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SUBSTRUCTURING PRECONDITIONERS FOR h-p MORTAR FEM.

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Abstract. We build and analyze a substructuring preconditioner for the mortar method in the h-p finite element framework. Particular attention is given to the construction of the coarse component of the preconditioner in this framework, in which continuity at the cross points is not required. Two variants are proposed: the first one is an improved version of a coarse preconditioner already presented in [12]. The second is new and is built by using a Discontinuous Galerkin interior penalty method as coarse problem. A bound of the condition number is proven for both variants and their efficiency and scalability is illustrated by numerical experiments.

1. Introduction

Introduced in the early nineties by Bernardi, Maday and Patera [8] as a tool to couple spectral and finite element method for the solution of second order elliptic PDE's, the *mortar method* has been quickly extended to treat many different application fields [2, 6, 5, 27, 25, 26], turning out to be well suited for parallel implementation and to the coupling of many different approximation spaces. The method has gained a wide popularity, since it offers the possibility to use different, non matching, possibly heterogeneous discretizations in different regions of the domain of definition of the problem at hand. However, in order to make such technique more competitive for real life applications, one has to deal with the problem of the efficient solution of the associated linear system of equations. The design of efficient preconditioners for such linear system is then a fundamental task. Different approaches were considered in the literature: iterative substructuring [1], additive Schwarz with overlap [23], FETI-DP [16, 20, 22] and BDDC [21]. To the best of our knowledge, all these methods deal the h-version of the mortar FEM, and the explicit dependence on the polynomial degree relative to the FEM space considered has never been analysed before.

Here we deal with the construction of preconditioners for the h-p mortar finite element method. We start by considering the approach proposed in the framework of conforming domain decomposition by J.H. Bramble, J.E. Pasciak and A.H. Schatz [14], which has already been extended to the h version of the Mortar method by Achdou, Maday, Widlund [1]. In doing this we will extend to the h-p version some tools that are common to the analysis of a wide range of substructuring preconditioner. This approach consists in considering a suitable splitting of the nonconforming discretization space in terms of "interior", "edge" and "vertex" degrees of freedom and then using the related block-Jacobi type preconditioners. While the "interior" and the "edge" blocks can be treated essentially as in the conforming case, the treatment of the vertex block deserves some additional considerations.

Indeed, a problem that, in our opinion, has not until now been tackled in a satifactory way for the mortar method is the design of the coarse vertex block of the preconditioner (which is responsible for the good scaling properties of the preconditioners considered). In fact, when building preconditioners for the Mortar method, we have to deal with the fact that the coarse space depends on the fine discretization, via the the action of the "mortar projection operator". Moreover, the design of such block is further complicated by the the presence of multiple degrees of freedom at each cross point (we recall, in fact, that in the definition of the mortar method, continuity at cross points is not required). The solution considered in [1] is to use as a coarse preconditioner the vertex block of the Schur complement. This is clearly not efficient, since it implies actually assembling at least a block of the Schur complement (which is a task that we would like to avoid) and, for a high number of subdomains, it is definitely not practically feasible. Here, we propose two different coarse preconditioners. The first one is the vertex block of the Schur complement for a fixed auxiliary order one mesh with a small number of degrees of freedom per subdomain. This idea was presented in [12] for the case of linear finite elements. We combine it, here, with a suitable balancing between vertex and edge component, yielding a better estimate for the condition number of the preconditioned matrix. This alternative allows to avoid the need of recomputing the coarse block of the preconditioner when refining the mesh. It still demands assembling a Schur complement matrix (though starting from a coarse mesh) and it is therefore quite expensive, at least when considering a large number of subdomains. In order to be able to tackle this kind of configuration, and obtain a feasible, scalable method even in a massively parallel environment we propose here, as a further alternative, to build the coarse preconditioner by giving up weak continuity and use, as a coarse preconditioner, a (non consistent) Discontinuous Galerkin type interior penalty method defined on the coarse mesh whose elements are the (quadrangular) subdomains. This approach turns out to be quite efficient even for a very a large number of subdomains (as we show in the numerical tests section).

By applying the theoretical approach first presented in [9], that allows us to provide a much more general analysis than [14, 1], we are able to prove, for both choices of the coarse preconditioner, a condition number bound for the preconditioned matrix of the form

$$Cond(P^{-1}S) \lesssim p^{3/2}(1 + \log\left(Hp^2/h\right)^2 ,$$

where H, h and p are the subdomain mesh-size, the fine mesh-size and the polynomial order respectively, see Corollary 4.2 and Theorem 4.5. Numerical experiments seem, however, to indicate that this bound is not optimal: the condition number appears to behave in a polylogarithmic way, and there is no numerical evidence of the presence of the factor $p^{3/2}$. The same kind of behavior (loss of a power of p in the theoretical estimate that does not appear in the numerical tests) was observed also for the first error estimates for the h-p mortar method [33]. Such estimate was then improved by applying an interpolation argument [7] that, unfortunately, cannot be applied for the type of bound that we are considering. The factor $p^{3/2}$ in the theoretical estimate derives from the boundedness estimates for the mortar projector (2.40,2.41), which were shown to be sharp in [32]. We observe that the norm of such projection operator also comes into play in the analysis of other preconditioners (like, for instance, the FETI method) so that a generalization of the related theoretical estimates to the h-p version would also suffer of the loss of a factor $p^{3/2}$.

The paper is organized as follows. The basic notation, functional setting and the description of the Mortar method are given in Section 2. Some technical tools required in the construction and analysis of the proposed preconditioners are revised in the same Section. The substructuring preconditioner is introduced and analyzed in Section 3 whereas two different choices for the vertex block of the preconditioner are presented in Section 4. Numerical experiments are presented in Section 5.

We are interested here in explicitly studying the dependence of the estimates that we are going to prove on the number and size of the subdomains and on the degree of the polynomial used. To this end, in the following we will employ the notation $A \lesssim B$ (resp. $A \gtrsim B$) to say that the quantity A is bounded from above (resp. from below) by cB, with a constant c independent of ℓ, of the H_ℓ's, as well as of any mesh size parameter and of the polynomial degree p_ℓ. The expression $A \simeq B$ will stand for $A \lesssim B \lesssim A$.

2. The Mortar Method

Let us at first recall the definition of the mortar method, see e.g. [34] and the literature therein. For simplicity we will consider the following simple model problem (though the results that we present here will very easily extend to a more general situation): letting $\Omega \subset \mathbb{R}^2$ be a polygonal domain and given $f \in L^2(\Omega)$, find u satisfying

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial \mathbf{x}_j}\left(a_{ij}(\mathbf{x})\frac{\partial u}{\partial \mathbf{x}_i}\right) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{2.1}$$

We assume that for almost all $\mathbf{x} \in \Omega$ the matrix $a(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1,2}$ is symmetric positive definite, with smallest eigenvalue $\geq \alpha > 0$ and largest eigenvalue $\leq \alpha'$, α, α' independent of $\mathbf{x}$.

In order to discretize the above problem, we start by considering a decomposition of Ω as the union of L subdomains Ω_ℓ,

$$\Omega = \bigcup_{\ell=1,\dots,L} \Omega_\ell \tag{2.2}$$

which, for simplicity, we assume to be quadrangles. We assume that each subdomain Ω_ℓ satisfies the following assumption: there exists orientation preserving bilinear mappings $B_\ell : [0,1]^2 \to \Omega_\ell$ such that there exist a constant H_ℓ with

$$H_\ell^{-1}|J(B_\ell)| \lesssim 1, \qquad H_\ell|J(B_\ell^{-1})| \lesssim 1,$$

where J denotes the Jacobian matrix and where H_ℓ is the diameter of the subdomain Ω_ℓ.

We set

$$\Gamma_{\ell n} = \partial\Omega_n \cap \partial\Omega_\ell, \qquad \mathcal{S} = \cup \Gamma_{\ell n} \tag{2.3}$$

and we denote by $\gamma_\ell^{(i)}$ $(i = 1, \dots, 4)$ the i-th side of the ℓ-th domain:

$$\partial\Omega_\ell = \bigcup_{i=1}^{4} \gamma_\ell^{(i)}.$$

For each subdomain Ω_ℓ, let x_i^ℓ, $i = 1, \cdots, 4$ be the vertices of the subdomain, which we assume to be ordered consecutively, so that each segment $\gamma_\ell^{(i)} = [x_i^\ell, x_{i+1}^\ell]$ (for notational simplicity we also introduce the notation $x_5 = x_1$).

Here we deal with the case of a geometrically conforming decomposition: each edge $\gamma_\ell^{(i)}$ coincides with $\Gamma_{\ell n}$ for some n. The extension to the case of a geometrically non–conforming decomposition will be considered in future works.

Functional spaces Let us at first introduce the necessary functional setting. For $\hat\Omega$ any domain in $\mathbb{R}^d$, $d = 1, 2$ we introduce the following unscaled norms and seminorms (with $0 < s < 1$):

$$\|\hat u\|^2_{0,\hat\Omega} = \int_{\hat\Omega} |\hat u|^2, \qquad |\hat u|^2_{1,\hat\Omega} = \int_{\hat\Omega} |\nabla u|^2, \qquad |\hat u|_{s,\hat\Omega} = \int_{\hat\Omega} dx \int_{\hat\Omega} dy \frac{|\hat u(x) - \hat u(y)|^2}{|x-y|^{d+2s}}.$$

We then introduce the following suitably scaled norms and seminorms: for two dimensional entities

$$\|u\|^2_{H^1(\Omega_\ell)} = H_\ell^{-2} \int_{\Omega_\ell} |u|^2\,dx + \int_{\Omega_\ell} |\nabla u|^2\,dx, \qquad |u|^2_{H^1(\Omega_\ell)} = \int_{\Omega_\ell} |\nabla u|^2\,dx, \tag{2.4}$$

and for one dimensional entities (γ being either $\gamma_\ell^{(i)}$ or $\partial\Omega_\ell$)

$$|\eta|^2_{H^s(\gamma)} = H_\ell^{2s-1} \int_\gamma \int_\gamma \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{2s+1}}\,dx\,dy, \qquad s \in (0,1) \tag{2.5}$$

$$\|\eta\|^2_{H^s(\gamma)} = |\eta|^2_{H^s(\gamma)} + H_\ell^{-1} \int_\gamma |\eta|^2\,ds, \qquad s \in (0,1). \tag{2.6}$$

Remark that the above norms are defined in such a way that they are scaling invariant, that is they are preserved when Ω_ℓ is rescaled to the reference domain $]0,1[^2$.

