

Viscoelastodynamics of swelling porous solids at large strains by an Eulerian approach

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Abstract. A model of saturated hyperelastic porous solids at large strains is formulated and analysed. The material response is assumed to be of a viscoelastic Kelvin-Voigt type and inertial effects are considered, too. The flow of the diffusant is driven by the gradient of the chemical potential and is coupled to the mechanics via the occurrence of swelling and squeezing. Buoyancy effects due to the evolving mass density in a gravity field are covered. Higher-order viscosity is also included, allowing for physically relevant stored energies and local invertibility of the deformation. The whole system is formulated in a fully Eulerian form in terms of rates. The energetics of the model is discussed and the existence and regularity of weak solutions is proved by a combined regularization-Galerkin approximation argument.

Keywords: poroelasticity, elastodynamics, finite strains, squeezing/swelling, multipolar continua, transport equations, Galerkin approximation.

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1 Introduction

The poromechanics of deformable media is a classical part of continuum mechanics of solids, bordering with fluid-solid mechanics and mixtures, and having a vast application ground, from petroleum engineering, to geology, soil and rock mechanics, polymers, etc. Correspondingly, literature is abundant, see, e.g., the monographs [10, 11, 33], and a whole hierarchy of models is available [10, 27], tailored to the description of different aspects at different scales.

In this paper we focus on a simple phenomenological model describing the flow of a diffusant in a saturated poroelastic permeable medium under the assumption that the diffusant

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flux is governed by Fick’s law (here, to be possibly referred as Darcy’s law, for the setting is purely mechanical). We assume the systems to be isothermal and consider the coupled evolution of the solid and of a single fluid, whose content plays the role of an internal variable. Indeed, the case of the phenomenological model under scrutiny here is justified if, among many other simplifications, the flow of a fluid through the solid is quasistatic and sufficiently slow (i.e., in particular no inertial effects in the diffusant are considered) and viscous effects in the fluid are neglected.

We focus on the case of possible large strains and use the *multiplicative decomposition* of the total deformation gradient into an elastic strain and a swelling distortion. This is indeed a classical assumption, especially in connection with swelling in soft materials (like gels) under large strains, see, e.g., [3, 7–9, 12–14, 20] or [2] for the coupling with inelastic strain (reflecting plasticity or creep). The swelling distortion can be modeled, alternatively to the multiplicative decomposition of the total strain, by adopting the Biot model [6] for the large strains, as used in [17, 28] for neo-Hookean material.

The liquid content may influence not only the stress-free configuration through the mentioned swelling distortion but also the elastic response. In particular, this amounts in modeling *elastic softening* effects. An everyday example of such phenomenon is the soaking of dried legumes which exhibit remarkable swelling accompanied with elastic softening by increasing wetting.

A specific feature of the model is that it is fully *Eulerian*, i.e., it is formulated in *actual coordinates* instead of a referential ones. In the frame of the analysis of hyperelastic solid response this is not common. Still, it allows for some simplification, for it avoids the need for implementing pullback/pushforward of fields from the reference to the actual configuration, ultimately simplifying transport coefficients. Let us note however, that alternative Lagrangian formulations have been considered in [30–32] or [19, Sect.9.6]. In spite of the above mentioned specific analytic intricacies, in contrast with our current Eulerian one, these Lagrangian models allow for a possible treatment of nonhomogeneous Dirichlet conditions on the solid.

Additional remarkable features of our model are its fully dynamical nature, including the description of inertial forces (thus allowing for elastic wave propagation), the attainment of local noninterpenetrability (in the sense that the deformation gradient is invertible everywhere), and the possibility of considering physically relevant stored energies (i.e., nonconvex and not necessarily bounded for degenerating Jacobian of the deformation).

Extensions to multi-porosity or multi-component flows, as well as combinations with additional processes featuring other evolving internal variables (as porosity or damage or an inelastic strain) is possible but not considered here. In addition, one could include thermal effects by considering also heat generation and transfer, possibly with phase transitions. For a metal-hydrid phase transition (coupled possibly with magnetic effects and ferro-to-paramagnetic phase transformation) within hydrogen diffusion in metals see [32].

The model is formulated and its energetics is presented in Section 2. The existence of weak solutions is then shown in Section 3. Here, we follow a regularization and Galerkin

approximation strategy. This is combined with transport theory by a regular velocity field.

2 The model

We devote this section to the presentation of the model and its energetics. After some preparation, these are to be found in Subsection 2.4 and 2.5, respectively.

Before going on, let us introduce the main notation used in this paper, as in the following table:

\mathbf{v} velocity (in m/s),	$\lambda(z)$ swelling stretch
ϱ mass density (in kg/m ³),	$\varphi = \varphi(\mathbf{F}_e, z)$ stored energy (in J/m ³ =Pa),
$\mathbf{F} = \mathbf{F}_e \mathbf{F}_s$ deformation gradient,	μ chemical potential (pore pressure, in Pa)
\mathbf{F}_e elastic strain,	m mobility (diffusion) coefficient (in m ³ s/kg)
$\mathbf{F}_s = \lambda(z)\mathbf{I}$ swelling distortion,	$\mathbf{e}(\mathbf{v}) = \frac{1}{2}\nabla\mathbf{v}^\top + \frac{1}{2}\nabla\mathbf{v}$ small strain rate (in s ⁻¹),
z diffusant content	$\zeta = \zeta(z, \cdot)$ viscosity dissipation potential,
$\mathbf{T}_{\text{tot}} = \mathbf{T} + \mathbf{D}$ Cauchy stress (in Pa)	\mathbf{g} external load (gravity acceleration in m/s ²),
\mathbf{T} elastic (conservative) stress	\mathbf{f} traction load (in N/m ²),
\mathbf{D} viscous (dissipative) stress	$(\cdot)^\cdot$ convective derivative

Table 1. Summary of the basic notation.

2.1 Geometric preliminaries

Let us start by recalling some basic notion from the general theory of large deformations in continuum mechanics. Note that we limit ourselves in introducing some minimal frame, to serve the sole purpose of presenting the model. In particular, no completeness is claimed and we refer the reader. e.g., to [16, 21] for additional material.

Assume to be given the *deformation* $\mathbf{y} : I \times \Omega \rightarrow \mathbb{R}^d$, $d = 1, 2, 3$, where $I = [0, T]$ and $T > 0$ is some final time. For all given times $t \in I$, the deformation maps the *reference configuration* $\Omega \subset \mathbb{R}^d$ of the deformable body to its *actual configuration* $\mathbf{y}(t, \Omega)$, a subset of the *physical space* \mathbb{R}^d . In what follows, we indicate *referential coordinates* by $\mathbf{X} \in \Omega$ and *actual coordinates* by $\mathbf{x} \in \mathbb{R}^d$. By assuming $\mathbf{y}(t, \cdot)$ to be globally invertible, we indicate the inverse by $\boldsymbol{\xi}(t, \cdot) = \mathbf{y}^{-1}(t, \cdot) : \mathbf{y}(t, \Omega) \rightarrow \Omega$; standardly $\boldsymbol{\xi}$ is called the *return* (or the reference) *mapping* or sometimes *inverse motion*.

Let Q indicate any physical quantity (scalar, vectorial, tensorial), supposed to be attached to a specific point \mathbf{x} of the deformed body at a specific time t . The quantity Q can be expressed in referential coordinates as $Q_R(t, \mathbf{X})$ by letting

$$(2.1) \quad Q_R(t, \mathbf{X}) = Q(t, \mathbf{y}(t, \mathbf{X})).$$

Equivalently, given any physical quantity Q_R (scalar, vectorial, tensorial), supposed to be attached to a specific referential position \mathbf{X} at a specific time t one can express it in actual coordinates as $Q(t, \mathbf{x})$ by posing

$$(2.2) \quad Q(t, \mathbf{x}) = Q_R(t, \boldsymbol{\xi}(t, \mathbf{x})).$$

We call Q and Q_{R} the *Eulerian* and the *Lagrangian or referential* representations of the quantity, respectively.

Given $\mathbf{y} = (y_1, \dots, y_d)$, we define the *deformation gradient* \mathbf{F}_{R} and the *referential velocity* \mathbf{v}_{R} as

$$(\mathbf{F}_{\text{R}}(t, \mathbf{X}))_{iK} = \frac{\partial y_i}{\partial X_K}(t, \mathbf{X}) \quad \text{and} \quad \mathbf{v}_{\text{R}}(t, \mathbf{X}) = \frac{d}{dt} \mathbf{y}(t, \mathbf{X})$$

for indices running from 1 to d . Here and in the following we indicate with d/dt the derivative with respect to time of a time dependent function, as opposed to the symbol $\frac{\partial}{\partial t}$ which denotes the partial time derivative. In the specific case of $\mathbf{v}_{\text{R}}(t, \mathbf{X})$ these two derivatives obviously coincide. The corresponding Eulerian representations from (2.2) are

$$(2.3) \quad \mathbf{F}(t, \mathbf{x}) = \mathbf{F}_{\text{R}}(t, \boldsymbol{\xi}(t, \mathbf{x})) \quad \text{and} \quad \mathbf{v}(t, \mathbf{x}) = \mathbf{v}_{\text{R}}(t, \boldsymbol{\xi}(t, \mathbf{x})).$$

The Eulerian velocity \mathbf{v} is then used to define the *material derivative* $\dot{q}(t, \mathbf{x})$ of any scalar Eulerian quantity $q(t, \mathbf{x})$ as

$$\dot{q}(t, \mathbf{x}) = \frac{\partial}{\partial t} q(t, \mathbf{x}) + \nabla q(t, \mathbf{x}) \cdot \mathbf{v}(t, \mathbf{x}) = \left(\frac{\partial}{\partial t} + (\mathbf{v}(t, \mathbf{x}) \cdot \nabla) \right) q(t, \mathbf{x}),$$

where the differentiation ∇ is, of course, taken with respect to actual coordinates. Similarly, one defines the material derivative of a vectorial or tensorial quantity by arguing on coordinates. In particular, this allows us to check that

$$(2.4) \quad \frac{d}{dt} Q_{\text{R}}(t, \mathbf{X}) \stackrel{(2.1)}{=} \frac{d}{dt} Q(t, \mathbf{y}(t, \mathbf{X})) = \dot{Q}(t, \mathbf{x})$$

For any sufficiently smooth quantity Q . In particular, we have that

$$(2.5) \quad \dot{\boldsymbol{\xi}}(t, \mathbf{x}) = \frac{d}{dt} \boldsymbol{\xi}(t, \mathbf{y}(t, \mathbf{X})) = \frac{d}{dt} \mathbf{X} = \mathbf{0}.$$

Note that property (2.4) in particular implies the product rule $(Q_1 Q_2) \cdot = \dot{Q}_1 Q_2 + Q_1 \dot{Q}_2$.

By applying the classical chain rule we get that

$$\begin{aligned} \frac{d}{dt} (\mathbf{F}_{\text{R}}(t, \mathbf{X}))_{iK} &= \frac{d}{dt} \frac{\partial y_i}{\partial X_K}(t, \mathbf{X}) = \frac{\partial}{\partial X_K} \frac{\partial y_i}{\partial t}(t, \mathbf{X}) \\ &= \frac{\partial v_{\text{R}i}}{\partial X_K}(t, \mathbf{X}) \stackrel{(2.1)}{=} \frac{\partial v_i}{\partial X_K}(t, \mathbf{y}(t, \mathbf{X})) \\ &= \frac{\partial v_i}{\partial x_j}(t, \mathbf{y}(t, \mathbf{X})) \frac{\partial y_j}{\partial X_K}(t, \mathbf{X}) = (\nabla \mathbf{v}(t, \mathbf{x}))_{ij} \mathbf{F}_{\text{R}}(t, \mathbf{X})_{jK} \end{aligned}$$

where, here and below, we use the summation convention over repeated indices. Owing to relation (2.4), the latter reads in Eulerian coordinates as

$$(2.6) \quad \dot{\mathbf{F}}(t, \mathbf{x}) = \nabla \mathbf{v}(t, \mathbf{x}) \mathbf{F}(t, \mathbf{x}).$$

From here on, we formulate the model in terms of the velocity \mathbf{v} and the deformation gradient \mathbf{F} only, without explicit reference to the deformation \mathbf{y} . Note that \mathbf{y} can be reconstructed by

taking the inverse of $\boldsymbol{\xi}$, which is solving equation (2.5) and is at least *locally* injective. The geometric relation (2.6) will then guarantee that $\mathbf{F} = \nabla \mathbf{y}$ for the reconstructed deformation. Both operations are admissible in our regularity frame, see Definition 3.1 later on, hinging on the invertibility of the return mapping $\boldsymbol{\xi}$. In case $\boldsymbol{\xi}$ happens to be *globally* injective, the reconstruction of \mathbf{v} and \mathbf{F} can be globally performed. Note however that such global injectivity is not granted by the model, given the assumed boundary conditions, which are not fixing tangential deformations. On the other hand, if boundary deformation were fixed and invertible, we could resort to the classical theory in [4,18] and deduce global injectivity.

By using the elementary identity $\mathbf{0} = \frac{d}{dt}(\mathbf{F}_R \mathbf{F}_R^{-1}) = (\frac{d}{dt} \mathbf{F}_R) \mathbf{F}_R^{-1} + \mathbf{F}_R (\frac{d}{dt} \mathbf{F}_R^{-1})$ and relation (2.6), we also get

$$(2.7) \quad (\mathbf{F}^{-1}(t, \mathbf{x})) \cdot = -\mathbf{F}^{-1}(t, \mathbf{x}) \nabla \mathbf{v}(t, \mathbf{x}).$$

We now use equations (2.6) and (2.7) with equivalence (2.4) and Jacobi's formula $\frac{d}{dt} \det A(t) = \det A(t) \operatorname{tr}(A^{-1}(t) \frac{d}{dt} A(t))$, valid for any sufficiently smooth map $t \mapsto A(t)$ with $A(t)$ invertible, in order to get

$$(2.8) \quad \begin{aligned} (\det \mathbf{F}) \cdot &= \frac{d}{dt} \det \mathbf{F}_R = (\det \mathbf{F}_R) \operatorname{tr} \left(\mathbf{F}_R^{-1} \frac{d}{dt} \mathbf{F}_R \right) \\ &\stackrel{(2.6)}{=} (\det \mathbf{F}) \operatorname{tr}(\mathbf{F}^{-1} \nabla \mathbf{v} \mathbf{F}) = (\det \mathbf{F}) \operatorname{div} \mathbf{v}. \end{aligned}$$

Moving from the latter, we also get that

$$(2.9) \quad \left(\frac{1}{\det \mathbf{F}} \right) \cdot = \frac{d}{dt} \left(\frac{1}{\det \mathbf{F}_R} \right) = -\frac{(\det \mathbf{F}_R) \operatorname{tr} \left(\mathbf{F}_R^{-1} \frac{d}{dt} \mathbf{F}_R \right)}{(\det \mathbf{F}_R)^2} = -\frac{\operatorname{div} \mathbf{v}}{\det \mathbf{F}}.$$

2.2 The governing equations

The state of the deformable body undergoing deformation and swelling is classically described in terms of the actual density $\varrho(t, \mathbf{x})$, the deformation $\mathbf{y}(t, \mathbf{x})$, and the scalar variable $z(t, \mathbf{x})$ expressing the pointwise solvent content in Eulerian coordinates. The evolution of the body is then described by the system

$$(2.10a) \quad \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0,$$

$$(2.10b) \quad \varrho \dot{\mathbf{v}} - \operatorname{div} \mathbf{T}_{\text{tot}} = \varrho \mathbf{g},$$

$$(2.10c) \quad \dot{z} - \operatorname{div}(m \nabla \mu) = 0.$$

Here, relations (2.10a)-(2.10b) are the classical conservation of mass and momentum, where \mathbf{T}_{tot} represents the total *Cauchy stress* and \mathbf{g} is the *gravity acceleration*. The kinetic relation (2.10c) describes the transport and diffusion of the solvent content, in dependence of the gradient of the *chemical potential* μ , which is additionally modulated by the *mobility*

coefficient m . Constitutive choices for the quantities \mathbf{T}_{tot} , μ , and m are made in Subsection 2.3 below.

