because CMC-bisimilarity is based on the fact that points reachable "in one step" are taken into consideration, as it is clear also from Definition 12. This, in turn, gives bisimilarity a sort of "counting" power, that goes against the idea that, for instance, all black points in the model could be considered spatially equivalent. In fact, they are black and can reach black or grey points. Furthermore, they could be considered equivalent to the black point of a smaller model consisting of just one black and one grey point mutually connected—that would in fact be minimal. In order to relax



(a) (b)

Fig. 6: A model (a); zones in paths (b).

such "counting" capability of bisimilarity, one could think of considering *paths* instead of single "steps"; and in fact in [18] we introduced such a bisimilarity, called path-bisimilarity. The latter requires that, in order for two points to be equivalent, for every bounded path rooted in one point there must be a bounded path rooted in the other point and the end-points of the two paths must be bisimilar.

As we have briefly discussed in Section 1, however, path-bisimilarity is too weak. A deeper insight into the structure of paths is desirable as well as some, relatively high-level, requirements over them. For that purpose we resort to a notion of "compatibility" between relevant paths that essentially requires each of them be composed of a non-empty sequence of non-empty, adjacent "zones". More precisely, both paths under consideration in a transfer condition should share the same structure, as follows (see Figure 6b):

- both paths are composed by a sequence of (non-empty) "zones";
- the number of zones should be the same in both paths, *but*
- the length of "corresponding" zones can be different, *as well as* the length of the two paths;
- *each* point in one zone of a path should be related by the bisimulation to *every* point in the corresponding zone of the other path.

This notion of compatibility gives rise to *Compatible Path bisimulation*, CoPa-bisimulation, defined below.

5.1 CoPa-bisimilarity

Definition 15. *Given CM* $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$ *and index space* $\mathcal{J} = (I, \mathcal{C}^{\mathcal{J}})$*, a symmetric relation* $B \subseteq X \times X$ *is a* CoPa-bisimulation for $\mathcal{M}$ *if, whenever* $(x_1, x_2) \in B$*, the following holds:*

1. *for all* $p \in \mathtt{AP}$ *we have* $x_1 \in \mathcal{V}(p)$ *in and only if* $x_2 \in \mathcal{V}(p)$*;*
2. *for all* $\pi_1 \in \mathtt{BPaths^F}_{\mathcal{J},\mathcal{M}}(x_1)$ *such that* $(\pi_1(i_1), x_2) \in B$ *for all* $i_1 \in [0, \mathtt{len}(\pi_1))$ *there is* $\pi_2 \in \mathtt{BPaths^F}_{\mathcal{J},\mathcal{M}}(x_2)$ *such that the following holds:* $(x_1, \pi_2(i_2)) \in B$ *for all* $i_2 \in [0, \mathtt{len}(\pi_2))$*, and* $(\pi_1(\mathtt{len}(\pi_1)), \pi_2(\mathtt{len}(\pi_2))) \in B$*;*
3. *for all* $\pi_1 \in \mathtt{BPaths^T}_{\mathcal{J},\mathcal{M}}(x_1)$ *such that* $(\pi_1(i_1), x_2) \in B$ *for all* $i_1 \in (0, \mathtt{len}(\pi_1)]$ *there is* $\pi_2 \in \mathtt{BPaths^T}_{\mathcal{J},\mathcal{M}}(x_2)$ *such that the following holds:* $(x_1, \pi_2(i_2)) \in B$ *for all* $i_2 \in (0, \mathtt{len}(\pi_2)]$*, and* $(\pi_1(0), \pi_2(0)) \in B$*.*

Two points $x_1, x_2 \in X$ *are called* CoPa-bisimilar *in* $\mathcal{M}$ $(x_1, x_2) \in B$ *for some CoPa-bisimulation* B *for* $\mathcal{M}$*. Notation,* $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CoPa}} x_2$*.* •

CoPa-bisimilarity is strictly weaker than CMC-bisimilarity, as shown below:

Proposition 2. *For* x_1, x_2 *in QdCM* $\mathcal{M}$*, if* $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CMC}} x_2$*, then* $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CoPa}} x_2$*.* □

The converse of Proposition 2 does not hold; with reference to Figure 7, with $\mathcal{V}(b) = \{x_{11}, x_{21}, x_{22}\}$ and $\mathcal{V}(g) = \{x_{12}, x_{23}\}$, it is easy to see that the symmetric closure of $B = \{(x_{11}, x_{21}), (x_{11}, x_{22}), (x_{12}, x_{23})\}$ is a CoPa-bisimulation, and so $x_{11} \rightleftharpoons_{\mathtt{CoPa}} x_{21}$ but $x_{11} \not\rightleftharpoons_{\mathtt{CMC}} x_{21}$ since $x_{12} \in \mathcal{V}(g)$ and $\vec{\mathcal{C}}(x_{21}) \cap \mathcal{V}(b) = \emptyset$.

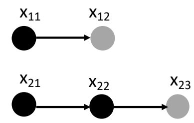


Fig. 7: $x_{11} \rightleftharpoons_{\mathtt{CoPa}} x_{21}$ but $x_{11} \not\rightleftharpoons_{\mathtt{CMC}} x_{21}$.

5.2 Logical characterisation of CoPa-bisimilarity

In order to provide a logical characterisation of CoPa-bisimilarity, we replace the proximity modalities $\vec{\mathcal{N}}$ and $\overleftarrow{\mathcal{N}}$ of `IMLC` by the *conditional reachability modalities* $\vec{\zeta}$ and $\overleftarrow{\zeta}$. Again with reference to the QdCM shown in Figure 5a, Figure 5c shows in black the points satisfying $\vec{\zeta}\,\mathtt{black}[\mathtt{white}]$, i.e. those white points from which a black point can be reached via a white path. We thus introduce the *Infinitary Compatible Reachability Logic* (`ICRL`).

Definition 16. *The abstract language of* `ICRL` *is defined as follows:*

$$\Phi ::= p \mid \neg\,\Phi \mid \bigwedge_{i \in I} \Phi_i \mid \vec{\zeta}\,\Phi_1[\Phi_2] \mid \overleftarrow{\zeta}\,\Phi_1[\Phi_2].$$

where p ranges over `AP` *and I ranges over a collection of index sets.*
The satisfaction relation for all CMs $\mathcal{M}$, points $x \in \mathcal{M}$, and `ICRL` *formulas Φ is defined recursively on the structure of Φ as follows:*

$\mathcal{M}, x \models_{\mathtt{ICRL}} p \quad \Leftrightarrow x \in \mathcal{V}(p)$;
$\mathcal{M}, x \models_{\mathtt{ICRL}} \neg\,\Phi \quad \Leftrightarrow \mathcal{M}, x \models_{\mathtt{ICRL}} \Phi$ *does not hold*;
$\mathcal{M}, x \models_{\mathtt{ICRL}} \bigwedge_{i \in I} \Phi_i \Leftrightarrow \mathcal{M}, x \models_{\mathtt{IRL}} \Phi_i$ *for all* $i \in I$;
$\mathcal{M}, x \models_{\mathtt{ICRL}} \vec{\zeta}\,\Phi_1[\Phi_2] \Leftrightarrow$ *path π and index ℓ exist such that* $\pi(0) = x$,
$\quad \pi(\ell) \models_{\mathtt{ICRL}} \Phi_1$, *and* $\pi(j) \models_{\mathtt{ICRL}} \Phi_2$ *for* $j \in [0, \ell)$
$\mathcal{M}, x \models_{\mathtt{ICRL}} \overleftarrow{\zeta}\,\Phi_1[\Phi_2] \Leftrightarrow$ *path π and index ℓ exist such that* $\pi(\ell) = x$,
$\quad \pi(0) \models_{\mathtt{ICRL}} \Phi_1$, *and* $\pi(j) \models_{\mathtt{ICRL}} \Phi_2$ *for* $j \in (0, \ell]$. •