In the following for $\gamma_\ell^{(i)}$ edge of Ω_ℓ we will also make explicit use of the spaces $H^s_0(\gamma_\ell^{(i)})$ and $H^{1/2}_{00}(\gamma_\ell^{(i)})$, which are defined as the subspaces of those functions η of $H^s(\gamma_\ell^{(i)})$ (resp. $H^{1/2}(\gamma_\ell^{(i)})$) such that the function $\hat\eta$ defined as $\hat\eta = \eta$ on $\gamma_\ell^{(i)}$ and $\hat\eta = 0$ on $\partial\Omega \setminus \gamma_\ell^{(i)}$ belongs to $H^s(\partial\Omega)$ (resp. to $H^{1/2}(\partial\Omega)$). The spaces $H^s_0(\gamma_\ell^{(i)})$ and $H^{1/2}_{00}(\gamma_\ell^{(i)})$ are endowed with the norms

$$\|\eta\|_{H^s_0(\gamma_\ell^{(i)})} = \|\hat\eta\|_{H^s(\partial\Omega^\ell)} \qquad \|\eta\|_{H^{1/2}_{00}(\gamma_\ell^{(i)})} = \|\hat\eta\|_{H^{1/2}(\partial\Omega^\ell)}.$$

Let the spaces X and T be defined as

$$X = \prod_\ell \{u_\ell \in H^1(\Omega_\ell) |\ \ u_\ell = 0 \text{ on } \partial\Omega \cap \partial\Omega_\ell\}, \qquad T = \prod_\ell H^{1/2}_*(\partial\Omega_\ell), \tag{2.7}$$

where $H^{1/2}_*(\Omega_\ell)$ is defined by

$$H^{1/2}_*(\partial\Omega_\ell) = H^{1/2}(\partial\Omega_\ell) \qquad \text{if } |\partial\Omega_\ell \cap \partial\Omega| = 0$$

and

$$H^{1/2}_*(\partial\Omega_\ell) = \{\eta \in H^{1/2}(\partial\Omega_\ell),\ \eta|_{\partial\Omega_\ell\cap\partial\Omega} \equiv 0\} \sim H^{1/2}_{00}(\partial\Omega_\ell \setminus \partial\Omega)$$

otherwise.

Discretizations We consider for each ℓ a family $\mathcal{K}_h^\ell$ of compatible quasi-uniform shape regular decompositions of Ω^k, each made of open elements K, which, to fix the ideas, we assume to be triangular (the extension to quadrilateral elements being trivial), depending on a parameter $h_\ell > 0$. We let $\mathcal{V}_h^\ell \subset H^1(\Omega_\ell)$ be the order p_ℓ finite element space defined on the decomposition $\mathcal{K}_h^\ell$ and satisfying an homogeneous boundary condition on $\partial\Omega \cap \partial\Omega_\ell$:

$$\mathcal{V}_h^\ell = \{v \in C^0(\bar{\Omega}_\ell) \text{ s.t. } v|_K \in P_{p_\ell}(K),\ K \in \mathcal{K}_h^\ell\} \cap H_0^1(\Omega_\ell),$$

where $P_{p_\ell}(K)$ stands for the space of polynomials of degree at most p_ℓ.

We set

$$T_h^\ell = \mathcal{V}_h^\ell|_{\partial\Omega_\ell}, \tag{2.8}$$

and, for each edge $\gamma_\ell^{(i)}$ of the subdomain Ω_ℓ, we define

$$T_{\ell,i} = \{\eta : \quad \eta \text{ is the trace on } \gamma_\ell^{(i)} \text{ of some } u_\ell \in \mathcal{V}_h^\ell\ \} \tag{2.9}$$

$$T_{\ell,i}^0 = \{\eta \in T_{\ell,i} : \quad \eta = 0 \text{ at the extrema of } \gamma_\ell^{(i)}\}. \tag{2.10}$$

Finally, we set

$$X_h = \prod_{\ell=1}^{L} \mathcal{V}_h^\ell\ \subset X, \qquad T_h = \prod_{\ell=1}^{L} T_h^\ell\ \subset T. \tag{2.11}$$

On X and T we introduce the following broken norm and semi-norm:

$$\|u\|_X = \left(\sum_\ell \|u\|^2_{H^1(\Omega_\ell)}\right)^{\frac{1}{2}}, \qquad |u|_X = \left(\sum_\ell |u|^2_{H^1(\Omega_\ell)}\right)^{\frac{1}{2}}, \tag{2.12}$$

$$\|\eta\|_T = \left(\sum_\ell \|\eta_\ell\|^2_{H^{1/2}(\partial\Omega_\ell)}\right)^{1/2} \qquad |\eta|_T = \left(\sum_\ell |\eta_\ell|^2_{H^{1/2}(\partial\Omega_\ell)}\right)^{1/2}. \tag{2.13}$$

The spaces considered satisfy classical direct and inverse inequalities (see e.g. [4, 15, 31]). In view of the scaling (2.4), the direct inequalities take the following form: for $0 \le s \le 1$, $s < r \le p_\ell + 1$

$$\inf_{\eta_h \in T_{\ell,i}} |\eta - \eta_h|_{H^s(\gamma_m)} \lesssim p_\ell^{s-r} \left(\frac{h_\ell}{H_\ell}\right)^{r-s} |\eta|_{H^r(\gamma_m)} \qquad \forall \eta \in H^r(\gamma_m) \tag{2.14}$$

$$\inf_{\eta_h \in T_{\ell,i}^0} |\eta - \eta_h|_{H^s(\gamma_m)} \lesssim p_\ell^{s-r} \left(\frac{h_\ell}{H_\ell}\right)^{r-s} |\eta|_{H^r(\gamma_m)} \qquad \forall \eta \in H^r(\gamma_m) \cap H_0^1(\gamma_m) \tag{2.15}$$

while the inverse inequalities take the form for all $\eta \in T_{\ell,i}$ and for all s, r such that $0 \le s < r \le 1$

$$\|\eta\|_{H^r(\gamma_m)} \lesssim {p_\ell}^{2(r-s)} \left(\frac{h_\ell}{H_\ell}\right)^{s-r} \|\eta\|_{H^s(\gamma_m)}, \qquad |\eta|_{H^r(\gamma_m)} \lesssim {p_\ell}^{2(r-s)} \left(\frac{h_\ell}{H_\ell}\right)^{s-r} |\eta|_{H^s(\gamma_m)} \tag{2.16}$$

and for all $\eta \in T_{\ell,i}^0$ and for all $s, r \ne 1/2$ such that $0 \le s < r \le 1$

$$\|\eta\|_{H_0^r(\gamma_m)} \lesssim {p_\ell}^{2(r-s)} \left(\frac{h_\ell}{H_\ell}\right)^{s-r} \|\eta\|_{H_0^s(\gamma_m)}, \qquad |\eta|_{H_0^r(\gamma_m)} \lesssim {p_\ell}^{2(r-s)} \left(\frac{h_\ell}{H_\ell}\right)^{s-r} |\eta|_{H_0^s(\gamma_m)}, \tag{2.17}$$

once again with constants independent of r, s. For $s = 1/2$ or $r = 1/2$ (2.17) holds with H^s_0 (resp. H^r_0) replaced by $H^{1/2}_{00}$.

In the following it will be convenient to introduce the following notation:

$$H = H_{\ell^*} \qquad h = h_{\ell^*} \qquad p = p_{\ell^*}$$

with

$$\ell^* = \arg\max_{\ell} \frac{H_\ell p_\ell^2}{h_\ell}$$

so that $\frac{H_\ell p_\ell^2}{h_\ell} \leq \frac{Hp^2}{h}$ for all ℓ. Remark that, depending on the different discretisation parameters, it might happen that for some ℓ we have $p_\ell > p_{\ell^*} = p$. In view of this remark we introduce also the notation

$$\hat{p} = \max_{\ell} p_\ell.$$

Classical bounds. With the chosen scaling, several classical bounds hold with constants independent of H_ℓ. In particular we have:

Trace bound For all $u \in H^1(\Omega_\ell)$ we have ([24])

$$\|u\|_{H^{1/2}(\partial\Omega_\ell)} \lesssim \|u\|_{H^1(\Omega_\ell)}, \qquad |u|_{H^{1/2}(\Omega_\ell)} \lesssim |u|_{H^1(\Omega_\ell)}. \tag{2.18}$$

Injection of H^s in L^∞ for $s > 1/2$ For all $\eta \in H^s(\gamma)$, $s > 1/2$, γ being either $\gamma_\ell^{(i)}$ or $\partial\Omega_\ell$, we have ([11])

$$\|\eta\|_{L^\infty(\gamma)} \lesssim \frac{1}{\sqrt{2s-1}} \|\eta\|_{H^s(\gamma)}. \tag{2.19}$$

Poincaré type inequalities. For all $\eta \in H^s_0(\gamma_\ell^{(i)})$ it holds that

$$\|\eta\|_{H^s_0(\gamma_\ell^{(i)})} \lesssim |\eta|_{H^s_0(\gamma_\ell^{(i)})} \tag{2.20}$$

and for all η with $\int_\gamma \eta = 0$, γ being either $\gamma_\ell^{(i)}$ or $\partial\Omega_\ell$, it holds that

$$\|\eta\|_{H^s(\gamma)} \lesssim |\eta|_{H^s(\gamma)}. \tag{2.21}$$

Injection of H^s in H^s_0 for $s < 1/2$. We recall that for $s < 1/2$ the spaces $H^s(\gamma_\ell^{(i)})$ and $H^s_0(\gamma_\ell^{(i)})$ coincide as sets and have equivalent norms. However, the constants in the norm equivalence goes to infinity as s tends to $1/2$. For all $\varphi \in H^s(\gamma_\ell^{(i)})$ the following bound can be shown (see [10]): for $\beta \in \mathbb{R}$ arbitrary it holds that

$$|\varphi|_{H^s_0(\gamma_\ell^{(i)})} \lesssim \frac{1}{1/2 - s} \|\varphi - \beta\|_{H^{1/2}(\gamma_\ell^{(i)})} + \frac{1}{\sqrt{1/2 - s}} |\beta|. \tag{2.22}$$

If ϕ is linear, bound (2.22) can be improved to

$$|\varphi|_{H^s_0(\gamma_\ell^{(i)})} \lesssim \frac{1}{\sqrt{1/2 - s}} (\|\varphi - \beta\|_{H^{1/2}(\gamma_\ell^{(i)})} + \frac{1}{\sqrt{1/2 - s}} |\beta|). \tag{2.23}$$

Technical tools. We now revise some technical tools that will be required in the construction and analysis of our preconditioner. We observe that the following results, are a generalisation to the hp-version of [9, Lemma 3.1] and of [14, Lemma 3.4], see e.g. [19] for the proof.

Lemma 2.1. *The following bounds hold:*

- *for all $\xi \in T_h^\ell$ and γ being either $\gamma_\ell^{(i)}$ or $\partial\Omega_\ell$, it holds*

$$\|\xi\|^2_{L^\infty(\gamma)} \lesssim \left(1 + \log\left(\frac{H_\ell p_\ell^2}{h_\ell}\right)\right) \|\xi\|^2_{H^{1/2}(\gamma)}; \tag{2.24}$$

- *for all $\xi \in T_h^\ell$ such that $\xi(P) = 0$ for some $P \in \gamma$, γ being either $\gamma_\ell^{(i)}$ or $\partial\Omega_\ell$, it holds*

$$\|\xi\|^2_{L^\infty(\gamma)} \lesssim \left(1 + \log\left(\frac{H_\ell p_\ell^2}{h_\ell}\right)\right) |\xi|^2_{1/2,\gamma}; \tag{2.25}$$

- *for all $\xi \in T_{\ell,i}^0$ it holds*

$$\|\xi\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})} \lesssim \left(1 + \log\left(\frac{H_\ell p_\ell^2}{h_\ell}\right)\right)^2 |\xi|^2_{H^{1/2}(\gamma_\ell^{(i)})}. \tag{2.26}$$

The following result is a generalization to the h-p version of Lemmas 3.2, 3.4 and 3.5 of [14].

