



# Nonlinear parabolic flows with dynamic flux on the boundary

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## Abstract

A nonlinear divergence parabolic equation with dynamic boundary conditions of Wentzell type is studied. The existence and uniqueness of a strong solution is obtained as the limit of a finite difference scheme, in the time dependent case and via a semigroup approach in the time-invariant case.

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## 1. Introduction

This paper deals with the well-posedness of a nonlinear parabolic equation posed in a bounded regular domain  $\Omega$  (of class  $C^2$ , for instance) of  $\mathbb{R}^N$ ,  $N \geq 1$ , coupled with a dynamic boundary condition of reaction–diffusion type. More exactly, we study the problem

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$$y_t - \nabla \cdot \beta(t, x, \nabla y) \ni f, \quad \text{in } Q := (0, T) \times \Omega, \tag{1.1}$$

$$\beta(t, x, \nabla y) \cdot \nu + y_t \ni g, \quad \text{on } \Sigma := (0, T) \times \Gamma, \tag{1.2}$$

$$y(0) = y_0, \quad \text{in } \Omega, \tag{1.3}$$

where  $t \in (0, T)$ ,  $T < \infty$ ,  $x \in \Omega$ ,  $\Gamma$  is the boundary of  $\Omega$ ,  $\nu$  is the outward normal to  $\Gamma$ ,  $y_t = \frac{\partial y}{\partial t}$  and  $\nabla y = \{ \frac{\partial y}{\partial x_i} \}_{i=1, \dots, N}$ .

In relation to various cases studied in this paper, certain combinations of the following hypotheses will be used:

(H<sub>1</sub>) For each  $(t, x) \in \bar{Q}$ ,  $\beta : \bar{Q} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a maximal monotone graph with respect to  $r$  on  $\mathbb{R}^N \times \mathbb{R}^N$ , and it is derived from a potential  $j(t, x, r)$ . The function  $j$  is continuous on  $\bar{Q} \times \mathbb{R}^N$ , and for each  $(t, x) \in \bar{Q}$ , it is convex with respect to  $r$ . We denote

$$\partial j(t, x, r) = \beta(t, x, r), \quad \text{for any } r \in \mathbb{R}^N, t \in [0, T], x \in \bar{\Omega}, \tag{1.4}$$

where  $\partial j(t, x, \cdot)$  denotes the subdifferential of  $j(t, x, \cdot)$ , that is

$$\partial j(t, x, r) = \{ w \in \mathbb{R}^N; j(t, x, r) - j(t, x, \bar{r}) \leq w(r - \bar{r}), \forall \bar{r} \in \mathbb{R}^N \}. \tag{1.5}$$

Moreover, there is

$$\xi_0 \in C(\bar{Q}; \mathbb{R}^N) \text{ with } \nabla \cdot \xi_0 \in L^2(\Omega), \quad \text{such that } \xi_0(t, x) \in \beta(t, x, 0), \forall (t, x) \in \bar{Q}. \tag{1.6}$$

(H<sub>2</sub>) (*Strong coercivity hypothesis*) There exist  $C_i, C_i^0 \in \mathbb{R}, C_1, C_2 > 0$ , such that

$$C_1 |r|_N^p + C_1^0 \leq j(t, x, r) \leq C_2 |r|_N^p + C_2^0, \quad \forall (t, x) \in \bar{Q}, \text{ for } 1 < p < \infty. \tag{1.7}$$

(All over,  $|\cdot|_N$  will denote the Euclidean norm in  $\mathbb{R}^N$ .)

(H<sub>3</sub>) (*Weak coercivity hypothesis*) The functions  $j$  and  $j^*$  satisfy

$$\lim_{|r|_N \rightarrow \infty} \frac{j(t, x, r)}{|r|_N} = +\infty, \quad \text{uniformly with respect to } t, x, \tag{1.8}$$

$$\lim_{|\omega|_N \rightarrow \infty} \frac{j^*(t, x, \omega)}{|\omega|_N} = +\infty, \quad \text{uniformly with respect to } t, x, \tag{1.9}$$

where  $j^* : \bar{Q} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is the conjugate of  $j$ , defined by

$$j^*(t, x, \omega) = \sup_{r \in \mathbb{R}^N} (\omega \cdot r - j(t, x, r)), \quad \text{for all } \omega \in \mathbb{R}^N, \forall (t, x) \in \bar{Q}. \tag{1.10}$$

We note that (1.8) and (1.9) are equivalent to

$$\sup \{ |r|_N; r \in \beta^{-1}(t, x, \omega), |\omega|_N \leq M \} \leq W_M, \tag{1.11}$$

$$\sup \{ |\omega|_N; \omega \in \beta(t, x, r), |r|_N \leq M \} \leq Y_M, \tag{1.12}$$

respectively, where  $M, W_M, Y_M$  are positive constants.

(H<sub>4</sub>) (Symmetry at infinity) There exist  $\gamma_1, \gamma_2 \geq 0$  such that

$$j(t, x, r) \leq \gamma_1 j(t, x, -r) + \gamma_2, \quad \gamma_1 > 0, \gamma_2 \geq 0. \tag{1.13}$$

(H<sub>5</sub>) (Regularity in  $t$ ) There exists  $L > 0$  such that

$$j(t, x, r) \leq j(s, x, r) + L|t - s|j(t, x, r), \quad \forall t, s \in [0, T], x \in \bar{Q}, r \in \mathbb{R}^N. \tag{1.14}$$

By (1.6) we see that subtracting  $\xi_0 \in \beta(t, x, 0)$  from  $\beta(t, x, r)$ , and redefining  $j(t, x, r)$  as  $j(t, x, r) - j(t, x, 0)$ , we may assume without loss of generality that

$$j(t, x, 0) = 0, \quad j(t, x, r) \geq 0, \quad j^*(t, x, r) \geq 0 \quad \text{for all } (t, x) \in \bar{Q}, r, \omega \in \mathbb{R}^N. \tag{1.15}$$

The strongly coercivity hypothesis (H<sub>2</sub>) includes, for instance, the situation

$$\begin{aligned} \beta : \bar{Q} \times \mathbb{R}^N &\rightarrow \mathbb{R}^N, & \beta(t, x, r_1, \dots, r_N) &= (\beta_1(t, x, r_1), \dots, \beta_N(t, x, r_N)), \\ \beta_i(t, x, r_i) &= \partial j_i(t, x, r_i), & i &= 1, \dots, N, \end{aligned}$$

where  $j_i : \bar{Q} \times \mathbb{R} \rightarrow \mathbb{R}$  are convex, continuous functions (with respect to  $r_i$ ), and continuous with respect to  $(t, x) \in \bar{Q}$ , and  $j : \bar{Q} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is given by

$$j(t, x, r_1, \dots, r_N) = j_1(t, x, r_1) + \dots + j_N(t, x, r_N).$$

For instance, one might take  $j_i$  of the form

$$j_i(t, x, r) = \alpha_i(t, x)|r|_N^p + \kappa_i(t, x) \log(|r|_N + 1) + \delta_i(t, x) \cdot r + \delta_i^1(t, x), \quad i = 1, \dots, N,$$

for  $\alpha_i, \kappa_i, \delta_i^1 \in C^1(\bar{Q})$ ,  $\kappa_i \geq 0$ ,  $\delta_i \in C^1(\bar{Q}; \mathbb{R}^N)$ .

In particular, we get the parabolic equation with a non-isotropic  $p$ -Laplacian

$$\begin{aligned} y_t - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \alpha_i(t, x) \left| \frac{\partial y}{\partial x_i} \right|^{p-2} \frac{\partial y}{\partial x_i} \right) &= f, & \text{in } Q, \\ y_t + \sum_{i=1}^N \left( \alpha_i(t, x) \left| \frac{\partial y}{\partial x_i} \right|^{p-2} \frac{\partial y}{\partial x_i} \right) \cdot \nu_i &= g, & \text{on } \Sigma. \end{aligned} \tag{1.16}$$

More generally, we can consider instead of (1.16), a model for a diffusion process in a fractured medium, described by the parabolic problem

$$\begin{aligned} y_t - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \left( \alpha_i(t, x) + H \left( \frac{\partial y}{\partial x_i} - r_i \right) \right) \left| \frac{\partial y}{\partial x_i} \right|^{p-2} \frac{\partial y}{\partial x_i} \right) &\ni f, & \text{in } Q, \\ y_t + \sum_{i=1}^N \left( \left( \alpha_i(t, x) + H \left( \frac{\partial y}{\partial x_i} - r_i \right) \right) \left| \frac{\partial y}{\partial x_i} \right|^{p-2} \frac{\partial y}{\partial x_i} \right) \cdot \nu_i &\ni g, & \text{on } \Sigma, \end{aligned} \tag{1.17}$$

where  $r_i \in \mathbb{R}$ , and  $H$  is the Heaviside multivalued function,  $H(s) = 0$  for  $s < 0$ ,  $H(s) = [0, 1]$  for  $s = 0$ ,  $H(s) = 1$  for  $s > 0$ . In fact, a discontinuous nondecreasing function  $r \rightarrow \beta(t, x, r)$  becomes a maximal monotone multivalued function by filling the jumps at the discontinuity points  $r_i$ , that is by taking  $\beta(t, x, r_i) = [\beta(t, x, r_i - 0), \beta(t, x, r_i + 0)]$  and this is the natural way of treating Eq. (1.1) with a discontinuous  $\beta(t, x, \cdot)$ .

Problem (1.1)–(1.3) extends the classical Wentzell boundary condition and models various phenomena in mathematical physics, and in particular, diffusion and reaction–diffusion processes, phase-transition, image restoring with observation on the boundary. If we view  $E(y) = \int_Q j(t, x, \nabla y) dx dt$  as the energy of the system, then hypothesis (H<sub>2</sub>) describes diffusion processes with coercive and differentiable energy, while (H<sub>3</sub>) refers to systems with  $W^{1,1}$  regular energy.

For various interpretations and treatment of the dynamic boundary conditions (1.2), we refer e.g., to the works [15,17,18,9,13,12]. In [19] there are studied equations of the form  $u_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + |u|^{p-2} u + \alpha_1(x) = f$  in  $Q$ , with the Wentzell boundary condition  $u_t - \nabla \cdot (|\nabla_\Gamma u|^{p-2} \nabla_\Gamma u) + |\nabla u|^{p-2} \partial u / \partial \nu + |u|^{p-2} u + \alpha_2(x, u) = g$  on  $\Sigma$ . Previously, in [11] and [10] there were studied problems with Wentzell boundary conditions of the form  $u_t - \nabla \cdot (a(|\nabla u|^2) \nabla u) + f(u) = h_1(x)$  (where  $a$  is a given nonnegative function), with the boundary condition  $u_t + b(x)a(|\nabla u|^2) \partial u / \partial \nu + g(u) = h_2(x)$ .

Compared with previous existence theory for problem (1.1)–(1.3), the novelty of the present work is two fold: the generality of the nonlinearity  $\beta$ , which is discontinuous (that is, multivalued) and the constructive approach based on a finite difference scheme, which permits to treat the time dependent case.

The content of the paper is the following. In Section 3 we deal with the strongly coercive case, under hypotheses (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>5</sub>). First, we prove the existence of a time-discretization solution to (1.1)–(1.3). Due to the generality assumed for  $j$  we use a variational principle involving an appropriate minimization problem. We also prove the stability of the finite difference scheme. Then, we get the existence of a weak solution as the strong limit of the  $h$ -discretized solution, with  $h$  the time step. On the basis of some further arguments, it turns out that this solution is strong and it is unique. In Section 4 we consider the situation when  $j$  is continuous and weakly coercive only, and exhibits a symmetry at infinity, following hypotheses (H<sub>1</sub>), (H<sub>3</sub>)–(H<sub>5</sub>). The latter case which provides a strong solution in the Sobolev space  $W^{1,1}$  is in particular of interest in image processing with observation on the boundary (see e.g. [3,5]). In Section 5 we present an alternative semigroup approach to the existence theory when the potential  $j$  is time independent.

## 2. Notation and functional framework

In the following we denote by  $|\cdot|_N$  the Euclidean norm in  $\mathbb{R}^N$ , by  $|\cdot|$  the norm in  $\mathbb{R}$ , and by  $u \cdot v$  the scalar product of  $u, v \in \mathbb{R}^N$ .

Let  $1 \leq p \leq \infty$ . By  $L^p(\Omega)$  we denote the space of  $L^p$ -Lebesgue integrable functions on  $\Omega$ , with the norm  $\|\cdot\|_{L^p(\Omega)}$ . Let  $T > 0$ . We set  $Q := (0, T) \times \Omega$ ,  $\Sigma := (0, T) \times \Gamma$  and denote by  $L^p(Q)$  and  $L^p(\Sigma)$  the corresponding  $L^p$  spaces. We denote by  $W^{1,p}(\Omega)$  the Sobolev spaces with the standard norm and  $H^{1,2}(\Omega) := W^{1,2}(\Omega)$ .

We also denote by  $p'$  the conjugate of  $p$ , that is  $1/p + 1/p' = 1$ .

We define for  $\sigma \in (0, 1)$  and  $p \in [1, +\infty)$  the fractional Sobolev spaces

$$W^{\sigma,p}(\Omega) = \left\{ z \in L^p(\Omega); \frac{|z(x) - z(x')|}{|x - x'|^{\sigma+N/p}} \in L^p(\Omega \times \Omega) \right\}$$

equipped with the natural norm (see, e.g., [7], p. 314).

Next, for  $s \in \mathbb{R}, s > 1$ , not an integer,  $s = m + \sigma, m$  being the integer part of  $s$ , one defines

$$W^{s,p}(\Omega) = \{z \in W^{m,p}(\Omega); D^\alpha z \in W^{\sigma,p}(\Omega), \forall \alpha \text{ with } |\alpha| = m\}.$$

If  $z \in W^{1,r}(\Omega)$ , with  $r > 1$ , it follows that the trace of  $z$  on  $\Gamma$ , denoted by  $\gamma(z)$ , is well defined,

$$\gamma(z) \in W^{1-\frac{1}{r},r}(\Gamma), \quad \|\gamma(z)\|_{W^{1-\frac{1}{r},r}(\Gamma)} \leq C \|z\|_{W^{1,r}(\Omega)}, \tag{2.1}$$

and the operator  $z \rightarrow \gamma(z)$  is surjective from  $W^{1,r}(\Omega)$  onto  $W^{1-\frac{1}{r},r}(\Gamma)$  (see e.g., [7], p. 315). We also have  $\gamma(z) \in L^1(\Gamma)$  for  $z \in W^{1,1}(\Omega)$ .

For simplicity, when no confusion can be made, we still write  $z$  instead of  $\gamma(z)$ .

Everywhere in the following, the gradient operator  $\nabla$ , as well as the divergence  $\nabla \cdot$  are considered in the sense of distributions on  $\Omega$ .

