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## Model parameter estimation using Bayesian and deterministic approaches: the case study of the Maddalena Bridge

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### Abstract

Finite element modeling has become common practice for assessing the structural health of historic constructions. However, because of the uncertainties typically affecting our knowledge of the geometrical dimensions, material properties and boundary conditions, numerical models can fail to predict the static and dynamic behavior of such structures. In order to achieve more reliable predictions, important information can be obtained measuring the structural response under ambient vibrations. This wholly non-destructive technique allows obtaining very accurate information on the structure's dynamic properties (Brincker and Ventura (2015)). Moreover, when experimental data is coupled with a finite element model, an estimate of the boundary conditions and the mechanical properties of the constituent materials can also be obtained via model updating procedures. This work presents two different model updating procedures. The first relies on construction of local parametric reduced-order models embedded in a trust region scheme to minimize the distance between the natural frequencies experimentally determined and the corresponding numerically evaluated ones (Girardi et al. (2018)). The second has been developed within a Bayesian statistical framework and uses both frequencies and mode shapes (Yuen (2015)). Both algorithms are used in conjunction with the NOSA-ITACA code for calculation of the eigenfrequencies and eigenvectors. These procedures are illustrated in the case study of the medieval Maddalena Bridge in Borgo a Mozzano (Italy). Experimental data, frequencies and mode shapes, acquired in 2015 (Azzara et al. (2017)) have enabled calibration of the bridge's constituent materials and boundary conditions.

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## 1. Introduction

Finite element model updating aims to estimate some unknown system properties on the basis of the system's behaviour observed during experimental tests. First developed for aerospace and mechanical engineering during the eighties (see Friswell and Mottershead (2013), Marwala, (2010) for a review), this procedure has been widely exploited in civil engineering, where it is generally applied to existing structures to obtain estimates of unknown material mechanical properties (Douglas and Reid (1982)) or for damage detection purposes (Rytter (1993)). It makes use of the results of vibration measurement campaigns. Applications to architectural heritage are more recent (see Gentile and Saisi, (2007), Ramos et al. (2011), Altunişik et al. (2018)) and rely on the recent extraordinary advancement of both sensor and computational technologies, which allow measuring very low-amplitude vibrations – such as the ambient vibrations of massive masonry structures - and managing very large numerical models. In this field, finite element model updating represents an important non-destructive tool which enables researchers and professionals to improve their knowledge of structures and provide estimates of their structural health.

In the present paper, two model updating procedures, the former using a deterministic approach, the latter developed in a Bayesian framework, are applied to the Maddalena Bridge in Borgo a Mozzano (Italy). For calculation of the eigenfrequencies and eigenvectors both procedures rely on the NOSA-ITACA code ([www.nositaca.it](http://www.nositaca.it)), a free finite element solver specifically developed for the structural analysis of ancient masonry buildings. In Azzara et al. (2017) a first attempt to update some parameters of the finite element model of the Maddalena Bridge was performed via the NOSA-ITACA code by matching experimental frequencies and mode shapes. In this work the two procedures have been applied to the Maddalena Bridge using the same experimental data processed in Azzara et al. (2016), Azzara et al. (2017).

## 2. The Maddalena Bridge

Also known as the Devil's Bridge, this famous, fascinating structure dating back to around the 11th century is the only one now remaining of the numerous bridges which once spanned the Serchio River (Fig. 1(a)). The total length of the bridge is about one hundred meter. The main arch of 38 m in span is just one meter high at the key, with a perfectly circular intrados profile and springs from the rock which forms the riverbed. The transverse section of the bridge ranges between 3.5 m and 3.7 m. The three piers in the riverbed have a peculiar raking form on the upstream side, while on the downstream side the pier between the second and third arches has been reinforced with a trapezoidal buttress. Currently several lines of communication run close by the bridge: in addition to the railway, two heavily trafficked thoroughfares stretch along both sides of the banks of the Serchio River and transmit road vibrations to the old bridge's structure.