Lemma 2.2. *Let $\xi \in T_h^\ell$ such that $\xi(x_i^\ell) = 0$, $i = 1, \cdots, 4$, and let $\zeta_L \in H^{1/2}(\partial\Omega_\ell)$, ζ_L linear on each edge of Ω_ℓ. Then it holds*

$$\sum_{i=1}^{4} \|\xi\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})} \lesssim \left(1 + \log\left(\frac{H_\ell p_\ell^2}{h_\ell}\right)\right)^2 \ |\xi + \zeta_L|^2_{H^{1/2}(\partial\Omega_\ell)}. \tag{2.27}$$

The following lemma holds.

Lemma 2.3. *Let $\sigma : \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}$ be defined as*

$$\sigma(\alpha, \beta) = \sum_{\ell,n:|\Gamma_{\ell n}|>0} (\alpha_\ell - \alpha_n)(\beta_\ell - \beta_n). \tag{2.28}$$

For $\eta \in T$ let $\bar\eta$ be defined by

$$\bar\eta = (\bar\eta_\ell)_{\ell=\ell=1,\cdots,L}, \qquad \bar\eta^\ell = |\partial\Omega_\ell|^{-1} \int_{\Omega_\ell} \eta^\ell \tag{2.29}$$

Then, if $\eta \in T$ verifies

$$\int_{\gamma_m} [\eta] = 0, \quad \forall m = (\ell, i) \in I, \tag{2.30}$$

we have

$$\sigma(\bar\eta, \bar\eta) \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) |\eta|^2_T. \tag{2.31}$$

Proof. For each edge $\Gamma_{\ell n}$ we introduce the constant

$$\bar{\eta}_{\ell,n} = \frac{1}{|\Gamma_{\ell n}|}\int_{\Gamma_{\ell n}} \eta_\ell = \frac{1}{|\Gamma_{\ell n}|}\int_{\Gamma_{\ell n}} \eta_n,$$

(the last identity is a consequence of (2.30)). For $\gamma_\ell^{(i)} = \Gamma_{\ell n}$ we also introduce the notation $\bar{\eta}_\ell^{(i)} = \bar{\eta}_{\ell,n}$. We have

$$\begin{aligned}\sigma(\bar{\eta},\bar{\eta}) &= \sum_{\ell,n:|\Gamma_{\ell n}|>0} |\bar{\eta}_\ell - \bar{\eta}_{\ell,n} - (\bar{\eta}_n - \bar{\eta}_{\ell,n})|^2 \lesssim \sum_\ell \sum_{n:|\Gamma_{\ell n}|>0} |\bar{\eta}_\ell - \bar{\eta}_{\ell,n}|^2 \\ &= \sum_\ell \sum_{i\in\mathcal{E}_\ell} |\bar{\eta}_\ell - \bar{\eta}_\ell^{(i)} + \eta(x_i^\ell) - \eta(x_i^\ell)|^2 \\ &\lesssim \sum_\ell \sum_{i\in\mathcal{E}_\ell} |\eta(x_i^\ell) - \bar{\eta}_\ell|^2 + \sum_\ell \sum_{i\in\mathcal{E}_\ell} |\eta(x_i^\ell) - \bar{\eta}_\ell^{(i)}|^2,\end{aligned}$$

where, for each ℓ, we let $\mathcal{E}_\ell = \{i : \ \gamma_\ell^{(i)} \text{ is an interior edge}\}$.

We have

$$|\bar{\eta}_\ell - \eta(x_i^\ell)|^2 \lesssim \|\eta - \bar{\eta}_\ell\|^2_{L^\infty(\Gamma_\ell)}.$$

We observe that $\int_{\partial\Omega_\ell} \eta_\ell - \bar{\eta}_\ell = 0$, which, since $\eta_\ell - \bar{\eta}_\ell \in C^0(\partial\Omega_\ell)$, implies that $\eta_\ell - \bar{\eta}_\ell$ vanishes at some point of $\partial\Omega_\ell$. We can then apply bound (2.25), which yields

$$|\bar{\eta}_\ell - \eta(x_i^\ell)|^2 \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) |\eta_\ell|^2_{1/2,\partial\Omega_\ell}.$$

The term $|\eta_\ell(x_i^\ell) - \bar{\eta}_\ell^{(i)}|^2$ is bound analogously. The thesis is obtained since the cardinality of the set $\mathcal{E}_\ell$ is bounded. $\square$

Mortar Problem. Let now a composite bilinear form $a_X : X\times X \longrightarrow \mathbb{R}$ be defined as follows:

$$a_X(u,v) = \sum_\ell a_\ell(u_\ell, v_\ell) \quad \text{with} \quad a_\ell(u_\ell,v_\ell) = \int_{\Omega_\ell} \sum_{i,j} a_{ij}(x) \frac{\partial u_\ell}{\partial x_i}\frac{\partial v_\ell}{\partial\, x_j}\, dx. \tag{2.32}$$

The bilinear form a_X is clearly not coercive on X. In order to obtain a well posed problem we will then consider proper subspaces of X, consisting of functions satisfying a suitable *weak continuity* constraint. For defining such constraint, according to the mortar method, we start by choosing for each segment $\Gamma_{\ell n} = \gamma_\ell^{(i)} = \gamma_n^{(j)}$, one side (let us say ℓ) to be the master side, while the other side will be the slave side. More precisely, we choose an index set $I \subset \{1,\dots,L\}\times\{1,\dots,4\}$ (which will individuate the slave sides), defined in such a way that,

$$\mathcal{S} = \bigcup_{(\ell,i)\in I} \gamma_\ell^{(i)}, \qquad \begin{array}{c}(\ell_1,i_1),(\ell_2,i_2)\in I,\\ (\ell_1,i_1)\neq(\ell_2,i_2)\end{array} \Rightarrow \gamma_{\ell_1}^{(i_1)} \cap \gamma_{\ell_2}^{(i_2)} = \emptyset. \tag{2.33}$$

Furthermore we will denote by $I^* \subset \{1,\cdots,L\}\times\{1,\cdots,4\}$ the index-set corresponding to master sides, which is defined in such a way that $I^*\cap I = \emptyset$ and $\mathcal{S} = \cup_{(\ell,i)\in I^*}\gamma_\ell^{(i)}$.

For each $m = (\ell, i) \in I$, let $a_0 = x_i^\ell < a_1 < \ldots < a_{M-1} < a_M = x_{i+1}^\ell$ denote the one dimensional mesh induced on γ_m by the two dimensional mesh $\mathcal{K}_\ell$. Let $e_i = (a_{i-1}, a_i)$ and let the finite dimensional multiplier space M_h^m on γ_m, be defined as

$$M_h^m = \{v \in C^0(\gamma_m),\ v|_{e_i} \in P_{p_\ell}(e_i), i \neq 1, M,\ v|_{e_1} \in P_{p_\ell - 1}(e_1),\ v|_{e_M} \in P_{p_\ell - 1}(e_M)\}. \tag{2.34}$$

Remark that $dim(M_h^m) = dim(T_m^0)$. We set:

$$M_h = \{\eta \in L^2(\mathcal{S}),\ \forall m \in I \quad \eta|_{\gamma_m} \in M_h^m\} \sim \prod_{m \in \mathcal{I}} M_m. \tag{2.35}$$

The *constrained* approximation and trace spaces $\mathcal{X}_h$ and $\mathcal{T}_h$ are then defined as follows:

$$\mathcal{X}_h = \{v_h \in X_h, \quad \int_S [v_h]\lambda\, ds = 0,\ \forall \lambda \in M_h\}, \tag{2.36}$$

$$\mathcal{T}_h = \{\eta \in T_h, \int_S [\eta]\lambda\, ds = 0,\ \forall \lambda \in M_h\}, \tag{2.37}$$

where, on $\gamma_\ell^{(i)} = \gamma_n^{(j)}$, $(\ell, i) \in I$ we set $[\eta] = \eta_\ell - \eta_n$.

We can now introduce the following discrete problem:

Problem 2.1. *Find $u_h \in \mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$*

$$a_X(u_h, v_h) = \int_\Omega f v_h\, dx. \tag{2.38}$$

It is known that Problem 2.1 admits a unique solution u_h. For an error estimate, see [7].

The mortar correction operator. For all $m = (\ell, i) \in I$ ($\gamma_\ell^{(i)}$ slave side), we let $\pi_m : L^2(\gamma_m) \longrightarrow T_m^0$ be the bounded projector defined as

$$\int_{\gamma_m} (\eta - \pi_m \eta)\lambda = 0, \quad \forall \lambda \in M_m. \tag{2.39}$$

The projection π_m is well defined and satisfies (see [33, 32]):

Theorem 2.4. *For $m = (\ell, i) \in I$ it holds:*

$$\|\pi_m \eta\|_{L^2(\gamma_m)} \lesssim p_\ell^{\frac{1}{2}} \|\eta\|_{L^2(\gamma_m)} \quad \forall \eta \in L^2(\gamma_m) \tag{2.40}$$

$$|\pi_m \eta|_{H^1(\gamma_m)} \lesssim p_\ell\, |\eta|_{H^1(\gamma_m)} \quad \forall \eta \in H_0^1(\gamma_m). \tag{2.41}$$

Remark 2.5. The problem of whether (2.40) and (2.41) are optimal was studied in [32], where, through an eigenvalue analysis the dependence on p to the power 1/2 and 1 of the norm of the projector appearing in (2.40) and (2.41) was confirmed. This dependence does not seem to affect the asymptotic rate of the error, which, as observed in [32] seems to be only slightly suboptimal (loss of a factor $C(\varepsilon)p^\varepsilon$ for ε arbitrarily small). In [7] this good behavior of the error was proven, for sufficiently smooth solutions, thanks to an interpolation argument.

By space interpolation and using the Poincaré inequality we immediately get the following corollary

Corollary 2.6. *For all s, $0 < s < 1$, $s \neq 1/2$, for all $\eta \in H^s_0(\gamma_m)$ we have*

$$|\pi_m \eta|_{H^s_0(\gamma_m)} \lesssim p_\ell^{(1+s)/2}\, |\eta|_{H^s_0(\gamma_m)}, \tag{2.42}$$

uniformly in s. For all $\eta \in H^{1/2}_{00}(\gamma_m)$ we have

$$|\pi_m \eta|_{H^{1/2}_{00}(\gamma_m)} \lesssim p_\ell^{3/4}\, |\eta|_{H^{1/2}_{00}(\gamma_m)}. \tag{2.43}$$

We now define a global linear operator

$$\pi_h : \prod_{\ell=1}^{L} L^2(\partial\Omega_\ell) \longrightarrow \prod_{\ell=1}^{L} L^2(\partial\Omega_\ell)$$

as follows: for $\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in \Pi_\ell L^2(\partial\Omega_\ell)$, we set $\pi_h(\eta) = (\eta^*_\ell)_{\ell=1,\cdots,L}$, where $\eta^*_\ell \in T^\ell_h$ is defined on multiplier sides as π_m applied to the jump of η, while it is set identically zero on trace sides and on the external boundary $\partial\Omega$: on $\gamma_m = \gamma^{(i)}_\ell = \gamma^{(j)}_n$, $(\ell, i) \in I$, $(n, j) \in I^*$ (ℓ slave side)

$$\eta^*_\ell|_{\gamma_m} = \pi_m([\eta]|_{\gamma_m}), \qquad \eta^*_n|_{\gamma_m} = 0,$$

and for all ℓ

$$\eta^*_\ell = 0 \text{ on } \partial\Omega_\ell \cap \partial\Omega.$$

The following bound holds.