If  $Y$  is a Banach space and  $1 \leq p \leq \infty$ , we denote by  $L^p(0, T; Y)$  the space of  $L^p$  measurable  $Y$ -valued functions on  $(0, T)$ . By  $W^{1,p}([0, T]; Y)$  we denote the space  $\{y \in L^p(0, T; Y); \frac{dy}{dt} \in L^p(0, T; Y)\}$ , where  $\frac{d}{dt}$  is considered in the sense of  $Y$ -valued distributions on  $(0, T)$ . Moreover, each  $y \in W^{1,p}([0, T]; Y)$  is  $Y$ -valued absolutely continuous on  $[0, T]$  and  $\frac{d}{dt}$  exists a.e. on  $(0, T)$  (see e.g. [4], p. 23).

### 3. The strongly coercive case

In this section we assume that hypotheses (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>5</sub>) hold, and  $p > 1$ . Let us define the space

$$U = \{z \in L^2(\Omega); \nabla z \in L^p(\Omega), \gamma(z) \in L^2(\Gamma)\},$$

endowed with the natural norm  $\|z\|_U = \|z\|_{L^2(\Omega)} + \|\nabla z\|_{L^p(\Omega)} + \|\gamma(z)\|_{L^2(\Gamma)}$ .

Recalling the Sobolev embeddings [7, p. 284],  $W^{1,2}(\Omega) \subset L^{p^*}(\Omega)$ , if  $N > 2$ , where  $p^* = \frac{2N}{N-2}$ ,  $W^{1,2}(\Omega) \subset L^q(\Omega)$ , if  $N = 2$ , for any  $q \in [2, +\infty)$ ,  $W^{1,2}(\Omega) \subset L^\infty(\Omega)$ , if  $N = 1$ , with continuous injections, we conclude that if  $z \in U$ , we have

$$z \in W^{1,\bar{p}}(\Omega), \quad \bar{p} = \begin{cases} p^*, & \text{if } N > 2, \ p > p^* > 2, \\ p, & \text{otherwise.} \end{cases} \tag{3.1}$$

In particular, if  $p \geq 2$ , it follows that  $U \subset H^1(\Omega)$  with a dense and continuous embedding.

We mention for later use, that under assumption (1.7), one can easily deduce that

$$|\xi| \leq C_3 |r|^{p-1} + C_3^0, \quad \text{for any } \xi \in \beta(t, x, r), \tag{3.2}$$

where  $C_3$  and  $C_3^0$  are positive constants.

We also assume that

$$y_0 \in U, \quad f \in L^2(Q), \quad g \in L^2(\Sigma). \tag{3.3}$$

### 3.1. Existence and stability of the solution to the time-discretized system

We consider an equidistant partition  $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = T$  of the interval  $[0, T]$ , with  $t_i = ih$  for  $i = 1, \dots, n$ ,  $h = T/n$ , and the finite sequences  $\{f_i^h\}_{i=1, \dots, n}$ ,  $\{g_i^h\}_{i=1, \dots, n}$ , defined by the time averages

$$f_i^h = \frac{1}{h} \int_{(i-1)h}^{ih} f(s) ds, \quad g_i^h = \frac{1}{h} \int_{(i-1)h}^{ih} g(s) ds. \tag{3.4}$$

We note that  $f_i^h \in L^2(\Omega)$ ,  $g_i^h \in L^2(\Gamma)$ . We consider the time discretized system

$$\frac{y_{i+1}^h - y_i^h}{h} - \nabla \cdot \beta(t_{i+1}, x, \nabla y_{i+1}^h) \ni f_{i+1}^h, \quad \text{in } \Omega, \quad i = 0, \dots, n - 1, \tag{3.5}$$

$$\beta(t_{i+1}, x, \nabla y_{i+1}^h) \cdot \nu + \frac{y_{i+1}^h - y_i^h}{h} \ni g_{i+1}^h, \quad \text{on } \Gamma, \tag{3.6}$$

$$y_0^h = y_0, \quad \text{in } \Omega. \tag{3.7}$$

**Definition 3.1.** We call a *weak solution* to the time-discretized system (3.5)–(3.6), a set of functions  $\{y_i^h\}_{i=1, \dots, n}$ ,  $y_i^h \in U$ , which satisfies (for each  $i = 1, \dots, n - 1$ )

$$\begin{aligned} & \int_{\Omega} y_{i+1}^h \psi dx + h \int_{\Omega} \eta_{i+1}^h \cdot \nabla \psi dx + \int_{\Gamma} y_{i+1}^h \psi d\sigma \\ &= \int_{\Omega} y_i^h \psi dx + \int_{\Gamma} y_i^h \psi d\sigma + h \int_{\Omega} f_{i+1}^h \psi dx + h \int_{\Gamma} g_{i+1}^h \psi d\sigma, \quad \text{for any } \psi \in U, \end{aligned} \tag{3.8}$$

for some measurable function  $\eta_{i+1}^h$ , such that  $\eta_{i+1}^h(x) \in \beta(t_{i+1}, x, \nabla y_{i+1}^h(x))$ , a.e.  $x \in \Omega$ .

We mention that in the third integral on the left-hand side in (3.8) we understand by  $y_{i+1}^h$  the trace of  $y_{i+1}^h \in U$  on  $\Gamma$ . If  $y_i^h$  is a solution, it follows by (1.7) and (3.2) that

$$j(t_i, \cdot, \nabla y_i^h) \in L^1(\Omega), \quad \eta_i^h \in (L^{p'}(\Omega))^N, \quad i = 0, \dots, n. \tag{3.9}$$

Proposition 3.2 is concerned with the stability of the finite difference scheme (3.5)–(3.6).

**Proposition 3.2.** Let us assume (3.3) and  $j(0, \cdot, \nabla y_0) \in L^1(\Omega)$ . System (3.5)–(3.6) has a unique weak solution satisfying

$$\|y_i^h\|_{L^2(\Omega)} \leq C, \quad i = 1, \dots, n, \tag{3.10}$$

$$\|\gamma(y_i^h)\|_{L^2(\Gamma)} \leq C, \quad i = 1, \dots, n, \tag{3.11}$$

$$h \sum_{i=0}^{m-1} \|\nabla y_{i+1}^h\|_{L^p(\Omega)}^p \leq C, \quad m = 1, \dots, n, \tag{3.12}$$

$$h \sum_{i=0}^{m-1} \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Omega)}^2 \leq C, \quad m = 1, \dots, n, \tag{3.13}$$

$$h \sum_{i=0}^{m-1} \left\| \frac{\gamma(y_{i+1}^h) - \gamma(y_i^h)}{h} \right\|_{L^2(\Gamma)}^2 \leq C, \quad m = 1, \dots, n, \tag{3.14}$$

$$h \sum_{i=0}^{m-1} \int_{\Omega} j(t_{i+1}, x, \nabla y_{i+1}^h) dx \leq C, \quad i = 1, \dots, n, \tag{3.15}$$

where  $C$  is a positive constant, independent of  $h$ .

**Proof.** Let us fix  $t \in [0, T]$ ,  $w_1 \in L^2(\Omega)$ ,  $w_2 \in L^2(\Gamma)$  and consider the intermediate problem

$$\begin{aligned} u - h \nabla \cdot \beta(t, x, \nabla u) &\ni w_1 && \text{in } \Omega, \\ u + h \beta(t, x, \nabla u) \cdot \nu &\ni w_2 && \text{on } \Gamma. \end{aligned} \tag{3.16}$$

We define  $b \in U'$  (the dual of the space  $U$ ) by

$$b(\psi) := \int_{\Omega} w_1(x) \psi(x) dx + \int_{\Gamma} w_2(\sigma) \psi(\sigma) d\sigma, \quad \text{for all } \psi \in U, \tag{3.17}$$

and note that

$$|b(\psi)| \leq \|w_1\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Gamma)} \|\psi\|_{L^2(\Gamma)} \quad \text{for all } \psi \in U. \tag{3.18}$$

For  $u \in U$  it is clear by (1.7) and (3.2) that, for  $t$  fixed,

$$j(t, \cdot, \nabla u) \in L^1(\Omega), \quad \eta \in (L^{p'}(\Omega))^N,$$

for all measurable sections  $\eta(x)$  of  $\beta(t, x, \nabla u(x))$ .

We call a weak solution to (3.16) a function  $u \in U$ , such that there is  $\eta \in (L^{p'}(\Omega))^N$ ,  $\eta(x) \in \beta(t, x, \nabla u(x))$  a.e.  $x \in \Omega$ , and

$$\int_{\Omega} (u \psi + h \eta \cdot \nabla \psi) dx + \int_{\Gamma} u \psi d\sigma = b(\psi), \quad \text{for all } \psi \in U. \tag{3.19}$$

To prove that (3.16) has a solution we use a variational argument, i.e., we show that a solution to this equation is retrieved as a solution to the minimization problem

$$\text{Min}\{\varphi(u); u \in U\}, \tag{3.20}$$

where  $\varphi : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$  is given by

$$\varphi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} u^2 dx + h \int_{\Omega} j(t, x, \nabla u) dx + \frac{1}{2} \int_{\Gamma} u^2 d\sigma - b(u), & \text{if } u \in U, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.21}$$

By (1.7) and (3.18) we have

$$\begin{aligned} \varphi(u) &\geq \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + hC_1 \|\nabla u\|_{L^p(\Omega)}^p + hC_1^0 + \frac{1}{2} \|u\|_{L^2(\Gamma)}^2 - |b(u)| \\ &\geq \frac{1}{4} \|u\|_{L^2(\Omega)}^2 + hC_1 \|\nabla u\|_{L^p(\Omega)}^p + \frac{1}{4} \|u\|_{L^2(\Gamma)}^2 \\ &\quad + hC_1^0 - 4\|w_1\|_{L^2(\Omega)}^2 - 4\|w_2\|_{L^2(\Gamma)}^2, \quad \forall u \in U. \end{aligned} \tag{3.22}$$

It is also easily seen that  $\varphi$  is proper, strictly convex and lower semicontinuous (l.s.c. for short) on  $L^2(\Omega)$ . Let us denote  $d = \inf_{u \in U} \varphi(u)$  and let us consider a minimizing sequence  $\{u_n\}_{n \geq 1}$  for  $\varphi$ . Then, we have

$$d \leq \varphi(u_n) = \frac{1}{2} \int_{\Omega} u_n^2 dx + h \int_{\Omega} j(t, x, \nabla u_n) dx + \frac{1}{2} \int_{\Gamma} u_n^2 d\sigma - b(u_n) \leq d + \frac{1}{n}. \tag{3.23}$$

By (3.22) it follows that

$$\|u_n\|_{L^2(\Omega)}^2 + h\|\nabla u_n\|_{L^p(\Omega)}^p + \|u_n\|_{L^2(\Gamma)}^2 \leq C, \quad \forall n \in \mathbb{N},$$

where  $C$  is a positive constant independent of  $n$ . Therefore we can select a subsequence ( $n \rightarrow \infty$ ) such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } L^2(\Omega), \\ \gamma(u_n) &\rightharpoonup \chi \quad \text{weakly in } L^2(\Gamma), \text{ as } n \rightarrow \infty, \\ \nabla u_n &\rightharpoonup \xi \quad \text{weakly in } (L^p(\Omega))^N, \text{ as } n \rightarrow \infty. \end{aligned}$$

It follows that  $\xi = \nabla u$ , a.e. in  $\Omega$ , and so  $u \in W^{1,\bar{p}}(\Omega)$  with  $\bar{p}$  given by (3.1). This implies that  $\gamma(u) \in W^{1-\frac{1}{\bar{p}},\bar{p}}(\Omega)$ , and it is clear that  $\chi = \gamma(u)$  a.e. on  $\Gamma$ . Then,  $u \in U$ . Moreover, by (3.2) we have also  $\eta \in (L^{p'}(\Omega))^N$ , where  $\eta(x) \in \beta(t, x, \nabla u(x))$  a.e.  $x \in \Omega$ . Since  $\varphi$  is convex and continuous it is also weakly l.s.c. on  $L^2(\Omega)$  and so  $\liminf_{n \rightarrow \infty} \varphi(u_n) \geq \varphi(u)$ . Passing to the limit in (3.23), as  $n \rightarrow \infty$ , we get that  $\varphi(u) = d$ , as claimed.

Next, we connect this solution to the solution to (3.16). Let  $\lambda > 0$  and define the variation  $u^\lambda = u + \lambda\psi$ , for all  $\psi \in C^\infty(\bar{Q})$ . We have  $\varphi(u) \leq \varphi(u^\lambda)$ , for any  $\lambda > 0$ . Replacing the expression of  $\varphi$ , dividing by  $\lambda$  and letting  $\lambda \rightarrow 0$ , we get

$$\int_{\Omega} (u\psi + h\eta \cdot \nabla \psi) dx + \int_{\Gamma} u\psi d\sigma - b(\psi) \leq 0,$$

for all  $\psi \in C^\infty(\bar{Q})$ . By density, this extends to all  $\psi \in U$ . Changing  $\psi$  to  $-\psi$  and making the same calculus we obtain the converse inequality, so that in conclusion we find that the solution to (3.20) satisfies (3.19).

Relying on this result we deduce in an iterative way that system (3.5)–(3.6) has a unique weak solution. We observe that it can be rewritten as

$$\int_{\Omega} y_{i+1}^h \psi dx + h \int_{\Omega} \eta_{i+1}^h \cdot \nabla \psi dx + \int_{\Gamma} y_{i+1}^h \psi d\sigma = b_{i+1}(\psi), \quad \forall \psi \in U, \tag{3.24}$$

for  $i = 0, \dots, n - 1$ , where  $\eta_{i+1}^h(x) \in \beta(t_{i+1}, x, \nabla y_{i+1}^h(x))$  a.e.  $x \in \Omega$ ,  $b_{i+1} \in U'$  is given by

$$b_{i+1}(\psi) := \int_{\Omega} (y_i^h \psi + h f_{i+1}^h \psi) dx + \int_{\Gamma} (y_i^h + h g_{i+1}^h) \psi d\sigma, \quad \forall \psi \in U, \tag{3.25}$$

and

$$\begin{aligned} |b_{i+1}(\psi)| &\leq (\|y_i^h\|_{L^2(\Omega)} + h \|f_{i+1}^h\|_{L^2(\Omega)}) \|\psi\|_{L^2(\Omega)} \\ &\quad + (\|y_i^h\|_{L^2(\Gamma)} + h \|g_{i+1}^h\|_{L^2(\Gamma)}) \|\psi\|_{L^2(\Gamma)}, \quad \text{for } i = 0, \dots, n - 1. \end{aligned}$$

We begin with Eq. (3.24) for  $i = 0$ , in which  $b_1(\psi)$  satisfies the previous relation for  $i = 0$ . Setting  $b = b_1$  in  $\varphi$  we get that the corresponding problem (3.20) has a unique weak solution  $y_1^h$  which verifies (3.24) for  $i = 0$ . Next, we set  $b = b_{i+1}$  in  $\varphi$  and by recurrence, we obtain a sequence of solutions  $y_{i+1}^h \in U$ , which satisfy (3.24) for all  $i = 1, \dots, n - 1$ . In particular,  $y_i^h \in W^{1,\bar{p}}(\Omega)$ , with  $\bar{p}$  given by (3.1).