Remark 3. With reference to Remark 2, we note that, clearly, $\vec{\zeta}$ can be derived from $\vec{\rho}$, namely: $\vec{\zeta}\,\Phi_1[\Phi_2] \equiv \Phi_1 \vee (\Phi_2 \wedge \vec{\rho}\,\Phi_1[\Phi_2])$ and similarly for $\overleftarrow{\zeta}\,\Phi_1[\Phi_2]$.

Also for `ICRL` we introduce the equivalence induced on $\mathcal{M}$:

Definition 17. *Given CM $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$, the equivalence relation $\simeq^{\mathcal{M}}_{\mathtt{ICRL}} \subseteq X \times X$ is defined as: $x_1 \simeq^{\mathcal{M}}_{\mathtt{ICRL}} x_2$ if and only if for all* `ICRL` *formulas Φ, the following holds: $\mathcal{M}, x_1 \models_{\mathtt{ICRL}} \Phi$ if and only if $\mathcal{M}, x_2 \models_{\mathtt{ICRL}} \Phi$.* •

Lemma 10. *For x_1, x_2 in QdCM $\mathcal{M}$,if $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CoPa}} x_2$ then $x_1 \simeq^{\mathcal{M}}_{\mathtt{ICRL}} x_2$.* □

The converse of Lemma 10 is given below.

Lemma 11. *For QdCM $\mathcal{M}$, $\simeq^{\mathcal{M}}_{\mathtt{ICRL}}$ is a CoPa-bisimulation for $\mathcal{M}$.* □

The correspondence between `ICRL`-equivalence and CoPa-bisimilarity is thus established by the following therorem.

Theorem 4. *For every QdCM $\mathcal{M}$ it holds that* `ICRL`*-equivalence $\simeq^{\mathcal{M}}_{\mathtt{ICRL}}$ coincides with CoPa-bisimilarity $\rightleftharpoons^{\mathcal{M}}_{\mathtt{CoPa}}$.* □

6 Conclusions

In this paper we have studied three main bisimilarities for closure spaces, namely CM-bisimilarity, its specialisation for QdCMs, CMC-bisimilarity, and CoPa-bisimilarity.

CM-bisimilarity is a generalisation for CMs of classical topo-bisimilarity for topological spaces. We can take into consideration the fact that, in QdCMs, there is a notion of "direction" given by the binary relation underlying the closure operator. This can be exploited in order to get an equivalence—namely CMC-bisimilarity—that, for QdCMs, refines CM-bisimilarity. Interestingly, the latter resembles *Strong Back-and-Forth bisimilarity* proposed by De Nicola, Montanari and Vaandrager in [19].

Both CM-bisimilarity and CMC-bisimilarity turn out to be too strong for expressing interesting properties of spaces. Therefore, we introduce CoPa-bisimilarity, that expresses a notion of path "compatibility" resembling the concept of *stuttering* equivalence for transition systems [9]. For each notion of bisimilarity we also provide an infinitary modal logic that characterises it.

Note that, in the context of space, and in particular when dealing with notions of directionality (e.g. one way roads, public area gates), it is essential to be able to distinguish between the concept of "reaching" and that of "being reached". A formula like $\vec{\zeta}(\texttt{rescue-area} \land \neg(\overleftarrow{\zeta}\,\texttt{danger-area})[\texttt{true}])[\texttt{safe-corridor}]$ expresses the fact that, via a safe corridor, a rescue area can be reached that cannot be reached from a dangerous area. This kind of situations have no obvious conterpart in the temporal domain, where there can be more than one future, like in the case of branching time logics, but there is typically only *one*, fixed past, i.e. the one that occurred[8]. The "back-and-forth" nature of CMC-bisimilarity and CoPa-bisimilarity, conceptually inherited from Back-and-Forth bisimilarity of [19], allows for such distinction in a natural way.

In this paper we did not address the problem of space minimisation explicitly. In [15] we have presented a minimisation algorithm for $\rightleftharpoons_{\texttt{CMC}}$[9]. We plan to investigate the applicability of the results presented in [21] for stuttering equivalence to minimisation modulo CoPa-bisimilarity.

Most of the results we have shown in this paper concern QdCMs. The investigation of their extension to continuous or general closure spaces is an interesting line of research. In [7] Ciancia et al. started this by approaching continuous multidimentional space using polyhedra and their representation as so-called simplicial complexes for which a model checking procedure and related tool have been developed. A similar approach is presented in [28], although the underlying model is based on an adjacency relation and the usage of simplicial complexes therein is aimed more at representing objects and higher-order relations between them than at the identification of properties of points / regions of volume meshes in a particular kind of topological model.

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[8] There are a few exception to this interpretation of past-tense operators, e.g. [26, 31].

[9] The implementation is available at `https://github.com/vincenzoml/MiniLogicA`.

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A Proofs of results in Section 2

A.1 Proof of Lemma 1

Lemma 1 *Let* $(X, \mathcal{C})$ *be a CS. For* $x \in X$, $A \subseteq X$, *it holds that* $x \in \mathcal{C}(A)$ *iff* $U \cap A \neq \emptyset$ *for each neighbourhood* U *of* x.

Proof. Suppose $x \in X$ and $A \subseteq X$ satisfy $x \in \mathcal{C}(A)$. Let U be a neighbourhood of x. Thus $x \in \mathcal{I}(U)$. Working towards a contradiction, assume $U \cap A = \emptyset$. Then $A \subseteq \overline{U}$ and $\mathcal{C}(A) \subseteq \mathcal{C}(\overline{U})$ by monotonicity of $\mathcal{C}$. Hence $\overline{\mathcal{C}(A)} \supseteq \overline{\mathcal{C}(\overline{U})}$, i.e. $\overline{\mathcal{C}(A)} \supseteq \mathcal{I}(U)$, and therefore $\mathcal{C}(A) \cap \mathcal{I}(U) = \emptyset$. However, $x \in \mathcal{C}(A)$ and $x \in \mathcal{I}(U)$, thus $x \in \mathcal{C}(A) \cap \mathcal{I}(U)$.