Relations (2.10) are to be fulfilled in the deformed domain $\mathbf{y}(t, \Omega)$ for $t \in I$ and have to be complemented by initial and boundary conditions, see Subsection 2.4 below. Let us anticipate that we impose the *impenetrability condition* $\mathbf{v} \cdot \mathbf{n} = 0$, where \mathbf{n} represents the outward unit normal at the boundary of the deformed domain. Note that this condition, although possibly being restrictive with respect to some applications, greatly expedites the analysis, for it guarantees that $\mathbf{y}(t, \Omega) \equiv \Omega$ for all times $t \in I$. In particular, one is actually asked to solve (2.10) on the cylinder $I \times \Omega$.

Before moving on, let us observe that the mass balance (2.10a) can be equivalently rewritten as $\dot{\varrho} + \varrho \operatorname{div} \mathbf{v} = 0$. One can hence use (2.8) in order to compute

$$(\varrho \det \mathbf{F})^\cdot = \dot{\varrho} \det \mathbf{F} + \varrho (\det \mathbf{F})^\cdot \stackrel{(2.8)}{=} (\dot{\varrho} + \varrho \operatorname{div} \mathbf{v}) \det \mathbf{F} = 0.$$

This in particular entails that $\varrho_{\text{R}}(\cdot, \mathbf{X}) \det \mathbf{F}_{\text{R}}(\cdot, \mathbf{X})$ is constant in time for all $\mathbf{X} \in \Omega$. Hence,

$$\varrho_{\text{R}}(t, \mathbf{X}) \det \mathbf{F}_{\text{R}}(t, \mathbf{X}) = \varrho_{\text{R}}(0, \mathbf{X}) \det \mathbf{F}_{\text{R}}(0, \mathbf{X}).$$

Passing to Eulerian variables the latter gives

$$\varrho(t, \mathbf{x}) \det \mathbf{F}(t, \mathbf{x}) = \varrho(0, \mathbf{x}) \det \mathbf{F}(0, \mathbf{x}).$$

In particular, provided that relation (2.6) holds one can equivalently replace the continuity equation (2.10a) and the initial condition $\varrho(0, \mathbf{x}) = \varrho_0(\mathbf{x})$ by

$$(2.11) \quad \varrho(t, \mathbf{x}) = \frac{\varrho_0(\mathbf{x}) \det \mathbf{F}(0, \mathbf{x})}{\det \mathbf{F}(t, \mathbf{x})}.$$

For the sake of later use, we define $\varrho_{\text{R}}(\mathbf{x}) := \varrho_0(\mathbf{x}) \det \mathbf{F}(0, \mathbf{x})$, which is given in terms of initial data only.

2.3 Constitutive relations

Let us now fix our constitutive choices in relations (2.10a)-(2.10c), leading to the final formulation of our model in (2.24), below.

We start by classically assuming that the deformation strain can be multiplicative decomposed as

$$(2.12) \quad \mathbf{F} = \mathbf{F}_{\text{e}} \mathbf{F}_{\text{s}}.$$

Here, \mathbf{F}_{e} denotes the elastic strain whereas \mathbf{F}_{s} is the strain associated with swelling. As swelling effects are usually assumed to be purely volumetric and isotropic, we let

$$(2.13) \quad \mathbf{F}_{\text{s}} = \lambda(z) \mathbf{I}$$

where the smooth scalar “swelling” function $\lambda : [0, 1] \rightarrow (0, +\infty)$ indicates the stress-free reference volume at solvent-content level z and \mathbf{I} is the identity second-order tensor.

We also assume the total stress \mathbf{T}_{tot} to be additively decomposed as

$$(2.14) \quad \mathbf{T}_{\text{tot}} = \mathbf{T} + \mathbf{D},$$

where \mathbf{D} and \mathbf{T} denote the viscous (dissipative) and the inviscid (conservative) stresses, respectively.

In order to specify constitutive relations, we introduce the stored energy in the actual configuration

$$(2.15) \quad (\mathbf{F}_e, z) \mapsto \int_{\Omega} \varphi(\mathbf{F}_e, z) + \delta_{[0,1]}(z) \, d\mathbf{x}.$$

By using (2.12) with (2.13) so that $\mathbf{F}_e = \mathbf{F}/\lambda(z)$ one can rewrite equivalently the stored energy in terms of \mathbf{F} as

$$(\mathbf{F}, z) \mapsto \int_{\Omega} \varphi\left(\frac{\mathbf{F}}{\lambda(z)}, z\right) + \delta_{[0,1]}(z) \, d\mathbf{x}.$$

Here, φ is the hyperelastic energy density and $\delta_{[0,1]} : \mathbb{R} \rightarrow \{0, +\infty\}$ denotes the indicator function of the interval $[0, 1]$ (namely, $\delta_{[0,1]}(z) = 0$ if $z \in [0, 1]$ and $\delta_{[0,1]}(z) = +\infty$ otherwise), which in particular forces z to take value in $[0, 1]$ only.

We define the chemical potential μ by taking the variation of the stored energy with respect to z , namely

$$(2.16) \quad \mu \in \varphi'_z\left(\frac{\mathbf{F}}{\lambda(z)}, z\right) - \varphi'_{\mathbf{F}_e}\left(\frac{\mathbf{F}}{\lambda(z)}, z\right) : \mathbf{F} \frac{\lambda'(z)}{\lambda^2(z)} + N_{[0,1]}(z).$$

Here, primes denote (partial) differentiation and $N_{[0,1]}$ is the subdifferential in the sense of convex analysis of $\delta_{[0,1]}$, namely the (multivalued) *normal cone* to $[0, 1]$ given by $N_{[0,1]}(z) = 0$ if $z \in (0, 1)$, $N_{[0,1]}(z) = [0, +\infty)$ if $z = 1$, $N_{[0,1]}(z) = (-\infty, 0]$ if $z = 0$, and $N_{[0,1]}(z) = \emptyset$ if $z \notin [0, 1]$.

In order to specify the conservative stress \mathbf{T} we start by computing the first Piola-Kirchhoff stress $\mathbf{P}_R(\mathbf{X})$ taking the variation with respect to \mathbf{F}_R of the stored energy in referential variables, namely,

$$(\mathbf{F}_R, z_R) \mapsto \int_{\Omega} \varphi\left(\frac{\mathbf{F}_R(\mathbf{X})}{\lambda(z_R(\mathbf{X}))}, z_R(\mathbf{X})\right) \det \mathbf{F}_R(\mathbf{X}) + \delta_{[0,1]}(z_R(\mathbf{X})) \, d\mathbf{X},$$

where we have used $z_R(\mathbf{X}) = z(\mathbf{y}(\mathbf{X}))$. We get

$$\begin{aligned} \mathbf{P}_R(\mathbf{X}) &= \frac{1}{\lambda(z_R(\mathbf{X}))} \varphi'_{\mathbf{F}_e}\left(\frac{\mathbf{F}_R(\mathbf{X})}{\lambda(z_R(\mathbf{X}))}, z_R(\mathbf{X})\right) \det \mathbf{F}_R(\mathbf{X}) \\ &\quad + \varphi\left(\frac{\mathbf{F}_R(\mathbf{X})}{\lambda(z_R(\mathbf{X}))}, z_R(\mathbf{X})\right) \text{Cof } \mathbf{F}_R(\mathbf{X}) \end{aligned}$$

where $\varphi'_{\mathbf{F}_e}$ indicates the derivative of φ in its first variable. Use now the classical position $\mathbf{T}_R = (\det \mathbf{F}_R)^{-1} \mathbf{P}_R \mathbf{F}_R^\top$ to conclude that

$$\mathbf{T}_R = \frac{1}{\lambda(z_R)} \varphi'_{\mathbf{F}_e} \left(\frac{\mathbf{F}_R}{\lambda(z_R)}, z_R \right) \mathbf{F}_R^\top + \varphi \left(\frac{\mathbf{F}_R}{\lambda(z_R)}, z_R \right) \mathbf{I}.$$

In actual variables, the latter reads

$$(2.17) \quad \mathbf{T} = \frac{1}{\lambda(z)} \varphi'_{\mathbf{F}_e} \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) \mathbf{F}^\top + \varphi \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) \mathbf{I}.$$

The constitutive equation for \mathbf{D} can be obtained deduced from a z -dependent dissipation potential

$$(2.18) \quad \mathbf{v} \mapsto \int_{\Omega} \zeta(z; \mathbf{e}(\mathbf{v})) + \frac{\nu}{p} |\nabla \mathbf{e}(\mathbf{v})|^p \, d\mathbf{x}$$

for some given dissipation density ζ by taking its variation with respect to $\mathbf{e}(\mathbf{v})$, namely,

$$(2.19) \quad \mathbf{D} = \zeta'_e(z; \mathbf{e}(\mathbf{v})) - \operatorname{div}(\nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v})).$$

The occurrence of the higher-order ν -term corresponds to assuming that the body behaves as a so-called *nonsimple material*. This follows the theory by E. Fried and M. Gurtin [15], as already anticipated in the general nonlinear context of multipolar fluids by J. Nečas at al. [23–25] or solids [26, 35], as inspired by R. A. Toupin [34] and R. D. Mindlin [22]. Such higher-order term in the dissipation ensures that $\nabla \mathbf{v}$ belongs to $L^1_{w*}(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))$ (weakly* measurable), which guarantees the Lipschitz continuity of $\mathbf{v}(t, \cdot)$, almost everywhere in time. This will turn out crucial in many technical points later on, in particular in the estimates (3.12), (3.14), and (3.18). Note that, by dropping such regularity requirement, the treatment of the transport problem becomes nontrivial due to the possible onset of singularities, whose occurrence in solids may be debatable [1].

Eventually, we assume the mobility m to be positive function depending on \mathbf{F}_e and z , namely, $m = m(\mathbf{F}_e, z)$.

2.4 The model

Following the discussion leading to relation (2.11), in the following we equivalently recast the system (2.10) in terms of the variables $(\mathbf{v}, \mathbf{F}, z)$ by dropping the mass conservation equation (2.10a) and requiring the geometric relation (2.6) instead. Taking also the constitutive relations (2.17)-(2.16) and (2.19) into account we get

$$(2.20a) \quad \varrho \dot{\mathbf{v}} = \operatorname{div}(\mathbf{T} + \mathbf{D}) + \varrho \mathbf{g} \quad \text{with} \quad \varrho = \frac{\varrho_R}{\det \mathbf{F}},$$

$$\mathbf{T} = \frac{1}{\lambda(z)} \varphi'_{\mathbf{F}_e} \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) \mathbf{F}^\top + \varphi \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) \mathbf{I},$$

and $\mathbf{D} = \zeta'_e(z; \mathbf{e}(\mathbf{v})) - \operatorname{div}(\nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v})),$

$$(2.20b) \quad \dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F},$$

$$(2.20c) \quad \dot{z} = \operatorname{div} \left(m \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) \nabla \mu \right)$$

$$\text{with } \mu \in \varphi'_z \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) - \varphi'_{\mathbf{F}_e} \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) : \mathbf{F} \frac{\lambda'(z)}{\lambda^2(z)} + N_{[0,1]}(z).$$

We complement the system with the boundary conditions

$$(2.21a) \quad \mathbf{v} \cdot \mathbf{n} = 0, \quad \left((\mathbf{T} + \mathbf{D}) \mathbf{n} - \operatorname{div}_s (\nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v}) \mathbf{n}) \right)_\top = \mathbf{f},$$

$$(2.21b) \quad \nabla \mathbf{e}(\mathbf{v}) : (\mathbf{n} \otimes \mathbf{n}) = \mathbf{0}, \quad \text{and} \quad m(\mathbf{F}/\lambda(z), z) \nabla \mu \cdot \mathbf{n} + \varkappa \mu = h,$$

where the $(d-1)$ -dimensional surface divergence is defined as

$$(2.22) \quad \operatorname{div}_s = \operatorname{tr}(\nabla_s) \quad \text{with} \quad \nabla_s \bullet = \nabla \bullet - \frac{\partial \bullet}{\partial \mathbf{n}} \mathbf{n},$$

where $\operatorname{tr}(\cdot)$ is the trace of a $(d-1) \times (d-1)$ -matrix and ∇_s denotes the surface gradient. Let us again remark the crucial role of the impenetrability boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$, indeed allowing system (2.20) to be formulated in the fixed set.

We introduce the short-hand notation

$$(2.23) \quad \widehat{\varphi}(\mathbf{F}, z) = \varphi \left(\frac{\mathbf{F}}{\lambda(z)}, z \right) \quad \text{and} \quad \widehat{m}(\mathbf{F}, z) = m \left(\frac{\mathbf{F}}{\lambda(z)}, z \right).$$

This allows to rewrite (2.20) in terms of \mathbf{F} instead of \mathbf{F}_e . Thus (2.20) can equivalently be written in terms of $(\mathbf{v}, \mathbf{F}, z, \mu)$ as

$$(2.24a) \quad \varrho \dot{\mathbf{v}} = \operatorname{div}(\mathbf{T} + \mathbf{D}) + \varrho \mathbf{g} \quad \text{with} \quad \varrho = \frac{\varrho_R}{\det \mathbf{F}}, \quad \mathbf{T} = \widehat{\varphi}'_{\mathbf{F}}(\mathbf{F}, z) \mathbf{F}^\top + \widehat{\varphi}(\mathbf{F}, z) \mathbf{I},$$

$$\text{and} \quad \mathbf{D} = \zeta'_e(z; \mathbf{e}(\mathbf{v})) - \operatorname{div}(\nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v})),$$

$$(2.24b) \quad \dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F},$$

$$(2.24c) \quad \dot{z} = \operatorname{div}(\widehat{m}(\mathbf{F}, z) \nabla \mu) \quad \text{with} \quad \mu \in \widehat{\varphi}'_z(\mathbf{F}, z) + N_{[0,1]}(z).$$

The last boundary condition in (2.21) can be rewritten correspondingly as

$$\widehat{m}(\mathbf{F}, z) \nabla \mu \cdot \mathbf{n} + \varkappa \mu = h.$$

2.5 Energy balance

Let us present the energy balance underlying system (2.24) by testing the three equations respectively by \mathbf{v} , $\mathbf{S} = \widehat{\varphi}'_{\mathbf{F}}(\mathbf{F}, z)$, and μ and adding up. After integrating by parts using $\mathbf{v} \cdot \mathbf{n} = 0$, The terms $-\operatorname{div} \mathbf{T} \cdot \mathbf{v}$ and $\mu \dot{z}$ are to be treated jointly as follows

$$(2.25) \quad \int_{\Omega} \mathbf{T} : \nabla \mathbf{v} + \mu \dot{z} \, d\mathbf{x} = \int_{\Omega} \mathbf{S} : (\nabla \mathbf{v}) \mathbf{F} + \widehat{\varphi}(\mathbf{F}, z) \operatorname{div} \mathbf{v} + \mu \dot{z} \, d\mathbf{x}$$

$$\stackrel{(2.24b)}{=} \int_{\Omega} \widehat{\varphi}'_{\mathbf{F}}(\mathbf{F}, z) : \dot{\mathbf{F}} + \widehat{\varphi}(\mathbf{F}, z) \operatorname{div} \mathbf{v} + \widehat{\varphi}'_z(\mathbf{F}, z) \dot{z} \, d\mathbf{x}$$

$$= \frac{d}{dt} \int_{\Omega} \widehat{\varphi}(\mathbf{F}, z) \, d\mathbf{x} + \underbrace{\int_{\Omega} \nabla \widehat{\varphi}(\mathbf{F}, z) \cdot \mathbf{v} + \widehat{\varphi}(\mathbf{F}, z) \operatorname{div} \mathbf{v} \, d\mathbf{x}}_{= \int_{\Gamma} \widehat{\varphi}(\mathbf{F}, z) \mathbf{v} \cdot \mathbf{n} \, dS = 0}.$$