To obtain the first estimate (3.10) we set  $\psi = y_{i+1}^h$  in (3.8) and use (1.4) getting

$$\begin{aligned} &\int_{\Omega} (y_{i+1}^h)^2 dx + h \int_{\Omega} (j(t_{i+1}, x, \nabla y_{i+1}^h) - j(t_{i+1}, x, 0)) dx + \int_{\Gamma} (y_{i+1}^h)^2 d\sigma \\ &\leq \int_{\Omega} (y_{i+1}^h)^2 dx + h \int_{\Omega} \eta_{i+1}^h \cdot \nabla y_{i+1}^h dx + \int_{\Gamma} (y_{i+1}^h)^2 d\sigma \\ &= \int_{\Omega} y_i^h y_{i+1}^h dx + \int_{\Gamma} y_i^h y_{i+1}^h d\sigma + h \int_{\Omega} f_{i+1}^h y_{i+1}^h dx + h \int_{\Gamma} g_{i+1}^h y_{i+1}^h d\sigma. \end{aligned} \tag{3.26}$$

Then, we sum up (3.26) from  $i = 0$  to  $i = m - 1 \leq n - 1$ , use (1.15),

$$\begin{aligned} &\frac{1}{2} \|y_m^h\|_{L^2(\Omega)}^2 + h \sum_{i=0}^{m-1} \int_{\Omega} j(t_{i+1}, x, \nabla y_{i+1}^h) dx + \frac{1}{2} \|y_m^h\|_{L^2(\Gamma)}^2 \\ &\leq \frac{h}{2} \sum_{i=0}^{m-1} \|y_{i+1}^h\|_{L^2(\Omega)}^2 + \frac{h}{2} \sum_{i=0}^{m-1} \|f_{i+1}^h\|_{L^2(\Omega)}^2 + \frac{h}{2} \sum_{i=0}^{m-1} \|y_{i+1}^h\|_{L^2(\Gamma)}^2 \\ &\quad + \frac{h}{2} \sum_{i=0}^{m-1} \|g_{i+1}^h\|_{L^2(\Gamma)}^2 + \frac{1}{2} \|y_0^h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|y_0^h\|_{L^2(\Gamma)}^2, \end{aligned} \tag{3.27}$$

and obtain (since  $j$  is continuous on  $\bar{Q}$ )

$$\begin{aligned} & \|y_m^h\|_{L^2(\Omega)}^2 + \|y_m^h\|_{L^2(\Gamma)}^2 + 2C_1h \sum_{i=0}^{m-1} \|\nabla y_{i+1}^h\|_{L^p(\Omega)}^p \\ & \leq C_0 + h \sum_{i=1}^m (\|y_i^h\|_{L^2(\Omega)}^2 + \|y_i^h\|_{L^2(\Gamma)}^2), \end{aligned} \tag{3.28}$$

where

$$C_0 = h \int_0^T (\|f(t)\|_{L^2(\Omega)}^2 + \|g(t)\|_{L^2(\Gamma)}^2) dt + \|y_0\|_{L^2(\Omega)}^2 + \|y_0\|_{L^2(\Gamma)}^2 + 2T|C_1^0| \text{meas}(\Omega).$$

Using a variant of the discrete Gronwall’s lemma (see e.g., [8]) we get

$$\|y_m^h\|_{L^2(\Omega)}^2 + \|y_m^h\|_{L^2(\Gamma)}^2 \leq 2e^T (\|y_0^h\|_{L^2(\Omega)}^2 + \|y_0^h\|_{L^2(\Gamma)}^2 + C_0), \tag{3.29}$$

$$h \sum_{i=1}^m \|y_i^h\|_{L^2(\Omega)}^2 + h \sum_{i=1}^m \|y_i^h\|_{L^2(\Gamma)}^2 \leq e^T (\|y_0\|_{L^2(\Omega)}^2 + \|y_0^h\|_{L^2(\Gamma)}^2 + C_0). \tag{3.30}$$

By the last two relations, (3.28) and (3.27) we obtain (3.10)–(3.12) and (3.15).

To get (3.13) we set  $\psi = \frac{y_{i+1}^h - y_i^h}{h}$  in (3.8) and we obtain

$$\begin{aligned} & \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Omega)}^2 + \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Gamma)}^2 + \int_{\Omega} \eta_{i+1}^h \cdot \nabla \frac{y_{i+1}^h - y_i^h}{h} dx \\ & = \int_{\Omega} f_{i+1}^h \frac{y_{i+1}^h - y_i^h}{h} dx + \int_{\Gamma} g_{i+1}^h \frac{y_{i+1}^h - y_i^h}{h} d\sigma \\ & \leq \frac{1}{2} \|f_{i+1}^h\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \|g_{i+1}^h\|_{L^2(\Gamma)}^2 + \frac{1}{2} \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Gamma)}^2. \end{aligned}$$

Further we use again (1.4) and sum up from  $i = 0$  to  $i = m - 1 \leq n - 1$ . We obtain

$$\begin{aligned} & h \sum_{i=0}^{m-1} \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Omega)}^2 + h \sum_{i=0}^{m-1} \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Gamma)}^2 \\ & \leq h \sum_{i=0}^{m-1} \|f_{i+1}^h\|_{L^2(\Omega)}^2 + h \sum_{i=0}^{m-1} \|g_{i+1}^h\|_{L^2(\Gamma)}^2 \\ & \quad + 2 \sum_{i=0}^{m-1} \int_{\Omega} (j(t_{i+1}, x, \nabla y_i^h) - j(t_{i+1}, x, \nabla y_{i+1}^h)) dx. \end{aligned} \tag{3.31}$$

By (1.14) we have

$$j(t, x, r) \leq \frac{j(s, x, r)}{1 - L|t - s|}, \quad \text{for } t, s \in [0, T], \quad |t - s| < 1/L.$$

Then, we compute

$$\begin{aligned} & \sum_{i=0}^{m-1} \int_{\Omega} (j(t_{i+1}, x, \nabla y_i^h) - j(t_{i+1}, x, \nabla y_{i+1}^h)) dx \\ &= \sum_{i=0}^{m-1} \int_{\Omega} (j(t_{i+1}, x, \nabla y_i^h) - j(t_i, x, \nabla y_i^h)) dx + \int_{\Omega} j(0, x, \nabla y_0) dx \\ &\leq Lh \sum_{i=0}^{m-1} \int_{\Omega} j(t_{i+1}, x, \nabla y_i^h) dx + \int_{\Omega} j(0, x, \nabla y_0) dx \\ &\leq \frac{Lh}{1 - Lh} \sum_{i=0}^{m-1} \int_{\Omega} j(t_i, x, \nabla y_i^h) dx + \int_{\Omega} j(0, x, \nabla y_0) dx \leq C, \end{aligned}$$

for  $h$  sufficiently small,  $h \ll 1/L$ , by (3.15), and since  $y_0 \in U$ . Then, (3.31) implies (3.13)–(3.14).  $\square$

### 3.2. Convergence of the discretization scheme

Let us define  $y^h : [0, T] \rightarrow L^2(\Omega)$  by

$$\begin{aligned} y^h(t) &= y_i^h, \quad t \in [(i - 1)h, ih), \quad i = 1, \dots, n, \\ y^h(0) &= y_0^h, \end{aligned} \tag{3.32}$$

and extend  $y^h$  by continuity to the right of  $T$  as

$$y^h(t) = y_n^h, \quad t \in [T, T + \delta], \quad \text{with } \delta \text{ arbitrary, } \delta > h.$$

The step function  $y^h$  defined by (3.32) is called an  $h$ -approximating solution to (1.1)–(1.3) (see [4], p. 129). Also, we set, for all  $i = 1, \dots, n$ ,

$$\begin{aligned} f^h(t) &= f_i^h, \quad t \in [(i - 1)h, ih), \\ g^h(t) &= g_i^h, \quad t \in [(i - 1)h, ih), \\ \beta(t, x, \nabla y^h(t)) &= \beta(t_i, x, \nabla y_i^h), \quad t \in [(i - 1)h, ih), \\ \eta^h(t) &= \eta_i^h, \quad t \in [(i - 1)h, ih), \\ j(t, x, \nabla y^h(t)) &= j(t_i, x, \nabla y_i^h), \quad t \in [(i - 1)h, ih). \end{aligned}$$

We see that  $\eta^h(t, x) \in \beta(t, x, \nabla y^h(t))$  a.e. on  $Q$ .

Then, we deduce from (3.10)–(3.15) the estimates

$$\|y^h(t)\|_{L^2(\Omega)} + \|\gamma(y^h(t))\|_{L^2(\Gamma)} \leq C, \text{ for } t \in [0, T], \tag{3.33}$$

$$\int_0^T \|\nabla y^h(t)\|_{L^p(\Omega)}^p dt \leq C, \tag{3.34}$$

$$\int_0^T \left\| \frac{y^h(t+h) - y^h(t)}{h} \right\|_{L^2(\Omega)}^2 dt \leq C, \tag{3.35}$$

$$\int_0^T \left\| \frac{\gamma(y^h(t+h)) - \gamma(y^h(t))}{h} \right\|_{L^2(\Gamma)}^2 dt \leq C, \tag{3.36}$$

$$\int_0^T \int_{\Omega} j(t, x, \nabla y^h(t)) dx dt \leq C, \tag{3.37}$$

with  $C$  independent of  $h$ . Also, (3.33) and (3.34) imply that

$$\int_0^T \|y^h(t)\|_{W^{1,\bar{p}}(\Omega)}^{\bar{p}} dt \leq C, \tag{3.38}$$

where  $\bar{p}$  given by (3.1).

**Definition 3.3.** We call a *weak solution* to problem (1.1)–(1.3) a function  $y \in L^2(Q)$ , with

$$\nabla y \in L^p(0, T; (L^p(\Omega))^N), \quad \gamma(y) \in L^2(\Sigma), \quad j(\cdot, \cdot, \nabla y) \in L^1(Q),$$

such that there exists  $\eta \in (L^p(Q))^N$ ,  $\eta(t, x) \in \beta(t, x, \nabla y(t, x))$  a.e.  $(t, x) \in Q$ , satisfying

$$\begin{aligned} & - \int_Q y \phi_t dx dt + \int_Q \eta \cdot \nabla \phi dx dt - \int_{\Sigma} y \phi_t d\sigma dt \\ & = \int_Q f \phi dx dt + \int_{\Sigma} g \phi d\sigma dt + \int_{\Omega} y_0 \phi(0) dx + \int_{\Gamma} y_0 \phi(0) d\sigma, \end{aligned} \tag{3.39}$$

for all  $\phi \in W^{1,2}([0, T]; L^2(\Omega))$ ,  $\nabla \phi \in (L^p(Q))^N$ ,  $\gamma(\phi) \in W^{1,2}([0, T]; L^2(\Gamma))$ ,  $\phi(T) = 0$ .

**Theorem 3.4** below is the main result of this section.

**Theorem 3.4.** *Let us assume (3.3). Then, under hypotheses (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>5</sub>), problem (1.1)–(1.3) has at least one weak solution  $y$ , which satisfies*

$$\begin{aligned} & y \in W^{1,2}([0, T]; L^2(\Omega)), \quad \gamma(y) \in W^{1,2}([0, T]; L^2(\Gamma)), \\ & \nabla \cdot \eta \in L^2(Q), \quad \eta(t, x) \in \beta(t, x, \nabla y(t, x)) \text{ a.e. } (t, x) \in Q. \end{aligned} \tag{3.40}$$

Moreover,  $y$  is a strong solution to (1.1)–(1.3), that is

$$y_t - \nabla \cdot \eta = f, \quad \text{a.e. in } Q, \tag{3.41}$$

$$\gamma(\eta) \cdot \nu + y_t = g, \quad \text{a.e. on } \Sigma, \tag{3.42}$$

$$y(0) = y_0, \quad \text{in } \Omega. \tag{3.43}$$

The solution  $y$  is given by

$$y = \lim_{h \rightarrow 0} y^h \quad \text{strongly in } L^r(Q), \tag{3.44}$$

with  $y^h$  defined by (3.32),  $r = 2$  if  $p \geq 2$  and  $r = p$ , if  $p \in (1, 2)$ .

The solution is unique in the class of functions satisfying (3.40)–(3.43) and the map  $(y_0, \gamma(y_0)) \rightarrow (y(t), \gamma(y(t)))$  is Lipschitz from  $L^2(\Omega) \times L^2(\Gamma)$  to  $C([0, T]; L^2(\Omega)) \times C([0, T]; L^2(\Gamma))$ .

**Proof.** The proof is done in three steps. First, we note that if  $y_0 \in U$  we have  $j(0, \cdot, \nabla y_0) \in L^1(\Omega)$ .