Suppose $x \in X$ and $A \subseteq X$ satisfy $x \notin \mathcal{C}(A)$. Then $x \in \overline{\mathcal{C}(A)} = \mathcal{I}(\overline{A})$. Note $\overline{A} \cap A = \emptyset$. Thus, $\overline{A}$ is a neighbourhood of x disjoint of A. □

A.2 Proof of Lemma 2

Lemma 2 *For all QdCSs* $(X, \vec{\mathcal{C}}), A, A_1, A_2 \subseteq X, x_1, x_2 \in X$, *and function* $\pi : \mathbb{N} \to X$ *the following holds:*

1. $\overleftarrow{\mathcal{C}}(A) = A \cup \{x \in X \mid \textit{there exists } a \in A \textit{ such that } (x, a) \in R\}$;
2. $x_1 \in \overleftarrow{\mathcal{C}}(\{x_2\})$ *if and only if* $x_2 \in \vec{\mathcal{C}}(\{x_1\})$;
3. $\overleftarrow{\mathcal{C}}(A) = \{x \mid x \in X \textit{ and exists } a \in A \textit{ such that } a \in \vec{\mathcal{C}}(\{x\})\}$;
4. *if* $A_1 \subseteq A_2$*, then* $\overleftarrow{\mathcal{C}}(A_1) \subseteq \overleftarrow{\mathcal{C}}(A_2)$ *and* $\overleftarrow{\mathcal{I}}(A_1) \subseteq \overleftarrow{\mathcal{I}}(A_2)$;
5. π *is a path over* X *if and only if for all* $j \neq 0$ *the following holds:* $\pi(j) \in \vec{\mathcal{C}}(\pi(j-1))$ *and* $\pi(j-1) \in \overleftarrow{\mathcal{C}}(\pi(j))$.

Proof. We prove only Point 5 of the lemma, the proof of the other points being trivial. We show that π is a path over X if and only if, for all $j \neq 0$, we have $\pi(j) \in \vec{\mathcal{C}}(\pi(j-1))$. Suppose π is a path over X; the following derivation:proves the assert:

$$
\begin{array}{ll}
 & \pi(j) \\
\in & \quad [\,\text{Set Theory}\,] \\
 & \{\pi(j-1), \pi(j)\} \\
= & \quad [\,\text{Definition of } \pi(N) \text{ for } N \subseteq \mathbb{N}\,] \\
 & \pi(\{j-1, j\}) \\
= & \quad [\,\text{Definition of } \mathcal{C}_{\mathsf{succ}}\,] \\
 & \pi(\mathcal{C}_{\mathsf{succ}}(\{j-1\})) \\
\subseteq & \quad [\,\text{Continuity of } \pi\,] \\
 & \vec{\mathcal{C}}(\pi(j-1))
\end{array}
$$

For proving the converse we have to show that for all sets $N \subseteq \mathbb{N} \setminus \{0\}$ we have $\pi(\mathcal{C}_{\mathsf{succ}}(N)) \subseteq \vec{\mathcal{C}}(\pi(N))$. By definition of $\mathcal{C}_{\mathsf{succ}}$ we have that $\mathcal{C}_{\mathsf{succ}}(N) = N \cup \{j \mid j-1 \in N\}$ and so $\pi(\mathcal{C}_{\mathsf{succ}}(N)) = \pi(N) \cup \pi(\{j \mid j-1 \in N\})$. By the second axiom of closure, we have $\pi(N) \subseteq \vec{\mathcal{C}}(\pi(N))$. We show that $\pi(\{j \mid j-1 \in N\}) \subseteq \vec{\mathcal{C}}(\pi(N))$ as well. Take any j such that $j-1 \in N$; we have $\{\pi(j-1)\} \subseteq \pi(N)$ since $j-1 \in N$, and, by monotonicity of $\vec{\mathcal{C}}$ it follows that $\vec{\mathcal{C}}(\{\pi(j-1)\}) \subseteq \vec{\mathcal{C}}(\pi(N))$ and since $\pi(j) \in \vec{\mathcal{C}}(\pi(j-1))$ by hypothesis, we also get $\pi(j) \in \vec{\mathcal{C}}(\pi(N))$. Since this holds for all elements of the set $\{j \mid j-1 \in N\}$ we also have $\pi(\{j \mid j-1 \in N\}) \subseteq \vec{\mathcal{C}}(\pi(N))$.

The proof for $\pi(j-1) \in \overleftarrow{\mathcal{C}}(\pi(j))$ is similar. □

A.3 Proof of Lemma 3

Lemma 3 *Let* $(X, \vec{\mathcal{C}})$ *be a QdCS. Then* $\vec{\mathcal{C}}(x) \subseteq A$ *iff* $x \in \overleftarrow{\mathcal{I}}(A)$ *and* $\overleftarrow{\mathcal{C}}(x) \subseteq A$ *iff* $x \in \vec{\mathcal{I}}(A)$*, for all* $x \in X$ *and* $A \subseteq X$.

Proof. We have the following derivation:

$$x \in \vec{\mathcal{I}}(A)$$

$\Leftrightarrow$ [Def. of $\vec{\mathcal{I}}$; Set Theory]

$x \notin \vec{\mathcal{C}}(\overline{A})$

$\Leftrightarrow$ [Def. of $\vec{\mathcal{C}}$]

$x \notin \overline{A}$ and there exixts no $\overline{a} \in \overline{A}$ such that $(\overline{a}, x) \in R$

$\Leftrightarrow$ [Logic]

$x \in A$ and for all $x' \in X$ we have that if $(x', x) \in R$ then $x' \in A$

$\Leftrightarrow$ [Set Theory]

$\overleftarrow{\mathcal{C}}(x) \subseteq A$

Symmetrically we obtain $x \in \overleftarrow{\mathcal{I}}(A)$ if and only if $\vec{\mathcal{C}}(x) \subseteq A$. $\square$

B Proofs of results in Section 3

B.1 Proof of Lemma 4

Lemma 4 *For all points x_1, x_2 in a CM $\mathcal{M}$, if $x_1 \rightleftharpoons_{\mathtt{CM}} x_2$ then $x_1 \simeq_{\mathtt{IML}} x_2$.*

Proof. Let $x_1, x_2 \in X$ be such that $x_1 \rightleftharpoons_{\mathtt{CM}} x_2$. We proceed by induction on the structure of Φ to prove $x_1 \models \Phi$ iff $x_2 \models \Phi$. We only consider the case for $\mathcal{N}\Phi'$, the others being straightforward. Suppose $x_1 \models \mathcal{N}\Phi$. Then by definition of satisfaction, $x_1 \in \mathcal{C}[\![\Phi]\!]$.

We verify that $x_2 \in \mathcal{C}[\![\Phi]\!]$ making use of Lemma 1. From this $x_2 \models \mathcal{N}\Phi$ is immediate: Let S_2 be a nbh of x_2. Since $x_1 \rightleftharpoons_{\mathtt{CM}} x_2$, exists a neighbourhood S_1 of x_1 such that for each point $s_1' \in S_1$ a point $s_2' \in S_2$ exists such that $s_1' \rightleftharpoons_{\mathtt{CM}} s_2'$. Because $x_1 \in \mathcal{C}[\![\Phi]\!]$, by Lemma 1 it holds that $S_1 \cap [\![\Phi]\!] \neq \emptyset$. Let $x_1' \in S_1 \cap [\![\Phi]\!]$. Since $x_1' \in S_1$ we can pick $x_2' \in S_2$ such that $x_1' \rightleftharpoons_{\mathtt{CM}} x_2'$. Because $x_1' \in [\![\Phi]\!]$ we have $x_1' \models \Phi$. By the induction hypothesis, we know that $x_1' \rightleftharpoons_{\mathtt{CM}} x_2'$, and so we get $x_2' \models \Phi$ as well. Thus $x_2' \in [\![\Phi]\!]$ and $x_2' \in S_2 \cap [\![\Phi]\!]$ as was to be shown. $\square$

B.2 Proof of Lemma 5

Lemma 5 *For a CM $\mathcal{M}$, it holds that $\simeq_{\mathtt{IML}}^{\mathcal{M}}$ is a CM-bisimulation for $\mathcal{M}$.*

The proof of this lemma has been inspired by the proof of an analogous result for topo-bisimulation in [6]. For this proof we need an auxilliary definition.