The dissipative terms $\operatorname{div} \mathbf{D} \cdot \mathbf{v}$ and $\mu \operatorname{div}(\widehat{m}(\mathbf{F}, z) \nabla \mu)$, resulting by testing (2.24c) by \mathbf{v} and (2.24c) by μ , can be treated by using twice the Green formula over Ω and once a surface Green formula over Γ . Specifically, using the short-hand notation $\mathfrak{H} = \nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v})$, we have

$$(2.26) \quad \begin{aligned} \int_{\Omega} \operatorname{div} \mathbf{D} \cdot \mathbf{v} + \dot{z} \mu \, d\mathbf{x} &= \int_{\Omega} \operatorname{div} \left(\zeta'_e(z; \mathbf{e}(\mathbf{v})) - \operatorname{div} \mathfrak{H} \right) \cdot \mathbf{v} + \mu \operatorname{div}(\widehat{m}(\mathbf{F}, z) \nabla \mu) \, d\mathbf{x} \\ &= \int_{\Gamma} \left(\zeta'_e(z; \mathbf{e}(\mathbf{v})) - \operatorname{div} \mathfrak{H} \right) : (\mathbf{v} \otimes \mathbf{n}) + \widehat{m}(\mathbf{F}, z) \nabla \mu \cdot \mathbf{n} \, dS \\ &\quad - \int_{\Omega} \left(\zeta'_e(z; \mathbf{e}(\mathbf{v})) - \operatorname{div} \mathfrak{H} \right) : \mathbf{e}(\mathbf{v}) + \widehat{m}(\mathbf{F}, z) |\nabla \mu|^2 \, d\mathbf{x} \\ &= \int_{\Gamma} \mathfrak{H} : (\mathbf{n} \otimes \mathbf{n}) + \left(\zeta'_e(z; \mathbf{e}(\mathbf{v})) - \operatorname{div} \mathfrak{H} \mathbf{n} - \operatorname{div}_s(\mathbf{n} \cdot \mathfrak{H}) \right) \cdot \mathbf{v} + \widehat{m}(\mathbf{F}, z) \nabla \mu \cdot \mathbf{n} \, dS \\ &\quad - \int_{\Omega} \zeta'_e(z; \mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) + \nu |\nabla \mathbf{e}(\mathbf{v})|^p + \widehat{m}(\mathbf{F}, z) |\nabla \mu|^2 \, d\mathbf{x}, \end{aligned}$$

where we also used the decomposition of $\nabla \mathbf{v} = (\mathbf{n} \cdot \nabla \mathbf{v}) \mathbf{n} + \nabla_s \mathbf{v}$ into its normal and tangential parts.

Since under (2.24) we have mass conservation (2.10a) as well, we can compute

$$(2.27) \quad \frac{\partial}{\partial t} \left(\frac{\varrho}{2} |\mathbf{v}|^2 \right) = \varrho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \varrho}{\partial t} \frac{|\mathbf{v}|^2}{2} = \varrho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div}(\varrho \mathbf{v}) \frac{|\mathbf{v}|^2}{2}.$$

By integrating this over Ω and using the Green formula and $\mathbf{v} \cdot \mathbf{n} = 0$, we obtain

$$(2.28) \quad \frac{d}{dt} \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2 \, d\mathbf{x} = \int_{\Omega} \varrho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \varrho \mathbf{v} \cdot (\mathbf{v} \cdot \nabla) \mathbf{v} \, d\mathbf{x} - \int_{\Gamma} \varrho |\mathbf{v}|^2 \mathbf{v} \cdot \mathbf{n} \, dS = \int_{\Omega} \varrho \dot{\mathbf{v}} \cdot \mathbf{v} \, d\mathbf{x}.$$

In particular, the inertial force $\varrho \dot{\mathbf{v}}$ tested by \mathbf{v} can be treated by (2.28) while the gravity force density $\varrho \mathbf{g}$ yields directly the power of the gravitational field $\varrho \mathbf{g} \cdot \mathbf{v}$.

Eventually, one obtains the energy balance

$$(2.29) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} \underbrace{\frac{\varrho}{2} |\mathbf{v}|^2}_{\text{kinetic energy}} + \underbrace{\widehat{\varphi}(\mathbf{F}, z)}_{\text{stored energy}} \, d\mathbf{x} + \int_{\Omega} \underbrace{\zeta'_e(z; \mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) + \nu |\nabla \mathbf{e}(\mathbf{v})|^p}_{\text{dissipation rate due to viscosity}} \, d\mathbf{x} \\ + \int_{\Omega} \underbrace{\widehat{m}(\mathbf{F}, z) |\nabla \mu|^2}_{\text{dissipation rate due to diffusion}} \, d\mathbf{x} + \int_{\Gamma} \underbrace{\varkappa \mu^2}_{\text{dissipation rate of influx}} \, dS = \int_{\Omega} \underbrace{\varrho \mathbf{g} \cdot \mathbf{v}}_{\text{power of gravity field}} \, d\mathbf{x} + \int_{\Gamma} \underbrace{\mathbf{f} \cdot \mathbf{v}}_{\text{power of traction}} + \underbrace{h \mu}_{\text{chemical influx}} \, dS. \end{aligned}$$

3 The analysis by Faedo-Galerkin semi-discretization

We consider the Cauchy problem for the system (2.20) with boundary conditions (2.21). For this, we prescribe the initial conditions

$$(3.1) \quad \mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \mathbf{F}|_{t=0} = \mathbf{F}_0, \quad \text{and} \quad z|_{t=0} = z_0.$$

In the following, we assume $\Omega \subset \mathbb{R}^n$ to be a nonempty, open, bounded, connected set with Lipschitz boundary $\Gamma := \partial\Omega$. We will use the following standard notation for Lebesgue and Sobolev spaces. Namely, $L^p(\Omega; \mathbb{R}^n)$ denotes the Banach space of Lebesgue measurable functions $\Omega \rightarrow \mathbb{R}^n$ whose p -power of the Euclidean norm is integrable and $W^{k,p}(\Omega; \mathbb{R}^n)$ is the space of $L^p(\Omega; \mathbb{R}^n)$ functions whose derivatives of order k are in $L^p(\Omega; \mathbb{R}^{n \times kd})$. We indicate $W_0^{2,p}(\Omega; \mathbb{R}^d) := \{\mathbf{v} \in W^{2,p}(\Omega; \mathbb{R}^d); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$ and use the short-hand notation $H^k = W^{k,2}$. Given a Banach space X and $I = [0, T]$, we use the notation $L^p(I; X)$ for the Bochner space of Bochner measurable functions $I \rightarrow X$ whose norm is in $L^p(I)$, and $H^1(I; X)$ for functions $I \rightarrow X$ whose distributional derivative is in $L^2(I; X)$. The spaces $C(I; X)$ and $C_w(I; X)$ indicate continuous and weakly continuous functions $I \rightarrow X$, respectively. Dual spaces are denoted by $(\cdot)^*$ and $p' = p/(p-1)$ indicates the conjugate exponent, with the convention $p' = \infty$ for $p = 1$ and $p' = 1$ for $p = \infty$. For $p < d$, we indicate by p^* the exponent from the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, i.e. $p^* = pd/(d-p)$. Occasionally, we will use $L_{w^*}^p(I; X)$ for weakly* measurable functions $I \rightarrow X$ for nonseparable spaces X which are duals to some other Banach spaces (specifically for $L^\infty(\Omega)$).

The energy balance (2.29) delivers formal a-priori estimates. Aiming at making this rigorous, we start by specifying our assumptions on the data. By indicating by $\text{GL}^+(d)$ the space of $d \times d$ matrices with positive determinant, we ask for the following.

$$(3.2a) \quad \varphi : \mathbb{R}^{d \times d} \rightarrow (-\infty, +\infty], \quad \varphi \in C^1(\text{GL}^+(d) \times \mathbb{R}) \quad \exists \kappa > 0 \quad \text{such that}$$

$$\varphi(\mathbf{F}_e, z) \geq \kappa / \det \mathbf{F}_e \quad \text{for all } \mathbf{F}_e \text{ with } \det \mathbf{F}_e > 0,$$

$$\varphi(\mathbf{F}_e, z) = +\infty \text{ for } \det \mathbf{F}_e \leq 0, \text{ and}$$

$$z \mapsto \widehat{\varphi}(\mathbf{F}, z) = \varphi\left(\frac{\mathbf{F}}{\lambda(z)}, z\right) \quad \text{strongly convex, uniformly w.r.t } \mathbf{F}, \text{ namely,}$$

$$\forall \mathbf{F} \in \text{GL}^+(d), \quad z_0, z_1 \in [0, 1], \quad \theta \in [0, 1] :$$

$$\widehat{\varphi}(\mathbf{F}, \theta z_1 + (1-\theta)z_0) \leq \theta \widehat{\varphi}(\mathbf{F}, z_1) + (1-\theta) \widehat{\varphi}(\mathbf{F}, z_0) - \frac{\kappa}{2} |z_1 - z_0|^2,$$

$$(3.2b) \quad p > d,$$

$$(3.2c) \quad \zeta : \mathbb{R} \times \mathbb{R}_{\text{sym}}^{d \times d} \rightarrow \mathbb{R} \quad \text{continuously differentiable, } \zeta(z, \cdot) \text{ convex,}$$

$$(3.2d) \quad \exists \bar{\varepsilon} > 0 \quad \forall (z, \mathbf{e}) \in \mathbb{R} \times \mathbb{R}_{\text{sym}}^{d \times d} : \quad \bar{\varepsilon} |\mathbf{e}|^2 \leq \zeta(z, \mathbf{e}) \leq (1 + |\mathbf{e}|^2) / \bar{\varepsilon},$$

$$(3.2e) \quad \lambda \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) \quad \text{and } \inf \lambda > 0,$$

$$(3.2f) \quad m : \mathbb{R}^{d \times d} \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{continuous and bounded with } \inf_{\mathbb{R}^{d \times d} \times \mathbb{R}} m > 0,$$

$$(3.2g) \quad \mathbf{g} \in L^1(I; L^\infty(\Omega; \mathbb{R}^d)), \quad \mathbf{f} \in L^{p'}(I; L^1(\Gamma; \mathbb{R}^d)), \quad h \in L^2(I; L^{4/3}(\Gamma)),$$

$$(3.2h) \quad \mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \mathbf{F}_0 \in W^{1,r}(\Omega; \mathbb{R}^{d \times d}), \quad z_0 \in W^{1,r}(\Omega), \quad \varrho_R \in W^{1,r}(\Omega), \quad r > d,$$

$$\text{with } \min_{\bar{\Omega}} \det \mathbf{F}_0 > 0 \quad \text{and} \quad \min_{\bar{\Omega}} \varrho_R > 0.$$

Note that the concrete form of $\varphi(\mathbf{F}, z)$ will actually be relevant only for $z \in [0, 1]$. Still, as in the proof of Proposition 3.4 such constraint is penalized, we are asked to define φ also for outside the interval $[0, 1]$ in (3.2a), and similarly also for ζ and m .

Before moving on, let us show that conditions (3.2a) can be realized in some physically

relevant situation. To this aim, let λ be positive. For all \mathbf{F}_e with $\det \mathbf{F}_e > 0$ let the *Ogden-type* energy density be defined as

$$\varphi(\mathbf{F}_e, z) = f_1(z)g_1(\mathbf{F}_e \mathbf{F}_e^\top) + f_2(z)g_2(\text{Cof}(\mathbf{F}_e \mathbf{F}_e^\top)) + f_3(z)g_3(\det \mathbf{F}_e) + \frac{\kappa}{\det \mathbf{F}_e} + h(z),$$

where h is uniformly convex, g_i, f_i are smooth, nonnegative, convex, with $g_i(0) = 0$, for $i = 1, 2, 3$. Indeed, the uniform convexity of $z \mapsto \widehat{\varphi}(\mathbf{F}, z)$ with respect to \mathbf{F} from (3.2a) follows from the uniform convexity of h and from the convexity in z of all other terms. Such convexity can be checked by noticing that all such terms have the form $\eta(\mathbf{F}_e, z) = f(z)g(H(\mathbf{F}_e))$ where H is a s -homogeneous function. In particular,

$$\eta(\mathbf{F}_e, z) = \eta\left(\frac{\mathbf{F}}{\lambda(z)}, z\right) = f(z)g\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right).$$

Under general assumptions on f and g , convexity can be directly checked by computing the second derivative with respect to z . To simplify notation, assume H to be scalar valued (which is the case for $i = 3$) and compute

$$\begin{aligned} \eta''_z(\mathbf{F}_e, z) &= \eta''_z\left(\frac{\mathbf{F}}{\lambda(z)}, z\right) = f''(z)g\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right) - 2sf'(z)g'\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right)\frac{H(\mathbf{F})\lambda'(z)}{(\lambda(z))^{s+1}} \\ &\quad - sf(z)g'\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right)\frac{H(\mathbf{F})\lambda''(z)}{(\lambda(z))^{s+1}} + s(s+1)f(z)g'\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right)\frac{H(\mathbf{F})(\lambda'(z))^2}{(\lambda(z))^{s+2}} \\ &\quad + s^2f(z)g''\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right)\frac{(H(\mathbf{F}))^2(\lambda'(z))^2}{(\lambda(z))^{2s+2}} \\ &\geq s(\lambda(z))^2g'\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right)\left(\frac{H(\mathbf{F})}{(\lambda(z))^s}\right)\left((s+1)f(z)(\lambda'(z))^2 - 2f'(z)\lambda'(z)\lambda(z) - f(z)\lambda''(z)\lambda(z)\right). \end{aligned}$$

As $g'(r)r \geq g(r) \geq 0$, the latter is nonnegative under appropriate assumptions on λ and f , for instance, if f is nonincreasing and λ is nondecreasing and concave.

In order to obtain a-priori estimates from (2.29), a number of technical points have to be faced. One first issue is estimation of the gravity force $\varrho \mathbf{g}$ when tested by the velocity \mathbf{v} , which can be estimated by the Hölder/Young inequality as

$$\begin{aligned} (3.3) \quad \int_{\Omega} \varrho \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x} &= \int_{\Omega} \sqrt{\frac{\varrho_R}{\det \mathbf{F}}} \sqrt{\varrho} \mathbf{v} \cdot \mathbf{g} \, d\mathbf{x} \leq \left\| \sqrt{\frac{\varrho_R}{\det \mathbf{F}}} \right\|_{L^2(\Omega)} \left\| \sqrt{\varrho} \mathbf{v} \right\|_{L^2(\Omega; \mathbb{R}^d)} \left\| \mathbf{g} \right\|_{L^\infty(\Omega; \mathbb{R}^d)} \\ &\leq \frac{1}{2} \left(\left\| \sqrt{\frac{\varrho_R}{\det \mathbf{F}}} \right\|_{L^2(\Omega)}^2 + \left\| \sqrt{\varrho} \mathbf{v} \right\|_{L^2(\Omega; \mathbb{R}^d)}^2 \right) \left\| \mathbf{g} \right\|_{L^\infty(\Omega; \mathbb{R}^d)} \\ &= \left\| \mathbf{g} \right\|_{L^\infty(\Omega; \mathbb{R}^d)} \int_{\Omega} \frac{\varrho_R}{2 \det \mathbf{F}} + \frac{\varrho}{2} |\mathbf{v}|^2 \, d\mathbf{x}. \end{aligned}$$

The integral on the right-hand side can be then treated by the Gronwall lemma, by relying on the kinetic-energy term and the fact that the stored energy controls $1/\det \mathbf{F}$, i.e., (3.2a). In order to apply the Gronwall lemma one needs the qualification (3.2g) for \mathbf{g} .

A second technical issue is estimation of the boundary term $\mathbf{f} \cdot \mathbf{v}$, which will follow along the lines of relation (3.34) below, which in turn hinges on a bound on $1/\varrho$, cf. (3.5b).