Lemma 2.7. *For all $\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in T$ and for all $\alpha = (\alpha_\ell)_{\ell=1,\cdots,L}$, α_ℓ constant in Ω_ℓ, it holds*

$$|(Id - \pi_h)(\eta)|^2_T \lesssim \hat{p}^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 \|\eta - \alpha\|^2_T + \hat{p}^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)\sigma(\alpha, \alpha) \tag{2.44}$$

where we recall that the bilinear form σ is defined in (2.28).

If, in addition, each η_ℓ is linear on each $\gamma^{(i)}_\ell$, then the bound can be improved to

$$|(Id - \pi_h)(\eta)|^2_T \lesssim \hat{p}^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)\left(\|\eta - \alpha\|_T + \sigma(\alpha, \alpha)\right). \tag{2.45}$$

Proof. We have

$$\begin{aligned} |\pi_h(\eta)|^2_T &\lesssim \sum_{m=(\ell,i)\in I} |\pi_m([\eta])|^2_{H^{1/2}_{00}(\gamma_m)} \\ &\lesssim \sum_{m=(\ell,i)\in I} H_\ell^{2\varepsilon} p_\ell^{4\varepsilon} h_\ell^{-2\varepsilon} |\pi_m([\eta])|^2_{H^{1/2-\varepsilon}_0(\gamma_m)} \\ &\lesssim \hat{p}^{3/2} \sum_{m=(\ell,i)\in I} h_\ell^{-2\varepsilon} H_\ell^{2\varepsilon} p_\ell^{4\varepsilon} |[\eta]|^2_{H^{1/2-\varepsilon}_0(\gamma_m)}. \end{aligned} \tag{2.46}$$

We now observe that, for $m = (\ell, i) \in I$, $\gamma^{(i)}_\ell = \Gamma_{\ell,n}$ we have (see (2.22))

$$|[\eta]|^2_{H^{1/2-\varepsilon}_0(\gamma_m)} \lesssim \frac{1}{\varepsilon^2}\|[\eta - \alpha]\|^2_{H^{1/2}(\gamma_m)} + \frac{1}{\varepsilon}|\alpha_\ell - \alpha_n|^2. \tag{2.47}$$

Then we obtain

$$|\pi_h(\eta)|_T^2 \lesssim \hat{p}^{3/2} \frac{H^{2\varepsilon} p^{4\varepsilon}}{h^{2\varepsilon}} \left(\frac{1}{\varepsilon^2} \sum_{m=(\ell,i)\in I} \|[\eta - \alpha]\|^2_{H^{1/2}(\gamma_m)} + \frac{1}{\varepsilon}\sigma(\alpha,\alpha) \right).$$

Observing that, for $\gamma_m = \Gamma_{\ell,n}$ it holds that

$$\|[\eta - \alpha]\|^2_{H^{1/2}(\gamma_m)} \leq \|\eta_\ell - \alpha_\ell\|^2_{H^{1/2}(\gamma_m)} + \|\eta_n - \alpha_n\|^2_{H^{1/2}(\gamma_m)},$$

by choosing $\varepsilon = 1/\log(Hp^2/h)$, we get

$$|\pi_h(\eta)|_T^2 \lesssim \hat{p}^{3/2} \left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 \|\eta - \alpha\|_T^2 + \hat{p}^{3/2} \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) \sigma(\alpha,\alpha).$$

The bound (2.44) follows easily by observing that

$$|(Id - \pi_h(\eta))|_T^2 \lesssim |\eta|_T^2 + |\pi_h(\eta)|_T^2 = |\eta - \alpha|_T^2 + |\pi_h(\eta)|_T^2 \lesssim \|\eta - \alpha\|_T^2 + |\pi_h(\eta)|_T^2.$$

The bound (2.45) is obtained by observing that, if each η_ℓ is linear on each $\gamma_\ell^{(i)}$, thanks to (2.23), the bound (2.47) can be improved to

$$|[\eta]|^2_{H_0^{1/2-\varepsilon}(\gamma_m)} \lesssim \frac{1}{\varepsilon}(\|[\eta - \alpha]\|^2_{H^{1/2}(\gamma_m)} + |\alpha_\ell - \alpha_n|^2).$$

□

Corollary 2.8. *Let $\eta \in T$ and let $\bar{\eta} = (\bar{\eta}_\ell)_{\ell\in\ell=1,\cdots,L}$ be defined by (2.29). Then*

$$|\pi_h(\eta)|_T^2 \lesssim \hat{p}^{3/2} \left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 |\eta|_T^2 + \hat{p}^{3/2} \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) \sigma(\bar{\eta},\bar{\eta}).$$

3. Substructuring Preconditioners for the Mortar Method

The main idea of substructuring preconditioners consists in splitting the functions $u \in \mathcal{X}_h$ as the sum of three suitably defined components: $u = u^0 + u^E + u^V$ identified respectively by *interior* degrees of freedom (corresponding to basis functions vanishing on the skeleton and supported on one sub-domain), *edge* degrees of freedom, and *vertex* degrees of freedom, and consider preconditioners that, when expressed in a basis related to such a splitting, are block diagonal.

More precisely, we start as usual by introducing the discrete lifting operator $R_h : T_h \to X_h$ defined as follows. For $\eta = (\eta_\ell)_{\ell=1,\ldots,L} \in T_h$ we let $R_h(\eta) = (R_h^\ell(\eta_\ell))_{\ell=1,\ldots,K} \in X_h$ with $R_h^\ell(\eta) \in \mathcal{V}_h^\ell$ solution of

$$R_h^\ell(\eta_\ell) = \eta_\ell \text{ on } \partial\Omega_\ell, \qquad a_\ell(R_h^\ell(\eta_\ell), v_h^\ell) = 0, \quad \forall v_h \in \mathcal{V}_h^\ell \cap H_0^1(\Omega_\ell).$$

It is immediate to check that the spaces X_h of unconstrained functions and $\mathcal{X}_h$ of constrained functions can be split as direct sums of an interior and of a (respectively unconstrained or constrained) trace component:

$$X_h = \mathcal{X}_h^0 \oplus R_h(T_h), \qquad \mathcal{X}_h = \mathcal{X}_h^0 \oplus R_h(\mathcal{T}_h), \tag{3.1}$$

with

$$\mathcal{X}_h^0 = \prod \mathcal{V}_h^\ell \cap H_0^1(\Omega_\ell).$$

We can easily verify that for $w = w^0 + R_h(\eta)$, $v = v^0 + R_h(\zeta)$ (with $w^0, v^0 \in \mathcal{X}_h^0$) $a_X : X_h \times X_h \to \mathbb{R}$ satisfies

$$a_X(w, v) = a_X(w^0, v^0) + a_X(R_h(\eta), R_h(\zeta)) := a_X(w^0, v^0) + s(\eta, \zeta), \tag{3.2}$$

where the *discrete Steklov-Poincaré* operator $s : T_h \times T_h \to \mathbb{R}$ is defined by

$$s(\xi, \eta) := \sum_\ell a_\ell(R_h^\ell(\xi)), R_h^\ell(\eta)). \tag{3.3}$$

Finally, it is well known that

$$\|R_h^\ell \eta\|_{H^1(\Omega_\ell)} \simeq \|\eta\|_{1/2,\partial\Omega_\ell}, \qquad |R_h^\ell \eta|_{H^1(\Omega_\ell)} \simeq |\eta|_{1/2,\partial\Omega_\ell}. \tag{3.4}$$

see [4, 33], whence

$$\|R_h(\eta)\|_X \simeq \|\eta\|_T, \qquad |R_h(\eta)|_X \simeq |\eta|_T. \tag{3.5}$$

The following result for the *Steklov–Poincaré* operator follows easily from the definition of $s(\cdot, \cdot)$, the continuity and coercivity of $a_X(\cdot, \cdot)$ and (3.5).

Corollary 3.1. *For all $\xi \in T_h$, it holds*

$$s(\xi, \xi) \simeq |\xi|_T^2, \tag{3.6}$$

The problem of preconditioning the matrix $\mathbf{A}$ associated to the discretization of a_X, reduces to finding good preconditioners for the matrices $\mathbf{A}_0$ and $\mathbf{S}$ corresponding respectively to the bilinear forms a_X restricted to $\mathcal{X}_h^0$ and s. Here we assume that we have good preconditioners for the stiffness matrix $\mathbf{A}_0$ and we concentrate therefore only on the discrete Steklov-Poincaré operator s.

We start by observing that the space of constrained skeleton functions $\mathcal{T}_h$ can be further split as the sum of *vertex* and *edge* functions. More specifically, if we denote by $\mathfrak{L}$ the space

$$\mathfrak{L} = \{(\eta_\ell)_{\ell=1,\cdots,L},\ \eta_\ell \in C^0(\partial\Omega_\ell) \text{ is linear on each edge of } \Omega_\ell\}, \tag{3.7}$$

then we can define the space of constrained *vertex* functions as

$$\mathcal{T}_h^V = (Id - \pi_h)\mathfrak{L}. \tag{3.8}$$

We observe that $\mathfrak{L} \subset T_h$, which yields $\mathcal{T}_h^V \subset \mathcal{T}_h$. We then introduce the space of constrained *edge* functions $\mathcal{T}_h^E \subset \mathcal{T}_h$ defined by

$$\mathcal{T}_h^E = \{\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in \mathcal{T}_h,\ \eta_\ell(x_i^\ell) = 0,\ i = 1, \cdots, 4\} \tag{3.9}$$

and we can easily verify that

$$\mathcal{T}_h = \mathcal{T}_h^V \oplus \mathcal{T}_h^E. \tag{3.10}$$

Moreover it is quite simple to check that a function in $\mathcal{T}_h^E$ is uniquely defined by its value on master edges, the value on slave edges being forced by the constraint.

It will be useful in the following to introduce the linear interpolation operator $\Lambda : T_h \to \mathfrak{L}$ defined as

$$\Lambda\eta = (\Lambda^\ell \eta_\ell)_{\ell=1,\ldots,L}, \qquad \Lambda^\ell \eta^\ell(x_i^\ell) = \eta(x_i^\ell),\ i = 1, \ldots, 4.$$

Observe that for $\eta \in \mathcal{T}_h$ we have $(1-\pi_h)\Lambda\eta \in \mathcal{T}_h^V$ and $\eta - (1-\pi_h)\Lambda\eta \in \mathcal{T}_h^E$. The following Lemma holds [19].