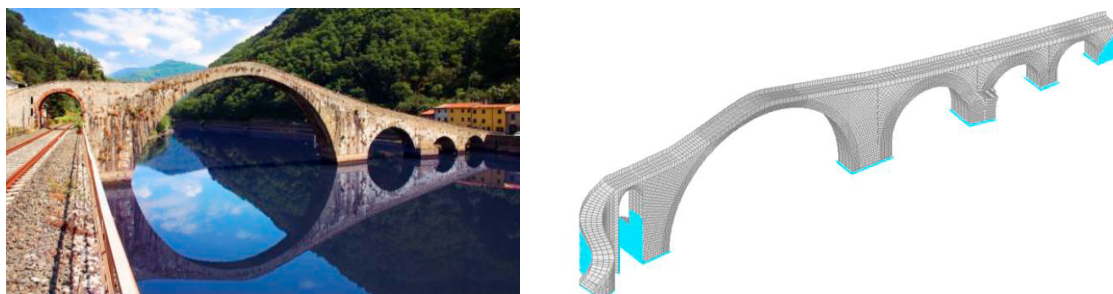


Fig. 1. (a) The Maddalena Bridge from the nearby railway; (b) Finite element model of the Maddalena Bridge.

In June 2015 an experimental campaign was conducted to measure the ambient vibrations acting on the bridge. To this purpose four SARA three-axial seismic stations were used, arranged in different layouts over the bridge during five tests (Azzara et al. (2017)). Table 1 reports the values of the first six natural frequencies of the bridge obtained from the experimental data via the Stochastic Subspace Identification method (implemented in Reynders et

al. (2014)) and averaged over the five tests. The corresponding experimental mode shapes were also extracted from the data and are reported in Azzara et al. (2017).

Table 1. Experimental values of the natural frequencies of the Maddalena bridge from Azzara et al. (2016).

	Exp. freq. [Hz]
<b>Mode shape 1</b>	<b>3.37</b>
<b>Mode shape 2</b>	<b>5.06</b>
<b>Mode shape 3</b>	<b>5.40</b>
<b>Mode shape 4</b>	<b>7.06</b>
<b>Mode shape 5</b>	<b>8.80</b>
<b>Mode shape 6</b>	<b>9.19</b>

### 3. Model parameter estimation through optimization: a deterministic approach

The model calibration procedure reported in Azzara et al. 2017 was performed by running the code for different values of the mechanical properties of the constituent materials and the soil supporting the bridge's piers to achieve a matching between numerical and experimental values of the bridge's dynamic characteristics. In the present paper a new algorithm is instead presented; it relies on construction of local parametric reduced-order models embedded in a trust region scheme to minimize the discrepancy between numerical and experimental frequencies. The algorithm has been coupled with the NOSA-ITACA code to calculate the eigenfrequencies of the finite-element model. The resulting procedure, which is completely automatic, turns out to be very efficient and reduces both the total computation time of the numerical process and user effort. The following provides a brief description of the proposed algorithm, which is described in detail in Girardi et al. (2018).

The model updating problem can be reformulated as an optimization problem by assuming that the stiffness and mass matrices  $\mathbf{K}$  and  $\mathbf{M}$  are functions of a parameter vector  $\mathbf{x} = (x_1, \dots, x_l)$  containing the geometrical data and material properties values. We use the notation

$$\mathbf{K} = \mathbf{K}(\mathbf{x}), \quad \mathbf{M} = \mathbf{M}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^l, \quad \mathbf{K}, \mathbf{M} \in \mathbb{R}^{n \times n}, \quad (1)$$

where  $n$  is the number of degrees of freedom of the FE model. The set of valid choices for the parameters is denoted by  $\Omega$ . Within this framework, we assume that the set  $\Omega$  is an  $l$ -dimensional box, that is

$$\Omega = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_l, b_l], \quad (2)$$

for certain values  $a_i < b_i$ ,  $i = 1 \dots l$ .