Definition 18. *Given a CM $\mathcal{M}$, define for $x, y \in X$ the* IML*-formula $\delta_{x,y}$ as follows: if $x \simeq_{\mathtt{IML}} y$, then $\delta_{x,y}$ is set to* `true`*; otherwise, $\delta_{x,y}$ is such that $x \models \delta_{x,y}$ and $y \models \neg\delta_{x,y}$. For a point x of $\mathcal{M}$ its formula $\chi(x)$ is given by $\chi(x) = \bigwedge_{y \in X} \delta_{x,y}$.* $\bullet$

The following lemma shows some useful properties of χ.

Lemma 12. *For all CMs $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$, $x, x_1, x_2 \in X$ and $S \subseteq X$, the following holds:*

1. $\mathcal{M}, x \models \chi_x$;
2. $\mathcal{M}, x_2 \models \chi_{x_1}$ *if and only if* $x_1 \simeq_{\mathtt{IML}} x_2$;
3. $S \subseteq [\![\bigvee_{s \in S} \chi_s]\!]$.

Proof. Points (1) and (2) follow directly from the relevant definitions. For what concerns Point (3) we proceed with the following derivation:

$$
\begin{array}{ll}
 & y \in S \\
\Rightarrow & \quad [\, S \subseteq X \text{ and Point (1) above} \,] \\
 & \mathcal{M}, y \models \chi_y \\
\Leftrightarrow & \quad [\, \text{Definition of } [\![\cdot]\!] \,] \\
 & y \in [\![\chi_y]\!] \\
\Rightarrow & \quad [\, y \in S \,] \\
 & y \in \bigcup_{s \in S} [\![\chi_s]\!] \\
\Leftrightarrow & \quad [\, \bigcup_{s \in S} [\![\chi_s]\!] = [\![\bigvee_{s \in S} \chi_s]\!] \,] \\
 & y \in [\![\bigvee_{s \in S} \chi_s]\!]
\end{array}
$$

□

Another ingredient for the proof of Lemma 5, is the property that, for a point x of a CM $\mathcal{M}$ and formula $\Phi \in \mathtt{IML}$ it holds that

$$x \in \mathcal{I}[\![\Phi]\!] \quad \text{iff} \quad x \models \neg \mathcal{N} \neg \Phi\,. \tag{1}$$

To see this, observe that $\overline{[\![\Phi]\!]} = [\![\neg \Phi]\!]$. Consequently, $x \in \mathcal{I}[\![\Phi]\!]$ iff $x \notin \mathcal{C}\overline{[\![\Phi]\!]}$ iff $x \notin \mathcal{C}[\![\neg \Phi]\!]$ iff $x \not\models \mathcal{N} \neg \Phi$ iff $x \models \neg \mathcal{N} \neg \Phi$.

We are now ready for proceeding with the proof of Lemma 5.

Proof. Let $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$. Suppose $x_1 \simeq_{\mathtt{IML}} x_2$. As to property (1) of Definition 7, let $p \in \mathtt{AP}$. Because $x_1 \simeq_{\mathtt{IML}} x_2$, we have $x_1 \models p \iff x_2 \models p$, i.e. $x_1 \in \mathcal{V}(p) \iff x_2 \in \mathcal{V}(p)$.

In order verify property (2), let $S_1 \subseteq X$ be a nbh of x_1, thus $x_1 \in \mathcal{I}(S_1)$. Put $S_2 = \{ s_2 \,|\, \exists s_1 \in S_1 \colon s_1 \simeq_{\mathtt{IML}} s_2 \}$. By definition of S_2, if $s_2 \in S_2$ then exists $s_1 \in S_1$ such that $s_1 \simeq_{\mathtt{IML}} s_2$. Therefore it suffices to verify that S_2 is a neighbourhood of x_2, i.e. $x_2 \in \mathcal{I}(S_2)$.

Put $\Phi = \bigvee_{s_1 \in S_1} \chi(s_1)$. (i) We claim that $S_2 = [\![\Phi]\!]$: If $s_2 \in S_2$, then $s_2 \simeq_{\mathtt{IML}} s_1$ for some $s_1 \in S_1$. Thus $s_2 \models \chi(s_1)$ and therefore $s_2 \in [\![\Phi]\!]$. If $s_2 \notin S_2$, then $s_2 \simeq_{\mathtt{IML}} s_1$ for no $s_1 \in S_1$. Thus $s_2 \not\models \chi(s_1)$ for all $s_1 \in S_1$ and $s_2 \not\models \Phi$. Hence $s_2 \notin [\![\Phi]\!]$. (ii) We claim that $x_1 \models \neg \mathcal{N} \neg \Phi$: It holds that $S_1 \subseteq [\![\Phi]\!]$ by definition of Φ and Lemma 12(3). Thus, $\mathcal{I}(S_1) \subseteq [\![\Phi]\!]$ and $\mathcal{I}(S_1) \cap \overline{[\![\Phi]\!]} = \emptyset$. So, x_1 has a neighbourhood, viz. S_1, missing $\overline{[\![\Phi]\!]}$, and therefore, by Lemma 1, $x_1 \notin \mathcal{C}\overline{[\![\Phi]\!]}$, $x_1 \notin \mathcal{C}[\![\neg\Phi]\!]$, $x_1 \not\models \mathcal{N}\neg\Phi$, and $x_1 \models \neg\mathcal{N}\neg\Phi$. Because $x_1 \simeq_{\mathtt{IML}} x_2$, it follows that $x_2 \models \neg\mathcal{N}\neg\Phi$. Therefore, by the property (1) above, $x_2 \in \mathcal{I}([\![\Phi]\!])$. Thus, $x_2 \in \mathcal{I}(S_2)$. ◻

C Proofs of results in Section 4

C.1 Proof of Lemma 6

Lemma 6 *Let $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$ be a QdCM and $B \subseteq X \times X$ a relation. It holds that B is a CM-bisimulation iff B is a closure-based CM-bisimulation.*

Proof. (*if*) Assume that B is a CM-bisimulation in the sense of Definition 7. Let $x_1, x_2 \in X$ such that $(x_1, x_2) \in B$. We verify condition (2) of Definition 10. Thus, let $x_1' \in X$ such that $x_1 \in \vec{\mathcal{C}}(x_1')$; i.e. $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$ by Lemma 2(2). For $S_2 := \overleftarrow{\mathcal{C}}(x_2)$ we have $x_2 \in \vec{\mathcal{I}}(S_2)$ by Lemma 3. By condition (2) of Definition 7, with the roles of x_1 and x_2, and of S_1 and S_2 interchanged, exists a subset $S_1 \subseteq X$ such that $x_1 \in \vec{\mathcal{I}}(S_1)$ and for each $s_1 \in S_1$ exists $s_2 \in S_2$ such that $(s_1, s_2) \in B$. In particular, exists $x_2' \in S_2 = \overleftarrow{\mathcal{C}}(x_2)$ such that $(x_1', x_2') \in B$. Thus exists $x_2' \in X$ such that $x_2 \in \vec{\mathcal{C}}(x_2')$ and $(x_1', x_2') \in B$.