Lemma 3.2. *For all $\eta = (\eta_\ell)_\ell \in T_h$, it holds*

$$|\Lambda\eta|_T^2 \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)|\eta|_T^2, \qquad \|\Lambda\eta\|_T^2 \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)\|\eta\|_T^2 \tag{3.11}$$

The preconditioner that we consider is built by introducing two bilinear forms:

$$b^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \to \mathbb{R} \qquad \text{and} \qquad b^V : \mathcal{T}_h^V \times \mathcal{T}_h^V \to \mathbb{R}.$$

Let us start by introducing the bilinear form relative to the edges: for any trace side $\gamma_n^{(j)}$, $m = (n,j) \in I^*$, let $b_{n,j} : T_{n,j}^0 \times T_{n,j}^0 \longrightarrow \mathbb{R}$ be a symmetric bilinear form satisfying for all $\eta \in T_{n,j}^0$

$$b_{n,j}(\eta,\eta) \simeq \|\eta\|^2_{H_{00}^{1/2}(\gamma_n^{(j)})}. \tag{3.12}$$

Then, the edge block diagonal global bilinear form $b^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \longrightarrow \mathbb{R}$ is defined by

$$b^E(\eta,\xi) = \sum_{(n,j)\in I^*} b_{n,j}(\eta_\ell,\xi_\ell). \tag{3.13}$$

Applying Lemma 2.2 we easily get

$$b^E(\eta^E,\eta^E) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta). \tag{3.14}$$

Moreover, using the fact that η^E verifies the weak continuity constraint and thar η_ℓ^E vanishes at the cross points we immediately get that for $m = (\ell,i) \in I$ and $k = (n,k) \in I^*$ we have $\eta_\ell^E|_{\gamma_\ell^{(i)}} = \pi_m(\eta_n^E|_{\gamma_n^{(j)}})$ and, by (2.43),

$$|\eta_\ell^E|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \lesssim \hat{p}^{3/2}|\eta_\ell^E|^2_{H_{00}^{1/2}(\gamma_n^{(j)})},$$

which allows us to write

$$|\eta^E|_T^2 \lesssim \sum_{m=(\ell,i)\in I^*} |\eta_\ell^E|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} + \sum_{m=(\ell,i)\in I} |\eta_\ell^E|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \tag{3.15}$$

$$\lesssim \hat{p}^{3/2} \sum_{m=(\ell,i)\in I^*} |\eta_\ell^E|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \lesssim \hat{p}^{3/2} b^E(\eta^E,\eta^E). \tag{3.16}$$

The construction of the vertex block of the preconditioner in the mortar method framework is not standard, since we need to take into account the week continuity constraint. In the P_1 framework, Achdou, Maday, Widlund in [1], propose to use

$$b_0^V(\eta^V,\zeta^V) = s(\eta^V,\zeta^V). \tag{3.17}$$

This choice immediately yields the bound

$$s(\eta,\eta) \lesssim b_0^V(\eta^V,\eta^V) + \hat{p}^{3/2} b^E(\eta^E,\eta^E).$$

Let us bound $b_0^V(\eta^V,\eta^V)$ in terms of $s(\eta,\eta)$. Let $\bar\eta = (\bar\eta_\ell)_{\ell=1,\cdots,L}$ be defined as in (2.29). Using Lemma 2.7 (and in particular (2.45)) we can write

$$b_0^V(\eta^V,\eta^V) \lesssim |(1-\pi_h)\Lambda\eta|_T^2 \lesssim \hat{p}^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\left(\|\Lambda(\eta-\bar\eta)\|_T^2 + \sigma(\bar\eta,\bar\eta)\right).$$

(where we used that $\Lambda\bar{\eta} = \bar{\eta}$). Now, thanks to a Poincaré inequality, Lemma 3.2 and Lemma 2.3, we obtain

$$b_0^V(\eta^V, \eta^V) \lesssim \hat{p}^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 |\eta|_T^2.$$

Then we have

$$b_0^V(\eta^V, \eta^V) + \hat{p}^{3/2} b^E(\eta^E, \eta^E) \lesssim \hat{p}^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta, \eta).$$

This bound would suggest to choose, as a preconditioner for the matrix $\mathbf{S}$, the matrix $\mathbf{P}_0$ corresponding to the bilinear form

$$s_0(\eta, \zeta) = b_0^V(\eta^V, \zeta^V) + \hat{p}^{3/2} b^E(\eta^E, \zeta^E).$$

With this choice we would have the bound

$$Cond(\mathbf{P}_0^{-1}\mathbf{S}) \lesssim \hat{p}^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2.$$

4. The vertex block of the preconditioner.

Building the vertex block of the preconditioner according to (3.17) for fine meshes turns out to be quite expensive, since it implies assembling at least a portion of the Schur complement matrix $\mathbf{S}$. In the the present section we propose two more efficient alternatives.

4.1. **A "coarse" vertex block preconditioner.** The first option that we considered is to build the vertex block of the preconditioner using a fixed auxiliary coarse mesh, independent of the space discretisation and of the polynomial degree. This idea was presented in [12] for the case of P1 finite elements. We combine it here with a suitable balancing between vertex and edge component, yielding a better estimate for the condition number of the preconditioned matrix.

Let $\mathtt{n}_c$ be a fixed small integer. We build coarse auxiliary quasi-uniform triangular meshes $\mathcal{K}_\delta^\ell$ with mesh size $\delta = \delta_\ell = \frac{H_\ell}{\mathtt{n}_c} \geq h_\ell$. We do not assume that $\mathcal{K}_\delta^\ell$ and $\mathcal{K}_h^\ell$ are nested. We define a coarse auxiliary P_1 discretization spaces $\mathcal{V}_\delta^\ell \subset H^1(\Omega_\ell) \cap C^0(\bar{\Omega}_\ell)$ defined by

$$\mathcal{V}_\delta^\ell = \{v \in C^0(\bar{\Omega}_\ell) \text{ s.t. } v|_K \in P_1(K),\ K \in \mathcal{T}_\delta^\ell\ \} \cap H_0^1(\Omega_\ell).$$

For each $m = (\ell, i) \in I$ we also consider the corresponding auxiliary multiplier space $M_\delta^m \subset L^2(\gamma_m)$, defined analogously to (2.34).

The spaces X_δ, M_δ, $\mathcal{X}_\delta$, and T_δ^ℓ, T_δ, $\mathcal{T}_\delta$ are built starting from the $\mathcal{V}_\delta^\ell$'s and the M_δ^m's in the same way as the spaces X_h, M_h, $\mathcal{X}_h$ and T_h^ℓ, T_h, $\mathcal{T}_h$ by using definitions similar to (2.9), (2.10), (2.11), (2.35) and (2.36). Analogously to π_h we can define the operator $\pi_\delta : \prod_{\ell=1}^L L^2(\partial\Omega_\ell) \longrightarrow T_\delta$. Using Lemma 2.7 we obtain for all $\eta \in T$ and $\alpha = (\alpha_\ell)_{\ell=\ell=1,\cdots,L} \in T$, with α_ℓ constant,

$$|(Id - \pi_\delta)\eta|_T^2 \lesssim (1 + \log(\mathtt{n}_c))^2 \|\eta - \alpha\|_T^2 + (1 + \log(\mathtt{n}_c))\,\sigma(\alpha, \alpha), \tag{4.1}$$

and for $\eta \in \mathfrak{L}$

$$|(Id - \pi_\delta)\eta|_T^2 \lesssim (1 + \log(\mathtt{n}_c))\,(\|\eta - \alpha\|_T^2 + \sigma(\alpha, \alpha)). \tag{4.2}$$

Moreover, Lemma 3.2 yields that for all $\eta \in T_\delta$

$$|\Lambda\eta|_T^2 \lesssim (1 + \log(\mathtt{n}_c))\, |\eta|_T^2. \tag{4.3}$$

Analogously to R_h^ℓ we can define a local coarse lifting operator R_δ^ℓ. By standard arguments this verifies, for all $\eta \in T_\delta$,

$$\|R_\delta \eta\|_X \simeq \|\eta\|_T, \qquad |R_\delta \eta|_X \simeq |\eta|_T. \tag{4.4}$$

We define the vertex block of the preconditioner as $b_1^V : \mathcal{T}_h^V \times \mathcal{T}_h^V \to \mathbb{R}$ as

$$b_1^V(\eta^V, \xi^V) := \sum_\ell \int_{\Omega_\ell} a(\mathbf{x}) \nabla (R_\delta^\ell (1 - \pi_\delta)\Lambda \eta^V) \cdot \nabla (R_\delta^\ell (1 - \pi_\delta)\Lambda \xi^V). \tag{4.5}$$

The second preconditioner we propose is then:

$$\begin{aligned} &s_1 : \mathcal{T}_h \times \mathcal{T}_h \longrightarrow \mathbb{R} \\ &s_1(\eta, \xi) = b^E(\eta^E, \xi^E) + \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) b_1^V(\eta^V, \xi^V). \end{aligned} \tag{4.6}$$

Remark that $(1 - \pi_\delta)\Lambda \mathcal{T}_h^V = \mathcal{T}_\delta^V$. In view of this identity it is not difficult to realize that computing the vertex block of this preconditioner only implies assembling the Schur complement matrix for an auxiliary mortar problem corresponding to the coarse dicretization. This is then independent of the mesh size h. More details will be given in the next section.

The following theorem holds:

Theorem 4.1. *For all $\eta \in \mathcal{T}_h$ we have:*

$$\hat{p}^{-3/2} s(\eta, \eta) \lesssim s_1(\eta, \eta) \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta, \eta). \tag{4.7}$$

Proof. By using (3.15) we get

$$s(\eta, \eta) \lesssim |\eta^E|_T^2 + |\eta^V|_T^2 \lesssim \hat{p}^{3/2} b^E(\eta^E, \eta^E) + |\eta^V|_T^2. \tag{4.8}$$

Concerning $|\eta^V|_T^2$, let $\eta_\delta^V = (1 - \pi_\delta)\Lambda\eta$. We have $\eta^V = (1 - \pi_h)\Lambda \eta_\delta^V$. We introduce $\tilde{\eta} = (\tilde{\eta}_\ell)_{\ell \in \ell = 1, \cdots, L} \in T$ with $\tilde{\eta}_\ell$ constant on $\partial\Omega_\ell$ defined as

$$\tilde{\eta}_\ell = |\partial\Omega_\ell|^{-1} \int_{\partial\Omega_\ell} \eta_\delta^V.$$

Using Lemma 2.3 and (4.3), as well as (2.21), we have ($\Lambda\tilde\eta = \tilde\eta$)

$$
\begin{aligned}
|\eta^V|_T^2 = |(Id-\pi_h)\Lambda\eta_\delta^V|_T^2 &\lesssim \hat p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\left(\|\Lambda(\eta_\delta^V-\tilde\eta)\|_T^2+\sigma(\tilde\eta,\tilde\eta)\right)\\
&\lesssim \hat p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)(1+\log(\mathtt{n}_c))\left(\|\eta_\delta^V-\tilde\eta\|_T^2+|\eta_\delta^V|_T^2\right)\\
&\lesssim \hat p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)(1+\log(\mathtt{n}_c))\,|\eta_\delta^V|_T^2\\
&\lesssim \hat p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)b_1^V(\eta^V,\eta^V),
\end{aligned}
$$

where the last bound holds since $\mathtt{n}_c$ is a constant independent of h, p and H. Then we have

$$
s(\eta,\eta)\lesssim|\eta^E|_T^2+|\eta^V|_T^2\lesssim\hat p^{3/2}b^E(\eta^E,\eta^E)+\hat p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)b_1^V(\eta^V,\eta^V)=\hat p^{3/2}s_1(\eta,\eta),
$$

that is the first part of the theorem.