Our ultimate aim is to determine the optimal value of  $\mathbf{x}$  that minimizes a certain cost functional  $\phi(\mathbf{x})$  within the box  $\Omega$ :

$$\min_{\mathbf{x} \in \Omega} \phi(\mathbf{x}) \quad (3)$$

The choice of the objective function  $\phi(\mathbf{x})$  is related to the frequencies that we wish to match. If we need to match  $s$  frequencies of the model, we choose a suitable weight vector  $\mathbf{w} = [w_1, \dots, w_s]$ , with  $w_i \geq 0$ , and define the functional  $\phi(\mathbf{x})$  as follows:

$$\phi(\mathbf{x}) = \left\| \frac{\sqrt{\Lambda_s(\mathbf{K}, \mathbf{M})}}{2\pi} - \mathbf{f} \right\|_{\mathbf{w}, 2}^2, \quad (4)$$

where  $\mathbf{f}$  is the vector of the measured frequencies and  $\Lambda_s(\mathbf{K}, \mathbf{M})$  the one containing the smallest  $s$  eigenvalues of the system, ordered according to their magnitude; the norm is defined as  $\|\mathbf{y}\|_{w,2} = \sqrt{\mathbf{y}^T \text{diag}(\mathbf{w})\mathbf{y}}$ . The vector  $\mathbf{w}$  encodes the weight that should be given to each frequency in the optimization scheme. If the goal is to minimize the distance between the vectors of the measured and the computed frequencies in the usual Euclidean norm, then  $\mathbf{w} = \mathbf{1}_s$ , and the vector of all ones should be chosen. If, instead, relative accuracy on the frequencies is desired,  $w_i = f_i^{-1}$  is a natural choice. If some frequencies need to be ignored, we can set the corresponding component of  $\mathbf{w}$  to zero. To avoid scaling issues, the weight vector is always normalized in order to have  $\|\mathbf{w}\|_2 = 1$ .

When the FE model is very large, model reduction techniques have to be used in order to decrease its size to a more manageable order. If, as in equation (1), the model depends on the parameters  $\mathbf{x}$ , obtaining an accurate reduced parametric model that reflects the behavior of the original one for all possible parameter values is not a trivial matter. To this end, an efficient Lanczos-based projection strategy tailored to the needs of the FE analysis of engineering structures has been implemented (Girardi et al, 2018). By modifying the Lanczos projection used to compute the structure's first eigenvalues (and corresponding eigenvectors), we obtain local parametric reduced-order models that, embedded in a trust region scheme, are the basis for an efficient algorithm that minimizes the objective function (4).

The new procedure is applied here to the finite element model of the Maddalena Bridge (Fig. 1(b)). We update three parameters, the Young's modulus  $E$  and the mass density  $\rho$  of the material constituting the bridge's structure (assumed to be homogeneous), and the stiffness  $k_s$  of the soil under the right abutment of the main arch. The remaining piers are assumed to be instead restrained from any displacements. As revealed by the numerical tests, the model seems to be quite insensitive to changes in the material Poisson's ratio, which was then set at 0.16. With regard to the masonry constituting the bridge's parapets, changes in its mechanical properties do not significantly affect the first natural frequencies (mode shapes involving the parapets' movements have been detected at about 17 Hz), which have therefore been fixed at  $E = 3800 \text{ MPa}$  and  $\rho = 2000 \text{ kg/m}^3$ , as per Azzara et al (2017).

The parameters are allowed to vary within the intervals

$$4000 \text{ MPa} \leq E \leq 7200 \text{ MPa}, \quad 1600 \text{ kg/m}^3 \leq \rho \leq 2200 \text{ kg/m}^3, \quad 1.9 \cdot 10^9 \text{ N/m}^3 \leq k_s \leq 2.9 \cdot 10^{11} \text{ N/m}^3$$

and the objective function built with  $w_i = f_i^{-1}$ .

The total computation time was 534 s, including the time for assembly of the parametric model as well as the optimization steps. The experiments have been performed on a server with an i7-920 CPU running at 2.67 GHz and 18GB of RAM. The starting components used for the parameters vector are  $E = 5000 \text{ MPa}$ ,  $\rho = 2100 \text{ kg/m}^3$ , and  $k_s = 2.1 \cdot 10^{10} \text{ N/m}^3$ , which represent the mean values of the prior distributions utilized for the Bayesian updating (Table 2).