Eventually, $\int_{\Gamma} h \mu \, dS$ can be estimated by $\|h\|_{L^{4/3}(\Gamma)} \|\mu\|_{L^4(\Gamma)} \leq \|h\|_{L^{4/3}(\Gamma)}^2 / \delta + \delta \|\mu\|_{L^4(\Gamma)}^2 \leq \|h\|_{L^{4/3}(\Gamma)}^2 / \delta + \delta N \|\nabla \mu\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \delta N \|\mu\|_{L^2(\Gamma)}^2$, where N indicates the square of the norm of the trace operator $H^1(\Omega) \rightarrow L^4(\Gamma)$.

Under assumptions (3.2), the energy balance (2.29) thus implies the a-priori estimates

$$\begin{aligned} (3.4a) \quad & \|\sqrt{\varrho} \mathbf{v}\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \leq C, \\ (3.4b) \quad & \|\varphi(\mathbf{F}_e, z)\|_{L^\infty(I; L^1(\Omega))} \leq C, \\ (3.4c) \quad & \|\mathbf{e}(\mathbf{v})\|_{L^2(I; W^{1,p}(\mathbb{R}^{d \times d}))} \leq C, \\ (3.4d) \quad & \|z\|_{L^\infty(I \times \Omega)} \leq C, \quad \text{and} \\ (3.4e) \quad & \|\mu\|_{L^2(I; H^1(\Omega))} \leq C, \end{aligned}$$

where, here and in the following, for the sake of notational simplicity the symbol C is used to indicate any positive constant just depending on data and possibly varying from line to line. In case of need, we will indicate the dependence of such constant on specific parameters by using subscripts.

As $p > d$, estimate (3.4c) prevents the onset of singularities for the quantities transported by the velocity field. In particular, due to qualification of \mathbf{F}_0 and $\varrho_0 = \varrho_R / \det \mathbf{F}_0$ in (3.2h), it yields the estimates

$$\begin{aligned} (3.5a) \quad & \|\mathbf{F}\|_{L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d}))} \leq C_r, \quad \left\| \frac{1}{\det \mathbf{F}} \right\|_{L^\infty(I; W^{1,r}(\Omega))} \leq C_r, \\ (3.5b) \quad & \|\varrho\|_{L^\infty(I; W^{1,r}(\Omega))} \leq C_r, \quad \text{and} \quad \left\| \frac{1}{\varrho} \right\|_{L^\infty(I; W^{1,r}(\Omega))} \leq C_r \quad \text{for any } 1 \leq r < +\infty, \end{aligned}$$

see the arguments in the proof of Lemmas 3.2 and 3.3 below. From (3.4a) and (3.5b), we then also have

$$(3.5c) \quad \|\mathbf{v}\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \leq \|\sqrt{\varrho} \mathbf{v}\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \left\| \frac{1}{\sqrt{\varrho}} \right\|_{L^\infty(I \times \Omega)} \leq C.$$

Based on the formal a-priori estimates (3.4)-(3.5) we now specify a notion of weak solution. In particular, we replace the inertial force $\varrho \dot{\mathbf{v}}$ in (2.20a) by using the equality

$$(3.6) \quad \varrho \dot{\mathbf{v}} = \frac{\partial}{\partial t}(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}),$$

as well as $\varrho(0) = \varrho_R / \det \mathbf{F}_0$. Noteworthy, this formula has exploited the continuity equation (2.10a).

Definition 3.1 (Weak solutions to (2.20)). *We call quintuple $(\varrho, \mathbf{v}, \mathbf{F}, z, \mu) \in L^\infty(I \times \Omega) \times (L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \cap C_w(I; L^2(\Omega))) \times L^\infty(I \times \Omega; \mathbb{R}^{d \times d}) \times C_w(I; L^2(\Omega)) \times L^2(I; H^1(\Omega))$ a weak solution to the system (2.20) with initial and boundary conditions (2.21) and (3.1) if $\mathbf{v} \cdot \mathbf{n} = 0$, $\det \mathbf{F} > 0$ and $\varrho = \varrho_R / \det \mathbf{F}$ a.e. on $I \times \Omega$, $\varphi(\mathbf{F}, z), \varphi'_z(\mathbf{F}, z) \in L^1(I \times \Omega)$, $\varphi'_F(\mathbf{F}, z) \in L^1(I \times \Omega; \mathbb{R}^{d \times d})$, z is valued in $[0, 1]$, $\mathbf{v}(0) = \mathbf{v}_0$,*

$$(3.7a) \quad \int_0^T \int_\Omega \left(\left(\widehat{\varphi}'_F(\mathbf{F}, z) \mathbf{F}^\top + \zeta'_e(z; \mathbf{e}(\mathbf{v})) - \varrho \mathbf{v} \otimes \mathbf{v} \right) : \nabla \tilde{\mathbf{v}} - \varrho \mathbf{v} \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \widehat{\varphi}(\mathbf{F}, z) (\operatorname{div} \tilde{\mathbf{v}}) \right)$$

$$+ \nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v}) : \nabla \mathbf{e}(\tilde{\mathbf{v}}) \Big) \, d\mathbf{x} dt = \int_0^T \int_{\Omega} \varrho \mathbf{g} \cdot \tilde{\mathbf{v}} \, d\mathbf{x} dt + \int_0^T \int_{\Gamma} \mathbf{f} \cdot \tilde{\mathbf{v}} \, dS dt$$

holds for any $\tilde{\mathbf{v}} \in C^\infty(I \times \bar{\Omega}; \mathbb{R}^d)$ with $\tilde{\mathbf{v}} \cdot \mathbf{n} = 0$ and $\tilde{\mathbf{v}}(T) = 0 = \tilde{\mathbf{v}}(0)$,

$$(3.7b) \quad \int_0^T \int_{\Omega} \mathbf{F} : \frac{\partial \tilde{\mathbf{S}}}{\partial t} + \left((\operatorname{div} \mathbf{v}) \mathbf{F} + (\nabla \mathbf{v}) \mathbf{F} \right) : \tilde{\mathbf{S}} + \mathbf{F} : ((\mathbf{v} \cdot \nabla) \tilde{\mathbf{S}}) \, d\mathbf{x} dt = - \int_{\Omega} \mathbf{F}_0 : \tilde{\mathbf{S}}(0) \, d\mathbf{x}$$

holds for any $\tilde{\mathbf{S}} \in C^\infty(I \times \bar{\Omega}; \mathbb{R}^{d \times d})$ with $\tilde{\mathbf{S}}(T) = \mathbf{0}$,

$$(3.7c) \quad \int_0^T \int_{\Omega} \hat{m}(\mathbf{F}, z) \nabla \mu \cdot \nabla \tilde{z} - z \frac{\partial \tilde{z}}{\partial t} - z \operatorname{div}(\mathbf{v} \tilde{z}) \, d\mathbf{x} dt + \int_0^T \int_{\Gamma} \varkappa \mu \tilde{z} \, dS dt \\ = \int_{\Omega} \mu_0 \tilde{z}(0) \, d\mathbf{x} + \int_0^T \int_{\Gamma} h \tilde{z} \, dS dt$$

holds for any $\tilde{z} \in C^\infty(I \times \bar{\Omega})$ with $\tilde{z}(T) = 0$, and

$$(3.7d) \quad \int_0^T \int_{\Omega} (\hat{\varphi}'_z(\mathbf{F}, z) - \mu)(\tilde{z} - z) \, d\mathbf{x} dt \geq 0$$

holds for any $\tilde{z} \in L^\infty(I \times \Omega)$ valued in $[0, 1]$.

If the velocity field \mathbf{v} is in $L^1(I; W^{1,\infty}(\Omega; \mathbb{R}^d))$, one classically obtain that regularity of the initial datum is preserved along the flow (2.24b). We provide a rigorous statement in the following lemma, as well as a proof based on Galerkin approximations.

Lemma 3.2 (Flow of \mathbf{F}). *Let $p > d$ and $r > 2$. Then, for any $\mathbf{v} \in L^1(I; W^{2,p}(\Omega; \mathbb{R}^d))$ with $\mathbf{v} \cdot \mathbf{n} = 0$ and any $\mathbf{F}_0 \in W^{1,r}(\Omega; \mathbb{R}^{d \times d})$, there exists a unique weak solution $\mathbf{F} \in C_w(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,1}(I; L^r(\Omega; \mathbb{R}^{d \times d}))$ to (2.24b) in the sense of (3.7b). The estimate*

$$(3.8) \quad \|\mathbf{F}\|_{L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,1}(I; L^r(\Omega; \mathbb{R}^{d \times d}))} \leq \mathfrak{C} \left(\|\nabla \mathbf{v}\|_{L^1(I; W^{1,p}(\Omega; \mathbb{R}^{d \times d}))}, \|\mathbf{F}_0\|_{W^{1,r}(\Omega; \mathbb{R}^{d \times d})} \right)$$

holds with some $\mathfrak{C} \in C(\mathbb{R}^2)$, equation (2.24b) holds a.e. on $I \times \Omega$, and $\mathbf{F} \in C(I \times \bar{\Omega}; \mathbb{R}^{d \times d})$. Moreover, the mapping

$$(3.9) \quad \mathbf{v} \mapsto \mathbf{F} : L^1(I; W^{2,p}(\Omega; \mathbb{R}^d)) \rightarrow L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d}))$$

is (weak, weak*)-continuous. If in addition $\det \mathbf{F}_0 > 0$ on $\bar{\Omega}$, then $\det \mathbf{F} > 0$ on $I \times \bar{\Omega}$; i.e., $\min_{I \times \bar{\Omega}} \det \mathbf{F} > 0$ uniformly with respect to bounded velocity fields \mathbf{v} , namely, for any $R > 0$ there exists $\delta > 0$ such that

$$(3.10) \quad \|\nabla \mathbf{v}\|_{L^1(I; W^{1,p}(\Omega; \mathbb{R}^{d \times d}))} \leq R \quad \Rightarrow \quad \min_{I \times \bar{\Omega}} \det \mathbf{F} \geq \delta.$$

Proof. Let us start by assuming $\mathbf{v} \in L^2(I; W^{2,p}(\Omega; \mathbb{R}^d))$; the weaker integrability setting of $\mathbf{v} \in L^1(I; W^{2,p}(\Omega; \mathbb{R}^d))$ will be recovered later in the proof.

Consider the following parabolic regularization of (2.24b)

$$(3.11) \quad \dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F} + \varepsilon \operatorname{div}(|\nabla \mathbf{F}|^{r-2} \nabla \mathbf{F}),$$

by complementing it by the additional boundary condition $(\nabla \mathbf{F}) \mathbf{n} = \mathbf{0}$. We tackle the regularized problem (3.11) by means of a Faedo-Galerkin approximation. Assume to be given a sequence of nested finite-dimensional subspaces $\{U_k\}_{k \in \mathbb{N}}$ whose union is dense in $W^{1,r}(\Omega; \mathbb{R}^{d \times d})$. Without loss of generality, we can ask for $\mathbf{F}_0 \in U_1$. The classical existence theory for systems of ordinary differential equations ensures that one can find a solution $t \in I \mapsto \mathbf{F}_k(t) \in U_k$ of the Galerkin-approximated problem for any k ; more precisely, local in time existence needs to be combined with maximal prolongation on the whole interval I , on the basis of the L^∞ -estimates below).

Testing (the Galerkin approximation of) (3.11) by \mathbf{F}_k we can estimate

$$(3.12) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\mathbf{F}_k|^2 \, d\mathbf{x} + \varepsilon \int_{\Omega} |\nabla \mathbf{F}_k|^r \, d\mathbf{x} &= \int_{\Omega} \left((\nabla \mathbf{v}) \mathbf{F}_k - (\mathbf{v} \cdot \nabla) \mathbf{F}_k \right) : \mathbf{F}_k \, d\mathbf{x} \\ &= \int_{\Omega} (\nabla \mathbf{v}) \mathbf{F}_k : \mathbf{F}_k + \frac{\operatorname{div} \mathbf{v}}{2} |\mathbf{F}_k|^2 \, d\mathbf{x} \leq \frac{3}{2} \|\nabla \mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \|\mathbf{F}_k\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2; \end{aligned}$$

where we used the calculus

$$\begin{aligned} \int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{F}_k : \mathbf{F}_k \, d\mathbf{x} &= \int_{\Gamma} |\mathbf{F}_k|^2 (\mathbf{v} \cdot \mathbf{n}) \, dS \\ &\quad - \int_{\Omega} \mathbf{F}_k : (\mathbf{v} \cdot \nabla) \mathbf{F}_k + (\operatorname{div} \mathbf{v}) |\mathbf{F}_k|^2 \, d\mathbf{x} = -\frac{1}{2} \int_{\Omega} (\operatorname{div} \mathbf{v}) |\mathbf{F}_k|^2 \, d\mathbf{x}, \end{aligned}$$

together with the boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$. Note in particular that, in order to perform the latter integration by parts, the integrability of \mathbf{v} is required, besides the regularity of $\nabla \mathbf{v}$. By the Gronwall inequality we obtain the estimate

$$(3.13) \quad \|\mathbf{F}_k\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{with} \quad \|\nabla \mathbf{F}_k\|_{L^r(I \times \Omega; \mathbb{R}^{d \times d \times d})} \leq C \varepsilon^{-1/r}.$$

At the Galerkin-discretization level, another legitimate test for (3.11) is $\frac{\partial}{\partial t} \mathbf{F}_k$. This allows us to estimate

$$(3.14) \quad \begin{aligned} \int_{\Omega} \left| \frac{\partial \mathbf{F}_k}{\partial t} \right|^2 \, d\mathbf{x} + \frac{\varepsilon}{r} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{F}_k|^r \, d\mathbf{x} &= \int_{\Omega} \left((\nabla \mathbf{v}) \mathbf{F}_k - (\mathbf{v} \cdot \nabla) \mathbf{F}_k \right) : \frac{\partial \mathbf{F}_k}{\partial t} \, d\mathbf{x} \\ &\leq \|\nabla \mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})}^2 \|\mathbf{F}_k\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 \\ &\quad + C_r \|\mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^d)}^2 \left(1 + \|\nabla \mathbf{F}_k\|_{L^r(\Omega; \mathbb{R}^{d \times d})}^r \right) + \frac{1}{2} \left\| \frac{\partial \mathbf{F}_k}{\partial t} \right\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2. \end{aligned}$$

Note that we used here that $r > 2$. As $\mathbf{v} \in L^2(I; W^{2,p}(\Omega; \mathbb{R}^d)) \subset L^2(I; L^\infty(\Omega; \mathbb{R}^d))$, the already obtained estimate (3.13), and the Gronwall inequality imply that

$$(3.15) \quad \left\| \frac{\partial \mathbf{F}_k}{\partial t} \right\|_{L^2(\Omega \times I; \mathbb{R}^{d \times d})} \leq C e^{1/(r\varepsilon)} \quad \text{and} \quad \|\nabla \mathbf{F}_k\|_{L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d \times d}))} \leq C e^{1/(r\varepsilon)}.$$

Keeping $\varepsilon > 0$ fixed, the above estimates allow us to pass to the limit as $k \rightarrow \infty$ by standard arguments for quasilinear parabolic equations; realize that all lower-order terms are linear while the last, highest-order quasilinear term in (3.11) is (even uniformly) monotone and its limit passage (exploiting compact embedding and Minty's trick or strong convergence) is easy, cf. e.g. [29, Ch.8]. The limit is a weak solution to the initial-boundary value problem for (3.11), which we denote by $\mathbf{F}_\varepsilon \in H^1(I; L^2(\Omega; \mathbb{R}^{d \times d})) \cap L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d}))$. As this solution is unique, no extraction of subsequences is actually needed and the whole sequence $\{\mathbf{F}_k\}_{k \in \mathbb{N}}$ converges to \mathbf{F}_ε ; the uniqueness for \mathbf{v} given is easy by the uniform monotonicity of the quasilinear term and by handling the lower-order terms by Green formula as in (3.12) and the Gronwall inequality.