Let us now bound $s_1(\eta,\eta)$ in terms of $s(\eta,\eta)$. We have, for $\bar\eta$ defined by (2.29),

$$
\begin{aligned}
b_1^V(\eta^V,\eta^V)&\lesssim|(Id-\pi_\delta)\Lambda\eta|_T^2\lesssim(1+\log(\mathtt{n}_c))\left(\|\Lambda(\eta-\bar\eta)\|_T^2+\sigma(\bar\eta,\bar\eta)\right)\\
&\lesssim(1+\log(\mathtt{n}_c))\left(1+\log\left(\frac{Hp^2}{h}\right)\right)|\eta|_T^2\lesssim\left(1+\log\left(\frac{Hp^2}{h}\right)\right)s(\eta,\eta),
\end{aligned}
$$

where we used (4.2) and (2.21).

Thanks to (3.14) and the definition of (4.6) we get that

$$
s_1(\eta,\eta)=b^E(\eta^E,\eta^E)+\left(1+\log\left(\frac{Hp^2}{h}\right)\right)b_1^V(\eta^V,\eta^V)\lesssim\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta)
$$

that concludes the proof of the Theorem 4.1. $\square$

Let $\mathbf{S}$ and $\mathbf{P}_1$ be the matrices obtained by discretizing respectively s and s_1 then, by using the lower and upper bounds for the eigenvalues of $\mathbf{P}_1^{-1}\mathbf{S}$ given by Theorem 4.1, we obtain:

Corollary 4.2. *The condition number of the preconditioned matrix $\mathbf{P}_1^{-1}\mathbf{S}$ satisfies:*

$$
Cond(\mathbf{P}_1^{-1}\mathbf{S})\lesssim\hat p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2. \tag{4.9}
$$

4.2. **A Discontinuous Galerkin vertex block preconditioner.** As a further alternative, we propose to construct the vertex block of the preconditioner, by completely giving up weak continuity and by using, instead, a Discontinuous Galerkin interior penalty method as coarse problem.

More precisely, letting $\mathcal{H}_\ell : H^{1/2}(\partial\Omega_\ell) \to H^1(\Omega_\ell)$ denote the harmonic lifting, we set

$$b^V_\#(\eta^V_\ell, \zeta^V_\ell) = \sum_\ell a_\ell(\mathcal{H}_\ell \Lambda^\ell \eta^V, \mathcal{H}_\ell \Lambda^\ell \zeta^V), \tag{4.10}$$

$$b^V_{[]}(\eta^V, \eta^V) = \sum_{m\in I} |\gamma_m|^{-1} \int_{\gamma_m} |[\Lambda\eta]|^2. \tag{4.11}$$

Then, as vertex block of the preconditioned, we consider:

$$b^V_2(\eta, \eta) = \beta\, b^V_\#(\eta^V_\ell, \eta^V_\ell) + \gamma\, b^V_{[]}(\eta^V_\ell, \eta^V_\ell) \tag{4.12}$$

with $\beta, \gamma > 0$ constant.

The global preconditioner is then assembled as

$$s_2(\eta, \eta) = b^E(\eta^E, \eta^E) + \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) b^V_2(\eta^V, \eta^V). \tag{4.13}$$

We have the following theorem.

Theorem 4.3. *For all $\eta \in \mathcal{T}_h$ we have:*

$$\hat{p}^{-3/2} s(\eta, \eta) \lesssim s_2(\eta, \eta) \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta, \eta). \tag{4.14}$$

Proof. Thanks to Lemma 3.2 we have

$$b^V_\#(\eta^V, \eta^V) \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) |\eta|^2_T.$$

Let us then bound $b^V_{[]}(\eta^V, \eta^V)$. For each slave side γ_m with $\gamma_m = \Gamma_{\ell n}$ we introduce the constant

$$\bar{\eta}_m = \frac{1}{|\gamma_m|} \int_{\gamma_m} \eta^V_\ell = \frac{1}{|\gamma_m|} \int_{\gamma_m} \eta^V_n$$

(the last identity is a consequence of the weak continuity constraint). For $\gamma_m = \gamma^{(i)}_\ell = \gamma^{(j)}_n$, we also introduce the notation $\bar{\eta}^{(i)}_\ell = \bar{\eta}^{(j)}_n = \bar{\eta}_m$.

Letting a_m and b_m denote the two extrema of γ_m we can write

$$b^V_{[]}(\eta^V, \eta^V) = \sum_{m\in I} |\gamma_m|^{-1} \int_{\gamma_m} |[\Lambda\eta]|^2 \simeq \sum_{m\in I} (|[\Lambda\eta](a_m)|^2 + |[\Lambda\eta](b_m)|^2).$$

Observing that for (ℓ, i), (n, j) such that $\gamma_m = \gamma^{(i)}_\ell = \gamma^{(j)}_n$ and for $x \in \bar{\gamma}_m$ we have that $|[\Lambda\eta](x)|^2 = |\eta_\ell(x) - \eta_n(x)|^2 = |\eta_\ell(x) - \bar{\eta}^{(i)}_\ell - (\eta_n(x) - \bar{\eta}^{(j)}_n)|^2 \lesssim |\eta_\ell(x) - \bar{\eta}^{(i)}_\ell|^2 + |\eta_n(x) - \bar{\eta}^{(j)}_n|^2$, we immediately obtain that

$$b^V_1(\eta^V, \eta^V) \lesssim \sum_\ell \sum_{i=1}^4 |\eta(x^\ell_i) - \bar{\eta}^{(i)}_\ell|^2.$$

Now, reasoning as in the proof of Lemma 2.3 we obtain

$$|\eta_\ell(x^\ell_i) - \bar{\eta}_\ell(i)|^2 \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) |\eta|^2_{H^{1/2}(\partial\Omega_\ell)}.$$

Putting all together we obtain

$$b_2^V(\eta^V,\eta^V) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right) s(\eta,\eta). \tag{4.15}$$

Combining (4.15),(3.14) with (4.13), we obtain

$$\hat{s}(\eta,\eta) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta).$$

Let us now bound $s(\eta,\eta)$. We let $\bar{\eta} \in L^2(\Sigma)$ denote the (single valued) function assuming the value $\bar{\eta}_m$ on γ_m for $m \in I$. We have

$$s(\eta,\eta) \lesssim |\eta^V|_T^2 + |\eta^E|_T^2.$$

Let us now consider $s(\eta^V,\eta^V)$. We have

$$s(\eta^V,\eta^V) = |\eta^V|_T^2 = |(1-\pi_h)\Lambda\eta|_T^2 = |\Lambda\eta|_T^2 + |\pi_h\Lambda\eta|_T^2.$$

We bound the two terms on the right hand side separately. We have (see [14])

$$|\Lambda\eta|_T^2 \lesssim \sum_\ell |\mathcal{H}_\ell\Lambda^\ell\eta|^2_{H^1(\Omega_\ell)} \lesssim b_\#^V(\eta^V,\eta^V).$$

As far as the second term is concerned, we can write

$$\begin{aligned}
|\pi_h(\Lambda\eta)|_T^2 &\lesssim \sum_{m=(\ell,i)\in I} |\pi_m([\Lambda\eta])|^2_{H^{1/2}_{00}(\gamma_m)} \\
&\lesssim \sum_{m=(\ell,i)\in I} H_\ell^{2\varepsilon}p^{4\varepsilon}h_\ell^{-2\varepsilon}|\pi_m([\Lambda\eta])|^2_{H_0^{1/2-\varepsilon}(\gamma_m)} \\
&\lesssim \hat{p}^{3/2}\sum_{m=(\ell,i)\in I} h^{-2\varepsilon}H^{2\varepsilon}p^{3\varepsilon}\|[\Lambda\eta]\|^2_{H_0^{1/2-\varepsilon}(\gamma_m)} \\
&\lesssim \hat{p}^{3/2}\sum_{m=(\ell,i)\in I} \frac{H^{2\varepsilon}p^{3\varepsilon}}{h^{2\varepsilon}}\frac{1}{\varepsilon}\|[\Lambda\eta]\|^2_{H^{1/2-\varepsilon}(\gamma_m)} \\
&\lesssim \hat{p}^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\sum_{m=(\ell,i)\in I}\|[\Lambda\eta]\|^2_{H^{1/2-\varepsilon}(\gamma_m)}.
\end{aligned} \tag{4.16}$$

Now we have (recall that $\|\cdot\|_{L^2(\Gamma_\ell)}$ is the scaled L^2 norm)

$$\begin{aligned}
\|[\Lambda\eta]\|^2_{H^{1/2-\varepsilon}(\gamma_m)} &= \|[\Lambda\eta]\|^2_{L^2(\gamma_m)} + |[\Lambda\eta]|^2_{H^{1/2-\varepsilon}(\gamma_m)} \\
&\lesssim \|[\Lambda\eta]\|^2_{L^2(\gamma_m)} + |[\Lambda\eta]|^2_{H^{1/2}(\gamma_m)} \lesssim |\gamma_m|^{-1}\|[\Lambda\eta]\|^2_{L^2(\gamma_m)},
\end{aligned}$$

where the last inverse type inequality is obtained by a scaling argument and using the linearity of $\Lambda\eta$ on γ_m.

Combining the bounds on the two contributions we obtain

$$s(\eta^V,\eta^V) \lesssim \hat{p}^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right) b_2^V(\eta^V,\eta^V).$$

which finally yields

$$s(\eta,\eta) \lesssim \hat{p}^{3/2}s_2(\eta,\eta).$$

□

Remark 4.4. We observe that if the Ω_ℓ's are rectangles, for $\eta \in \mathfrak{L}$ we have that $\mathcal{H}_\ell \eta_\ell$ is the Q_1 function (polynomial of degree ≤ 1 in each of the two unknowns) coinciding with η_ℓ at the four vertices of Ω_ℓ. The local matrix corresponding to the block b_1^V can then be replaced by the elementary Q1 stifness matrix for the problem considered.

Let $\mathbf{S}$ and $\mathbf{P}_2$ be the matrices obtained by discretizing respectively s and $\hat{s}$ then, by using the lower and upper bounds for the eigenvalues of $\mathbf{P}_3^{-1}\mathbf{S}$ given by Theorem 4.3, we obtain:

Corollary 4.5. *The condition number of the preconditioned matrix* $\mathbf{P}_2^{-1}\mathbf{S}$ *satisfies:*

$$Cond(\mathbf{P}_2^{-1}\mathbf{S}) \lesssim \hat{p}^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2. \tag{4.17}$$

5. Implementation and numerical results

5.1. **Implementation.** In this section, we test the properties of the preconditioners previously proposed, by performing a p-, H- and h-convergence study. We consider the model problem

$$-\Delta u = f \quad \text{in } \Omega =]0,1[^2, \qquad u = 0 \quad \text{on } \partial\Omega$$

and for all tests we set $f = 1$. A geometrically conforming, domain decomposition of Ω in $N = 2^\ell \times 2^\ell$ subdomains, $\ell = 2, 3, 4, \ldots$ with a quasiuniform mesh of order $n \times n$ in each subdomain, is considered.