(*only if*) Assume that B is a closure-based CM-bisimulation in the sense of Definition 10. Let $x_1, x_2 \in X$ such that $(x_1, x_2) \in B$. We verify condition (2) of Definition 7. Suppose subset $S_1 \subseteq X$ is such that $x_1 \in \vec{\mathcal{I}}(S_1)$. By Lemma 3 we have $\overleftarrow{\mathcal{C}}(x_1) \subseteq S_1$. Let $S_2 := \overleftarrow{\mathcal{C}}(x_2)$. Then $x_2 \in \vec{\mathcal{I}}(S_2)$, again by Lemma 3. By the reformulation of condition (2) of Definition 10 in terms of $\overleftarrow{\mathcal{C}}$, exists for each $x_2' \in \overleftarrow{\mathcal{C}}(x_2)$ a point $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$ such that $(x_1', x_2') \in B$. Since $S_2 = \overleftarrow{\mathcal{C}}(x_2)$ it follows that for each $s_2 \in S_2$ exists $s_1 \in S_1$ such that $(s_1, s_2) \in B$. ◻

C.2 Proof of Lemma 7

Lemma 7 *Let $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$ be a QdCM and $B \subseteq X \times X$ a relation. It holds that B is a CMC-bisimulation iff B is a closure-based CMC-bisimulation.*

Proof. Clearly, requirements (1) of Definition 11 and (1) of Definition 12 are equivalent. In view of the correspondence of criterium (3) of Definition 12 and criterium (2) of Definition 10 mentioned in Remark 1 on page 9, one can prove, along the same lines as in the proof of Lemma 6, that requirements (2) of Definition 11 and (3) of Definition 12 are equivalent. Symmetrically, one shows the equivalence of requirements (3) of Definition 11 and (2) of Definition 12. ◻

C.3 Proof of Lemma 8

Lemma 8 *For all points x_1, x_2 in a QdCM $\mathcal{M}$, if $x_1 \rightleftharpoons_{\mathtt{CMC}} x_2$ then $x_1 \simeq_{\mathtt{IMLC}} x_2$.*

Proof. Let $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$. We verify by induction on the structure of the formula Φ that $x_1 \models \Phi$ if and only if $x_2 \models \Phi$ for $x_1, x_2 \in X$ such that $x_1 \rightleftharpoons_{\texttt{CMC}} x_2$.

The cases for proposition letter $p \in \texttt{AP}$, negation $\neg\Phi$, and conjunction $\bigwedge_{i \in I} \Phi_i$ are straightforward.

For the case of $\vec{\mathcal{N}}\Phi$ we will exploit condition (3) of Definition 12. Suppose $x_1 \rightleftharpoons_{\texttt{CMC}} x_2$ and $x_1 \models \vec{\mathcal{N}}\Phi$. Then $x_1 \in \vec{\mathcal{C}}[\![\Phi]\!]$ by Definition 13. Thus exists $x_1' \in X$ such that $x_1' \models \Phi$ and $x_1 \in \vec{\mathcal{C}}(x_1')$. By definition of $\vec{\mathcal{C}}$ and $\overleftarrow{\mathcal{C}}$, we also have $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$. Since $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$ and $x_1 \rightleftharpoons_{\texttt{CMC}} x_2$ we obtain, from condition (3) of Definition 12, that $x_2' \in \overleftarrow{\mathcal{C}}(x_2)$ exists such that $x_1' \rightleftharpoons_{\texttt{CMC}} x_2'$. From $x_1' \models \Phi$, we obtain $x_2' \models \Phi$ by induction hypothesis for Φ. Therefore, $x_2' \in [\![\Phi]\!]$ and $x_2 \in \vec{\mathcal{C}}[\![\Phi]\!]$, which implies $x_2 \models \vec{\mathcal{N}}\Phi$.

The case for $\overleftarrow{\mathcal{N}}\Phi$ is similar and is proven with appeal to condition (2) of Definition 12. Suppose we have $x_1 \rightleftharpoons_{\texttt{CMC}} x_2$ and $x_1 \models \overleftarrow{\mathcal{N}}\Phi$. Then $x_1 \in \overleftarrow{\mathcal{C}}[\![\Phi]\!]$ by Definition 13. Thus $x_1 \in \overleftarrow{\mathcal{C}}(x_1')$ for some $x_1' \in X$ such that $x_1' \models \Phi$. Note, $x_1' \in \vec{\mathcal{C}}(x_1)$. Therefore, by condition (2) of Definition 12, exists $x_2' \in \vec{\mathcal{C}}(x_2)$ such that $x_1' \rightleftharpoons_{\texttt{CMC}} x_2'$. By induction hypothesis, $x_2' \models \Phi$ since $x_1' \models \Phi$. Hence $x_2' \in [\![\Phi]\!]$. Now $x_2 \in \overleftarrow{\mathcal{C}}(x_2')$ and therefore $x_2 \in \overleftarrow{\mathcal{C}}[\![\Phi]\!]$, i.e. $x_2 \models \overleftarrow{\mathcal{N}}\Phi$. □

C.4 Proof of Lemma 9

Lemma 9 *For a QdCM $\mathcal{M}$, it holds that $\simeq^{\mathcal{M}}_{\texttt{IMLC}}$ is a CMC-bisimulation for $\mathcal{M}$.*

Proof. Let $\mathcal{M} = (X, \mathcal{C}, \mathcal{V})$. Define, for points $x, y \in X$, the IMLC-formula $\delta_{x,y}$ as follows: if $x \simeq_{\texttt{IMLC}} y$, then $\delta_{x,y}$ is set to **true**; otherwise, $\delta_{x,y}$ is such that $x \models \delta_{x,y}$ and $y \models \neg\delta_{x,y}$. Next, put $\chi(x) = \bigwedge_{y \in X} \delta_{x,y}$. Note, for $x, y \in X$, it holds that $y \in [\![\chi(x)]\!]$ if and only if $x \simeq_{\texttt{IMLC}} y$.

Suppose $x_1 \simeq_{\texttt{IMLC}} x_2$ for $x_1, x_2 \in X$. It is immediate that condition (1) of Definition 12 is fulfilled: For $p \in \texttt{AP}$ we have $x_1 \models p$ if and only if $x_2 \models p$. Thus, $x_1 \in \mathcal{V}(p)$ if and only if $x_2 \in \mathcal{V}(p)$.

For condition (2) of Definition 12, let $x_1' \in \vec{\mathcal{C}}(x_1)$. Since $x_1 \in \overleftarrow{\mathcal{C}}(x_1')$ and $x_1' \in [\![\chi(x_1')]\!]$ it holds that $x_1 \models \overleftarrow{\mathcal{N}}\chi(x_1')$. By assumption $x_2 \models \overleftarrow{\mathcal{N}}\chi(x_1')$. Hence, for some $x_2' \in X$ we have $x_2 \in \overleftarrow{\mathcal{C}}(x_2')$ and $x_2' \in [\![\chi(x_1')]\!]$. Thus $x_2' \in \vec{\mathcal{C}}(x_2)$ and $x_1' \simeq_{\texttt{IMLC}} x_2'$.