**Weak solution.** By (3.33)–(3.37) it follows that one can select a subsequence such that as  $h \rightarrow 0$ , we have,

$$\begin{aligned} y^h &\rightharpoonup y \quad \text{weak-star in } L^\infty(0, T; L^2(\Omega)), \\ y^h &\rightharpoonup y \quad \text{weak-star in } L^\infty(0, T; L^2(\Gamma)), \\ \nabla y^h &\rightharpoonup \nabla y \quad \text{weakly in } (L^p(Q))^N, \\ \frac{y^h(t+h) - y^h(t)}{h} &\rightharpoonup l \quad \text{weakly in } L^2(Q), \\ \frac{\gamma(y^h(t+h)) - \gamma(y^h(t))}{h} &\rightharpoonup l_1 \quad \text{weakly in } L^2(\Sigma). \end{aligned}$$

To prove the last two assertions we proceed by a direct calculus. For some  $\delta > 0$ , we take  $\phi \in M_\delta$ , where

$$M_\delta = \{ \phi \in C^\infty(\bar{Q}); \phi(t, x) = 0 \text{ on } t \in [T - \delta, T] \}$$

and compute (without writing the argument  $x$  for the functions  $y^h$  and  $\phi$ )

$$\begin{aligned} &\int_0^T \int_\Omega \frac{y^h(t+h) - y^h(t)}{h} \phi(t) dx dt \\ &= \int_0^{T-h} \int_\Omega \frac{1}{h} y^h(t+h) \phi(t) dx dt - \int_0^{T-h} \int_\Omega \frac{1}{h} y^h(t) \phi(t) dx dt \end{aligned}$$

$$\begin{aligned}
 &= \int_h^T \int_{\Omega} \frac{1}{h} y^h(s) \phi(s-h) dx ds - \int_0^{T-h} \int_{\Omega} \frac{1}{h} y^h(t) \phi(t) dx dt \\
 &= \int_h^{T-h} \int_{\Omega} \frac{1}{h} y^h(s) \phi(s-h) dx ds + \int_{T-h}^T \int_{\Omega} \frac{1}{h} y^h(s) \phi(s-h) dx ds \\
 &\quad - \int_0^h \int_{\Omega} \frac{1}{h} y^h(s) \phi(s) dx ds - \int_h^{T-h} \int_{\Omega} \frac{1}{h} y^h(s) \phi(s) dx ds \\
 &= - \int_h^{T-h} \int_{\Omega} \frac{\phi(s) - \phi(s-h)}{h} y^h(s) dx ds - \int_0^h \int_{\Omega} \frac{1}{h} y^h(s) \phi(s) dx ds \\
 &\quad + \int_{T-h}^T \int_{\Omega} \frac{1}{h} y^h(s) \phi(s-h) dx ds.
 \end{aligned}$$

In the calculus above  $\phi(t, x) = 0$  for  $t \in [T - h, T]$  since we can take  $\delta > h$ . Next,

$$\begin{aligned}
 & - \int_0^h \int_{\Omega} \frac{1}{h} y^h(s) \phi(s) dx ds \\
 &= - \frac{1}{h} \int_0^h \int_{\Omega} y^h(s) (\phi(s) - \phi(0)) dx ds - \frac{1}{h} \int_0^h \int_{\Omega} y^h(s) \phi(0) dx ds \\
 &\leq \frac{1}{h} \int_0^h \|y^h(s)\|_{L^2(\Omega)} \|\phi(s) - \phi(0)\|_{L^2(\Omega)} ds - \int_{\Omega} \phi(0) \frac{1}{h} \int_0^h y^h(s) ds dx \\
 &\leq C \frac{1}{h} \int_0^h \|\phi(s) - \phi(0)\|_{L^2(\Omega)} ds - \int_{\Omega} \phi(0) y_0 dx + \epsilon(h) \rightarrow - \int_{\Omega} \phi(0) y_0 dx,
 \end{aligned}$$

where  $\epsilon(h) \rightarrow 0$  as  $h \rightarrow 0$ .

Proceeding in the same way for the last term we get

$$\begin{aligned}
 & \int_{T-h}^T \int_{\Omega} \frac{1}{h} y^h(s) \phi(s-h) dx ds \\
 &= \int_{T-h}^T \int_{\Omega} \frac{1}{h} y^h(s) (\phi(s-h) - \phi(T-h)) dx ds
 \end{aligned}$$

$$+ \int_{T-h}^T \int_{\Omega} \frac{1}{h} y^h(s) \phi(T-h) dx ds \leq C \frac{1}{h} \int_{T-h}^T \|\phi(s-h) - \phi(T-h)\|_{L^2(\Omega)} ds,$$

as  $\phi(T-h) = 0$ . Again by the continuity of  $\phi$  we obtain that

$$\int_{T-h}^T \int_{\Omega} \frac{1}{h} y^h(s) \phi(s-h) dx ds \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

In conclusion, all these yield

$$\lim_{h \rightarrow 0} \int_0^T \int_{\Omega} \frac{y^h(t+h) - y^h(t)}{h} \phi(t) dx dt = - \int_0^T \int_{\Omega} y(t) \frac{d\phi}{dt}(t) dx dt - \int_{\Omega} \phi(0) y_0 dx,$$

for any  $\phi \in M_{\delta}$ . Therefore, we get, in the sense of distributions, that

$$\lim_{h \rightarrow 0} \int_0^T \int_{\Omega} \frac{y^h(t+h) - y^h(t)}{h} \phi(t) dx dt = \frac{dy}{dt}(\phi), \quad \text{for any } \phi \in C_0^{\infty}(Q), \tag{3.45}$$

and so,  $l = \frac{dy}{dt}$  in  $\mathcal{D}'(Q)$  (the space of Schwartz distributions on  $Q$ ). Moreover, by (3.35) we still have

$$\left| \frac{dy}{dt}(\phi) \right| \leq C \|\phi\|_{L^2(Q)}, \quad \text{for } \phi \in C_0^{\infty}(Q) \cap M_{\delta},$$

with  $C$  independent of  $\delta$ , and so  $\frac{dy}{dt} \in L^2(Q)$  and  $l = \frac{dy}{dt}$  a.e. on  $Q$ . Therefore,  $y \in W^{1,2}([0, T - \delta]; L^2(\Omega))$  and since  $\delta$  is arbitrary we infer that  $y \in W^{1,2}([0, T]; L^2(\Omega))$ .

Proceeding in the same way for the time differences on  $\Gamma$  we get that

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_0^T \int_{\Gamma} \frac{\gamma(y^h(t+h)) - \gamma(y^h(t))}{h} \phi(t) d\sigma dt \\ &= - \int_0^T \int_{\Gamma} \gamma(y(t)) \frac{d\phi}{dt}(t) d\sigma dt - \int_{\Gamma} \phi(0) y_0 d\sigma, \quad \text{for } \phi \in M_{\delta}. \end{aligned} \tag{3.46}$$

Therefore, we obtain that

$$l_1 = \frac{d\gamma(y)}{dt} \quad \text{a.e. on } \Sigma, \quad \frac{d\gamma(y)}{dt} \in L^2(0, T - \delta; L^2(\Gamma))$$

and so, finally  $\gamma(y) \in W^{1,2}([0, T]; L^2(\Gamma))$ .

We deduce that  $\xi = \nabla y$  a.e. on  $Q$ , by the same argument used in Proposition 3.2 and by passing to the limit in (3.37) and using the weak lower semicontinuity of the convex integrand we get  $j(t, \cdot, \nabla y) \in L^1(Q)$ .

The next step is to prove (3.44). A simple way to show it is to use a compactness argument in the space of vectorial functions with bounded variations on  $[0, T]$ .

We have that  $y^h \in BV([0, T]; L^2(\Omega))$ , the space of functions with bounded variation from  $[0, T]$  to  $L^2(\Omega)$ , i.e.,

$$V_0^T(y^h) = \sup_{P \in \mathcal{P}} \sum_{i=1}^{n_p} \|y^h(s_i) - y^h(s_{i-1})\|_{L^2(\Omega)} \leq C, \tag{3.47}$$

where  $C$  is a constant and  $\mathcal{P} = \{P = (s_0, \dots, s_{n_p}); P \text{ is a partition of } [0, T]\}$  is the set of all partitions of  $[0, T]$ . Indeed, if we consider an equidistant partition (e.g., with  $s_i = t_i$ ) we have by (3.13) that

$$\begin{aligned} & \left( \sum_{i=0}^{n-1} \|y^h(t_{i+1}) - y^h(t_i)\|_{L^2(\Omega)} \right)^2 \\ & \leq n \sum_{i=0}^{n-1} \|y_{i+1}^h - y_i^h\|_{L^2(\Omega)}^2 = nh \cdot h \sum_{i=1}^n \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Omega)}^2 \leq TC. \end{aligned} \tag{3.48}$$

Now, we discuss separately the cases  $p \geq 2$  and  $p \in (1, 2)$ .

Let  $p \geq 2$ . By (3.48) we also have  $y^h \in BV([0, T]; (H^1(\Omega))')$ .

On the basis of this relation, (3.33), and since  $L^2(\Omega)$  is compact in  $(H^1(\Omega))'$  we can apply the strong version of Helly theorem for the infinite dimensional case (see [6], Remark 1.127, p. 48). We deduce that

$$y^h(t) \rightarrow y(t) \text{ strongly in } (H^1(\Omega))' \text{ uniformly in } t \in [0, T]. \tag{3.49}$$

Next, applying Lemma 5.1 in [14], p. 58, we have that for any  $\varepsilon > 0$  there exists a constant  $C_\varepsilon$  such that

$$\|w\|_{L^2(\Omega)} \leq \varepsilon \|w\|_{H^1(\Omega)} + C_\varepsilon \|w\|_{(H^1(\Omega))'}, \quad \forall w \in H^1(\Omega). \tag{3.50}$$

This lemma applied for  $w = y^h(t) - y(t)$  yields

$$\frac{1}{2} \|y^h(t) - y(t)\|_{L^2(\Omega)}^2 \leq \varepsilon \|y^h(t) - y(t)\|_{H^1(\Omega)}^2 + C_\varepsilon \|y^h(t) - y(t)\|_{(H^1(\Omega))'}^2.$$

Integrating with respect to  $t$  on  $(0, T)$  we obtain that

$$\begin{aligned} & \frac{1}{2} \int_0^T \|y^h(t) - y(t)\|_{L^2(\Omega)}^2 dt \\ & \leq \varepsilon \int_0^T \|y^h(t) - y(t)\|_{H^1(\Omega)}^2 dt + C_\varepsilon \int_0^T \|y^h(t) - y(t)\|_{(H^1(\Omega))'}^2 dt. \end{aligned}$$

Then, the last term on the right-hand side tends to 0 as  $h \rightarrow 0$ , by (3.49), and the coefficient of  $\varepsilon$  is bounded, by (3.38). Hence

$$\limsup_{h \rightarrow 0} \int_0^T \|y^h(t) - y(t)\|_{L^2(\Omega)}^2 dt \leq C\varepsilon \quad \text{for any } \varepsilon > 0,$$

and since  $\varepsilon$  is arbitrary we get (3.44), with  $r = 2$ .

Let  $p \in (1, 2)$ . We shall prove that

$$y^h \rightarrow y \quad \text{strongly in } L^p(Q), \text{ as } h \rightarrow 0. \tag{3.51}$$

From (3.48) it follows that  $y^h \in BV([0, T]; L^p(\Omega))$ . We assert that there exists a Banach space  $X$  such that  $L^p(\Omega) \subset X$ , with compact injection. For example,  $X = (W^{1,r}(\Omega))'$  (i.e., if  $W^{1,r}(\Omega) \subset L^{p'}(\Omega)$ , with  $p' < \frac{rN}{N-r}$  if  $N > r$ , and  $p' = r$  if  $N \leq r$ ).

Therefore, once again by the strong Helly theorem and since  $\|y^h(t)\|_{L^p(\Omega)} \leq C$  we get

$$y^h(t) \rightarrow y(t) \quad \text{strongly in } X, \text{ uniformly with } t \in [0, T]. \tag{3.52}$$

Then, we apply the argument before for the triplet  $W^{1,p}(\Omega) \subset L^p(\Omega) \subset X$  and use (3.52) and (3.38). We get

$$\begin{aligned} & \frac{1}{2} \int_0^T \|y^h(t) - y(t)\|_{L^p(\Omega)}^p dt \\ & \leq \varepsilon \int_0^T \|y^h(t) - y(t)\|_{W^{1,p}(\Omega)}^p dt + C_\varepsilon \int_0^T \|y^h(t) - y(t)\|_X^p dt \rightarrow 0, \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , whence (3.51) follows.

Let us fix  $t \in [0, T]$ . By (3.33), we have that on a subsequence,

$$y^h(t) \rightarrow \vartheta(t) \quad \text{weakly in } L^2(\Omega), \text{ as } h \rightarrow 0, \text{ for each fixed } t \in [0, T],$$

and since we have either (3.49) or (3.52) we get by the limit uniqueness that  $\vartheta(t) = y(t)$ , a.e. on  $\Omega$ . In particular, it follows that

$$y^h(T) \rightarrow y(T) \quad \text{weakly in } L^2(\Omega), \text{ as } h \rightarrow 0.$$

We mention, that as  $h \rightarrow 0$ ,

$$f^h \rightarrow f \quad \text{strongly in } L^2(0, T; L^2(\Omega)), \tag{3.53}$$

$$g^h \rightarrow g \quad \text{strongly in } L^2(0, T; L^2(\Gamma)). \tag{3.54}$$

Now, relation (3.2) yields

$$|\eta^h(t, x)| \leq C_3 |\nabla y^h(t, x)|^{p-1} + C_3^0,$$

for  $\eta^h(t, x) \in \beta(t, x, \nabla y^h(t, x))$  a.e. on  $Q$ , and so we conclude that  $\{\eta^h\}_h$  is bounded in  $(L^{p'}(Q))^N$ . Therefore, on a subsequence, we have

$$\eta^h \rightharpoonup \eta \text{ weakly in } (L^{p'}(Q))^N, \text{ as } h \rightarrow 0,$$

and it remains to prove that  $\eta(t, x) \in \beta(t, x, \nabla y(t, x))$ , a.e.  $(t, x) \in Q$ .

Summing up (3.26) from  $i = 0$  to  $n - 1$  we get

$$\begin{aligned} & \frac{1}{2} \|y_n^h\|_{L^2(\Omega)}^2 + h \sum_{i=0}^{n-1} \int_{\Omega} \eta_{i+1}^h \cdot \nabla y_{i+1}^h dx + \frac{1}{2} \|y_n^h\|_{L^2(\Gamma)}^2 \\ & \leq h \sum_{i=0}^{n-1} \int_{\Omega} f_{i+1}^h y_{i+1}^h dx + h \sum_{i=0}^{n-1} \int_{\Gamma} g_{i+1}^h y_{i+1}^h d\sigma + \frac{1}{2} \|y_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \|y_0\|_{L^2(\Gamma)}^2, \end{aligned}$$

whence, replacing the definition of  $y^h(t)$  we obtain

$$\begin{aligned} & \frac{1}{2} \|y^h(T)\|_{L^2(\Omega)}^2 + h \int_0^T \int_{\Omega} \eta^h \cdot \nabla y^h dx dt + \frac{1}{2} \|y^h(T)\|_{L^2(\Gamma)}^2 \\ & \leq h \int_0^T \int_{\Omega} f^h y^h dx dt + h \int_0^T \int_{\Gamma} g^h y^h d\sigma dt + \frac{1}{2} \|y_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \|y_0\|_{L^2(\Gamma)}^2. \end{aligned}$$

Passing to the limit as  $h \rightarrow 0$ , this yields

$$\begin{aligned} \limsup_{h \rightarrow 0} \int_Q \eta^h \cdot \nabla y^h dx dt & \leq -\frac{1}{2} \|y(T)\|_{L^2(\Omega)}^2 + \int_Q f y dx dt + \int_{\Sigma} g y d\sigma dt + \frac{1}{2} \|y_0\|_{L^2(\Omega)}^2 \\ & \quad + \frac{1}{2} \|y_0\|_{L^2(\Gamma)}^2 - \frac{1}{2} \|y(T)\|_{L^2(\Gamma)}^2. \end{aligned} \tag{3.55}$$

Now we write (3.5)–(3.6) in the following form, after replacing the functions  $y_i^h$  by  $y^h$ , and integrating with respect to  $t \in (0, T)$

$$\begin{aligned} & \int_Q \frac{y^h(t+h, x) - y^h(t, x)}{h} \phi(t, x) dx dt + \int_Q \eta^h(t+h, x) \cdot \nabla \phi(t, x) dx dt \\ & \quad + \int_{\Sigma} \frac{\gamma(y^h(t+h, x)) - \gamma(y^h(t, x))}{h} \phi(t, x) d\sigma dt \\ & = \int_Q f^h(t+h, x) \phi(t, x) dx dt + \int_{\Sigma} g^h(t+h, x) \phi(t, x) d\sigma dt \end{aligned}$$

for any  $\phi \in C^\infty(\bar{Q})$ , with  $\phi(T, x) = 0$ . We pass to the limit as  $h \rightarrow 0$ , using the convergences previously deduced and get

$$\begin{aligned}
 & - \int_Q y \phi_t dx dt + \int_Q \eta \cdot \nabla \phi dx dt - \int_\Sigma y \phi_t d\sigma dt \\
 & = \int_Q f \phi dx dt + \int_\Sigma g \phi d\sigma dt + \int_\Omega y_0 \phi(0) dx + \int_\Gamma y_0 \phi(0) d\sigma,
 \end{aligned} \tag{3.56}$$

where  $\eta = \lim_{h \rightarrow 0} \eta^h$  (weakly in  $(L^{p'}(\Omega))^N$ ).