Let $\mathbf{S}$ be the matrix associated to the discrete Steklov–Poincaré operator $s(\cdot,\cdot)$ defined in (3.3) and let $\widehat{\mathbf{S}}$ be the matrix obtained after the change of basis corresponding to switching from the standard nodal basis to the basis related to the splitting (3.10). From now on, we focus on testing the efficiency of the preconditioners for the transformed Schur complement system

$$\widehat{\mathbf{S}}\,\widehat{\mathbf{u}} = \widehat{\mathbf{g}} \tag{5.1}$$

where the matrix $\widehat{\mathbf{S}}$, after ordering of the indices as nodes lying on the edges and on the vertices, can be written as:

$$\widehat{\mathbf{S}} = \begin{pmatrix} \widehat{\mathbf{S}}_{\mathsf{ee}} & \widehat{\mathbf{S}}_{\mathsf{ev}} \\ \widehat{\mathbf{S}}_{\mathsf{ev}}^T & \widehat{\mathbf{S}}_{\mathsf{vv}} \end{pmatrix}.$$

We solved the transformed Schur complement system (5.1) by the Preconditioned Conjugate Gradient (PCG) method with a relative tolerance set equal to 10^{-6}. The condition number of the (preconditioned) Schur complement matrix has been estimated within the PCG iteration by exploiting the analogies between the Lanczos technique and the PCG method (see [18, Sects. 9.3, 10.2] for more details).

Remark 5.1. Only the action of $\widehat{\mathbf{S}}$ on a vector is needed and $\widehat{\mathbf{S}}$ is never explicitly assembled, see [30] for a detailed description of the efficient implementation that we carried out.

The preconditioner for $\widehat{\mathbf{S}}$ will be of block-Jacobi type: one block for each one of the master edges and an additional block for the vertices. For the edge block of the preconditioner, we need the matrix counterpart of (3.13). In the literature it is possible to find different ways to build bilinear forms $b^E(\cdot,\cdot)$ that satisfy (3.13)-(3.12). The choice we followed here for defining $b^E(\cdot,\cdot)$ is the one proposed in [14] and it is based on an equivalence result for the

$H^{1/2}_{00}$ norm, see [13] and [3] for a detailed description of its construction. We denote by $\boldsymbol{\eta}^E$ the vector representation of $\eta^E \in T^0_{\ell,i}$. Then it can be verified that, for each $\gamma_\ell^{(i)} \subset \partial\Omega_\ell$, we have (see [13] pag. 1110 and [17])

$$|\eta^E|^2_{H^{1/2}(\gamma_\ell^{(i)})} \simeq (l_0^{1/2}\eta^E, \eta^E)_{\gamma_\ell^{(i)}} = \boldsymbol{\eta}^{E^T}\widehat{\mathbf{K}}_E\boldsymbol{\eta}^E,$$

with $\widehat{\mathbf{K}}_E = \mathbf{M}_E^{1/2}(\mathbf{M}_E^{-1/2}\mathbf{R}_E\mathbf{M}_E^{-1/2})^{1/2}\mathbf{M}_E^{1/2}$, where $\mathbf{M}_E$ and $\mathbf{R}_E$ are the mass and stiffness matrices associated to the discretization of the operator $-\mathrm{d}^2/\mathrm{ds}^2$ (in $T^0_{\ell,i}$) with homogeneous Dirichlet boundary conditions at the extrema a and b of $\gamma_\ell^{(i)}$. Thus, the edge block of the preconditioner can be written as:

$$\widehat{\mathbf{K}}_{\mathsf{ee}} = \begin{pmatrix} \widehat{\mathbf{K}}_{E_1} & 0 & 0 & 0 \\ 0 & \widehat{\mathbf{K}}_{E_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \widehat{\mathbf{K}}_{E_M} \end{pmatrix} \tag{5.2}$$

with one block for each master edge where M is the number of masters.

The preconditioner $\mathbf{P}_1$. Concerning the vertex block of our preconditioner, following section 4.1, we introduce a coarse auxiliary mesh in each subdomain made of 3×3 elements and we fix the polynomial order $p = 1$. Let $\widehat{\mathbf{S}}^c$ be the matrix obtained after applying the change of basis to the associated Schur complement system. $\widehat{\mathbf{S}}^c$ takes the form

$$\widehat{\mathbf{S}}^c = \begin{pmatrix} \widehat{\mathbf{S}}^c_{\mathsf{ee}} & \widehat{\mathbf{S}}^c_{\mathsf{ve}} \\ \widehat{\mathbf{S}}^{c\,T}_{\mathsf{ve}} & \widehat{\mathbf{S}}^c_{\mathsf{vv}} \end{pmatrix}. \tag{5.3}$$

The preconditioner $\mathbf{P}_1$, described in section 4.1, can then be written as:

$$\mathbf{P}_1 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \mathbf{P}^c_{\mathsf{v}} \end{pmatrix}, \qquad \text{with } \mathbf{P}^c_{\mathsf{v}} = \left(1 + \log\left(\frac{Hp^2}{h}\right)\right)\widehat{\mathbf{S}}^c_{\mathsf{vv}}. \tag{5.4}$$

The Preconditioner $\mathbf{P}_2$. Let $\mathbf{P}_\#$ and $\mathbf{P}_{[]}$ be the matrix counterparts of (4.10) and of (4.11) respectively and let

$$\mathbf{P}_v^{DG} = \left(1 + \log\left(\frac{Hp^2}{h}\right)\right)(\beta\mathbf{P}_\# + \gamma\mathbf{P}_{[]}).$$

Then the new preconditioner we propose is:

$$\mathbf{P}_2 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \mathbf{P}_v^{DG} \end{pmatrix}. \tag{5.5}$$

All the tests presented relate to $\beta = 1/10$ and $\gamma = 2$. These values of β and γ were obtained by trial-and-error on small tests problems. Remark that the ratio between β and γ is consistent with the choice that is usually done in the framework of interior penalty DG methods.

5.2. **Computation platforms.** For the implementation of the methodology described in this paper, we developed the code in C++11 using the library Feel++ [29, 28], which allows for a wide variety of numerical methods including continuous and discontinuous Galerkin methods from 1D to 3D and, of course, the h-p mortar method we are dealing with. Feel++ uses MPI for parallel computing and its data structures can be customized with respect to MPI communicators, which allows to implement the various preconditioners presented in this paper. Finally linear algebra is handled by PETSc both in sequential in the subdomains, and in parallel for the coarse preconditioner. The implementation details as well as more extensive results with respect to strong and weak scalability are presented in [30].

The simulations presented in the next sections were partly performed on `hpc-login` at MesoCentre@Strasbourg. MesoCentre is a supercomputer with 288 compute nodes interconnected by an infiniband QDR network. The system is Scientific Linux based on Intel Xeon Ivy Bridge processors with 16 cores and 64GO of RAM running at 2.6 Ghz. MesoCentre has a theoretical peak performance of 70 TFLOP/s. The simulations on a large number of cores, 1024, 4096, 16384, 22500 and 40000, were done on Curie at the TGCC, a TIER-0 system which is part of PRACE. Curie has 5040 B510 bullx nodes and for each node a 2 eight-core Intel processors Sandy Bridge cadenced at 2.7 GHz with 64 GB.

5.3. **Numerical tests.** In summary, the numerical tests relate the following two preconditioners for the transformed Schur complement system:

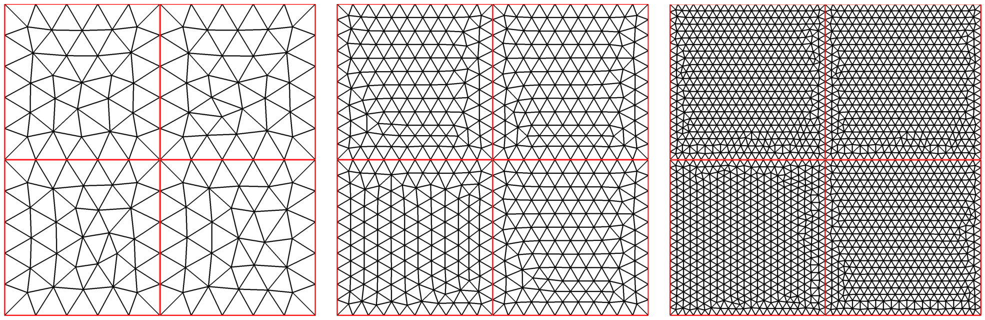
$$\mathbf{P}_1 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \widehat{\mathbf{S}}^c_{\mathsf{vv}} \end{pmatrix} \qquad \text{and} \qquad \mathbf{P}_2 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \mathbf{P}_v^{DG} \end{pmatrix}. \tag{5.6}$$

We report the condition number estimates of the preconditioned schur complement matrix $\kappa(\widehat{\mathbf{P}}^{-1}\widehat{\mathbf{S}})$ where $\widehat{\mathbf{P}}$ is either one of the preconditioners defined in (5.6), the number of iterations and the following two ratios:

$$R_2 = \frac{\kappa(\widehat{\mathbf{P}}^{-1}\widehat{\mathbf{S}})}{\left(1 + \log\left(\dfrac{Hp^2}{h}\right)\right)^2} \qquad R_{2p} = \frac{R_2}{p^{3/2}} \tag{5.7}$$

where H is the coarse mesh-size, h the fine mesh-size and p the polynomial order.

Linear elements. In the first set of experiments, we consider piecewise linear elements ($p = 1$), and compute the estimated condition number when varying the number of subdomains and the mesh size. We split the domain Ω in $N = 4^\ell$ subdomains, $\ell = 2, 3, 4$ with a quasiuniform mesh of order $n \times n$ in each subdomain. These results were obtained on a sequence of triangular grids like the ones shown in FIGURE 1.



1.1. Level 1 1.2. Level 2 1.3. Level 3

FIGURE 1. First three levels of refinements for unstructured triangular grids on a subdomain partition made of 4 squares.

We start by showing, in TABLE 1., the number of iterations required by the solution of the transformed Schur complement system without preconditioning as a function of N and of n. TABLE 2. shows the number of iterations to solve the system (5.1), preconditioned with $\mathbf{P}_1$, when increasing N and n. Analogous results obtained with preconditioned $\mathbf{P}_2$ are reported in TABLE 3.. As expected, a logarithmic growth is clearly observed for both preconditioner $\mathbf{P}_1$ and $\mathbf{P}_2$.