Recalling now that

$$(3.16) \quad \left\| \frac{\partial \mathbf{F}_\varepsilon}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{F}_\varepsilon - (\nabla \mathbf{v}) \mathbf{F}_\varepsilon \right\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq C e^{1/(r\varepsilon)},$$

by comparison in (3.11) we also obtain

$$(3.17) \quad \|\varepsilon \operatorname{div}(|\nabla \mathbf{F}_\varepsilon|^{r-2} \nabla \mathbf{F}_\varepsilon)\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq C e^{1/(r\varepsilon)}.$$

Note that this estimate degenerates for $\varepsilon \rightarrow 0$. Still, we have that $\operatorname{div}(|\nabla \mathbf{F}_\varepsilon|^{r-2} \nabla \mathbf{F}_\varepsilon) \in L^2(I \times \Omega; \mathbb{R}^{d \times d})$, so that (3.11) is solved almost everywhere. In particular, we can legitimately test it by $\operatorname{div}(|\nabla \mathbf{F}_\varepsilon|^{r-2} \nabla \mathbf{F}_\varepsilon)$. Since $p > d$, we have $p^{-1} + (r^*)^{-1} + (r')^{-1} \leq 1$, and thus by the Hölder and Young inequalities, we can estimate

$$(3.18) \quad \begin{aligned} \frac{d}{dt} \int_\Omega \frac{1}{r} |\nabla \mathbf{F}_\varepsilon|^r \, d\mathbf{x} &\leq \frac{d}{dt} \int_\Omega \frac{1}{r} |\nabla \mathbf{F}_\varepsilon|^r \, d\mathbf{x} + \varepsilon \int_\Omega |\operatorname{div}(|\nabla \mathbf{F}_\varepsilon|^{r-2} \nabla \mathbf{F}_\varepsilon)|^2 \, d\mathbf{x} \\ &= - \int_\Omega \nabla((\mathbf{v} \cdot \nabla) \mathbf{F}_\varepsilon - (\nabla \mathbf{v}) \mathbf{F}_\varepsilon) : (|\nabla \mathbf{F}_\varepsilon|^{r-2} \nabla \mathbf{F}_\varepsilon) \, d\mathbf{x} \\ &= - \int_\Omega |\nabla \mathbf{F}_\varepsilon|^{r-2} (\nabla \mathbf{F}_\varepsilon \otimes \nabla \mathbf{F}_\varepsilon) : \mathbf{e}(\mathbf{v}) - \frac{1}{r} |\nabla \mathbf{F}_\varepsilon|^r \operatorname{div} \mathbf{v} \\ &\quad - ((\nabla \mathbf{v}) \nabla \mathbf{F}_\varepsilon + (\nabla^2 \mathbf{v}) \mathbf{F}_\varepsilon) : (|\nabla \mathbf{F}_\varepsilon|^{r-2} \nabla \mathbf{F}_\varepsilon) \, d\mathbf{x} \\ &\leq C_r \|\nabla \mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \|\nabla \mathbf{F}_\varepsilon\|_{L^r(\Omega; \mathbb{R}^{d \times d})}^r \\ &\quad + C_r \|\nabla^2 \mathbf{v}\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} \|\mathbf{F}_\varepsilon\|_{L^{r^*}(\Omega; \mathbb{R}^{d \times d})} \|\nabla \mathbf{F}_\varepsilon\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^{r-1} \\ &\leq C_r \|\nabla \mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \|\nabla \mathbf{F}_\varepsilon\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^r \\ &\quad + C_r N \|\nabla^2 \mathbf{v}\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} \|\mathbf{F}_\varepsilon\|_{L^2(\Omega; \mathbb{R}^{d \times d})} (1 + \|\nabla \mathbf{F}_\varepsilon\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^r) \\ &\quad + C_r N \|\nabla^2 \mathbf{v}\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} \|\nabla \mathbf{F}_\varepsilon\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^r, \end{aligned}$$

where we used $p > d$ in order to get $\nabla \mathbf{v} \in W^{1,p}(\Omega; \mathbb{R}^{d \times d}) \subset L^\infty(\Omega; \mathbb{R}^{d \times d})$, as well as the computation

$$\int_\Omega \nabla((\mathbf{v} \cdot \nabla) \mathbf{F}_\varepsilon) : |\nabla \mathbf{F}_\varepsilon|^{r-2} \nabla \mathbf{F}_\varepsilon \, d\mathbf{x}$$

$$\begin{aligned}
&= \int_{\Omega} |\nabla \mathbf{F}_{\varepsilon}|^{r-2} (\nabla \mathbf{F}_{\varepsilon} \otimes \nabla \mathbf{F}_{\varepsilon}) : \mathbf{e}(\mathbf{v}) + (\mathbf{v} \cdot \nabla) \nabla \mathbf{F}_{\varepsilon} : |\nabla \mathbf{F}_{\varepsilon}|^{r-2} \nabla \mathbf{F}_{\varepsilon} \, d\mathbf{x} \\
&= \int_{\Gamma} |\nabla \mathbf{F}_{\varepsilon}|^r \mathbf{v} \cdot \mathbf{n} \, dS + \int_{\Omega} \left(|\nabla \mathbf{F}_{\varepsilon}|^{r-2} (\nabla \mathbf{F}_{\varepsilon} \otimes \nabla \mathbf{F}_{\varepsilon}) : \mathbf{e}(\mathbf{v}) \right. \\
&\quad \left. - (\operatorname{div} \mathbf{v}) |\nabla \mathbf{F}_{\varepsilon}|^r - (r-1) |\nabla \mathbf{F}_{\varepsilon}|^{r-2} \nabla \mathbf{F}_{\varepsilon} : (\mathbf{v} \cdot \nabla) \nabla \mathbf{F}_{\varepsilon} \right) d\mathbf{x} \\
&= \int_{\Gamma} \frac{|\nabla \mathbf{F}_{\varepsilon}|^r}{r} \mathbf{v} \cdot \mathbf{n} \, dS + \int_{\Omega} |\nabla \mathbf{F}_{\varepsilon}|^{r-2} (\nabla \mathbf{F}_{\varepsilon} \otimes \nabla \mathbf{F}_{\varepsilon}) : \mathbf{e}(\mathbf{v}) - (\operatorname{div} \mathbf{v}) \frac{|\nabla \mathbf{F}_{\varepsilon}|^r}{r} \, d\mathbf{x}.
\end{aligned}$$

Again, the boundary integral above vanishes since

$\mathbf{v} \cdot \mathbf{n} = 0$. For the last inequality in (3.18), we have used $\|\mathbf{F}_{\varepsilon}\|_{L^{r^*}(\Omega; \mathbb{R}^{d \times d})} \leq N(\|\mathbf{F}_{\varepsilon}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} + \|\nabla \mathbf{F}_{\varepsilon}\|_{L^r(\Omega; \mathbb{R}^{d \times d})})$, where N is the norm of the embedding $W^{1,r}(\Omega) \subset L^{r^*}(\Omega)$ if $W^{1,r}(\Omega)$ is endowed with the norm $\|\cdot\|_{L^2(\Omega)} + \|\nabla \cdot\|_{L^r(\Omega; \mathbb{R}^d)}$.

One can thus apply the Gronwall inequality to (3.18). Correspondingly, by using the former estimate in (3.13) and the regularity of the initial datum $\mathbf{F}_0 \in W^{1,r}(\Omega; \mathbb{R}^{d \times d})$ one obtains the estimates

$$(3.19a) \quad \|\nabla \mathbf{F}_{\varepsilon}\|_{L^{\infty}(I; L^r(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{and}$$

$$(3.19b) \quad \|\operatorname{div}(|\nabla \mathbf{F}_{\varepsilon}|^{r-2} \nabla \mathbf{F}_{\varepsilon})\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq C \varepsilon^{-1/2}.$$

The limit passage for $\varepsilon \rightarrow 0$ in the linear terms is then straightforward and the quasilinear regularizing term in (3.11) converges to 0 as $\mathcal{O}(\varepsilon^{1/2})$ for $\varepsilon \rightarrow 0$ due to (3.19b). Alternatively, one can observe that when tested by $\tilde{\mathbf{S}} \in L^r(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d}))$ and by using (3.19a), quasilinear regularizing term converges to 0 even faster as

$$\left| \int_0^T \int_{\Omega} \varepsilon |\nabla \mathbf{F}_{\varepsilon}|^{r-2} \nabla \mathbf{F}_{\varepsilon} : \nabla \tilde{\mathbf{S}} \, d\mathbf{x} dt \right| \leq \varepsilon \|\nabla \mathbf{F}_{\varepsilon}\|_{L^r(I \times \Omega; \mathbb{R}^{d \times d})}^{r-1} \|\nabla \tilde{\mathbf{S}}\|_{L^r(I \times \Omega; \mathbb{R}^{d \times d})} = \mathcal{O}(\varepsilon).$$

In any case, the limit for $\varepsilon \rightarrow 0$ solves the original nonregularized initial-boundary value problem for (2.24b). As this equation is linear, the solution is unique and no extraction of subsequences is needed in the $\varepsilon \rightarrow 0$ limit passage.

Estimate (3.15) on $\frac{\partial}{\partial t} \mathbf{F}_{\varepsilon}$ does not pass to the limit as $\varepsilon \rightarrow 0$. Still, we can argue by comparison in $\frac{\partial}{\partial t} \mathbf{F} = (\nabla \mathbf{v}) \mathbf{F} - (\mathbf{v} \cdot \nabla) \mathbf{F}$ and get the estimate

$$(3.20) \quad \left\| \frac{\partial \mathbf{F}}{\partial t} \right\|_{L^1(I; L^r(\Omega; \mathbb{R}^{d \times d}))} \leq C.$$

In particular, (2.24b) holds a.e. on $I \times \Omega$. By the embedding

$$L^{\infty}(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,1}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \subset C(I \times \bar{\Omega}; \mathbb{R}^{d \times d}),$$

we also have that $\mathbf{F} \in C(I \times \bar{\Omega}; \mathbb{R}^{d \times d})$.

The (weak, weak*)-continuity of the mapping $\mathbf{v} \mapsto \mathbf{F}$ is easy to obtain. Let $\mathbf{v}_n \rightarrow \mathbf{v}$ weakly in $L^1(I; W^{2,p}(\Omega; \mathbb{R}^{d \times d}))$ and let \mathbf{F}_n be the corresponding unique solutions of (2.24b). Starting from the bound (3.20) on \mathbf{F}_n in $W^{1,1}(I; L^r(\Omega; \mathbb{R}^{d \times d}))$ (which indeed depends on $\|\mathbf{v}_n\|_{L^1(I; W^{2,p}(\Omega; \mathbb{R}^{d \times d}))}$), one applies the Aubin-Lions theorem obtaining strong convergence of

\mathbf{F}_n in $L^{1/\epsilon}(I; L^{r^*-\epsilon}(\Omega; \mathbb{R}^{d \times d}))$ for any $0 < \epsilon \leq 1$. Then, we simply pass to the limit in (2.24b) in its weak formulation (3.7b) as $n \rightarrow \infty$.

Recall that $\mathbf{F}_0 \in W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \subset L^\infty(\Omega; \mathbb{R}^{d \times d})$. If $\det \mathbf{F}_0 > 0$ on $\bar{\Omega}$, \mathbf{F}_0^{-1} exists and is bounded on $\bar{\Omega}$. In fact we have that $\mathbf{F}_0^{-1} \in W^{1,r}(\Omega; \mathbb{R}^{d \times d})$ as

$$\nabla \mathbf{F}_0^{-1} = \nabla \left(\frac{\text{Cof} \mathbf{F}_0}{\det \mathbf{F}_0} \right) = \left(\frac{\text{Cof}'(\mathbf{F}_0)}{\det \mathbf{F}_0} - \frac{\text{Cof}(\mathbf{F}_0) \text{Cof}'(\mathbf{F}_0)}{\det \mathbf{F}_0^2} \right) \nabla \mathbf{F}_0 \in L^r(\Omega; \mathbb{R}^{d \times d \times d}).$$

We can then apply the above arguments to the flow equation (2.7) for the inverse \mathbf{F}^{-1} , as well. In particular, we obtain that \mathbf{F}^{-1} is bounded on $I \times \Omega$, so that $1/\det \mathbf{F}$ stays positive and bounded away from 0. \square

A scalar-valued variant of Lemma 3.2 holds for the continuity equation (2.10a). Its weak formulation corresponds to the integral identity

$$(3.21) \quad \int_0^T \int_\Omega \varrho \frac{\partial v}{\partial t} + \varrho \mathbf{v} \cdot \nabla v \, d\mathbf{x} dt = \int_\Omega \varrho_0 v(0) \, d\mathbf{x}$$

for any $v \in C^1(I \times \bar{\Omega})$ with $v(T) = 0$. We state this Lemma without proof, for the sake of completeness and later reference.

Lemma 3.3 (Flow of ϱ). *Let $p > d$ and $r > 2$. Then, for any $\mathbf{v} \in L^1(I; W^{2,p}(\Omega; \mathbb{R}^d))$ with $\mathbf{v} \cdot \mathbf{n} = 0$ and any $\varrho_0 \in W^{1,r}(\Omega)$, there exists a unique weak solution $\varrho \in C_w(I; W^{1,r}(\Omega)) \cap W^{1,1}(I; L^r(\Omega))$ to (2.10a) in the sense of (3.21) and the estimate*

$$(3.22) \quad \|\varrho\|_{L^\infty(I; W^{1,r}(\Omega)) \cap W^{1,1}(I; L^r(\Omega))} \leq \mathfrak{C} \left(\|\nabla \mathbf{v}\|_{L^1(I; W^{1,p}(\Omega; \mathbb{R}^{d \times d}))}, \|\varrho_0\|_{W^{1,r}(\Omega)} \right)$$

holds with some $\mathfrak{C} \in C(\mathbb{R}^2)$. Moreover, $\varrho \in C(I \times \bar{\Omega})$ and the mapping

$$(3.23) \quad \mathbf{v} \mapsto \varrho : L^1(I; W^{2,p}(\Omega; \mathbb{R}^d)) \rightarrow L^\infty(I; W^{1,r}(\Omega))$$

is (weak, weak*)-continuous. If in addition $\varrho_0 > 0$ on $\bar{\Omega}$, then $\varrho > 0$ on $I \times \bar{\Omega}$ uniformly with respect to bounded velocity fields \mathbf{v} , namely, for any $R > 0$ there exists $\delta > 0$ such that

$$(3.24) \quad \|\nabla \mathbf{v}\|_{L^1(I; W^{1,p}(\Omega; \mathbb{R}^{d \times d}))} \leq R \quad \Rightarrow \quad \min_{I \times \bar{\Omega}} \varrho \geq \delta.$$

We are now in the position of stating the main result of this section.

Proposition 3.4 (Existence and regularity of weak solutions). *Under assumptions (3.2) there exists a weak solution $(\varrho, \mathbf{v}, \mathbf{F}, z, \mu)$ to the initial-boundary-value problem (2.20)–(2.21) with (3.1) in the sense of Definition 3.1. Moreover,*

$$\begin{aligned} \mathbf{F} &\in H^1(I; L^r(\Omega; \mathbb{R}^{d \times d})), \\ z &\in L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*), \\ \varrho &= \varrho_{\mathbb{R}} / \det \mathbf{F} \in H^1(I; L^r(\Omega)), \\ \varrho \mathbf{v} &\in L^4(I; W^{1,4}(\Omega; \mathbb{R}^d)) \cap W^{1,p'}(I; W^{2,p}(\Omega; \mathbb{R}^d)^*). \end{aligned}$$

Eventually, the energy dissipation balance (2.29) holds integrated on the time interval $[0, t]$ for any $t \in I$.

The remainder of the paper is devoted to a proof of the latter existence statement. This hinges upon a nested regularization and Galerkin space-approximation procedure. In particular, nonlinearities firstly are replaced by regularizations. Then, the PDE problem is reduced to an ODE system by resorting to finite dimensional subspaces. The crucial point here is that the (weak formulations of the) momentum equation (2.24a) and of the diffusion equation (2.24c) will be space discretized. The continuity equation (3.21) and the flow equation for \mathbf{F} will not be space discretized, in order to take advantage of Lemmas 3.2 and 3.3.