Taking into account that  $y \in W^{1,2}([0, T]; L^2(\Omega))$  we have

$$\int_Q y \phi_t dx dt = - \int_Q y_t \phi dx dt - \int_\Omega y_0 \phi(0) dx$$

and so we obtain

$$\int_Q y_t \phi dx dt + \int_Q \eta \cdot \nabla \phi dx dt + \int_\Sigma y_t \phi d\sigma dt = \int_Q f \phi dx dt + \int_\Sigma g \phi d\sigma dt. \tag{3.57}$$

By density, this extends to all  $\phi \in W^{1,2}([0, T]; L^2(\Omega))$  with  $\nabla \phi \in (L^p(Q))^N$ , and in particular, for  $\phi = y$ . Finally, we have got

$$\begin{aligned}
 \int_Q \eta \cdot \nabla y dx dt & = -\frac{1}{2} \|y(T)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|y_0\|_{L^2(\Omega)}^2 \\
 & + \int_Q f y dx dt + \int_\Sigma g \phi d\sigma dt - \frac{1}{2} \|y(T)\|_{L^2(\Gamma)}^2 + \frac{1}{2} \|y_0\|_{L^2(\Gamma)}^2.
 \end{aligned} \tag{3.58}$$

Comparing with (3.55) we deduce that

$$\limsup_{h \rightarrow 0} \int_0^T \int_\Omega \eta^h \cdot \nabla y^h dx dt \leq \int_Q \eta \cdot \nabla y dx dt$$

and since the operator  $z \rightarrow \beta(t, x, z)$  is maximal monotone in the dual pair  $(L^p(Q))^N - (L^{p'}(Q))^N$ , we get  $\eta(t, x) \in \beta(t, x, \nabla y(t, x))$ , a.e. on  $Q$  (see e.g., [4], p. 41). Hence  $y$  is a weak solution to (1.1)–(1.3).

**Strong solution.** By (3.57) we see that

$$\int_Q y_t \phi dx dt + \int_Q \eta \cdot \nabla \phi dx dt = \int_Q f \phi dx dt, \quad \forall \phi \in C_0^\infty(Q).$$

Since we have

$$(-\nabla \cdot \eta)(\phi) = \int_Q \eta \cdot \nabla \phi dx dt, \quad \forall \phi \in C_0^\infty(Q),$$

and recalling that  $y_t, f \in L^2(Q), \eta \in (L^{p'}(Q))^N$ , we get (3.41) in  $D'(Q)$ , as claimed.

We note that since  $\nabla \cdot \eta \in L^2(Q)$  it follows that

$$\gamma(\eta) \cdot \nu \in L^2(0, T; H^{-1/2}(\Gamma))$$

is well-defined (see e.g., [1], or Theorem 1.2 in [16]) and the following formula holds

$$\begin{aligned} \int_\Omega \phi(t) \nabla \cdot \eta(t) dx &= - \int_\Omega \eta(t) \cdot \nabla \phi(t) dx \\ &+ \int_\Gamma \phi(t) (\gamma(\eta(t)) \cdot \nu) d\sigma, \quad \text{a.e. } t \in (0, T). \end{aligned} \tag{3.59}$$

Next, we multiply (3.41) by  $\phi \in W^{1,2}([0, T]; L^2(\Omega))$  with  $\nabla \phi \in (L^p(Q))^N, \gamma(\phi) \in W^{1,2}([0, T]; L^2(\Gamma)), \phi(T) = 0$  and by (3.59) we get that

$$- \int_Q y \phi_t dx dt - \int_\Omega y_0 \phi(0) dx - \int_\Sigma \phi \eta \cdot \nu d\sigma dt + \int_Q \eta \cdot \nabla \phi dx dt = \int_Q f \phi dx dt. \tag{3.60}$$

After replacing (3.60) in (3.39), we obtain that

$$\int_\Sigma \gamma(y) \phi_t d\sigma dt + \int_\Gamma y_0 \phi(0) d\sigma - \int_\Sigma \phi \eta \cdot \nu d\sigma dt + \int_\Sigma g \phi d\sigma dt = 0,$$

whence we obtain (3.42) in the sense of distributions and also a.e. on  $\Sigma$ , since, as seen earlier,  $\frac{d}{dt} \gamma(y) \in L^2(\Sigma)$ .

**Continuous dependence on data.** Let us consider two solutions  $y$  and  $\bar{y}$  to (1.1)–(1.3), corresponding to the data  $(y_0, f, g)$  and  $(\bar{y}_0, \bar{f}, \bar{g})$ , respectively, in the class of functions satisfying (3.40)–(3.43). We make the difference of the two equations (3.41), corresponding to these data, multiply the difference by  $(y - \bar{y})(t)$  and integrate on  $\Omega$ . We get, a.e.  $t \in [0, T]$

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|y(t) - \bar{y}(t)\|_{L^2(\Omega)}^2 + \int_\Omega (\eta(t) - \bar{\eta}(t)) \cdot (\nabla y(t) - \nabla \bar{y}(t)) dx \\ &+ \frac{1}{2} \frac{d}{dt} \|\gamma(y(t)) - \gamma(\bar{y}(t))\|_{L^2(\Gamma)}^2 \\ &= \int_\Omega (f(t) - \bar{f}(t))(y(t) - \bar{y}(t)) dt, \end{aligned}$$

where  $\eta(t, x) \in \beta(t, x, \nabla y(t, x))$ ,  $\bar{\eta}(t, x) \in \beta(t, x, \nabla \bar{y}(t, x))$  a.e.  $(t, x) \in Q$ . Using the monotonicity of  $\beta$  and integrating with respect to  $t$  we get that

$$\begin{aligned} & \|y(t) - \bar{y}(t)\|_{L^2(\Omega)}^2 + \|\gamma(y(t)) - \gamma(\bar{y}(t))\|_{L^2(\Gamma)}^2 \\ & \leq C \left( \|y_0 - \bar{y}_0\|_{L^2(\Omega)}^2 + \|\gamma(y_0) - \gamma(\bar{y}_0)\|_{L^2(\Gamma)}^2 \right. \\ & \quad \left. + \int_0^T \|f(t) - \bar{f}(t)\|_{L^2(\Omega)}^2 dt + \int_0^T \|g(t) - \bar{g}(t)\|_{L^2(\Gamma)}^2 dt \right), \quad \forall t \in [0, T], \end{aligned}$$

as claimed.  $\square$

#### 4. The weakly coercive case

In this section we assume that hypotheses  $(H_1)$ ,  $(H_3)$ – $(H_5)$  hold. Also, as mentioned earlier, without loss of generality we may assume (1.15).

We note that here no polynomial growth or coercivity on  $j$  are assumed whatever.

A standard example in this case is

$$\beta(t, x, r) = a(t, x) \log(|r|_N + 1) \operatorname{sgn} r + a(t, x) \frac{r}{|r|_N + 1},$$

where  $a \in C^1(\bar{Q})$ ,  $a > 0$ ,  $\operatorname{sgn} r = r/|r|_N$  for  $r \neq 0$ ,  $\operatorname{sgn} 0 = \{r; |r|_N \leq 1\}$ .

On the other hand, monotone functions  $r \rightarrow \beta(t, x, r)$  with exponential growth and symmetric at  $\pm\infty$ , in the sense of (1.13), are accepted by the current hypotheses.

First, we note down for later use the following simple lemma.

**Lemma 4.1.** *Let*

$$u \in (L^1(\Omega))^N, \quad w \in (L^1(\Omega))^N, \quad j(\cdot, \cdot, u) \in L^1(\Omega), \quad j^*(\cdot, \cdot, w) \in L^1(\Omega).$$

Then, under assumption (1.13) we have

$$u \cdot w \in L^1(\Omega). \tag{4.1}$$

**Proof.** First, we recall the relations (see e.g., [4, p. 8])

$$j(t, x, r) + j^*(t, x, \omega) \geq \omega \cdot r, \quad \forall r, \omega \in \mathbb{R}^N, \quad \forall (t, x) \in \bar{Q} \tag{4.2}$$

$$j(t, x, r) + j^*(t, x, \omega) = \omega \cdot r \quad \text{iff } \omega \in \partial j(t, x, r), \quad \forall (t, x) \in \bar{Q}. \tag{4.3}$$

By (4.2),

$$j(t, x, u(x)) + j^*(t, x, w(x)) \geq u(x) \cdot w(x), \quad \forall (t, x) \in \bar{Q},$$

and this yields

$$\int_{\Omega} u(x) \cdot w(x) dx < \infty.$$

We write (4.2) for  $(-u^*)$

$$j(t, x, -u^*(x)) + j^*(t, x, w(x)) \geq -u^*(x) \cdot w(x), \quad \forall (t, x) \in \bar{Q},$$

and use (1.13), obtaining

$$\int_{\Omega} (-u \cdot w) dx \leq \gamma_1 \int_{\Omega} j(t, x, u) dx + \gamma_2 \text{meas}(\Omega) + \int_{\Omega} j^*(t, x, w) dx < \infty.$$

Therefore we get (4.1), as claimed.  $\square$

Let us define the space

$$U_1 = \{z \in L^2(\Omega); z \in W^{1,1}(\Omega), \gamma(z) \in L^2(\Gamma)\}. \tag{4.4}$$

For  $t$  fixed in  $[0, T]$ ,  $h > 0$ , and  $w_1 \in L^2(\Omega)$ ,  $w_2 \in L^2(\Gamma)$ , let us consider the problem

$$\begin{aligned} u - h \nabla \cdot \beta(t, x, \nabla u) &\ni w_1 && \text{in } \Omega, \\ u + h \beta(t, x, \nabla u) \cdot \nu &\ni w_2 && \text{on } \Gamma. \end{aligned} \tag{4.5}$$

As in the previous case, we call a *weak solution* to problem (4.5) a function  $u \in U_1$ , such that  $j(t, \cdot, \nabla u) \in L^1(\Omega)$ , and there exists

$$\eta \in (L^1(\Omega))^N, \quad \eta(x) \in \beta(t, x, \nabla u(x)) \quad \text{a.e. } x \in \Omega, \quad j^*(t, \cdot, \eta) \in L^1(\Omega), \tag{4.6}$$

satisfying

$$\int_{\Omega} (u \psi + h \eta \cdot \nabla \psi) dx + \int_{\Gamma} u \psi d\sigma = b(\psi), \quad \forall \psi \in C^1(\bar{\Omega}), \tag{4.7}$$

with  $b$  given by (3.17) for all  $u \in U_1$ .

Problem (4.5) has a unique solution, namely given by the unique minimizer of the functional  $\varphi : L^2(\Omega) \rightarrow \mathbb{R}$ ,

$$\varphi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} u^2 dx + h \int_{\Omega} j(t, x, \nabla u) dx + \frac{1}{2} \int_{\Gamma} u^2 dx - b(u), & \text{if } u \in U_1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.8}$$

Actually, we have the equivalence between (4.5) and the minimization problem

$$\text{Min}\{\varphi(u); u \in U_1\}. \tag{4.9}$$

**Proposition 4.2.** *Problem (4.5) has a unique solution which is the minimizer of  $\varphi$ .*

**Proof.** Let  $\lambda > 0$  and consider the approximating regularized problem

$$\begin{aligned} u - h \nabla \cdot (\beta_\lambda(t, x, \nabla u) + \lambda \nabla u) &= w_1 \quad \text{in } \Omega, \\ u + h(\beta_\lambda(t, x, \nabla u) + \lambda \nabla u) \cdot \nu &= w_2 \quad \text{on } \Gamma, \end{aligned} \tag{4.10}$$

where  $\beta_\lambda$  is the Yosida approximation of  $\beta$ ,

$$\beta_\lambda(t, x, r) = \frac{1}{\lambda} (1 - (1 + \lambda \beta(t, x, \cdot))^{-1}) r, \quad \forall r \in \mathbb{R}^N. \tag{4.11}$$

Its potential (i.e., the Moreau regularization of  $j$ ) is given by

$$\begin{aligned} j_\lambda(t, x, r) &= \inf_{s \in \mathbb{R}^N} \left\{ \frac{|r - s|_N^2}{2\lambda} + j(t, x, s) \right\} \\ &= \frac{1}{2\lambda} |(1 + \lambda \beta(t, x, \cdot))^{-1} r - r|_N^2 + j(t, x, (1 + \lambda \beta(t, x, \cdot))^{-1} r), \end{aligned} \tag{4.12}$$

and the function  $j_\lambda$  has the following properties

$$\begin{aligned} j_\lambda(t, x, r) &\leq j(t, x, r) \quad \text{for all } r \in \mathbb{R}^N, (t, x) \in \overline{Q}, \lambda > 0, \\ \lim_{\lambda \rightarrow 0} j_\lambda(t, x, r) &= j(t, x, r), \quad \text{for all } r \in \mathbb{R}^N, (t, x) \in \overline{Q}. \end{aligned} \tag{4.13}$$

As in [Proposition 3.2](#) we deduce that the solution to (4.10) is provided by the unique minimizer of the problem

$$\text{Min} \{ \varphi_\lambda(u); u \in L^2(\Omega) \}, \tag{4.14}$$

where  $\varphi_\lambda : L^2(\Omega) \rightarrow \mathbb{R}$ ,

$$\varphi_\lambda(u) = \begin{cases} \frac{1}{2} \int_\Omega u^2 dx + h \int_\Omega j_\lambda(t, x, \nabla u) dx + \frac{1}{2} \int_\Gamma u^2 d\sigma + \lambda \int_\Omega |\nabla u|_N^2 dx - b(u), \\ \quad \text{if } u \in W^{1,2}(\Omega), \\ +\infty, \quad \text{otherwise.} \end{cases} \tag{4.15}$$

Namely, we have, following [Proposition 3.2](#), that in this case the weak solution  $u_\lambda \in W^{1,2}(\Omega)$  and it satisfies

$$\begin{aligned} \int_\Omega u_\lambda \psi dx + h \int_\Omega \beta_\lambda(t, x, \nabla u_\lambda) \cdot \nabla \psi dx + \lambda \int_\Omega \nabla u_\lambda \cdot \nabla \psi dx + \int_\Gamma u_\lambda \psi d\sigma \\ = \int_\Omega w_1 \psi dx + \int_\Gamma w_2 \psi d\sigma, \quad \forall \psi \in W^{1,2}(\Omega). \end{aligned} \tag{4.16}$$

In particular for  $\psi = u_\lambda$ , this yields

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} u_{\lambda}^2 dx + \int_{\Omega} \beta_{\lambda}(t, x, \nabla u_{\lambda}) \cdot \nabla u_{\lambda} dx + \lambda h \int_{\Omega} |\nabla u_{\lambda}|_N^2 dx + \int_{\Gamma} u_{\lambda}^2 d\sigma \\ & = \int_{\Omega} w_1 u_{\lambda} dx + \int_{\Gamma} w_2 u_{\lambda} d\sigma, \end{aligned} \tag{4.17}$$

whence we obtain the estimate

$$\int_{\Omega} u_{\lambda}^2 dx + 2h \int_{\Omega} j_{\lambda}(t, x, \nabla u_{\lambda}) dx + 2\lambda h \int_{\Omega} |\nabla u_{\lambda}|_N^2 dx + \int_{\Gamma} u_{\lambda}^2 d\sigma \leq C. \tag{4.18}$$

(By  $C$  we denote a positive constant independent of  $\lambda$ .)

