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Interlocking Factorization

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A NOTE ON THE QUADRANT INTERLOCKING FACTORIZATION

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ABSTRACT

It was recently shown that the Quadrant Interlocking Factorization (QIF) techniques proposed by D.J. Evans and several co-workers, are equivalent to block-LU factorization, with the exception of centrosymmetric matrices.

In this work the complete equivalence between Quadrant Interlocking and LU factorization techniques is proved for centrosymmetric matrices as well. As a consequence all the questions addressed on the QIF can be stated in terms of block-LU factorization.

1. INTRODUCTION

Let us consider a linear system $A \underline{x} = \underline{b}$, where $A = (a_{ij})$ is a real nonsingular $n \times n$ matrix.

The Quadrant Interlocking Factorization (QIF) has been presented as an attractive alternative to LU factorization, well suitable for the implementation on Parallel Computers. Several versions of this factorization were presented, namely: QIF1 (EVANS & HATZOPOULOS (1979), EVANS & HADJIDIMOS (1981)), QIF2 (EVANS & HADJIDIMOS (1980)), HQIF2 (EVANS, HADJIDIMOS & NOUSIOS (1981a) and (1981b)), see also SHANBECHI & EVANS (1981).

Afterwards, HADJIDIMOS & NOUSIOS (1981) proved the equivalence of QIF1 with block-LU factorization, QIF2 with LU factorization and HQIF2 with LDU factorization. Their work, published on the Bulletin of Greek Mathematical Society is not very known and was followed by various other papers on the QIF (EVANS (1982), EVANS & SOJODI-HAGHIGHI (1982), SHANBECHI & EVANS (1982), HATZOPOULOS & MISSIRLIS (1985)), see also ORTEGA & VOIGT (1985).

In their work HADJIDIMOS & NOUSIOS (1981), after having proved the equivalence relationships, state that: "Unless there are strong arguments to the contrary, it is preferable to use the parallel LU, LDU and block-LU, triangular decomposition methods which seem to be easier to program than the corresponding QIF techniques". Moreover they claim that "the only case for which we can say for sure that the parallel

block-LU method cannot replace the QIF technique is the case when A is a centrosymmetric matrix. This is because the parallel block-LU method cannot take advantage of the property of centrosymmetry".

In this work we prove that block-LU factorization can take advantage of the centrosymmetry as well, and then the equivalence between LU and QIF techniques is complete.

The main idea of the QIF method consists in factorizing A in the form

$$A = W Z,$$

with $W = (w_{ij})$, $Z = (z_{ij})$ $n \times n$ matrices which have the structure

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & * \\ * & * & 1 & 0 & * & * \\ * & * & 0 & 1 & * & * \\ * & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & * & * & * & * & 0 \\ * & * & 1 & * & * & * \end{bmatrix}$$

where "*" denotes a generally nonzero entry.

2. RESULTS

In the following we ^{first} assume n to be even.

Let us consider the permutation matrix P corresponding to the permutation $(1, n, 2, n-1, \dots)$, namely

$$P = (p_{ij}), \quad p_{ij} = \begin{cases} 1 & i \text{ odd, } j = (i+1)/2, \\ 1 & i \text{ even, } j = n - i/2 + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Given two matrices W and Z with the structure shown in the previous section, one can readily verify that, if n is even

$$P W P^T = \begin{bmatrix} I & 0 & & & 0 \\ \$ & I & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & I & 0 \\ \$ & & & & \$ & I \end{bmatrix},$$

where " I " denotes the 2×2 identity matrix, and " $\$$ " denotes a generally nonzero 2×2 block.

Analogously, it is easy to see that the matrix $P Z P^T$ is 2×2 block upper triangular.

The linear system $A \underline{x} = \underline{b}$, can be written $W Z \underline{x} = \underline{b}$,

from which: $P W P^T P Z \underline{x} = P \underline{b}$,

and $P W P^T P Z P^T P \underline{x} = P \underline{b}$,

i.e. $L U \underline{y} = P \underline{b}$, with $\underline{y} = P \underline{x}$.

Thus the decomposition $A = W Z$ is equivalent to an LU decomposition where $L = P W P^T$ is 2×2 block unit lower triangular, and $U = P Z P^T$ is 2×2 block upper triangular.

It readily follows that any structural property of W and Z implies a consequent structural property on L and U .