Moreover, as the frequencies remain almost unchanged when the ratio  $E/\rho$  is kept constant, the Jacobian in the optimization method might be badly conditioned. In order to overcome this problem, standard regularization procedures have been employed (Wright and Nocedal, 1999). This made the method robust to the presence of noise in the data. The optimal parameters found through this approach are

$$E = 6889 \text{ MPa}, \quad \rho = 1845 \text{ kg/m}^3, \quad k_s = 1.929 \cdot 10^{10} \text{ N/m}^3$$

These values of the parameters substantially coincide with those found in Azzara et al. (2017), which pointed out that the high resulting value of the homogenized Young's modulus can be regarded as an initial, dynamic value of the high quality masonry making up the Maddalena bridge. The value of  $k_s$  corresponds to a very compact cohesive soil or a very dense sand, while the remaining part of the bridge lies on rock.

Fig. 2a shows the objective function  $\phi(\mathbf{x})$  versus  $E$  and  $k_s$ , evaluated for  $w_i = f_i^{-1}$ . The chosen solution, which is plotted in the figure, represents the absolute minimum of  $\phi(\mathbf{x})$  in  $\Omega$ . Fig. 2b instead shows a section of  $\phi(\mathbf{x})$  passing through the function's minimum point. Finally, Fig. 3 shows, on the left, the convergence of the objective function to the minimum for each iteration corresponding to a new reduced model (dashed line is the tolerance), and

on the right, the convergence of the model’s frequencies to the experimental values.

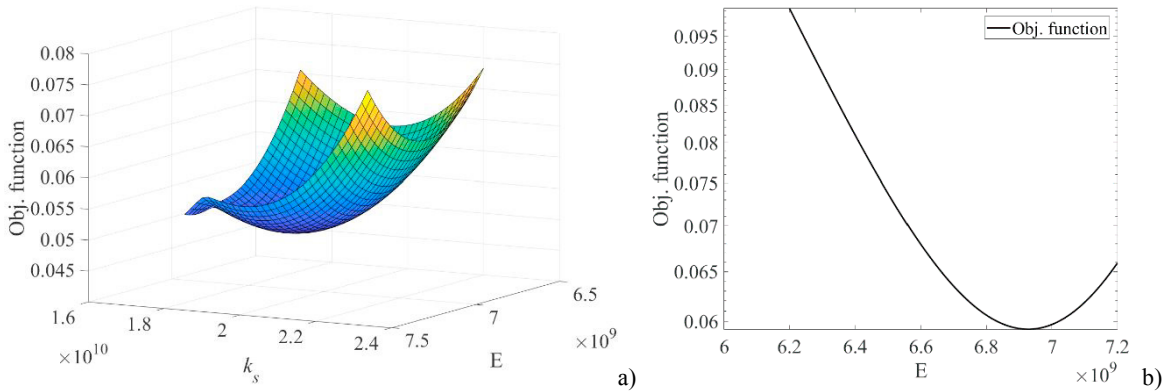


Fig. 2. (a) The objective function  $\phi$  vs. the Young’s modulus  $E$  [N/m<sup>2</sup>] and the stiffness  $k_s$  [N/m<sup>3</sup>] of the soil. (b) A section of the objective function  $\phi$  passing through the minimum point. Young’s modulus in [N/m<sup>2</sup>].