Proof. As mentioned, the proof relies on subsequent approximations and is here divided into steps, for better clarity.

Step 1: Regularization. Since $r > d$, we can choose $\varepsilon > 0$ small enough so that all fields \mathbf{F} fulfilling the formal estimate (3.5a) satisfy

$$(3.25) \quad \det \mathbf{F} > \varepsilon \quad \text{and} \quad |\mathbf{F}| < \frac{1}{\varepsilon} \quad \text{a.e. on } I \times \Omega.$$

Correspondingly, we may perform a regularization of the stress in (2.20) by considering a smooth cut-off $\varphi_\varepsilon(\cdot, z)$ of the original stored energy density φ defined as

$$(3.26) \quad \varphi_\varepsilon(\mathbf{F}_e, z) = \chi_\varepsilon(\lambda(z)\mathbf{F}_e)\varphi(\mathbf{F}_e, z)$$

$$\text{with } \chi_\varepsilon(\mathbf{F}) = \begin{cases} 1 & \text{for } \det \mathbf{F} \geq \varepsilon \text{ and } |\mathbf{F}| \leq 1/\varepsilon, \\ 0 & \text{for } \det \mathbf{F} \leq \varepsilon/2 \text{ or } |\mathbf{F}| \geq 2/\varepsilon, \\ \left(\frac{3}{\varepsilon^2} (2 \det \mathbf{F} - \varepsilon)^2 - \frac{2}{\varepsilon^3} (2 \det \mathbf{F} - \varepsilon)^3 \right) \times \\ \quad \times (3(\varepsilon|\mathbf{F}| - 1)^2 - 2(\varepsilon|\mathbf{F}| - 1)^3) & \text{otherwise.} \end{cases}$$

We moreover make use of the notation $\widehat{\varphi}_\varepsilon(\mathbf{F}, z) = \chi_\varepsilon(\mathbf{F})\widehat{\varphi}(\mathbf{F}, z)$. Note that also $\varphi_\varepsilon, \widehat{\varphi}_\varepsilon \in C^1(\mathbb{R}^{d \times d} \times \mathbb{R})$ if $\varphi \in C^1(\mathbb{R}^{d \times d} \times \mathbb{R})$. Moreover, $[\widehat{\varphi}_\varepsilon]_{\mathbf{F}}$, the Cauchy stress $(\mathbf{F}, z) \mapsto \mathbf{T}_\varepsilon = [\widehat{\varphi}_\varepsilon]_{\mathbf{F}}'(\mathbf{F}, z)\mathbf{F}^\top + \widehat{\varphi}_\varepsilon(\mathbf{F}, z)\mathbf{I}$, and the driving pressure $\pi_\varepsilon = [\widehat{\varphi}_\varepsilon]_z'$ are bounded, continuous. In fact, \mathbf{T}_ε and π_ε vanish as an effect of the choice of χ_ε if \mathbf{F} “substantially” violates the bounds (3.25), specifically if $\det \mathbf{F} \leq \varepsilon/2$ or $|\mathbf{F}| \geq 2/\varepsilon$. It is also important to notice that the strong convexity of $\widehat{\varphi}(\mathbf{F}, \cdot)$ is *not* inherited by $\widehat{\varphi}_\varepsilon(\mathbf{F}, \cdot)$, which is why we are forced to resort to a regularization of the diffusion equation (3.29d) below.

The multivalued mapping $N_{[0,1]}(\cdot)$ in (2.24c) is approximated via the standard Yosida approximation

$$(3.27) \quad \mathcal{N}_k(z) = \begin{cases} k(z-1) & \text{if } z > 1, \\ 0 & \text{if } 0 \leq z \leq 1, \\ kz & \text{if } z < 0. \end{cases}$$

Note that $k \in \mathbb{N}$ is the index of the Galerkin approximation of the momentum equation as well, see Step2 below.

We moreover regularize the singular nonlinearity $1/\det(\cdot)$, showing up in the right-hand-side of the momentum equation, although simultaneously the mass-density continuity equation is considered for the inertial term. To this aim, we introduce the short-hand notation

$$(3.28) \quad \det_\varepsilon \mathbf{F} := \max(\det \mathbf{F}, \varepsilon),$$

Eventually, we regularize also the diffusion equation for z . Altogether, the regularized system reads as follows

$$(3.29a) \quad \frac{\partial \varrho}{\partial t} = -\operatorname{div}(\varrho \mathbf{v}),$$

$$(3.29b) \quad \frac{\partial}{\partial t}(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div}(\mathbf{T}_\varepsilon + \mathbf{D}) + \frac{\varrho_R \mathbf{g}}{\det_\varepsilon \mathbf{F}}$$

$$\begin{aligned} & \text{with } \mathbf{T}_\varepsilon = [\widehat{\varphi}_\varepsilon]'_{\mathbf{F}}(\mathbf{F}, z) \mathbf{F}^\top + \widehat{\varphi}_\varepsilon(\mathbf{F}, z) \mathbf{I} \\ & \text{and } \mathbf{D} = \zeta'_\varepsilon(z; \mathbf{e}(\mathbf{v})) - \operatorname{div}(\nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v})), \end{aligned}$$

$$(3.29c) \quad \dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F},$$

$$(3.29d) \quad \dot{z} = \operatorname{div}(\widehat{m}(\mathbf{F}, z) \nabla \mu + (1 - \chi_\varepsilon(\mathbf{F})) \nabla z) \quad \text{with } \mu = [\widehat{\varphi}_\varepsilon]'_z(\mathbf{F}, z) + \mathcal{N}_k(z).$$

The boundary conditions for (3.29b) are as in (2.21) while the condition for the diffusion equation (i.e., the last condition in (2.21)) is now modified as

$$(3.30) \quad (\widehat{m}(\mathbf{F}, z) \nabla \mu + (1 - \chi_\varepsilon(\mathbf{F})) \nabla z) \cdot \mathbf{n} + \varkappa \mu + (1 - \chi_\varepsilon(\mathbf{F})) z = h.$$

Note that the terms with factor $1 - \chi_\varepsilon(\mathbf{F})$ in (3.29d) and (3.30) vanish if \mathbf{F} complies with the bounds (3.25). On the other hand, they ensure the strong monotonicity of the diffusion operators and the coercivity of the boundary conditions, even when the approximate solution violates these bounds and thus the cut-off $[\widehat{\varphi}_\varepsilon]'_z$ may degenerate.

Step 2: Galerkin approximation. We perform a Galerkin approximation separately of the momentum equation (3.29b) and of the diffusion equation for z (3.29d). On the other hand, we do not approximate in space the continuity equation (3.29a) and the flow equation (3.29c) for \mathbf{F} but rather rely respectively on Lemmas 3.3 and 3.2 for their weak solutions. The Galerkin approximations of equations (3.29b) and (3.29d) are kept independent, in order to be able to pass separately to the limit in Steps 6 and 4, respectively.

Specifically, we use a nested finite-dimensional subspaces $\{V_k\}_{k \in \mathbb{N}}$ whose union is dense in $W^{2,p}(\Omega; \mathbb{R}^d)$ for the momentum equation (3.29b). Note that these spaces are indexed by the same $k \in \mathbb{N}$ used in (3.29d) for the regularization of the normal-cone mapping. In addition, we perform a Galerkin approximation of the diffusion equation (3.29d) by using a second collection of nested finite-dimensional subspaces $\{Z_l\}_{l \in \mathbb{N}}$ whose union is dense in $H^1(\Omega)$. Without loss of generality, we may assume $\mathbf{v}_0 \in V_1$ and $z_0 \in Z_1$.

The space approximation of the solution of the regularized system (3.29) will be denoted by

$$(\varrho_{kl}, \mathbf{v}_{kl}, \mathbf{F}_{kl}, z_{kl}) : I \rightarrow W^{1,r}(\Omega) \times V_k \times W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \times Z_l.$$

Existence of such space-approximated solution can be obtained via the standard existence theory for first-order systems of ordinary differential equations: local-in-time existence follows from smoothness, also in connection with Lemmas 3.2 and 3.3. Then, global existence on the whole time interval $[0, T]$ results from the standard successive-prolongation argument, on the basis of the uniform-in-time estimates proved below.

Let us once again stress that the continuity equation (3.29a) is not space discretized. This allows us to test it by $|\mathbf{v}_{kl}|^2$ so that identity (3.6) is at disposal also at the Galerkin level. On the other hand, it is to be emphasized that also the equation for μ in (3.29d) is not space discretized: the corresponding μ_{kl} is therefore not valued in Z_l and thus is not a legitimate test function for the diffusion equation (3.29d).

Step 3: First a-priori estimates. A basic estimate follows by testing the Galerkin approximation of the momentum equation (3.29b) by \mathbf{v}_{kl} , taking advantage of the (not discretized!) continuity equation (3.29a) tested by $|\mathbf{v}_{kl}|^2/2$, and by testing the Galerkin approximation of the diffusion equation (3.29d) by z_{kl} .

The continuity equation (3.29a) tested by $|\mathbf{v}_{kl}|^2/2$ can be used in (3.6), here written in terms of ϱ_{kl} and \mathbf{v}_{kl} , in order to exploit the formulas (2.27)–(2.28) to obtain the rate of kinetic energy. A crucial observation is that, due to the presence of the cut-offs \det_ε and φ_ε , the equations (3.29a–c) can be estimated independently of z , i.e., independently from the estimate of the diffusion equation (3.29d). Specifically, from the Galerkin approximation of (3.29b) tested by \mathbf{v}_{kl} we obtain the identity

$$(3.31) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{\varrho_{kl}}{2} |\mathbf{v}_{kl}|^2 d\mathbf{x} + \int_{\Omega} \zeta'_\varepsilon(z; \mathbf{e}(\mathbf{v}_{kl})) : \mathbf{e}(\mathbf{v}_{kl}) + \nu |\nabla \mathbf{e}(\mathbf{v})|^p d\mathbf{x} \\ = \int_{\Omega} \frac{\varrho_{kl} \mathbf{g}}{\det_\varepsilon \mathbf{F}_{kl}} \cdot \mathbf{v}_{kl} - \mathbf{T}_{\varepsilon, kl} : \mathbf{e}(\mathbf{v}_{kl}) + \int_{\Gamma} \mathbf{f} \cdot \mathbf{v}_{kl} dS \end{aligned}$$

where

$$(3.32) \quad \mathbf{T}_{\varepsilon, kl} = [\widehat{\varphi}_\varepsilon]_{\mathbf{F}}'(\mathbf{F}_{kl}, z_{kl}) \mathbf{F}_{kl}^\top + \widehat{\varphi}_\varepsilon(\mathbf{F}_{kl}, z_{kl}) \mathbf{I}.$$

Due to Lemmas 3.2 and 3.3 with $\mathbf{v} = \mathbf{v}_{kl}$ and with the fixed initial conditions \mathbf{F}_0 and ϱ_0 , we may define the nonlinear operators $\mathfrak{F} : I \times L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \rightarrow W^{1,r}(\Omega; \mathbb{R}^{d \times d})$ and $\mathfrak{R} : I \times L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \rightarrow W^{1,r}(\Omega)$ by

$$(3.33) \quad \mathbf{F}_{kl}(t) = \mathfrak{F}(t, \mathbf{v}_{kl}) \quad \text{and} \quad \varrho_{kl}(t) = \mathfrak{R}(t, \mathbf{v}_{kl}).$$

Since we have that $p \geq 2$, we can estimate

$$(3.34) \quad \begin{aligned} \int_{\Gamma} \mathbf{f} \cdot \mathbf{v}_{kl} dS &\leq \|\mathbf{f}\|_{L^1(\Gamma; \mathbb{R}^d)} \|\mathbf{v}_{kl}\|_{L^\infty(\Gamma; \mathbb{R}^d)} \\ &\leq N \|\mathbf{f}\|_{L^1(\Gamma; \mathbb{R}^d)} \left(\|\mathbf{v}_{kl}\|_{L^2(\Gamma; \mathbb{R}^d)} + \|\nabla \mathbf{e}(\mathbf{v}_{kl})\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} \right) \\ &\leq C \|\mathbf{f}\|_{L^1(\Gamma; \mathbb{R}^d)}^{p'} + \|\mathbf{f}\|_{L^1(\Gamma; \mathbb{R}^d)} \left(1 + \|\mathbf{v}_{kl}\|_{L^2(\Omega; \mathbb{R}^d)}^2 \right) + \delta \|\nabla \mathbf{e}(\mathbf{v}_{kl})\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})}^p \\ &\leq C \|\mathbf{f}\|_{L^1(\Gamma; \mathbb{R}^d)}^{p'} + \|\mathbf{f}\|_{L^1(\Gamma; \mathbb{R}^d)} \left(1 + \frac{\|\sqrt{\varrho_{kl}} \mathbf{v}_{kl}\|_{L^2(\Omega; \mathbb{R}^d)}^2}{\sqrt{\min \varrho_{kl}}} \right) + \delta \|\nabla \mathbf{e}(\mathbf{v}_{kl})\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})}^p, \end{aligned}$$

where N depends on the norm of the trace operator $W^{2,p}(\Omega) \rightarrow L^\infty(\Gamma)$ and the Korn-inequality constant, while C depends on N and $\delta > 0$, which can be chosen arbitrarily.

By the Gronwall inequality, we obtain the estimates

$$(3.35a) \quad \|\mathbf{e}(\mathbf{v}_{kl})\|_{L^2(I;W^{1,p}(\Omega;\mathbb{R}^{d \times d}))} \leq C \quad \text{and} \quad \|\sqrt{\varrho_{kl}}\mathbf{v}_{kl}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \leq C,$$

and, since ϱ_{kl} is uniformly bounded away from 0, from (3.24) together with (3.35a), we also have that

$$(3.35b) \quad \|\mathbf{v}_{kl}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \leq C.$$

Next, we use the strong convexity of $\widehat{\varphi}(\mathbf{F}, \cdot)$, cf. (3.2a), in order to drop momentarily equation $\mu_{kl} = [\widehat{\varphi}_\varepsilon]'_z(\mathbf{F}_{kl}, z_{kl}) + \mathcal{N}_k(z_{kl})$ which holds a.e. on $I \times \Omega$. Indeed, this equation should otherwise be tested by \dot{z}_{kl} , which would not be a legitimate test at the Galerkin-approximation level. By computing the gradient, we have

$$(3.36) \quad \nabla \mu_{kl} = ([\widehat{\varphi}_\varepsilon]''_{zz}(\mathbf{F}_{kl}, z_{kl}) + \xi_{kl}) \nabla z_{kl} + [\widehat{\varphi}_\varepsilon]''_{\mathbf{F}z}(\mathbf{F}_{kl}, z_{kl}) \nabla \mathbf{F}_{kl} \quad \text{with } \xi_{kl} \in \mathcal{N}'_k(z_{kl}).$$