For condition (3) of Definition 12, we reason symmetrically. If $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$ then $x_1 \in \vec{\mathcal{C}}(x_1')$ and $x_1 \models \vec{\mathcal{N}}\chi(x_1')$. So, $x_2 \models \vec{\mathcal{N}}\chi(x_1')$ and $x_2 \in \vec{\mathcal{C}}(x_2')$ for some $x_2' \in X$ such that $x_2' \in [\![\chi(x_1')]\!]$. For x_2' we have $x_2' \in \overleftarrow{\mathcal{C}}(x_2)$ and $x_1' \simeq_{\texttt{IMLC}} x_2'$. □

D Proofs of results in Section 5

In the sequel, we let R^{rt} denote the *reflexive* and *transitive* closure of (symmetric) binary relation R.

D.1 Proof of Proposition 2

Proposition 2 *For all points x_1, x_2 in QdCM $\mathcal{M}$, if $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CMC}} x_2$, then $x_1 \rightleftharpoons^{\mathcal{M}}_{\mathtt{CoPa}}$ x_2.*

Proof. Suppose $x_1 \rightleftharpoons_{\mathtt{CMC}} x_2$. Then, by there exists a CMC-bisimulation $B \subseteq X \times X$ such that $(x_1, x_2) \in B$. By Lemma 13 below we know that $B^{rt} \subseteq X \times X$ is a CoPa-bisimulation and since $B \subseteq B^{rt}$ we have $(x_1, x_2) \in B^{rt}$, i.e. $x_1 \rightleftharpoons_{\mathtt{CoPa}} x_2$.

Lemma 13. *For all QdCMs $(X, \vec{\mathcal{C}}, \mathcal{V})$ and relations $B \subseteq X \times X$ the following holds: if B is a CMC-bisimulation, then B^{rt} is a CoPa-bisimulation.*

Proof. We have to prove that B^{rt} satisfies the conditions of Definition 15, under the assumption that B is a CMC-bisimulation. We consider only condition 1 and condition 2, since the proof for condition 3 is similar. Suppose $(x_1, x_2) \in B^{rt}$. For what concerns condition 1 there are three cases to consider:

1. $x_1 = x_2$: trivial;
2. $(x_1, x_2) \in B$: in this case $x_1 \in \mathcal{V}(p)$ if and only if $x_2 \in \mathcal{V}(p)$, for all $p \in \mathtt{AP}$ since B is a CMC-bisimulation;
3. there are $y_1, \ldots, y_n \in X$ such that $y_1 = x_1, y_n = x_2$ and for all $i \in \{1, \ldots, n-1\}$ we have $(y_i, y_{i+1}) \in B$: in this case $y_i \in \mathcal{V}(p)$ if and only if $y_{i+1} \in \mathcal{V}(p)$ for all $i \in \{1, \ldots, n-1\}$ since B is a CMC-bisimulation and so also $x_1 \in \mathcal{V}(p)$ if and only if $x_2 \in \mathcal{V}(p)$, for all $p \in \mathtt{AP}$.

For what concerns Condition 2, let π_1 any path in $\mathtt{BPaths}^{\mathtt{F}}(x_1)$ such that $(\pi_1(i_1), x_2) \in B^{rt}$ for all $i_1 < \mathtt{len}(\pi_1)$, and assume $\mathtt{len}(\pi_1) > 0$—the case $\mathtt{len}(\pi_1) = 0$ being trivial by choosing π_2 such that $\pi(i_2) = x_2$ for all i_2. By Lemma 2(5) we know that $\pi_1(i_1) \in \vec{\mathcal{C}}(\pi_1(i_1 - 1))$ for all $i_1 = 1, \ldots, \mathtt{len}(\pi_1)$. We build π_2, such that $\mathtt{len}(\pi_2) = \mathtt{len}(\pi_1)$, as follows. We let $\pi_2(0) = x_2$; since $(\pi_1(0), \pi_2(0)) = (x_1, x_2) \in B^{rt}$ and $\pi_1(1) \in \vec{\mathcal{C}}(\pi_1(0))$, there is, by Lemma 14 below, $\eta \in \vec{\mathcal{C}}(\pi_2(0))$ s.t. $(\pi_1(0), \eta) \in B^{rt}$. We let $\pi_2(1) = \eta$ and we proceed in a similar way for defining $\pi_2(i_2) \in \vec{\mathcal{C}}(\pi_2(i_2 - 1))$ for all $i_2 < \mathtt{len}(\pi_2)$, ensuring that for all such i_2, $(\pi_1(0), \pi_2(i_2)) \in B^{rt}$.

Now, by hypothesis and since $\pi_2(0) = x_2$ by definition, we know that $(\pi_1(\mathtt{len}(\pi_1) - 1), \pi_2(0)) \in B^{rt}$ and $(\pi_1(0), \pi_2(0)) \in B^{rt}$, and, by symmetry of ($B$ and thus of) B^{rt}, also $(\pi_2(0), \pi_1(0)) \in B^{rt}$. By construction of π_2, we have also $(\pi_1(0), \pi_2(\mathtt{len}(\pi_2) - 1)) \in B^{rt}$. Thence, by transitivity of B^{rt}, we finally get $(\pi_1(\mathtt{len}(\pi_1) - 1), \pi_2(\mathtt{len}(\pi_2) - 1)) \in B^{rt}$. But then, by Lemma 2(5) we know that $\pi_1(\mathtt{len}(\pi_1)) \in \vec{\mathcal{C}}(\pi_1(\mathtt{len}(\pi_1) - 1))$ and so, again by Lemma 14, we know that there exists $\xi \in \vec{\mathcal{C}}(\pi_2(\mathtt{len}(\pi_2) - 1))$ such that $(\pi_1(\mathtt{len}(\pi_1)), \xi) \in B^{rt}$. We define $\pi_2(\mathtt{len}(\pi_2)) = \xi$; so $(\pi_1(\mathtt{len}(\pi_1)), \pi_2(\mathtt{len}(\pi_2))) \in B^{rt}$ and, noting that, again by Lemma 2(5), the resulting function π_2 is continuous, i.e. it is a path, we get the assert.

Lemma 14. *For all QdCMs $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$, CMC-bisimulation B and $(x_1, x_2) \in B^{rt}$ the following holds: for all $x'_1 \in \vec{\mathcal{C}}(x_1)$ there exists $x'_2 \in \vec{\mathcal{C}}(x_2)$ such that $(x'_1, x'_2) \in B^{rt}$*

Proof. There are three cases to consider:

1. $x_1 = x_2$: trivial;
2. $(x_1, x_2) \in B$: in this case the assert follows directly from the fact that B is a CMC-bisimulation, Lemma 7, and $B \subseteq B^{rt}$;
3. there are $y_1, \ldots, y_n \in X$ such that $y_1 = x_1$, $y_n = x_2$ and for all $i \in \{1, \ldots, n-1\}$ we have $(y_i, y_{i+1}) \in B$: in this case—by applying the same reasoning as for case (2) above—we have that for all $y'_i \in \vec{\mathcal{C}}(y_i)$ there is $y'_{i+1} \in \vec{\mathcal{C}}(y_{i+1})$ with $(y'_i, y'_{i+1}) \in B \subseteq B^{rt}$, for all $i \in \{1, \ldots, n-1\}$; the assert then follows by transitivity of B^{rt}.