Replacing the definition (4.12) (second line) of  $j_{\lambda}$  we get

$$\begin{aligned} & \int_{\Omega} u_{\lambda}^2 dx + h \int_{\Omega} \frac{1}{\lambda} |(1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_{\lambda} - \nabla u_{\lambda}|_N^2 dx \\ & + 2h \int_{\Omega} j(t, x, (1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_{\lambda}) dx + 2\lambda \int_{\Omega} |\nabla u_{\lambda}|_N^2 dx + \int_{\Gamma} u_{\lambda}^2 d\sigma \\ & \leq C. \end{aligned} \tag{4.19}$$

Consequently, each term on the left-hand side in (4.19) is bounded independently of  $\lambda$  and in particular

$$\int_{\Omega} j(t, x, (1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_{\lambda}) dx \leq C, \quad \forall \lambda > 0. \tag{4.20}$$

By (4.20), (1.8) and the Dunford–Pettis theorem we can deduce that the sequence  $\{(1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_{\lambda}\}_{\lambda > 0}$  is weakly compact in  $L^1(\Omega)$ . Indeed, denoting  $z_{\lambda} = (1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_{\lambda}$ , we have to show that the integrals  $\int_S |z_{\lambda}|_N dx$ , with  $S \subset \Omega$ , are equi-absolutely continuous, meaning that for every  $\varepsilon > 0$  there exists  $\delta$  such that  $\int_S |z_{\lambda}|_N dx dt < \varepsilon$  whenever  $\text{meas}(S) < \delta$ . Let  $M_{\varepsilon} > \frac{2C}{\varepsilon}$ , where  $C$  is the constant in (4.20), and let  $R_M$  be such that  $\frac{j(t, x, z_{\lambda})}{|z_{\lambda}|_N} \geq M_{\varepsilon}$  for  $|z_{\lambda}|_N > R_M$ , by (1.8). If  $\delta < \frac{\varepsilon}{2R_M}$  then

$$\begin{aligned} \int_S |z_{\lambda}| dx dt & \leq \int_{\{x; |z_{\lambda}(x)|_N \geq R_M\}} |z_{\lambda}| dx + \int_{\{x; |z_{\lambda}(x)|_N < R_M\}} |z_{\lambda}| dx \\ & \leq M_{\varepsilon}^{-1} \int_{\Omega} j(t, x, z_{\lambda}(x)) dx + R_M \delta < \varepsilon. \end{aligned}$$

Then, we select a subsequence (again denoted  $\lambda$ ) such that as  $\lambda \rightarrow 0$ , we have

$$u_\lambda \rightharpoonup u^* \quad \text{weakly in } L^2(\Omega), \tag{4.21}$$

$$\gamma(u_\lambda) \rightharpoonup \gamma(u^*) \quad \text{weakly in } L^2(\Gamma), \tag{4.22}$$

$$(1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_\lambda \rightharpoonup \zeta_1 \quad \text{weakly in } (L^1(\Omega))^N, \tag{4.23}$$

$$(1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_\lambda - \nabla u_\lambda \rightarrow 0 \quad \text{strongly in } (L^2(\Omega))^N, \tag{4.24}$$

$$\sqrt{\lambda} \nabla u_\lambda \rightharpoonup \zeta_2 \quad \text{weakly in } (L^2(\Omega))^N. \tag{4.25}$$

By (4.23), (4.24) and (4.21) we get that

$$\nabla u_\lambda \rightharpoonup \zeta_1 = \nabla u^* \quad \text{weakly in } (L^1(\Omega))^N. \tag{4.26}$$

By (4.2) we can write

$$\int_\Omega (j_\lambda(t, x, \nabla u_\lambda) dx + j_\lambda^*(t, x, \beta_\lambda(t, x, \nabla u_\lambda)) - \beta_\lambda(t, x, \nabla u_\lambda) \cdot \nabla u_\lambda) dx = 0,$$

whence, by (4.12) and (4.17), we get that

$$\begin{aligned} & \int_\Omega (j(t, x, (1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_\lambda) + j_\lambda^*(t, x, \beta_\lambda(t, x, \nabla u_\lambda))) dx \\ & \leq \int_\Omega \beta_\lambda(t, x, \nabla u_\lambda) \cdot \nabla u_\lambda dx \\ & \leq \frac{1}{h} \left\{ \int_\Omega w_1 u_\lambda dx + \int_\Gamma w_2 u_\lambda d\sigma - \int_\Omega u_\lambda^2 dx - \int_\Gamma u_\lambda^2 d\sigma - \lambda \int_\Omega |\nabla u_\lambda|_N^2 dx \right\} \leq C. \end{aligned} \tag{4.27}$$

Passing to the limit in (4.20), recalling that  $(1 + \lambda\beta(t, x, \cdot))^{-1} \nabla u_\lambda \rightharpoonup \nabla u^*$  weakly, we obtain on the basis of the weak lower semicontinuity of the convex integrand that

$$j(t, \cdot, \nabla u^*) \in L^1(\Omega). \tag{4.28}$$

Also, by (4.27) it follows that

$$\int_\Omega j_\lambda^*(t, x, \beta_\lambda(t, x, \nabla u_\lambda)) dx \leq C, \quad \forall \lambda > 0. \tag{4.29}$$

This yields (by the definition (4.12) for  $j_\lambda^*$ )

$$\begin{aligned} & \int_\Omega \frac{1}{2\lambda} |(1 + \lambda\beta^{-1}(t, x, \cdot))^{-1} \beta_\lambda(t, x, \nabla u_\lambda) - \beta_\lambda(t, x, \nabla u_\lambda)|_N^2 \\ & \quad + \int_\Omega j^*(t, x, (1 + \lambda\beta^{-1}(t, x, \cdot))^{-1} \beta_\lambda(t, x, \nabla u_\lambda)) dx \end{aligned}$$

$$\leq \int_{\Omega} j_{\lambda}^*(t, x, \beta_{\lambda}(t, x, \nabla u_{\lambda})) dx \leq C.$$

Arguing as above, on the basis of (1.9) and the Dunford–Pettis theorem we deduce that the sequence  $\{(1 + \lambda\beta^{-1}(t, x, \cdot))^{-1}\beta_{\lambda}(t, x, \nabla u_{\lambda})\}_{\lambda>0}$  is weakly compact in  $(L^1(\Omega))^N$ , and so, on a subsequence, as  $\lambda \rightarrow 0$ , we get

$$\begin{aligned} (1 + \lambda\beta^{-1}(t, x, \cdot))^{-1}\beta_{\lambda}(t, x, \nabla u_{\lambda}) &\rightarrow \eta \quad \text{weakly in } (L^1(\Omega))^N, \\ (1 + \lambda\beta^{-1}(t, x, \cdot))^{-1}\beta_{\lambda}(t, x, \nabla u_{\lambda}) - \beta_{\lambda}(t, x, \nabla u_{\lambda}) &\rightarrow 0 \quad \text{strongly in } (L^2(\Omega))^N, \end{aligned} \tag{4.30}$$

which implies

$$\beta_{\lambda}(t, x, \nabla u_{\lambda}) \rightarrow \eta \quad \text{weakly in } (L^1(\Omega))^N. \tag{4.31}$$

Then, by (4.29) and the weak lower semicontinuity of the convex integrand we infer that

$$j^*(t, \cdot, \eta) \in L^1(\Omega). \tag{4.32}$$

Now, we pass to the limit in (4.27), taking into account (4.19) and (4.30) and we get

$$\begin{aligned} \limsup_{\lambda \rightarrow 0} \int_{\Omega} \beta_{\lambda}(t, x, \nabla u_{\lambda}) \cdot \nabla u_{\lambda} dx \\ \leq \frac{1}{h} \left\{ \int_{\Omega} w_1 u^* dx + \int_{\Gamma} w_2 u^* d\sigma - \int_{\Omega} (u^*)^2 dx - \int_{\Gamma} (u^*)^2 d\sigma \right\}. \end{aligned} \tag{4.33}$$

Then, letting  $\lambda \rightarrow 0$  in (4.16) and recalling (4.1) we obtain

$$\int_{\Omega} u^* \psi dx + h \int_{\Omega} \eta \cdot \nabla \psi dx + \int_{\Gamma} u^* \psi d\sigma = \int_{\Omega} w_1 \psi dx + \int_{\Gamma} w_2 \psi d\sigma, \quad \forall \psi \in C^1(\overline{\Omega}).$$

This is extended by density for all  $\psi \in L^2(\Omega) \cap W^{1,1}(\Omega)$ ,  $\gamma(\psi) \in L^2(\Gamma)$ , and in particular for  $\psi = u^*$ . We obtain

$$\int_{\Omega} \eta \cdot \nabla u^* dx = \frac{1}{h} \left\{ \int_{\Omega} w_1 u^* dx + \int_{\Gamma} w_2 u^* d\sigma - \int_{\Omega} (u^*)^2 dx - \int_{\Gamma} (u^*)^2 d\sigma \right\}. \tag{4.34}$$

By (4.33) and (4.34) we finally obtain that

$$\limsup_{\lambda \rightarrow 0} \int_{\Omega} \beta_{\lambda}(t, x, \nabla u_{\lambda}) \cdot \nabla u_{\lambda} dx \leq \int_{\Omega} \eta \cdot \nabla u^* dx. \tag{4.35}$$

Since  $\nabla u_{\lambda} \rightarrow \nabla u^*$  weakly in  $(L^1(\Omega))^N$ ,  $\beta_{\lambda}(t, x, \nabla u_{\lambda}) \rightarrow \eta$  weakly in  $(L^1(\Omega))^N$ , we deduce that

$$\eta(x) \in \beta(t, x, \nabla u^*(x)) \quad \text{a.e. } x \in \overline{\Omega}.$$

As a matter of fact, to get the latter we note that by (4.35) we have

$$\int_{\Omega} (j_{\lambda}(t, x, \nabla u_{\lambda}) - j_{\lambda}(t, x, \theta)) dx \leq \int_{\Omega} \eta \cdot (\nabla u^* - \theta) dx, \quad \forall \theta \in (L^1(\Omega))^N,$$

and letting  $\lambda \rightarrow 0$  we get by (4.12), (4.26) and (4.13)

$$\int_{\Omega} (j(t, x, \nabla u^*) - j(t, x, \theta)) dx \leq \int_{\Omega} \eta \cdot (\nabla u^* - \theta) dx, \quad \forall \theta \in (L^1(\Omega))^N.$$

Since  $\theta$  is arbitrary we get that  $\eta(x) \in \partial j(t, x, \nabla u^*(x))$ , as desired.

Passing to the limit in (4.16) we get

$$\int_{\Omega} (u^* \psi + h \eta \cdot \nabla \psi) dx + \int_{\Gamma} u^* \psi d\sigma = b(\psi), \quad \forall \psi \in W^{1,2}(\Omega),$$

where  $b$  is defined by (3.17) for all  $u \in U_1$ . By density this extends to all  $\psi \in L^2(\Omega) \cap W^{1,1}(\Omega)$ , with  $\gamma(\psi) \in L^2(\Gamma)$ . Hence  $u^*$  is the weak solution to (4.5). Uniqueness of  $u^*$ , as weak solution, is immediate.

Moreover, it also follows that  $\nabla \cdot \eta \in L^2(\Omega)$ .  $\square$

**Definition 4.3.** Let

$$y_0 \in U_1, \quad f \in L^2(Q), \quad g \in L^2(\Sigma). \tag{4.36}$$

We call a *weak solution* to problem (1.1)–(1.3) a function  $y \in L^2(Q)$ , such that

$$y \in L^1(0, T; W^{1,1}(\Omega)), \quad \gamma(y) \in L^2(\Sigma), \quad j(\cdot, \cdot, \nabla y) \in L^1(Q), \tag{4.37}$$

and there exists  $\eta \in (L^1(Q))^N$ ,  $\eta(t, x) \in \beta(t, x, \nabla y(t, x))$ , a.e.  $(t, x) \in Q$ ,  $j^*(\cdot, \cdot, \eta) \in L^1(Q)$ , such that

$$\begin{aligned} & - \int_Q y \phi_t dx dt + \int_Q \eta \cdot \nabla \phi dx dt - \int_{\Sigma} y \phi_t d\sigma dt \\ & = \int_Q f \phi dx dt + \int_{\Omega} y_0 \phi(0) dx + \int_{\Gamma} y_0 \phi(0) d\sigma + \int_{\Sigma} g \phi d\sigma dt, \end{aligned} \tag{4.38}$$

for all  $\phi \in W^{1,2}([0, T]; L^2(\Omega)) \cap L^1(0, T; W^{1,1}(\Omega))$ ,  $\gamma(\phi) \in W^{1,2}([0, T]; L^2(\Gamma))$ , with  $\phi(T) = 0$ .

By Lemma 4.1 it is clear that the second term on the left-hand side of (4.38) makes sense.

**Theorem 4.4.** *Let us assume (4.36) and  $j(0, \cdot, \nabla y_0) \in L^1(\Omega)$ . Then, under hypotheses (H<sub>1</sub>), (H<sub>3</sub>)–(H<sub>5</sub>), problem (1.1)–(1.3) has at least one weak solution. Moreover,  $y$  is a strong solution to (1.1)–(1.3), that is, it satisfies (3.41)–(3.43). Finally,  $y$  is given by*

$$y = \lim_{h \rightarrow 0} y^h \quad \text{strongly in } L^1(Q), \tag{4.39}$$

where  $y^h$  is defined by (3.32). The solution is unique in the class of functions satisfying (4.37), (3.41)–(3.43).