Note that $\mathcal{N}_k \in W^{2,\infty}(\mathbb{R})$ and the (generalized) derivative \mathcal{N}'_k indeed jumps (i.e., is set-valued) at $z = 0$ and $z = 1$. On the other hand, we nevertheless have that $0 \leq \xi_{kl} \leq 1/k$. Substituting this into (3.29d), we obtain an initial-boundary-value problem for z_{kl} , namely, (the Galerkin approximation of)

$$(3.37) \quad \dot{z}_{kl} = \operatorname{div} \mathbf{j}_{kl} \quad \text{with}$$

$$\mathbf{j}_{kl} = \underbrace{\left(\widehat{m}(\mathbf{F}_{kl}, z_{kl}) ([\widehat{\varphi}_\varepsilon]''_{zz}(\mathbf{F}_{kl}, z_{kl}) + \xi_{kl}) + 1 - \chi_\varepsilon(\mathbf{F}_{kl}) \right)}_{=: \mathbf{m}(\mathbf{F}_{kl}, z_{kl}), \text{ "uniformly" positive with respect to } (\mathbf{F}_{kl}, z_{kl})} \nabla z_{kl} + [\widehat{\varphi}_\varepsilon]''_{\mathbf{F},z}(\mathbf{F}_{kl}, z_{kl}) \nabla \mathbf{F}_{kl}$$

and with the boundary condition $\mathbf{j}_{kl} \cdot \mathbf{n} + \varkappa \mu_{kl} = h$. It is now allowed to test (3.37) in its Galerkin approximation by z_{kl} , which leads to the identity

$$(3.38) \quad \frac{1}{2} \frac{d}{dt} \|z_{kl}\|_{L^2(\Omega)}^2 + \int_\Omega \mathbf{m}(\mathbf{F}_{kl}, z_{kl}) |\nabla z_{kl}|^2 d\mathbf{x} + \int_\Gamma \varkappa \mu_{kl} z_{kl} dS$$

$$= \int_\Omega [\widehat{\varphi}_\varepsilon]''_{\mathbf{F},z}(\mathbf{F}_{kl}, z_{kl}) : (\nabla \mathbf{F}_{kl} \otimes \nabla z_{kl}) - (\mathbf{v}_{kl} \cdot \nabla z_{kl}) z_{kl} d\mathbf{x} + \int_\Gamma \varkappa h z_{kl} + (1 - \chi_\varepsilon(\mathbf{F}_{kl})) z_{kl}^2 dS$$

$$= \int_\Omega [\widehat{\varphi}_\varepsilon]''_{\mathbf{F},z}(\mathbf{F}_{kl}, z_{kl}) : (\nabla \mathbf{F}_{kl} \otimes \nabla z_{kl}) + \frac{|z_{kl}|^2}{2} \operatorname{div} \mathbf{v}_{kl} d\mathbf{x} + \int_\Gamma \varkappa h z_{kl} - \frac{|z_{kl}|^2}{2} \mathbf{v}_{kl} \cdot \mathbf{n} dS,$$

where also the Green formula in Ω has been used. It is important that the term $\varkappa \mu_{kl} z_{kl} = \varkappa \chi_\varepsilon(\mathbf{F}_{kl}) \widehat{\varphi}'_z(\mathbf{F}_{kl}, z_{kl}) z_{kl} + (1 - \chi_\varepsilon(\mathbf{F}_{kl})) z_{kl}^2$ can be estimated from below by $\delta |z_{kl}|^2 - 1/\delta$ for sufficiently small $\delta > 0$, depending on the strong convexity of $\widehat{\varphi}(\mathbf{F}, \cdot)$, cf. (3.2a), so that the boundary term $\int_\Gamma \varkappa h z_{kl} dS$ in (3.38) can be estimated by using also the coercive left-hand-side term $\int_\Gamma \varkappa \mu_{kl} z_{kl} dS$. Using the boundary condition $\mathbf{v}_{kl} \cdot \mathbf{n} = 0$ and the Gronwall and the Hölder inequalities, we obtain the estimate

$$(3.39) \quad \|z_{kl}\|_{L^\infty(I;L^2(\Omega)) \cap L^2(I;H^1(\Omega))} \leq C.$$

From this, we also obtain an information about $\mu_{kl} = [\widehat{\varphi}_\varepsilon]'_z(\mathbf{F}_{kl}, z_{kl}) + \mathcal{N}_k(z_{kl})$:

$$(3.40) \quad \|\mu_{kl}\|_{L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega))} \leq Ck.$$

Step 4: Limit passage for $l \rightarrow \infty$. By the obtained a-priori estimates and the sequential weak* compactness of balls in the involved spaces, we can standardly use the Banach selection principle [5, Chap. III, Thm. 3] (i.e., a special form of the Alaoglu-Bourbaki principle devised later for nonmetrizable situations) and extract some not relabeled subsequence and $(\varrho_k, \mathbf{v}_k, \mathbf{F}_k, z_k, \mu_k) : I \rightarrow W^{1,r}(\Omega) \times V_k \times W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \times H^1(\Omega)^2$ such that

$$(3.41a) \quad \varrho_{kl} \rightarrow \varrho_k \quad \text{weakly* in } L^\infty(I; W^{1,r}(\Omega)) \cap W^{1,p}(I; L^r(\Omega)),$$

$$(3.41b) \quad \mathbf{v}_{kl} \rightarrow \mathbf{v}_k \quad \text{weakly* in } L^\infty(I; L^2(\Omega; \mathbb{R}^d)) \cap L^2(I; W^{2,p}(\Omega; \mathbb{R}^d)),$$

$$(3.41c) \quad \mathbf{F}_{kl} \rightarrow \mathbf{F}_k \quad \text{weakly* in } L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap H^1(I; L^2(\Omega; \mathbb{R}^{d \times d})),$$

$$(3.41d) \quad z_{kl} \rightarrow z_k \quad \text{weakly* in } L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)),$$

$$(3.41e) \quad \mu_{kl} \rightarrow \mu_k \quad \text{weakly* in } L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)).$$

Recalling that $r > d$, by the Aubin-Lions Lemma we also have that

$$(3.42a) \quad \varrho_{kl} \rightarrow \varrho_k \quad \text{strongly in } C(I \times \overline{\Omega})$$

and $\mathbf{F}_{kl} \rightarrow \mathbf{F}_k$ strongly in $C(I \times \overline{\Omega}; \mathbb{R}^{d \times d})$. By comparison in the equation in (3.29d) we obtain a bound on $\frac{\partial}{\partial t} z_{kl}$, implying that

$$(3.42b) \quad z_{kl} \rightarrow z_k \quad \text{strongly in } L^s(I \times \Omega) \text{ for any } 1 \leq s < 2 + 4/d,$$

cf. [29, Ch.8]. Thus, by the continuity of the corresponding Nemytskiĭ (or here simply superposition) mappings, also the conservative part of the regularized Cauchy stress and the diffusivity and the regularized pore pressure in the diffusion equation converge, namely,

$$(3.42c) \quad \mathbf{T}_{\varepsilon,kl} \rightarrow \mathbf{T}_{\varepsilon,k} = [\widehat{\varphi}_\varepsilon]'_{\mathbf{F}}(\mathbf{F}_k, z_k) \mathbf{F}_k^\top + \widehat{\varphi}_\varepsilon(\mathbf{F}_k, z_k) \mathbf{I} \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^{d \times d}),$$

$$(3.42d) \quad \widehat{m}(\mathbf{F}_{kl}, z_{kl}) \rightarrow \widehat{m}(\mathbf{F}_k, z_k) \quad \text{strongly in } L^c(I \times \Omega),$$

$$(3.42e) \quad [\widehat{\varphi}_\varepsilon]'_z(\mathbf{F}_{kl}, z_{kl}) \rightarrow [\widehat{\varphi}_\varepsilon]'_z(\mathbf{F}_k, z_k) \quad \text{strongly in } L^c(I \times \Omega),$$

for any $1 \leq c < \infty$. It is important to notice that

$$(3.43) \quad \nabla(\varrho_{kl} \mathbf{v}_{kl}) = \nabla \varrho_{kl} \otimes \mathbf{v}_{kl} + \varrho_{kl} \nabla \mathbf{v}_{kl}$$

is bounded in $L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d}))$ due to the already obtained bounds (3.13) and (3.22). Therefore, $\varrho_{kl} \mathbf{v}_{kl}$ converges weakly* in $L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^d))$. In fact, the limit of $\varrho_{kl} \mathbf{v}_{kl}$ can be identified as $\varrho_k \mathbf{v}_k$ because we already showed that ϱ_{kl} converges strongly in (3.42a) and \mathbf{v}_{kl} converges weakly due to (3.41b).

By comparison, we also obtain some information about $\frac{\partial}{\partial t}(\varrho_{kl} \mathbf{v}_{kl})$. Note indeed that (3.6) still holds for the semi-discretized system since the continuity equation has not been space-discretized. Specifically, we have

$$(3.44) \quad \frac{\partial}{\partial t}(\varrho_{kl} \mathbf{v}_{kl}) = \varrho_{kl} \dot{\mathbf{v}}_{kl} - \text{div}(\varrho_{kl} \mathbf{v}_{kl} \otimes \mathbf{v}_{kl})$$

$$= \varrho_{kl} \dot{\mathbf{v}}_{kl} - \varrho_{kl} (\mathbf{v}_{kl} \cdot \nabla) \mathbf{v}_{kl} - \varrho_{kl} (\operatorname{div} \mathbf{v}_{kl}) \mathbf{v}_{kl} - (\mathbf{v}_{kl} \cdot \nabla \varrho_{kl}) \mathbf{v}_{kl}.$$

We may hence compare in (3.29b) in order to obtain a bound on $\varrho_{kl} \dot{\mathbf{v}}_{kl}$. By the compact embedding $L^\infty(I; V_k) \cap W^{1,p'}(I; V_k) \subset L^\infty(I; V_k)$, we have

$$(3.45) \quad \varrho_{kl} \mathbf{v}_{kl} \rightarrow \varrho_k \mathbf{v}_k \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^d) \quad \text{for all } 1 \leq c < 4.$$

Since obviously $\mathbf{v}_{kl} = (\varrho_{kl} \mathbf{v}_{kl})(1/\varrho_{kl})$, thanks to (3.13), (3.42a), and (3.45), we also have that

$$(3.46) \quad \mathbf{v}_{kl} \rightarrow \mathbf{v}_k \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^d) \quad \text{with any } 1 \leq c < 4, .$$

The convergences (3.42d,e) allow to pass to the limit for $l \rightarrow \infty$ in the regularized diffusion equation (3.29d). The limit passage in the evolution equations (3.29a) and (3.29c) follows from Lemmas 3.2 and 3.3.

For the limit passage in the momentum equation, one uses the monotonicity of the dissipative stress \mathbf{D} , i.e., the monotonicity of the quasilinear operator

$$\mathbf{v} \mapsto \operatorname{div}(\operatorname{div}(|\nabla \mathbf{e}(\cdot)|^{p-2} \nabla \mathbf{e}(\mathbf{v})) - \zeta'_e(z, \mathbf{e}(\mathbf{v}))),$$

and one employs weak convergence, in combination with the so-called Minty trick. We take $\tilde{\mathbf{v}} \in H^1(I; V_k)$ and test the momentum equation by $\mathbf{v}_{kl} - \tilde{\mathbf{v}}$. Note that one has

$$(3.47) \quad \begin{aligned} \int_0^T \int_\Omega \varrho_{kl} \dot{\mathbf{v}}_{kl} \cdot \tilde{\mathbf{v}} \, d\mathbf{x} dt &= \int_0^T \int_\Omega \left(\frac{\partial}{\partial t} (\varrho_{kl} \mathbf{v}_{kl}) + \operatorname{div}(\varrho_{kl} \mathbf{v}_{kl} \otimes \mathbf{v}_{kl}) \right) \cdot \tilde{\mathbf{v}} \, d\mathbf{x} dt \\ &= \int_\Omega \varrho_{kl}(T) \mathbf{v}_{kl}(T) \cdot \tilde{\mathbf{v}}(T) - \varrho_0 \mathbf{v}_0 \cdot \tilde{\mathbf{v}}(0) \, d\mathbf{x} - \int_0^T \int_\Omega \varrho_{kl} \mathbf{v}_{kl} \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\varrho_{kl} \mathbf{v}_{kl} \otimes \mathbf{v}_{kl}) : \nabla \tilde{\mathbf{v}} \, d\mathbf{x} dt \\ &\rightarrow \int_\Omega \varrho_k(T) \mathbf{v}_k(T) \cdot \tilde{\mathbf{v}}(T) - \varrho_0 \mathbf{v}_0 \cdot \tilde{\mathbf{v}}(0) \, d\mathbf{x} - \int_0^T \int_\Omega \varrho_k \mathbf{v}_k \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\varrho_k \mathbf{v}_k \otimes \mathbf{v}_k) : \nabla \tilde{\mathbf{v}} \, d\mathbf{x} dt \\ &= \int_0^T \int_\Omega \varrho_k \dot{\mathbf{v}}_k \cdot \tilde{\mathbf{v}} \, d\mathbf{x} dt. \end{aligned}$$

Here, we have used the fact that the term $\varrho_{kl}(T)$ is also bounded in $W^{1,r}(\Omega)$ and $\mathbf{v}_{kl}(T)$ is bounded in $L^2(\Omega; \mathbb{R}^d)$, together with some information about the time derivatives $\frac{\partial}{\partial t} \varrho_{kl} = -\operatorname{div}(\varrho_{kl} \mathbf{v}_{kl})$ and $\frac{\partial}{\partial t} (\varrho_{kl} \mathbf{v}_{kl})$, cf. (3.44), so that we can identify the weak limit of $\varrho_{kl}(T) \mathbf{v}_{kl}(T)$. We have hence obtained that

$$(3.48) \quad \varrho_{kl}(T) \mathbf{v}_{kl}(T) \rightarrow \varrho_k(T) \mathbf{v}_k(T) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d).$$

In (3.47), we have relied on (3.45) and on the fact that $\frac{\partial}{\partial t} \tilde{\mathbf{v}}$ is well defined at the Galerkin level and that the continuity equation is not discretized, so that the identity (3.6) holds even for the semi-discrete problem. This is to be used in the following calculations

$$(3.49) \quad 0 \leq \limsup_{l \rightarrow \infty} \left(\int_0^T \int_\Omega \left(\nu (|\nabla \mathbf{e}(\mathbf{v}_{kl})|^{p-2} \nabla \mathbf{e}(\mathbf{v}_{kl}) - |\nabla \mathbf{e}(\tilde{\mathbf{v}})|^{p-2} \nabla \mathbf{e}(\tilde{\mathbf{v}})) : \nabla \mathbf{e}(\mathbf{v}_{kl} - \tilde{\mathbf{v}}) \right) \right)$$