□

D.2 Proof of Lemma 10

Lemma 10 *For all points x_1, x_2 in a QdCM $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$, if $x_1 \rightleftharpoons_{\texttt{CoPa}} x_2$ then $x_1 \simeq_{\texttt{ICRL}} x_2$.*

Proof. We proceed by induction on the structure of formulas and consider only the case $\vec{\zeta}\Phi_1[\Phi_2]$, the case for $\overleftarrow{\zeta}\Phi_1[\Phi_2]$ being similar, and the others being trivial. So, let us assume that for all x_1, x_2, if $x_1 \rightleftharpoons_{\texttt{CoPa}} x_2$, then $\mathcal{M}, x_1 \models \Phi$ if and only if $\mathcal{M}, x_2 \models \Phi$ and prove the assert for $\vec{\zeta}\Phi_1[\Phi_2]$.

Suppose that $\mathcal{M}, x_1 \models \vec{\zeta}\Phi_1[\Phi_2]$. This means there exist π, ℓ s.t. $\pi(0) = x_1, \mathcal{M}, \pi(\ell) \models \Phi_1$ and, for $j \in \{\iota \,|\, 0 \leq \iota < \ell\}$ we have $\mathcal{M}, \pi(j) \models \Phi_2$. If $\ell = 0$, then, by definition of $\vec{\zeta}$, we know that $\mathcal{M}, x_1 \models \Phi_1$ and, by the I.H. we get that also $\mathcal{M}, x_2 \models \Phi_1$ and, again by definition of $\vec{\zeta}$ we get $\mathcal{M}, x_2 \models \vec{\zeta}\Phi_1[\Phi_2]$. Suppose now that $\ell > 0$, and let path π_1 be defined as follows:

$$\pi_1(i_1) = \begin{cases} \pi(i_1), & \text{if } i_1 \leq \ell, \\ \pi(\ell), & \text{if } i_1 > \ell. \end{cases}$$

Clearly, $\pi_1 \in \texttt{BPaths}^{\texttt{F}}(x_1)$, $\texttt{len}(\pi_1) = \ell$, $\mathcal{M}, \pi(\texttt{len}(\pi_1)) \models \Phi_1$ and, for $j \in \{\iota \,|\, 0 \leq \iota < \texttt{len}(\pi_1)\}$ we have $\mathcal{M}, \pi_1(j) \models \Phi_2$. Let B be a CoPa-bisimulation such that $(x_1, x_2) \in B$; such a B exists since $x_1 \rightleftharpoons_{\texttt{CoPa}} x_2$. In the sequel, we will construct a path $\pi_2 \in \texttt{BPaths}^{\texttt{F}}(x_2)$ such that $\pi_2(0) = x_2$ and we also have $\mathcal{M}, \pi_2(\texttt{len}(\pi_2)) \models \Phi_1$ and for all $i_2 \in \{\iota \,|\, 0 \leq \iota < \texttt{len}(\pi_2)\}$ we have $\mathcal{M}, \pi_2(i_2) \models \Phi_2$ thus showing that $\mathcal{M}, x_2 \models \vec{\zeta}\Phi_1[\Phi_2]$ (see Figure 8).

Let $M_0 = 0$, $x_{21} = x_2$. Now let M_1 be the greatest m_1 such that $m_1 \leq \texttt{len}(\pi_1)$ and $(\pi_1(i_1), x_{21}) \in B$ for all $i_1 \in \{\iota \,|\, M_0 \leq \iota < m_1\}$, recalling that $(\pi_1(M_0), x_{21}) \in B$ by hypothesis. Moreover, since $(\pi_1(M_0), x_{21}) \in B$ and B is a CoPa-bisimulation, by condition 2 of Definition 15, there exists $\pi_{21} \in \texttt{BPaths}^{\texttt{F}}(x_{21})$ such that $(\pi_1(M_0), \pi_{21}(i_2)) \in B$ for all $i_2 \in \{\iota \,|\, 0 \leq \iota < \texttt{len}(\pi_{21})\}$ and $(\pi_1(M_1), \pi_{21}(x_{22})) \in B$, where $x_{22} = \pi_{21}(\texttt{len}(\pi_{21}))$. Furthermore, since $\mathcal{M}, \pi_1(M_0) \models \Phi_2$, by the I.H. we get that also $\mathcal{M}, \pi_{21}(i_2) \models \Phi_2$ for all $i_2 \in \{\iota \,|\, 0 \leq \iota < \texttt{len}(\pi_{21})\}$.

For $j > 1$, let M_j be the greatest m_j such that $m_j \leq \texttt{len}(\pi_1)$ and $(\pi_1(i_1), x_{2j}) \in B$ for all $i_1 \in \{\iota \,|\, Z_{j-1} \leq \iota < z_j\}$ recalling that $(\pi_1(M_{j-1}), x_{2j}) \in B$ by

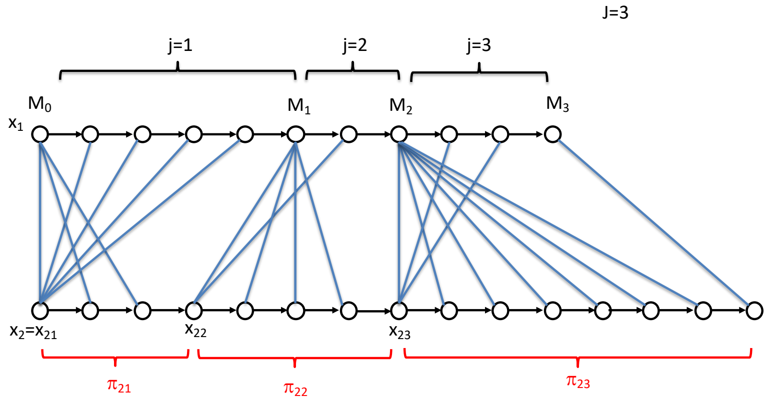

Fig. 8: Example of schema for the Proof of Lemma 10, for $J = 3$. Relation B is shown as blue segments.

definition of π_{2j-1}. Moreover, since $(\pi_1(M_{j-1}), x_{2j}) \in B$ and B is a CoPa-bisimulation, by condition 2 of Definition 15, there exists $\pi_{2j} \in \mathtt{BPaths}^{\mathtt{F}}(x_{2j})$ such that $(\pi_1(M_{j-1}), \pi_{2j}(i_2)) \in B$ for all $i_2 \in \{\iota \,|\, 0 \leq \iota < \mathtt{len}(\pi_{2j})\}$ and $(\pi_1(M_j), \pi_{2j}(x_{2(j+1)})) \in B$, where $x_{2(j+1)} = \pi_{2j}(\mathtt{len}(\pi_{2j}))$. Furthermore, since $\mathcal{M}, \pi_1(M_{j-1}) \models \Phi_2$, by the I.H. we get that also $\mathcal{M}, \pi_{2j}(i_2) \models \Phi_2$ for all $i_2 \in \{\iota \,|\, 0 \leq \iota < \mathtt{len}(\pi_{2j})\}$.