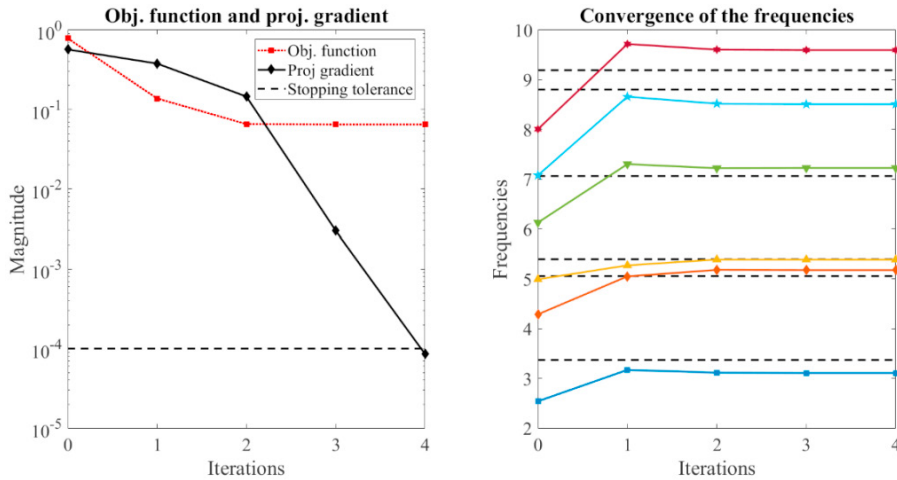


Fig. 3. Convergence history of the algorithm.

#### 4. Bayesian approach for model parameter estimation

The proposed procedure, developed in a Bayesian framework, aims at identifying the material parameters of the Maddalena Bridge, the material elastic moduli and density, and the soil stiffness  $k_s$  under the right abutment of the main arch. Parameters updating is carried out by matching the experimental natural frequencies with the corresponding model output related to the same mode shape.

Bayes' rule in statistical analysis (Box and Tiao, 1992) describes the probability of an event based on prior knowledge of the conditions that might be related to the event. Bayes’ theorem is expressed as follows

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)} \tag{5}$$

where  $P(A|B)$  expresses the conditional probability of event A, when event B has occurred,  $P(A)$  and  $P(B)$  are the probabilities of events A and B, respectively, and  $P(B|A)$  is the conditional probability of event B, when event A has

occurred. By interpreting event  $A$  as the parameters vector to be updated and  $B$  as one of the available data measurements,  $P(A)$  is the initial state of knowledge of the parameters and  $P(A|B)$  is the updated state of the knowledge. The term  $P(B|A)/P(B)$ , which summarizes the available information on the structure, transforms the knowledge of the system and can be expressed in the form given by Fisher (1922) as the product of a normalization factor  $\kappa$  and the likelihood function.

By substituting  $A$  and  $B$  with  $\mathbf{x}$  and  $\mathbf{y}$ , where  $\mathbf{x}$  is the vector of the  $l$  random variables,  $\mathbf{y}$  the vector of the  $q$  measurements, the updating rule is expressed by

$$p(\mathbf{x}|\mathbf{y}) = \kappa L(\mathbf{x}|\mathbf{y}) p(\mathbf{x}). \quad (6)$$

$p(\mathbf{x}|\mathbf{y})$  represents the posterior distribution that denotes the updated state of knowledge about the random variable  $\mathbf{x}$ , and  $L(\mathbf{x}|\mathbf{y})$  is the likelihood function, which transforms the prior distribution into the posterior distribution by updating the model parameters once the new  $q$  data collected in  $\mathbf{y}$  are gathered. Finally,  $p(\mathbf{x})$  is the prior distribution, which represents the state of knowledge before introduction of new data  $\mathbf{y}$ , and  $\kappa$  is the normalizing factor

$$\kappa = \left[ \int L(\mathbf{x}|\mathbf{y}) p(\mathbf{x}) d\mathbf{x} \right]^{-1}. \quad (7)$$

Considering an additive probabilistic model and normally distributed model error, the likelihood function in the  $q$ -variate form can be written as (Gardoni, 2002), (Box and Tiao, 1992)

$$L(\mathbf{x}, \sigma_k) \propto \prod_{i=1}^q P[\sigma_k \varepsilon_{ki} = r_{ki}(\mathbf{x})] \quad k = 1, \dots, s \quad (8)$$

where  $\varepsilon_{ki}$  is a normal random variable having zero mean value and unit standard deviation,  $\sigma_k$  is the  $k$ -th component of the covariance matrix  $\Sigma$  of the error, and  $r_{ki}(\mathbf{x})$  is the  $i$ -th residual, which represents the discrepancy between the measurement and the prediction of the  $k$ -th experimental frequency. The following expression thus holds

$$r_{ki}(\mathbf{x}) = C_{ki} - \hat{c}_{ki}(\mathbf{x}), \quad (9)$$

where  $C_{ki}$  is the term related to the measurements and  $\hat{c}_{ki}(\mathbf{x})$  is the term related to the model output.