**Proof.** Let us consider the time discretized system (3.5)–(3.7) whose weak solution is defined as in Definition 3.1, by replacing  $U$  by  $U_1$ . We claim that system (3.5)–(3.6) has a unique weak solution which satisfies

$$\begin{aligned} & \|y_m^h\|_{L^2(\Omega)} + \|\gamma(y_m^h)\|_{L^2(\Gamma)} \\ & + h \sum_{i=0}^{m-1} \int_{\Omega} j(t_{i+1}, x, \nabla y_{i+1}^h) dx + h \sum_{i=0}^{m-1} \|\nabla y_{i+1}^h\|_{L^1(\Omega)} \leq C, \quad m = 1, \dots, n, \end{aligned} \tag{4.40}$$

$$h \sum_{i=0}^{m-1} \left\| \frac{y_{i+1}^h - y_i^h}{h} \right\|_{L^2(\Omega)}^2 + h \sum_{i=0}^{m-1} \left\| \frac{\gamma(y_{i+1}^h) - \gamma(y_i^h)}{h} \right\|_{L^2(\Gamma)}^2 \leq C, \quad m = 1, \dots, n, \tag{4.41}$$

where  $C$  is a positive constant, independent of  $h$ . The proof follows as in Proposition 3.2 (see (3.26)), using the hypothesis corresponding to the weakly coercive case.

Next, we define  $y^h$  by (3.32) and on the basis of (4.40) and (4.41) we write

$$\|y^h(t)\|_{L^2(\Omega)} + \|\gamma(y^h(t))\|_{L^2(\Gamma)} + \int_Q j(t, x, \nabla y^h(t)) dx dt \leq C, \quad \text{for } t \in [0, T], \tag{4.42}$$

$$\int_0^T \left\| \frac{y^h(t+h) - y^h(t)}{h} \right\|_{L^2(\Omega)}^2 dt + \int_0^T \left\| \frac{\gamma(y^h(t+h)) - \gamma(y^h(t))}{h} \right\|_{L^2(\Gamma)}^2 dt \leq C. \tag{4.43}$$

By these estimates and the Dunford–Pettis compactness theorem in  $L^1(Q)$ , we can select a subsequence such that, as  $h \rightarrow 0$ ,

$$\begin{aligned} y^h &\rightharpoonup y \quad \text{weak-star in } L^\infty(0, T; L^2(\Omega)), \\ y^h &\rightharpoonup y \quad \text{weak-star in } L^\infty(0, T; L^2(\Gamma)), \\ \nabla y^h &\rightharpoonup \nabla y \quad \text{weakly in } (L^1(Q))^N, \\ \frac{y^h(t+h) - y^h(t)}{h} &\rightharpoonup \frac{dy}{dt} \quad \text{weakly in } L^2(Q), \\ \frac{\gamma(y^h(t+h)) - \gamma(y^h(t))}{h} &\rightharpoonup \frac{dy}{dt} \quad \text{weakly in } L^2(\Sigma). \end{aligned}$$

We denote  $X = W^{1,r}(\Omega)$  with  $r > N$ . Then  $W^{1,r}(\Omega)$  is compact in  $L^\infty(\Omega)$  and  $L^1(\Omega)$  is compact in  $X' = (W^{1,r}(\Omega))'$ . We have that  $y^h \in BV([0, T]; L^2(\Omega))$  which implies that  $y^h \in BV([0, T]; X')$ . By the Helly theorem it follows that

$$y^h(t) \rightarrow y(t) \quad \text{strongly in } X', \text{ uniformly with } t \in [0, T].$$

Using again Lemma 5.1 in [6] and taking into account that  $W^{1,1}(\Omega) \subset L^1(\Omega) \subset X'$ , we deduce that

$$y^h \rightarrow y \quad \text{strongly in } L^1(Q), \text{ as } h \rightarrow 0. \tag{4.44}$$

The remainder of the proof follows as in Theorem 3.4 and so it will be omitted.  $\square$

**Remark 4.5.** The singular case  $\beta(t, x, r) \equiv \rho \operatorname{sgn} r$  (which is relevant in the study of diffusion systems with singular energy) is ruled out by the present approach, but, as seen later, the corresponding problem (1.1)–(1.3) is well posed, however, in the space of functions with bounded variation on  $\Omega$ .

### 5. The semigroup approach

Everywhere in the following we assume that either hypotheses  $(H_1)$ ,  $(H_2)$ ,  $(H_5)$ , or  $(H_1)$ ,  $(H_3)$ – $(H_5)$  are satisfied. In other words, we are in one of the cases considered before: strongly coercive or weakly coercive. Moreover, we assume that  $\beta$  is independent of  $t$ ,  $\beta \equiv \beta(x, r)$ . It shall turn out that in this time-invariant case Theorems 3.4 and 4.4 can be derived by the nonlinear contraction semigroup theory which leads to sharper regularity results for the solution  $y$ .

Namely, on the space  $X = L^2(\Omega) \times L^2(\Gamma)$ , endowed with the standard Hilbertian structure, we consider the operator  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ , defined by

$$\mathcal{A} \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} -\nabla \cdot \beta(x, \nabla u) \\ \beta(x, \nabla u) \cdot \nu \end{pmatrix}, \quad \forall \begin{pmatrix} u \\ z \end{pmatrix} \in D(\mathcal{A}), \tag{5.1}$$

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ z \end{pmatrix} \in X; u \in \tilde{U}, z = \gamma(u), \exists \eta(x) \in \beta(x, \nabla u(x)) \text{ a.e. } x \in \Omega, \right. \\ \left. \eta \in (L^1(\Omega))^N, \nabla \cdot \eta \in L^2(\Omega), \eta \cdot \nu \in L^2(\Gamma) \right\}. \tag{5.2}$$

Here,  $\tilde{U} = U$  in the strongly coercive case, that is under hypothesis  $(H_2)$  and  $\tilde{U} = U_1$  in the weakly coercive case under the hypotheses  $(H_3)$ – $(H_4)$ .

In (5.1), by  $\beta(x, \nabla u)$  we mean, as usually, any measurable section  $\eta$  of  $\beta(x, \nabla u)$  satisfying (5.2). Then, the system

$$y_t - \nabla \cdot \beta(x, \nabla y) \ni f \quad \text{in } Q, \tag{5.3}$$

$$\beta(x, \nabla y) \cdot \nu + y_t \ni g \quad \text{on } \Sigma, \tag{5.4}$$

$$y(0) = y_0 \quad \text{in } \Omega, \tag{5.5}$$

can be written as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \mathcal{A} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} &\ni \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad \text{a.e. } t \in (0, T), \\ \begin{pmatrix} y \\ z \end{pmatrix} (0) &= \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}. \end{aligned} \tag{5.6}$$

**Lemma 5.1.** *The operator  $\mathcal{A}$  is maximal monotone in  $X$ .*

**Proof.** It is easily seen that  $\mathcal{A}$  is monotone, that is,

$$\left( \mathcal{A} \begin{pmatrix} u \\ z \end{pmatrix} - \mathcal{A} \begin{pmatrix} \bar{u} \\ \bar{z} \end{pmatrix}, \begin{pmatrix} u - \bar{u} \\ z - \bar{z} \end{pmatrix} \right)_X \geq 0, \quad \forall \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{z} \end{pmatrix} \in D(\mathcal{A}).$$

In fact, this follows by the Gauss–Ostrogradski formula

$$-\int_{\Omega} v \nabla \cdot \eta dx = -\int_{\Gamma} \gamma(v) \eta \cdot \nu d\sigma + \int_{\Omega} \eta \cdot \nabla v dx, \quad \eta(x) \in \beta(x, \nabla u(x)), \quad \text{a.e. } x \in \Omega,$$

for all  $u, v \in U$ , in the strongly coercive case, or  $u, v \in U_1$  in the weakly coercive case.

On the other hand, the range  $R(I + \mathcal{A})$  is all of  $X$ , for all  $\lambda > 0$ . Indeed, equation

$$(I + \mathcal{A}) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{for all } \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in X$$

reduces to Eq. (3.16) (or (4.5)), for which existence has been previously proved.  $\square$

Then, by the standard existence theorem for the Cauchy problem associated with non-linear maximal monotone operators (see, e.g., [4, p. 151]), for  $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in D(\mathcal{A})$  and  $f \in W^{1,1}([0, T]; L^2(\Omega))$ ,  $g \in W^{1,1}([0, T]; L^2(\Gamma))$ , there is a unique function  $\begin{pmatrix} y \\ z \end{pmatrix} \in W^{1,\infty}([0, T]; L^2(\Omega)) \times W^{1,\infty}([0, T]; L^2(\Gamma))$  which satisfies (5.6) a.e., and also in the following stronger sense,

$$\frac{d^+}{dt} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \left( \mathcal{A} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} - \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \right)^\circ = 0, \quad \forall t \in [0, T],$$

where, for each closed convex set  $C$ ,  $C^\circ$  stands for the minimal section of  $C$ . Moreover, if  $f = 0$ ,  $g = 0$ , we have the exponential formula

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \lim_{n \rightarrow \infty} \left( I + \frac{t}{n} \mathcal{A} \right)^{-n} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \quad \text{in } L^2(\Omega) \times L^2(\Gamma),$$

uniformly in  $t$  on compact intervals.

We are led therefore to the following sharper versions of Theorems 3.4 and 4.4.

**Theorem 5.2.** *Let  $y_0 \in U$  (respectively  $U_1$  in the weakly coercive case) be such that  $\nabla \cdot \beta(\cdot, \nabla y_0) \in L^2(\Omega)$ ,  $\beta(\cdot, \nabla y_0) \cdot \nu \in L^2(\Gamma)$ , and let  $f \in W^{1,1}([0, T]; L^2(\Omega))$ ,  $g \in W^{1,1}([0, T]; L^2(\Gamma))$ . Then, there is a unique  $y \in W^{1,\infty}([0, T]; L^2(\Omega))$  with  $\gamma(y) \in W^{1,\infty}([0, T]; L^2(\Gamma))$  which satisfies*

$$\begin{aligned} \frac{d^+}{dt}y(t, x) - (\nabla \cdot \beta(x, \nabla y(t, x)) - f(t, x))^\circ &= 0 \quad \text{in } [0, T] \times \Omega, \\ \frac{d^+}{dt}y(t, x) + (\beta(x, \nabla y(t, x)) \cdot \nu - g(t, x))^\circ &= 0 \quad \text{on } [0, T] \times \Gamma, \\ y(0, x) &= y_0 \quad \text{in } \Omega. \end{aligned}$$

Moreover, (3.44) holds.

Condition  $\nabla \cdot \beta(\cdot, \nabla y_0) \in L^2(\Omega)$  means, of course, that there is  $\eta_0$  measurable, such that  $\eta_0(x) \in \beta(x, \nabla y_0(x))$ , a.e.  $x \in \Omega$ ,  $\nabla \cdot \eta_0 \in L^2(\Omega)$ .

We also note that the operator  $\mathcal{A}$  is the subdifferential of the function  $\Phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ ,

$$\Phi \left( \begin{matrix} u \\ z \end{matrix} \right) = \begin{cases} \int_{\Omega} j(x, \nabla u(x)) dx, & z = \gamma(u) \in L^2(\Gamma) \quad \text{if } u \in W^{1,1}(\Omega), j(\cdot, \nabla u) \in L^1(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

This is the energy functional associated with system (5.3)–(5.5).

Indeed, for  $\begin{pmatrix} u \\ z \end{pmatrix} \in D(\mathcal{A})$ ,  $\begin{pmatrix} \bar{u} \\ \bar{z} \end{pmatrix} \in D(\Phi)$  we have

$$\begin{aligned} \Phi \left( \begin{matrix} u \\ z \end{matrix} \right) - \Phi \left( \begin{matrix} \bar{u} \\ \bar{z} \end{matrix} \right) &= \int_{\Omega} (j(x, \nabla u) - j(x, \nabla \bar{u})) dx \\ &\leq \int_{\Omega} \eta \cdot (\nabla u - \nabla \bar{u}) dx = - \int_{\Omega} (u - \bar{u}) \nabla \cdot \eta dx + \int_{\Gamma} (\eta \cdot \nu)(z - \bar{z}) d\sigma, \end{aligned}$$

for  $\eta \in (L^1(\Omega))^N$ ,  $\eta(x) \in \beta(x, \nabla u(x))$  a.e.  $x \in \Omega$ .

Here we have used the Gauss–Ostrogradski formula

$$\int_{\Omega} v \cdot \nabla u dx = - \int_{\Omega} u \nabla \cdot v dx + \int_{\Gamma} \gamma(u) v \cdot \nu d\sigma,$$

which is valid for all  $u \in W^{1,1}(\Omega) \cap L^2(\Omega)$  and  $v \in (L^1(\Omega))^N$  such that  $\gamma(u)(v \cdot \nu) \in L^1(\Gamma)$  and  $u \nabla \cdot v \in L^1(\Omega)$ . In virtue of Lemma 4.1,  $v = \eta$  satisfies this condition.

This implies that  $\mathcal{A} \subset \partial\Phi$  and since  $\mathcal{A}$  is maximal monotone we infer that  $A = \partial\Phi$ , as claimed.

Then, by Theorem 4.11 in [4], p. 158, it follows that for all  $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in D(\Phi)$ ,  $f \in L^2(Q)$ ,  $g \in L^2(\Sigma)$ , problem (5.6) has a unique solution  $\begin{pmatrix} y \\ z \end{pmatrix} \in W^{1,2}([0, T]; X)$ . Hence, under the above assumptions there is a unique solution  $y \in W^{1,2}([0, T]; L^2(\Omega))$  with  $\gamma(y) \in W^{1,2}([0, T]; L^2(\Gamma))$  to (5.3)–(5.5).

By Theorem 4.13 in [4], p. 164, we have also in this case the following asymptotic result for the solution  $y$  to (5.3)–(5.5).

**Theorem 5.3.** *Let  $y_0 \in W^{1,p}(\Omega)$ ,  $2 \leq p < \infty$  and  $f(t) \equiv f \in L^2(\Omega)$ ,  $g(t) \equiv g \in L^2(\Gamma)$ . Assume that the set of equilibrium states for (1.1),*

$$\mathcal{K} = \{y \in U; \nabla \cdot \beta(\cdot, \nabla y) \in L^2(\Omega), \beta(\cdot, \nabla y) \cdot \nu \in L^2(\Gamma), \nabla \cdot \beta(x, \nabla y) = f \text{ in } \Omega, \beta(x, \nabla y) \cdot \nu = g \text{ on } \Gamma\}$$

is non-empty. Then, for  $t \rightarrow \infty$ , we have

$$\begin{aligned} y(t) &\rightarrow y_\infty \text{ weakly in } L^2(\Omega), \\ \gamma(y(t)) &\rightarrow \gamma(y_\infty) \text{ weakly in } L^2(\Gamma), \end{aligned} \tag{5.7}$$

where  $y_\infty \in \mathcal{K}$ .