Let us consider now the special case of A centrosymmetric. If the factorization can be carried out, then W and Z are centrosymmetric, and it is straightforward to prove that the 2×2 blocks of L and U are centrosymmetric as well. Therefore, any algorithm which exploits the centrosymmetry of W and Z can be modified in order to exploit the centrosymmetry of the 2×2 blocks of L and U . A similar analysis can be performed when n is odd.

However, there is another well known technique fully exploiting the centrosymmetry of A by reducing the original linear system into two independent systems of size $n/2$, see ANDREW (1973). Let $m=n/2$ and let J be the $m \times m$ permutation matrix corresponding to the permutation $(m, m-1, \dots, 1)$. Then A can be written as

$$A = \begin{bmatrix} B & CJ \\ JC & JBJ \end{bmatrix}.$$

Let Q be the $n \times n$ matrix

$$Q = \begin{bmatrix} I & J \\ I & -J \end{bmatrix},$$

and it is easy to show that

$$Q A Q^T = \begin{bmatrix} I & J \\ I & -J \end{bmatrix} \begin{bmatrix} B & CJ \\ JC & JBJ \end{bmatrix} \begin{bmatrix} I & I \\ J & -J \end{bmatrix} = \begin{bmatrix} 2B+2C & 0 \\ 0 & 2B-2C \end{bmatrix}.$$

EXAMPLE

Let A be the 6×6 centrosymmetric matrix

$$A = \begin{bmatrix} 2 & 3 & 4 & 3 & 2 & -1 \\ 9 & 15 & 19 & 15 & 11 & -6 \\ 5 & 18 & 31 & 32 & 20 & -4 \\ -4 & 20 & 32 & 31 & 18 & 5 \\ -6 & 11 & 15 & 19 & 15 & 9 \\ -1 & 2 & 3 & 4 & 3 & 2 \end{bmatrix}$$

then $A = XZ$, where

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & -1 \\ 2 & 4 & 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 & 4 & 2 \\ -1 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 2 & 3 & 4 & 3 & 2 & -1 \\ 0 & 5 & 6 & 7 & 6 & 0 \\ 0 & 0 & 9 & 8 & 0 & 0 \\ 0 & 0 & 8 & 9 & 0 & 0 \\ 0 & 6 & 7 & 6 & 5 & 0 \\ -1 & 2 & 3 & 4 & 3 & 2 \end{bmatrix}$$

The matrix PAP^T is

$$PAP^T = \begin{bmatrix} 2 & -1 & 3 & 2 & 4 & 3 \\ -1 & 2 & 2 & 3 & 3 & 4 \\ 9 & -6 & 15 & 11 & 19 & 15 \\ -6 & 9 & 11 & 15 & 15 & 19 \\ 5 & -4 & 18 & 20 & 31 & 32 \\ -4 & 5 & 20 & 18 & 32 & 31 \end{bmatrix}$$

and L and U are

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & -1 & 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 1 & 0 & 0 \\ 2 & -1 & 4 & -1 & 1 & 0 \\ -1 & 2 & -1 & 4 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 3 & 2 & 4 & 3 \\ -1 & 2 & 2 & 3 & 3 & 4 \\ 0 & 0 & 5 & 6 & 6 & 7 \\ 0 & 0 & 6 & 5 & 7 & 6 \\ 0 & 0 & 0 & 6 & 9 & 8 \\ 0 & 0 & 0 & 0 & 8 & 9 \end{bmatrix}$$

Finally

$$Q^T A Q = \begin{bmatrix} 2 & 10 & 14 & 0 & 0 & 0 \\ 6 & 52 & 68 & 0 & 0 & 0 \\ 2 & 76 & 126 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 2 & 2 \\ 0 & 0 & 0 & 30 & 8 & 8 \\ 0 & 0 & 0 & 18 & -4 & -2 \end{bmatrix}$$

3. CONCLUSIONS

The arguments of the previous section show that QIF factorization is a special case of the general technique of applying LU decomposition to a row and column permutation of the matrix A. As a consequence, it turns out that all questions concerning: conditions on the existence of QIF without pivoting, numerical stability of the method, parallel implementation, block extensions of QIF, rate of convergence of SOR methods based on QIF, eigenvalues computations based on QIF, can be stated in terms of LU factorization, and there are no special classes of matrices for which QIF techniques are favourable.

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