Determination of the likelihood, and hence of the posterior distribution of random variables cannot generally be solved in closed form, so numerical techniques are needed. The Markov Chain Monte Carlo (MCMC) method is one of the most commonly adopted techniques (Medova, (2007)) when stochastic FE are involved.

In the present case, the reference measure is represented by natural frequencies. Unfortunately, when parameters vary within the confidence interval, the order of the mode shapes in the model may also vary, so the procedure must be able to recognize these changes and correctly link the numerical frequencies to the corresponding experimental ones. In order to compare numerical and experimental mode shapes, the Metropolis-Hastings version of the Monte Carlo Markov Chain (MCMC) algorithm has been modified by introducing the Modal Assurance Criterion (MAC), to associate frequencies with mode shapes. More specifically, at each step of the algorithm, when sampling is carried out, the numerical model's mode shapes are compared with the experimental ones, and the MAC matrix calculated, thereby allowing us to identify the correspondence between experimental and numerical mode shapes. Moreover, in order to reduce the computational load, the finite element model response in terms of frequencies and mode shapes is reproduced through a proxy model (Marwala, 2010). The technique used for this purpose is the General Polynomial Chaos Expansion (GPCE), (Xiu, 2010), which allows creating a response surface that depends on the parameters and makes the uncertainties propagation computationally simple. It also enables assessing the influence of each parameter on the results of the FE model through synthetic indices. Finally, this model provides an estimate of the covariance matrix of the error distribution containing the error variance for each frequency.

The procedure described above has been implemented for the case study based on the same experimental results (Azzara et al. (2017)), using the NOSA-ITACA FE code for eigenfrequencies and eigenvectors calculation (Fig. 1b).

The material parameters to be identified are the two elastic moduli (bulk modulus  $K$  and shear modulus  $G$ ) and the density  $\rho$  and the constant  $k_s$  of the elastic supports under the right abutment to model the soil stiffness.

Construction of the proxy model through the GPCE technique has been performed using Hermitian Polynomials of grade up to 4, whose coefficients have been calculated through a regression of 625 FE analyses. The number of analyses was determined via the full tensor grid method (Xiu, 2010). The updating procedure required about 43000 runs of the proxy model. As a first result stemming from use of the GPCE technique, it can be deduced that the most significant parameters are  $G$ ,  $\rho$  and  $k_s$ .

Six mode shapes have been identified: the first strongly influenced by  $G$ , the second involving vertical displacements of the main arcade, and the third regarding its out-of-plane displacements. Some results obtained are shown in Fig. 4 in the form of probability density functions of the three parameters mentioned above (4a, 4b) and of comparison between the experimental and the numerical values of the bridge's natural frequencies (4c). Table 2 reports the results in terms of mean values and standard deviations of the parameters after updating.