Taking into account that, as easily seen by (5.6) we have

$$\Phi \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \leq \Phi \begin{pmatrix} y_0 \\ \gamma(y_0) \end{pmatrix}, \quad \forall t \geq 0,$$

it follows by (5.7) and by the compactness of  $W^{1,p}(\Omega)$  in  $L^q(\Omega)$ , that for  $t \rightarrow \infty$ ,

$$y(t) \rightarrow y_\infty \text{ strongly in } L^q(\Omega),$$

where  $1 \leq q < \frac{Np}{N-p}$  if  $N > p$ ,  $q = p$  if  $N \leq p$  in the strongly coercive case, and  $q = 1$  in the weakly coercive case.

In other words, the solution  $y$  is strongly convergent to an equilibrium solution  $y_\infty$  to system (5.3)–(5.5).

One of the main advantages of the semigroup approach is its flexibility to incorporate other nonlinear terms in the basic equations (5.3)–(5.4). We shall consider two such extensions. The first is the problem studied in [19], already mentioned in Introduction,

$$\frac{\partial y}{\partial t} - \nabla \cdot \beta(x, \nabla y) + a_1(x, y) \ni f(t, x) \quad \text{in } Q, \tag{5.8}$$

$$\frac{\partial y}{\partial t} - \nabla \cdot (|\nabla_\Gamma y|_{N-1}^{p-2} \nabla_\Gamma y) + \beta(x, \nabla y) \cdot \nu + a_2(x, y) \ni g(t, x) \quad \text{on } \Sigma, \tag{5.9}$$

$$y(0, x) = y_0 \quad \text{in } \Omega, \tag{5.10}$$

where  $\beta$  satisfies assumption (1.7),  $a_i : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $i = 1, 2$  are continuous and  $p \geq 2$ . Here,  $\nabla_\Gamma y$  is the Riemannian gradient of  $y$ , that is  $\nabla_\Gamma y = (\partial_{\tau_1} y, \dots, \partial_{\tau_{N-1}} y)$ , where  $\partial_{\tau_i} y$  is the directional derivative of  $y$  along the tangential directions  $\tau_i$  at each point on  $\Gamma$  (see [19]) and

$$-\int_\Gamma \nu \nabla \cdot (|\nabla_\Gamma u|^{p-2} \nabla_\Gamma u) d\sigma = \int_\Gamma |\nabla_\Gamma u|^{p-2} \nabla_\Gamma y \cdot \nabla_\Gamma \nu d\sigma.$$

Problem (5.8)–(5.10) can be written as (5.6), where

$$\mathcal{A} \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} -\nabla \cdot \beta(x, \nabla u) + a_1(x, u) \\ -\nabla \cdot (|\nabla_\Gamma u|_{N-1}^{p-2} \nabla_\Gamma u) + \beta(x, \nabla u) \cdot \nu + a_2(x, u) \end{pmatrix}, \tag{5.11}$$

for all  $\begin{pmatrix} u \\ z \end{pmatrix} \in D(\mathcal{A})$ , where

$$D(\mathcal{A}) = \left\{ \begin{array}{l} \begin{pmatrix} u \\ z \end{pmatrix} \in X; u \in U, z = \gamma(u), \exists \eta(x) \in \beta(x, \nabla u(x)) \text{ a.e. } x \in \Omega, \text{ such that} \\ \eta \in (L^1(\Omega))^N, \nabla \cdot \eta + a_1(\cdot, u) \in L^2(\Omega), |\nabla_{\Gamma} u|_{N-1} \in L^p(\Gamma), \\ -\nabla \cdot (|\nabla_{\Gamma} u|_{N-1}^{p-2} \nabla_{\Gamma} u) + \eta \cdot \nu + a_2(\cdot, u) \in L^2(\Gamma) \end{array} \right\}. \tag{5.12}$$

If  $y \rightarrow a_i(x, y), i = 1, 2$ , are monotone (or more generally quasi-monotone, that is,  $\lambda y + a_i(x, y)$  are monotone for some  $\lambda > 0$ ) and  $|a_i(x, r)| \leq C|r|^{q-1}, \forall r \in \mathbb{R}$ , where  $q$  is as before, then, arguing as above, it follows that the operator  $\mathcal{A}$  is maximal monotone in  $X$  (or quasi- $m$ -accretive if  $a_i$  are quasi-monotone), and in fact it is a subdifferential operator.

Then, we get for problem (5.8)–(5.10) the following existence result: *let  $y_0 \in U$ , such that  $\nabla_{\Gamma} y_0 \in (L^p(\Gamma))^{N-1}$  and let  $f \in L^2(0, T; L^2(\Omega)), g \in L^2(0, T; L^2(\Gamma))$ . Then, there is a unique solution  $y \in W^{1,2}([0, T]; L^2(\Omega))$  to (5.8)–(5.10), such that  $\gamma(y) \in W^{1,2}([0, T]; L^2(\Gamma))$  and  $\nabla_{\Gamma} y \in (L^p(\Sigma))^{N-1}$ .*

**The obstacle problem.** Consider the following free boundary problem associated with the Wentzell boundary condition, namely,

$$\begin{aligned} y_t - \nabla \cdot \beta(x, \nabla y) &\geq f, & y &\geq 0, \text{ in } Q, \\ y_t - \nabla \cdot \beta(x, \nabla y) &\ni f, & \text{ in } \{(t, x) \in Q; y(t, x) > 0\}, \\ y_t + \beta(x, \nabla y) \cdot \nu &\ni g, & \text{ on } \Sigma, \\ y(0, x) &= y_0, & \text{ in } \Omega. \end{aligned} \tag{5.13}$$

This problem can be written as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \mathcal{A} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \mathcal{B} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} &\ni \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, & \text{ a.e. } t \in (0, T), \\ \begin{pmatrix} y \\ z \end{pmatrix} (0) &= \begin{pmatrix} y_0 \\ \gamma(y_0) \end{pmatrix}, \end{aligned} \tag{5.14}$$

where

$$\mathcal{B} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} a(y) \\ 0 \end{pmatrix}, \quad \forall \begin{pmatrix} y \\ z \end{pmatrix} \in X = L^2(\Omega) \times L^2(\Gamma)$$

and  $a : \mathbb{R} \rightarrow \mathbb{R}$  is the multivalued function  $a(s) = 0$  for  $s > 0, a(0) = (-\infty, 0], a(s) = \emptyset$  for  $s < 0$ . Taking into account that  $(\mathcal{A} \begin{pmatrix} y \\ z \end{pmatrix}, \mathcal{B} \begin{pmatrix} y \\ z \end{pmatrix})_X \geq 0$  for all  $\begin{pmatrix} y \\ z \end{pmatrix} \in D(\mathcal{A}) \cap D(\mathcal{B})$ , it follows that  $\mathcal{A} + \mathcal{B}$  is maximal monotone and therefore  $\mathcal{A} + \mathcal{B} = \partial \Phi_1$ , where

$$\Phi_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{cases} \Phi \begin{pmatrix} y \\ z \end{pmatrix}, & \text{for } y \in L^2(\Omega), y \geq 0 \text{ a.e. in } \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, applying the general existence theory, we infer that for  $y_0 \in W^{1,1}(\Omega)$ , such that  $y_0 \geq 0$  a.e. in  $\Omega$ , and  $j(\cdot, \nabla y_0) \in L^1(\Omega)$ , problem (5.14) has a unique solution  $y \in W^{1,2}([0, T]; L^2(\Omega))$ .

More generally, one might take instead of  $a$  a general maximal monotone graph in  $\mathbb{R} \times \mathbb{R}$ . This case is studied in [12].

**The total variation Wentzell flow.** Let us consider now the singular case  $j(x, r) \equiv \rho|r|, r \in \mathbb{R}^N$ , or equivalently

$$\beta(x, r) = \rho \operatorname{sgn} r = \begin{cases} \rho \frac{r}{|r|^N}, & \text{if } r \neq 0, \\ \{r; |r|_N \leq \rho\}, & \text{if } r = 0. \end{cases}$$

Then, problem (5.3)–(5.5) reduces to

$$y_t - \rho \nabla \cdot \operatorname{sgn}(\nabla y) \ni f, \quad \text{in } Q, \tag{5.15}$$

$$\rho \operatorname{sgn}(\nabla y) \cdot \nu + y_t \ni g, \quad \text{on } \Sigma, \tag{5.16}$$

$$y(0) = y_0, \quad \text{in } \Omega. \tag{5.17}$$

As mentioned earlier, this problem is not covered by the previous weakly coercive case and, as a matter of fact, it cannot be treated in the  $W^{1,1}(\Omega)$  space, but in the space  $BV(\Omega)$  of functions with bounded variation on  $\Omega$ , that is

$$BV(\Omega) = \left\{ u \in L^1(\Omega); \|Du\| = \sup_{\|\varphi\|_{L^\infty(\Omega; \mathbb{R}^N)} \leq 1} \left\{ \int_{\Omega} u \nabla \cdot \varphi dx; \varphi \in C_0^\infty(\Omega; \mathbb{R}^N) \right\} < \infty \right\}.$$

We recall (see e.g., [2]) that for each  $u \in BV(\Omega)$  there is the trace  $\gamma(u) \in L^1(\Gamma; d\mathcal{H}^{N-1})$ , where  $d\mathcal{H}^{N-1}$  is the Hausdorff measure on  $\Gamma$ , defined by

$$\int_{\Omega} u \nabla \cdot \psi dx = - \int_{\Omega} \psi d(\nabla u) + \int_{\Gamma} u \psi \cdot \nu d\mathcal{H}^{N-1}, \quad \forall \psi \in C^1(\mathbb{R}^N, \mathbb{R}^N).$$

Here,  $\nabla u$  (the gradient of  $u$  in the sense of distributions) is a Radon measure on  $\Omega$ .

Let us define the energy functional  $\Phi : L^2(\Omega) \times L^2(\Gamma) \rightarrow (-\infty, +\infty]$ ,

$$\Phi \left( \begin{matrix} u \\ z \end{matrix} \right) = \begin{cases} \rho \|Du\|, & \text{if } u \in BV(\Omega) \cap L^2(\Omega), z = \gamma(u) \in L^2(\Gamma), \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easily seen that  $\Phi$  is convex and l.s.c. on  $X = L^2(\Omega) \times L^2(\Gamma)$ . Let  $\partial\Phi : X \rightarrow X$  be its subdifferential. Then, for each  $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in D(\Phi)$  and  $f \in L^2(0, T; L^2(\Omega)), g \in L^2(0, T; L^2(\Gamma))$  the problem

$$\frac{d}{dt} \begin{pmatrix} y \\ z \end{pmatrix} (t) + \begin{pmatrix} \xi \\ \eta \end{pmatrix} (t) = \begin{pmatrix} f \\ g \end{pmatrix} (t), \quad \text{a.e. } t \in (0, T), \tag{5.18}$$

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \in \partial\Phi \left( \begin{matrix} y(t) \\ z(t) \end{matrix} \right), \quad \text{a.e. } t \in (0, T), \tag{5.19}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} (0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \quad \text{in } \Omega, \tag{5.20}$$

has a unique solution  $\begin{pmatrix} y \\ z \end{pmatrix} \in W^{1,2}([0, T]; X), \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in L^2(0, T; X)$ .

Taking into account (see e.g. [1]) that for all  $u, v \in BV(\Omega) \cap L^2(\Omega)$  and  $\zeta \in (L^\infty(\Omega))^N$ ,

$$\|Du\| \leq \|Dv\| - \int_{\Omega} (u - v) \nabla \cdot \zeta \, dx - \int_{\Gamma} (\zeta \cdot \nu)(u - v) \, d\mathcal{H}^{N-1},$$

where  $\|\zeta\|_{(L^\infty(\Omega))^N} \leq 1$ ,  $\nabla \cdot \zeta \in L^2(\Omega)$ , we may interpret  $t \rightarrow y(t)$  as a solution to system (5.15)–(5.17). This is the total variation Wentzell flow.

The operator  $\partial\Phi$  (and implicitly system (5.18)–(5.20)) is however hardly to be described in explicit terms, so that a better insight into problem (5.15)–(5.17) can be gained by taking into account that the solution to (5.18)–(5.20) is the limit of the finite difference scheme provided by the iteration process

$$\begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} + h\partial\Phi \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} \ni \begin{pmatrix} y_i \\ z_i \end{pmatrix}, \quad i = 0, 1,$$

or equivalently

$$y_{i+1} = \arg \min_u \left\{ \rho h \|Du\| + \frac{1}{2} \int_{\Omega} |u - y_i|^2 \, dx + \frac{1}{2} \int_{\Gamma} |\gamma(u) - \gamma(y_i)|^2 \, d\sigma \right\}.$$

**Final remarks.** The previous results naturally extend to the case of nonlinear functions  $\beta : \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ , which are not of gradient type with respect to  $r \in \mathbb{R}^N$ . Namely, it suffices to assume that  $\beta \equiv \beta(x, r)$  is continuous on  $\bar{\Omega} \times \mathbb{R}^N$ , monotone with respect to  $r$ , that is

$$(\beta(x, r) - \beta(x, \bar{r})) \cdot (r - \bar{r}) \geq 0, \quad \forall r, \bar{r} \in \mathbb{R}^N, \tag{5.21}$$

and that it satisfies

$$\beta(x, r) \cdot r \geq \alpha_1 |r|_N^p, \quad \forall r \in \mathbb{R}^N, \tag{5.22}$$

$$|\beta(x, r)|_N \leq \alpha_2 |r|_N^{p-1} + \alpha_3, \quad \forall r \in \mathbb{R}^N, \tag{5.23}$$

where  $2 \leq p < \infty$ ,  $\alpha_1, \alpha_2 > 0$ ,  $\alpha_3 \in \mathbb{R}$ .

Let us consider

$$\mathcal{U} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in U \times L^2(\Gamma); \gamma(u) = z \right\}$$

endowed with the natural norm and denote by  $\mathcal{U}'$  the dual space, in the duality induced by the pivot space  $X$ . Then, the operator  $\tilde{\mathcal{A}} : \mathcal{U} \rightarrow \mathcal{U}'$ , defined by

$$\left\langle \tilde{\mathcal{A}} \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\rangle_{\mathcal{U}', \mathcal{U}} = \int_{\Omega} \beta(x, \nabla u) \cdot \nabla \varphi_1 \, dx, \quad \forall \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathcal{U}$$

is, by the Browder theory (see e.g., [4], p. 81), maximal monotone in  $\mathcal{U} \times \mathcal{U}'$  and so its restriction

$$\mathcal{A}_X \begin{pmatrix} u \\ z \end{pmatrix} = \tilde{\mathcal{A}} \begin{pmatrix} u \\ z \end{pmatrix} \cap X$$

to  $X$  is maximal monotone in  $X \times X$ . Then, [Theorem 5.2](#) remains true in the present situation.

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