$$\begin{aligned}
& + \left(\partial'_e \zeta(z_{kl}, \mathbf{e}(\mathbf{v}_{kl})) - \partial'_e \zeta(z_{kl}, \mathbf{e}(\tilde{\mathbf{v}})) : \mathbf{e}(\mathbf{v}_{kl} - \tilde{\mathbf{v}}) \right) d\mathbf{x} dt \Big) \\
& = \limsup_{l \rightarrow \infty} \left(\int_0^T \int_{\Omega} \left(\varrho_{kl} (\mathbf{g} - \dot{\mathbf{v}}_{kl}) \cdot (\mathbf{v}_{kl} - \tilde{\mathbf{v}}) - \mathbf{T}_{\varepsilon, kl} : \nabla (\mathbf{v}_{kl} - \tilde{\mathbf{v}}) - \partial'_e \zeta(z_{kl}, \mathbf{e}(\tilde{\mathbf{v}})) : \mathbf{e}(\mathbf{v}_{kl} - \tilde{\mathbf{v}}) \right. \right. \\
& \quad \left. \left. - \nu (|\nabla \mathbf{e}(\tilde{\mathbf{v}})|^{p-2} \nabla \mathbf{e}(\tilde{\mathbf{v}})) : \nabla \mathbf{e}(\mathbf{v}_{kl} - \tilde{\mathbf{v}}) \right) d\mathbf{x} dt + \int_0^T \int_{\Gamma} \mathbf{f} \cdot (\mathbf{v}_{kl} - \tilde{\mathbf{v}}) dS dt \right) \\
& = \limsup_{l \rightarrow \infty} \left(\int_0^T \int_{\Omega} \left(\varrho_{kl} \mathbf{g} \cdot (\mathbf{v}_{kl} - \tilde{\mathbf{v}}) + \varrho_{kl} \dot{\mathbf{v}}_{kl} \cdot \tilde{\mathbf{v}} - \mathbf{T}_{\varepsilon, kl} : \nabla (\mathbf{v}_{kl} - \tilde{\mathbf{v}}) \right. \right. \\
& \quad \left. \left. - \partial'_e \zeta(z_{kl}, \mathbf{e}(\tilde{\mathbf{v}})) : \mathbf{e}(\mathbf{v}_{kl} - \tilde{\mathbf{v}}) - \nu (|\nabla \mathbf{e}(\tilde{\mathbf{v}})|^{p-2} \nabla \mathbf{e}(\tilde{\mathbf{v}})) : \nabla \mathbf{e}(\mathbf{v}_{kl} - \tilde{\mathbf{v}}) \right) d\mathbf{x} dt \right. \\
& \quad \left. + \int_0^T \int_{\Gamma} \mathbf{f} \cdot (\mathbf{v}_{kl} - \tilde{\mathbf{v}}) dS dt - \int_{\Omega} \frac{\varrho_{kl}(T)}{2} |\mathbf{v}_{kl}(T)|^2 - \frac{\varrho_0}{2} |\mathbf{v}_0|^2 d\mathbf{x} \right) \\
& \leq \int_0^T \int_{\Omega} \left(\varrho_k \mathbf{g} \cdot (\mathbf{v}_k - \tilde{\mathbf{v}}) + \varrho_k \dot{\mathbf{v}}_k \cdot \tilde{\mathbf{v}} - \mathbf{T}_{\varepsilon, k} : \nabla (\mathbf{v}_k - \tilde{\mathbf{v}}) - \partial'_e \zeta(z_k, \mathbf{e}(\tilde{\mathbf{v}})) : \mathbf{e}(\mathbf{v}_k - \tilde{\mathbf{v}}) \right. \\
& \quad \left. - \nu (|\nabla \mathbf{e}(\tilde{\mathbf{v}})|^{p-2} \nabla \mathbf{e}(\tilde{\mathbf{v}})) : \nabla \mathbf{e}(\mathbf{v}_k - \tilde{\mathbf{v}}) \right) d\mathbf{x} dt \\
& \quad + \int_0^T \int_{\Gamma} \mathbf{f} \cdot (\mathbf{v}_k - \tilde{\mathbf{v}}) dS dt - \int_{\Omega} \frac{\varrho_k(T)}{2} |\mathbf{v}_k(T)|^2 - \frac{\varrho_0}{2} |\mathbf{v}_0|^2 d\mathbf{x} \\
& = \int_0^T \int_{\Omega} \left(\varrho_k (\mathbf{g} - \dot{\mathbf{v}}_k) \cdot (\mathbf{v}_k - \tilde{\mathbf{v}}) - \mathbf{T}_{\varepsilon, k} : \nabla (\mathbf{v}_k - \tilde{\mathbf{v}}) - \partial'_e \zeta(z_k, \mathbf{e}(\tilde{\mathbf{v}})) : \mathbf{e}(\mathbf{v}_k - \tilde{\mathbf{v}}) \right. \\
& \quad \left. - \nu (|\nabla \mathbf{e}(\tilde{\mathbf{v}})|^{p-2} \nabla \mathbf{e}(\tilde{\mathbf{v}})) : \nabla \mathbf{e}(\mathbf{v}_k - \tilde{\mathbf{v}}) \right) d\mathbf{x} dt + \int_0^T \int_{\Gamma} \mathbf{f} \cdot (\mathbf{v}_k - \tilde{\mathbf{v}}) dS dt.
\end{aligned}$$

Here, we used also $\nabla \mathbf{v}_{kl} \rightarrow \nabla \mathbf{v}_k$ weakly in $L^2(I \times \Omega; \mathbb{R}^{d \times d})$. Besides (3.47), we also used the weak upper semicontinuity of $\int_{\Omega} -\frac{1}{2} \varrho_{kl}(T) |\mathbf{v}_{kl}(T)|^2 d\mathbf{x} = \int_{\Omega} -\frac{1}{2} |\sqrt{\varrho_{kl}(T)} \mathbf{v}_{kl}(T)|^2 d\mathbf{x}$, together with the fact that, like (3.48), we have also

$$(3.50) \quad \sqrt{\varrho_{kl}(T)} \mathbf{v}_{kl}(T) \rightarrow \sqrt{\varrho_k(T)} \mathbf{v}_k(T) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d).$$

This follows since $\sqrt{\varrho_{kl}(T)}$ is bounded in $W^{1,r}(\Omega)$ and hinges on the boundedness of the term $\frac{\partial}{\partial t} \sqrt{\varrho_{kl}} = (\operatorname{div} \mathbf{v}_{kl}) \sqrt{\varrho_{kl}} + \mathbf{v}_{kl} \cdot \nabla \varrho_{kl} / \sqrt{\varrho_{kl}}$.

By density arguments, inequality (3.49) holds for any $\tilde{\mathbf{v}} \in L^p(I; V_k)$ so that we can substitute $\tilde{\mathbf{v}} = \mathbf{v}_k \pm \epsilon \mathbf{w}$ for $\mathbf{w} \in L^p(I; V_k)$ with $\mathbf{w}(T) = 0 = \mathbf{w}(0)$. This gives equality in (3.49) and, dividing this equality by $\epsilon \neq 0$ passing with $\epsilon \rightarrow 0$, we obtain the weak formulation of the momentum equation (3.7a), here still at its Galerkin-approximation level. The initial condition $\mathbf{v}_{kl}(0) = \mathbf{v}_0$ is kept in the limit, too.

Step 5: Further a-priori estimates. At this point, the only equation which is still discretized is the momentum equation (3.29b). We can perform the ‘‘physical’’ test of the six equations in (3.29) respectively by $|\mathbf{v}_k|^2/2$, \mathbf{v}_k , $[\widehat{\varphi}_{\varepsilon}]'_{\mathbf{F}}(\mathbf{F}_k, z_k) \mathbf{F}_k^{\top}$, μ_k , and $\frac{\partial}{\partial t} z_k + \nabla \mathbf{v}_k \cdot z_k$, thus obtaining estimates (3.4a–c,e) and (3.5) written now for the weak solution $(\varrho_k, \mathbf{v}_k, \mathbf{F}_k, z_k, \mu_k)$ of the (still semidiscretized) system (3.29). By comparison, we also obtain an estimate for $\mathcal{N}_k(z_k) = \mu_k - [\widehat{\varphi}_{\varepsilon}]'_z(\mathbf{F}_k, z_k)$. Specifically, relying on (3.39) and on the estimates (3.4e) and (3.5a), we obtain

$$(3.51) \quad \|\mathcal{N}_k(z_k)\|_{L^2(I; H^1(\Omega))} \leq C.$$

Step 6: Limit passage for $k \rightarrow \infty$. We use sequential weak* compactness and the Banach selection principle as in Step 4, now also taking (3.19a) into account instead of the estimate in (3.15) which was not uniform in k . For some not relabeled subsequence and some $(\varrho, \mathbf{v}, \mathbf{F}, z, \mu)$, we now have

$$\begin{aligned}
(3.52a) \quad \varrho_k &\rightarrow \varrho && \text{strongly in } C(I \times \overline{\Omega}), \\
(3.52b) \quad \mathbf{v}_k &\rightarrow \mathbf{v} && \text{weakly* in } L^\infty(I; L^2(\Omega; \mathbb{R}^d)) \cap L^2(I; W^{2,p}(\Omega; \mathbb{R}^d)), \\
(3.52c) \quad \mathbf{F}_k &\rightarrow \mathbf{F} && \text{weakly* in } L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap H^1(I; L^2(\Omega; \mathbb{R}^{d \times d})), \\
&&& \text{and strongly in } C(I \times \overline{\Omega}; \mathbb{R}^{d \times d}), \\
(3.52d) \quad z_k &\rightarrow z && \text{weakly* in } L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)) \\
&&& \text{and strongly in } L^c(I \times \Omega) \text{ for any } 1 \leq c < 2+4/d, \\
(3.52e) \quad \mu_k &\rightarrow \mu && \text{weakly* in } L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)), \\
(3.52f) \quad \mathbf{T}_{\varepsilon,k} &\rightarrow \mathbf{T}_\varepsilon = [\widehat{\varphi}_\varepsilon]_{\mathbf{F}}'(\mathbf{F}, z) \mathbf{F}^\top + \widehat{\varphi}_\varepsilon(\mathbf{F}, z) \mathbf{I} && \text{strongly in } L^c(I \times \Omega; \mathbb{R}^{d \times d}), \\
(3.52g) \quad \widehat{m}(\mathbf{F}_k, z_k) &\rightarrow \widehat{m}(\mathbf{F}, z) && \text{strongly in } L^c(I \times \Omega) \text{ for any } 1 \leq c < \infty, \\
(3.52h) \quad [\widehat{\varphi}_\varepsilon]_z'(\mathbf{F}_k, z_k) &\rightarrow [\widehat{\varphi}_\varepsilon]_z'(\mathbf{F}, z) && \text{strongly in } L^c(I \times \Omega) \text{ for any } 1 \leq c < \infty.
\end{aligned}$$

The momentum equation (3.29b) (still regularized by ε) is to be treated like in Step 4. The argument which led to (3.45) is to be now based on the the information about the time derivative $\frac{\partial}{\partial t}(\varrho_k \mathbf{v}_k)$ in a seminorm on $L^{p'}(I; W^{2,p}(\Omega; \mathbb{R}^d)^*)$ induced by a test by $L^p(I; V_{k_0})$ with $k \geq k_0$, $k_0 \in \mathbb{N}$, or by a Hahn-Banach extension of such time derivatives, cf. [29, Ch. 8]. The other terms in (3.44) are bounded in $L^{4/3}(I; L^2(\Omega; \mathbb{R}^d))$. By a generalization of the Aubin-Lions compact-embedding theorem, cf. [29, Lemma 7.7], we then obtain

$$(3.53) \quad \varrho_{kl} \mathbf{v}_{kl} \rightarrow \varrho_k \mathbf{v}_k \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^d) \quad \text{with any } 1 \leq c < 4,$$

In fact, the treatment of (3.49) is to be slightly modified by using first $\tilde{\mathbf{v}} \in H^1(I; V_{k_0})$ and then, for and $k \geq k_0$, can be used for $(\varrho_k, \mathbf{v}_k, \mathbf{T}_{\varepsilon,k}, z_k)$ in place of $(\varrho_{kl}, \mathbf{v}_{kl}, \mathbf{T}_{\varepsilon,kl}, z_{kl})$ and $(\varrho, \mathbf{v}, \mathbf{T}_\varepsilon, z)$ in place of $(\varrho_k, \mathbf{v}_k, \mathbf{T}_{\varepsilon,k}, z_k)$. Then, by density arguments, we can resort to some arbitrary $\tilde{\mathbf{v}} \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d))$.

The limit passage in the semilinear equation

$$(3.54) \quad \frac{\partial z_k}{\partial t} + \mathbf{v}_k \cdot \nabla z_k = \operatorname{div} \left(\widehat{m}(\mathbf{F}_k, z_k) \nabla \mu_k + (1 - \chi_\varepsilon(\mathbf{F}_k)) \nabla z_k \right)$$

towards the former equation in (3.29d) formulated weakly is straightforward due to (3.52b–e,g). The limit passage in the equation

$$(3.55) \quad \mu_k = [\widehat{\varphi}_\varepsilon]_z'(\mathbf{F}_k, z_k) + \mathcal{N}_k(z_k)$$

towards the variational inequality (3.7d) is simple by writing the monotone function \mathcal{N}_k as the derivative of the Yosida approximation n_k of the indicator function $\delta_{[0,1]}$, i.e. $n_k(z) = \min_{0 \leq \tilde{z} \leq 1} |z - \tilde{z}|^2 / 2$. Thus, using convexity of n_k , (3.55) can be written as the variational inequality $\int_0^T \int_\Omega n_k(\tilde{z}) + (\mu_k - [\widehat{\varphi}_\varepsilon]_z'(\mathbf{F}_k, z_k))(\tilde{z} - z_k) \, d\mathbf{x} dt \geq \int_0^T \int_\Omega n_k(z_k) \, d\mathbf{x} dt$ for \tilde{z} valued in

$[0, 1]$. The limit passage is by the convergence (3.52d,e,h) and the Γ -convergence of n_k to $\delta_{[0,1]}$ for $k \rightarrow \infty$.

From the calculus in (3.44), we can also see the information $\frac{\partial}{\partial t}(\varrho \mathbf{v}) \in L^{p'}(I; W^{2,p}(\Omega; \mathbb{R}^d)^*)$ while $\nabla(\varrho \mathbf{v}) \in L^2(I; L^r(\Omega; \mathbb{R}^{d \times d}))$ is like in (3.43).

Step 7: Removing the regularization. Since $L^\infty(I; W^{1,r}(\Omega)) \cap H^1(I; L^2(\Omega))$ is embedded in $C(I \times \bar{\Omega})$ for $r > d$, \mathbf{F} and its determinant evolve continuously in time, being valued respectively in $C(\bar{\Omega}; \mathbb{R}^{d \times d})$ and $C(\bar{\Omega})$. Let us recall that, due to (3.2h) and to the choice of $\varepsilon > 0$, the initial condition \mathbf{F}_0 (which is the initial state for the ε -regularized system as well) complies with the bounds in (3.25). Therefore, \mathbf{F} satisfies these bounds in (3.25) not only at $t = 0$ but also up to a small positive time. Indeed, the ε -regularization of $1/\det(\cdot)$ and of φ is not active, $(\varrho, \mathbf{v}, \mathbf{F}, z, \mu)$ solves the original nonregularized system for some small time, and a-priori bounds (3.25) hold. By a continuation argument, such local-in-time solution can hence be extended to the whole time interval I . In particular, the ε -regularization remains not active for all times.

Step 8: Energy balance. Let us conclude by checking that the tests of equations (2.24) respectively by \mathbf{v} , \mathbf{S} , and μ and of (2.10a) by $|\mathbf{v}|^2$ are legitimate, i.e., rigorously justifiable. These in turn allow to prove the energy balance (2.29) integrated over a current time interval $[0, t]$ via (2.25), (2.26), and (2.28).

The already obtained estimates ensure that $\mathbf{F} \in L^\infty(I, W^{1,r}(\Omega; \mathbb{R}^{d \times d}))$, as well as $\mathbf{v} \in L^\infty(I; L^2(\Omega; \mathbb{R}^d)) \cap L^2(I; L^\infty(\Omega; \mathbb{R}^d))$. From this we deduce $(\nabla \mathbf{v})\mathbf{F} \in L^2(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))$ and $(\mathbf{v} \cdot \nabla)\mathbf{F} \in L^2(I; L^r(\Omega; \mathbb{R}^{d \times d}))$, and from these two we get $\frac{\partial}{\partial t}\mathbf{F} = (\nabla \mathbf{v})\mathbf{F} - (\mathbf{v} \cdot \nabla)\mathbf{F} \in L^2(I; L^r(\Omega; \mathbb{R}^{d \times d}))$. Thus, the particular terms in (2.24b) are in duality with $\mathbf{S} = \tilde{\varphi}'_{\mathbf{F}}(\mathbf{F}, z) \in L^\infty(I \times \Omega; \mathbb{R}^{d \times d})$. On the other hand, we have that $\varphi(\mathbf{F}, z) \in L^\infty(I \times \Omega)$ is in duality with $\operatorname{div} \mathbf{v} \in L^4(I \times \Omega)$ and $\mu \in L^2(I; H^1(\Omega))$ is in duality with $\dot{z} \in L^2(I; H^1(\Omega)^*)$. Thus the tests (2.25) and (2.26) can be legitimately performed.

Similarly, we can see that $\frac{\partial}{\partial t}\varrho = -(\operatorname{div} \mathbf{v})\varrho - \mathbf{v} \cdot \nabla \varrho \in L^2(I; L^r(\Omega))$ is in duality with $|\mathbf{v}|^2 \in L^2(I \times \Omega)$ and $\varrho \dot{\mathbf{v}} \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)^*) + L^1(I; L^\infty(\Omega; \mathbb{R}^d))$ is in duality with $\mathbf{v} \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \cap L^\infty(I; L^2(\Omega; \mathbb{R}^d))$. Hence, also the test (2.28) can be rigorously performed. \square

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