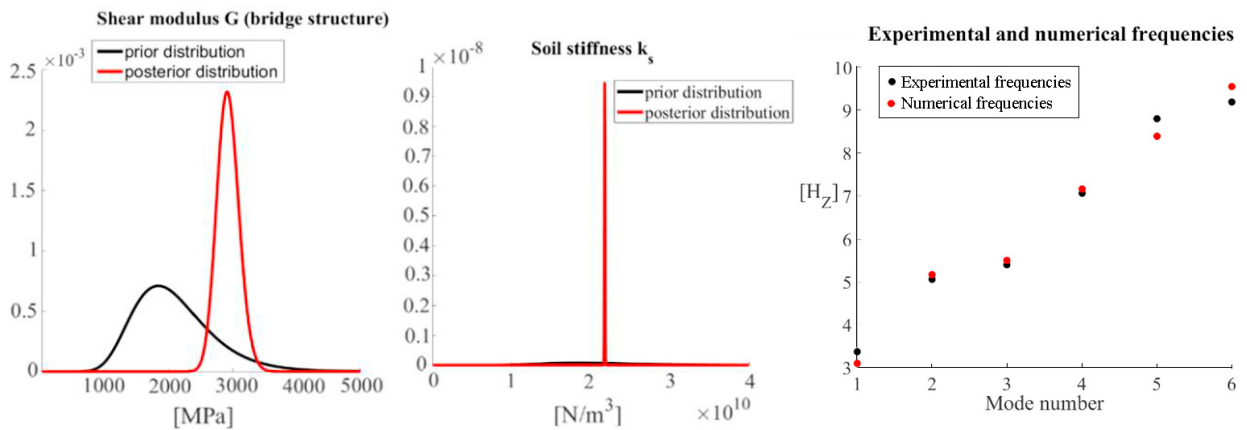


Fig. 4. (a) probability density functions of  $G$ ; (b) probability density functions of soil stiffness  $k_s$ ; (c) Comparison between experimental and numerical frequencies - Bayesian approach.

Table 2. Parameters identified through the Bayesian approach

	$G$ [MPa]	$\rho$ [kg/m <sup>3</sup> ]	$K$ [MPa]	$k_s$ [N/m <sup>3</sup> ]
Prior mean	2083	2100	2777	$2.1 \cdot 10^{10}$
Prior standard deviation	625	300	833	$6.30 \cdot 10^9$
Posterior mean	2922	1976	3308	$2.1731 \cdot 10^{10}$
Posterior standard deviation	172.81	8.3539	320	$4.0721 \cdot 10^7$

## 5. Conclusion

The paper has presented two model updating procedures, one developed within a deterministic framework and the other following a Bayesian probabilistic approach. These procedures have been applied to the finite element model of the Maddalena Bridge in Borgo a Mozzano and furnish an estimate of the mechanical properties of the bridge's constituent materials and foundation soil stiffness. The finite element model has been calibrated using some experimental results, natural frequencies and mode shapes of the bridge obtained via Operational Modal Analysis (Azzara et al, 2017). Tables 3 and 4 compare the procedures in terms of identified parameters and relative errors between numerical and experimental frequency values. The two methods give very similar results in terms of the numerical frequencies and the parameter values – the most significant difference arising in the evaluation of the

bridge's homogenized mass density. In both cases, the greatest error with respect to experimental data is found in the evaluation of the fundamental frequency, whose value is underestimated by both algorithms. The reason for this probably lies in the lack of knowledge on the real distribution of mass and stiffness throughout the bridge.

For both approaches, special attention has been devoted to the computational efficiency of the methods proposed by using simplified models to approximate the dynamic properties of the original finite element model. With regard to the Bayesian approach, the use of a proxy model seems to be convenient in order to reduce the computational burden of the MCMC technique, also in view of performing sensitivity analyses that can direct the choice of the most significant parameters.

Table 3. Bayesian vs. deterministic approach: identified parameters

	$G$ [MPa]	$\rho$ [kg/m <sup>3</sup> ]	$K$ [MPa]	$k_g$ [N/m <sup>3</sup> ]
Bayesian approach	2922 $\pm$ 173	1976 $\pm$ 8	3308 $\pm$ 320	2.17 10 <sup>10</sup> $\pm$ 4.07 10 <sup>7</sup>
Deterministic approach	2969	1845	3377	1.93 10 <sup>10</sup>

Table 4. Bayesian and deterministic approaches: matching of the experimental frequencies

Experimental frequencies [Hz]	Bayesian approach [Hz]	Rel. errors [%]	Deterministic approach	Rel. errors [%]
<b>3.37</b>	3.098	8.08	3.108	7.79
<b>5.06</b>	5.170	2.18	5.179	2.35
<b>5.40</b>	5.503	1.91	5.391	0.17
<b>7.06</b>	7.159	1.40	7.223	2.31
<b>8.80</b>	8.391	4.64	8.507	3.33
<b>9.19</b>	9.550	3.91	9.596	4.42

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