Finally, letting J be the greatest j as above, since $\mathcal{M}, \pi_1(M_J) \models \Phi_1$, by the I.H. we get that also $\mathcal{M}, \pi_{2J}(\mathtt{len}(\pi_{2J})) \models \Phi_1$.

We note that $\pi_{2j}(0) = \pi_{2(j-1)}(\mathtt{len}(\pi_{2(j-1)}))$ for $j = 1 \ldots J$. Thus we can build the following path π_2:

$$\pi_2(n) = \begin{cases} \pi_{21}(n), \text{ if } n \in [0, \mathtt{len}(\pi_{21})), \\ \vdots \\ \pi_{2j}(n - \sum_{i=1}^{j-1} \mathtt{len}(\pi_{2i})), \text{ if } n \in \left[\sum_{i=1}^{j-1} \mathtt{len}(\pi_{2i}), \sum_{i=1}^{j} \mathtt{len}(\pi_{2i})\right), \\ \vdots \\ \pi_{2J}(n - \sum_{i=1}^{J-1} \mathtt{len}(\pi_{2i})), \text{ if } n \geq \sum_{i=1}^{j} \mathtt{len}(\pi_{2i}). \end{cases}$$

Clearly, $\pi_2 \in \mathtt{BPaths}^{\mathtt{F}}(x_2)$ since $\pi_2(0) = \pi_{2,1}(0) = x_2$ because $\pi_{21} \in \mathtt{BPaths}^{\mathtt{F}}(x_2)$ and π_{2J} is bounded. Moreover, by construction, $\mathcal{M}, \pi_2(i_2) \models \Phi_2$ for all $i_2 \in \{\iota \,|\, 0 \leq \iota < \mathtt{len}(\pi_2)\}$ and $\mathcal{M}, \pi_2(\mathtt{len}(\pi_2)) \models \Phi_1$. Thus $\mathcal{M}, x_2 \models \vec{\zeta}\Phi_1[\Phi_2]$. □

D.3 Proof of Lemma 11

Lemma 11 *For a QdCM $\mathcal{M}$ it holds that $\simeq^{\mathcal{M}}_{\mathtt{ICRL}}$ is a CoPa-bisimulation for $\mathcal{M}$.*

Proof. We have to prove that the conditions of Definition 15 are fullfilled. We consider only condition 2, since the proof for conditions 3 is similar and that of

condition 1 is trivial. We proceed by contradition. Suppose condition 2 is not satisfied; this means that there exists $\bar{\pi} \in \mathtt{BPaths}^{\mathtt{F}}(x_1)$ such that $(\bar{\pi}(i), x_2) \in \simeq_{\mathtt{ICRL}}$ for all $i \in \{\iota \,|\, 0 \leq \iota < \mathtt{len}(\bar{\pi})\}$ and, for all $\pi \in \mathtt{BPaths}^{\mathtt{F}}(x_2)$, having considered that $\pi(0) = x_2 \simeq_{\mathtt{ICRL}} x_1$, the following holds:
$(\bar{\pi}(\mathtt{len}(\bar{\pi})), \pi(\mathtt{len}(\pi))) \notin \simeq_{\mathtt{ICRL}}$ or there exists h_π such that $0 < h_\pi < \mathtt{len}(\pi)$ and $(x_1, \pi(h_\pi)) \notin \simeq_{\mathtt{ICRL}}$. Let set I be defined as

$$I = \{\pi \in \mathtt{BPaths}^{\mathtt{F}}(x_2) \,|\, \text{there exists } h_\pi \text{ such that } 0 < h_\pi < \mathtt{len}(\pi) \text{ and } (x_1, \pi(h_\pi)) \notin \simeq_{\mathtt{ICRL}}\}$$

and, for each $\pi \in I$, let Ω^I_π be a formula such that $\mathcal{M}, x_1 \models \Omega^I_\pi$ and $\mathcal{M}, \pi(h_\pi) \not\models \Omega^I_\pi$—such a formula exists because $(x_1, \pi(h_\pi)) \notin \simeq_{\mathtt{ICRL}}$.

Let furthermore set L be defined as

$$L = \{\pi \in \mathtt{BPaths}^{\mathtt{F}}(x_2) \,|\, (\bar{\pi}(\mathtt{len}(\bar{\pi})), \pi(\mathtt{len}(\pi))) \notin \simeq_{\mathtt{ICRL}}\}$$

and, for each $\pi \in L$, let Ω^L_π be a formula such that $\mathcal{M}, \bar{\pi}(\mathtt{len}(\bar{\pi})) \models \Omega^L_\pi$ and $\mathcal{M}, \pi(\mathtt{len}(\pi)) \not\models \Omega^L_\pi$—such a formula exists because $(\bar{\pi}(\mathtt{len}(\bar{\pi})), \pi(\mathtt{len}(\pi))) \notin \simeq_{\mathtt{ICRL}}$. Note that $I \cup L = \mathtt{BPaths}^{\mathtt{F}}(x_2)$ by hypothesis. Clearly, $\mathcal{M}, x_1 \models \bigwedge_{\pi \in I} \Omega^I_\pi$ and, since $(\bar{\pi}(i), x_2) \in \simeq_{\mathtt{ICRL}}$ for all $i \in \{\iota \,|\, 0 \leq \iota < \mathtt{len}(\bar{\pi})\}$, we also get $\mathcal{M}, \bar{\pi}(i) \models \bigwedge_{\pi \in I} \Omega^I_\pi$ for all $i \in \{\iota \,|\, 0 \leq \iota < \mathtt{len}(\bar{\pi})\}$—recall that $\bar{\pi}(0) = x_1$. Also, $\mathcal{M}, \bar{\pi}(\mathtt{len}(\bar{\pi})) \models \bigwedge_{\pi \in L} \Omega^L_\pi$.

Thus, we get $\mathcal{M}, x_1 \models \Psi$, where Ψ is the formula $\vec{\zeta}(\bigwedge_{\pi \in L} \Omega^L_\pi)[\bigwedge_{\pi \in I} \Omega^I_\pi]$. On the other hand, $\mathcal{M}, x_2 \not\models \Psi$, since, for every path $\pi \in \mathtt{BPaths}^{\mathtt{F}}(x_2)$, $\pi(h_\pi)$ does not satisfy $\bigwedge_{\pi \in I} \Omega^I_\pi$ for some h_π with $0 < h_\pi < \mathtt{len}(\pi)$—by construction of $\bigwedge_{\pi \in I} \Omega^I_\pi$—or $\pi(\mathtt{len}(\pi))$ does not satisfy $\bigwedge_{\pi \in L} \Omega^L_\pi$—by construction of $\bigwedge_{\pi \in L} \Omega^L_\pi$. In conclusion, we have found a formula, Ψ, such that $\mathcal{M}, x_1 \models \Psi$ whereas $\mathcal{M}, x_2 \not\models \Psi$ and this contradicts $x_1 \simeq_{\mathtt{ICRL}} x_2$.