TABLE 1. Unpreconditioned system. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	44	59	84	105	155	240	354
64	59	77	109	150	213	298	468
256	81	99	127	178	250	327	484

TABLE 2. Preconditioner $\mathbf{P}_1$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	26	27	28	31	33	34	36
64	24	27	29	31	33	35	36
256	21	23	25	28	30	33	35

TABLE 3. Preconditioner $\mathbf{P}_2$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	23	24	26	28	31	33	35
64	22	23	26	29	31	33	35
256	20	21	23	26	28	30	33

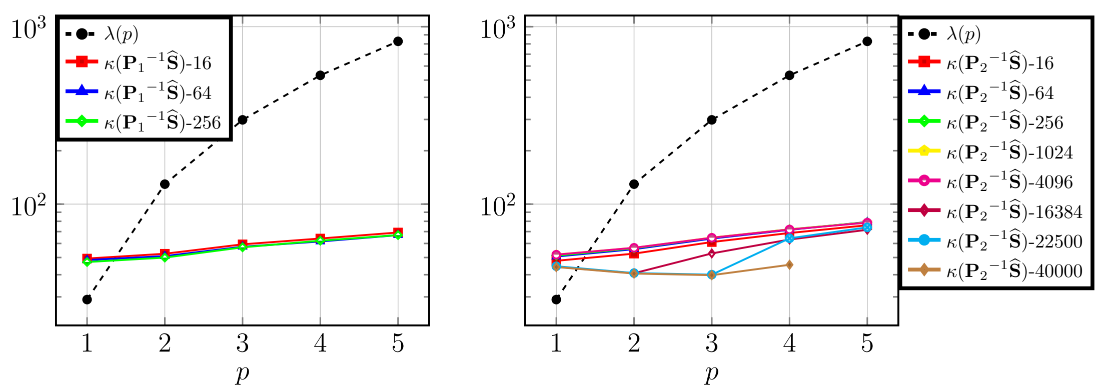
High-order elements. We now present some computations obtained with high-order elements. We run the same set of experiments carried out for linear FEM, but now we increase the polynomial order p up to 5.

In Table 5.3, for $H/h \simeq n/N = 80$ constant, we report the condition number estimates $\kappa(\widehat{\mathbf{S}})$ and the number of iterations (between parenthesis) to solve system (5.1) without preconditioning for increasing values of the polynomial order p.

TABLE 4. Condition number estimate $\kappa(\widehat{S})$ and number of iterations (between parenthesis) for $n/N = 80$.

$N\backslash p$	1	2	3	4	5
16	1.78e+3 (155)	5.34e+3 (256)	1.04e+4 (344)	1.78e+4 (444)	2.9e+4 (533)
64	2.02e+3 (213)	5.43e+3 (330)	1.18e+4 (468)	2.01e+4 (613)	3.28e+4 (765)
256	2.09e+3 (250)	6.23e+3 (356)	1.22e+4 (495)	2.07e+4 (631)	3.35e+4 (787)

To study the dependence on p of our preconditioners, we report the condition number estimate for the preconditioned system, as function of p with H/h constant. Let the function λ be defined as $\lambda(p) = p^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2$. In FIGURE 2., we plot the condition number of the transformed Schur system, preconditioned with $\mathbf{P}_1$ and $\mathbf{P}_2$, and $\lambda(p)$ as function of p.



2.1. Preconditioner $\mathbf{P}_1$ 2.2. Preconditioner $\mathbf{P}_2$

FIGURE 2. Condition number of the preconditioned system as function of p with 16, 64, 256, 1024, 4096 subdomains and $n/N = 80$

To highlight the dependence on p of our preconditioners, we report in (TABLE 5.) and (TABLE 6.), the ratio R_2 defined in (5.7) for $n/N = 80$ fixed and for increasing values of the polynomial order p. We clearly do not see the factor $\hat{p}^{3/2}$ which appears in the theoretical estimates (4.5) and (4.9) (which, we recall, stems the mortar projector operator), and it would seem that the condition number depends on p only poly-logarithmically. Indeed, the numerical results seem to show an even better behaviour than the polylogarithmic dependence on Hp^2/h. In particular, in table TABLE 5., for fixed H/h and p the ratio R_2 seems to be slightly decreasing rather than constant. We believe that there are different causes for this behaviour. First of all the problem chosen has a quite regular solution which, for a large number of subdomains, is already well approximated at the coarse level. Moreover, as the coarse mesh becomes finer and finer and the polynomial degree increases, round-off errors might become more significant and they might pollute the numerical results. We plan to investigate this issue in a forthcoming paper.

Finally, in FIGURE 3. we plot the number of degrees of freedom associated with the Schur complement as well as the total number of degrees of freedom as a function of p and of the number of subdomain/cores, at $n/N = 80$ fixed. At $p = 4$ and with 40000 cores, we reach about 5 billions of unknowns and the system is solved in less than three minutes.

TABLE 5. Ratio R_2 for $n/N = 80$, preconditioner $\mathbf{P}_1$ and increasing values of the polynomial order p. Between parenthesis the number of iterations.

$N \backslash p$	1	2	3	4	5
16	1.70 (33)	1.14 (31)	1.03 (32)	0.96 (33)	0.93 (34)
64	1.67 (33)	1.11 (32)	1.00 (34)	0.92 (34)	0.90 (34)
256	1.62 (30)	1.09 (31)	0.99 (32)	0.93 (33)	0.90 (33)

TABLE 6. Ratio R_2 for $n/N = 80$, preconditioner $\mathbf{P}_2$ and increasing values of the polynomial order p. Between parenthesis the number of iterations. The results at 40000 cores and $p = 5$ are not available.

$N \backslash p$	1	2	3	4	5
16	1.65 (31)	1.14 (32)	1.06 (33)	1.03 (38)	1.02 (39)
64	1.74 (31)	1.21 (33)	1.11 (35)	1.07 (40)	1.07 (42)
256	1.76 (28)	1.23 (32)	1.12 (34)	1.08 (36)	1.06 (40)
1,024	1.78 (27)	1.23 (29)	1.12 (31)	1.08 (32)	1.06 (34)
4,096	1.79 (25)	1.23 (28)	1.12 (29)	1.08 (31)	1.06 (31)
16,384	1.52 (20)	0.88 (22)	0.91 (26)	0.94 (27)	0.96 (28)
22,500	1.52 (19)	0.88 (20)	0.69 (22)	0.95 (26)	0.99 (27)
40,000	1.52 (17)	0.88 (20)	0.69 (22)	0.68 (23)	0.00 (0)

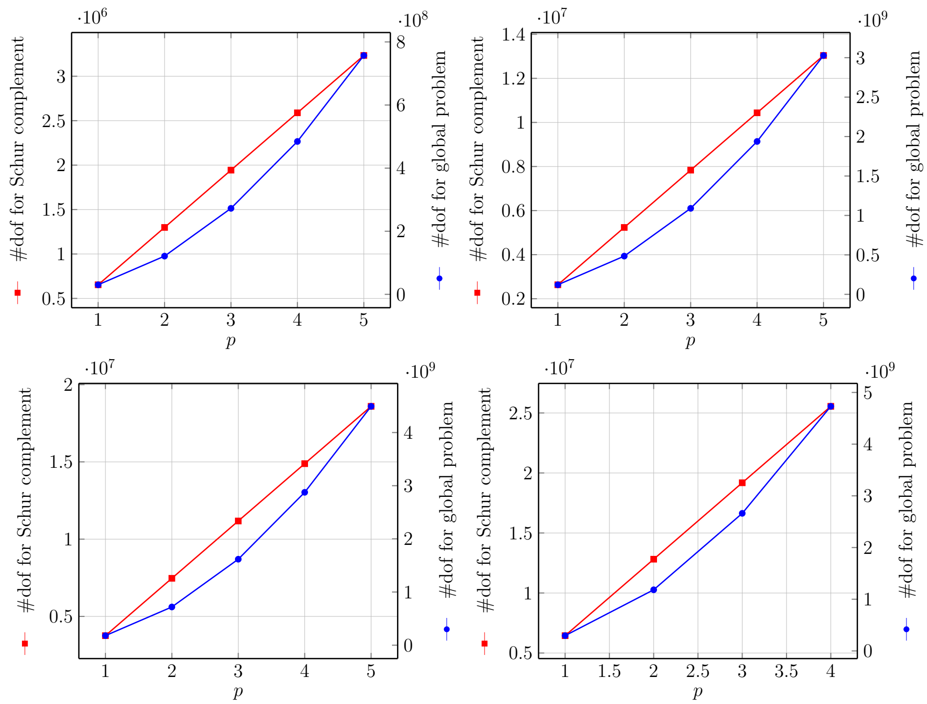
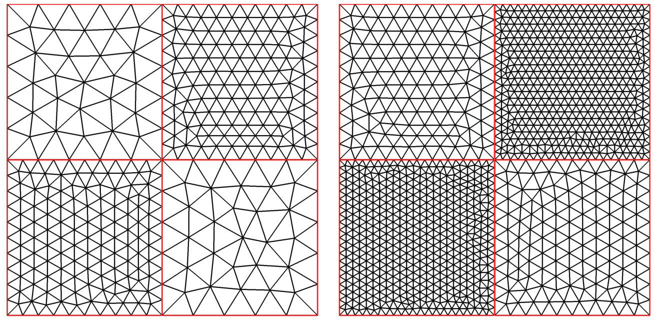


FIGURE 3. Number of degrees of freedom as function of p with 4096, 16384, 22500 and 40000 subdomains

Nonmatching grids. The tests performed until now deal with decomposition with matching grid (though the solution is non conforming, due to the lack of continuity at the cross points). We now turn to the numerical results for nonconforming decompositions. As before, we split the domain Ω in $N = 2^\ell \times 2^\ell$ subdomains, $\ell = 2, 3, 4$ but now we take quasiuniform meshes with two different mesh sizes: $h_{\text{fine}} = 1/(2n)$ and $h_{\text{coarse}} = 1/n$. We deliberately choose embedded grids in order to ensure exact numerical integration for the constraints. On the interface the master subdomains are chosen to be the ones corresponding to the coarser mesh.

We start as before with the linear case, $p = 1$, and we report the number of iterations when increasing the number of subdomains N and the number of elements n of the fine mesh. Then, for $n/N = 80$ constant and increasing values of p we report, for preconditioners $\mathbf{P}_1$ and $\mathbf{P}_2$, the ratio R_2 introduced in (5.7).



4.1. Level 2 4.2. Level 3

FIGURE 4. Nonconforming decompositions with unstructured meshes in the case of 4 subdomains

TABLE 7. Polynomial order $p = 1$. Preconditioner $\mathbf{P}_1$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	12	17	19	20	20	21	21
64	14	18	20	22	24	27	30
256	12	15	17	19	21	21	23

TABLE 8. Polynomial order $p = 1$. Preconditioner $\mathbf{P}_2$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	14	17	18	18	20	22	23
64	18	18	19	20	23	26	28
256	16	16	17	19	22	24	27

TABLE 9. Ratio R_2 for $n/N = 80$, preconditioner $\mathbf{P}_1$ and increasing values of the polynomial order p. Between parenthesis the number of iterations.

$N\backslash p$	1	2	3	4	5
16	0.82 (22)	0.65 (24)	0.63 (24)	0.62 (24)	0.62 (26)
64	0.82 (25)	0.65 (26)	0.63 (27)	0.62 (29)	0.61 (29)
256	0.77 (21)	0.63 (21)	0.60 (21)	0.60 (22)	0.59 (23)

TABLE 10. Ratio R_2 for $n/N = 80$, preconditioner $\mathbf{P}_2$ and increasing values of the polynomial order p. Between parenthesis the number of iterations.

$N\backslash p$	1	2	3	4	5
16	0.74 (22)	0.70 (27)	0.71 (28)	0.73 (28)	0.74 (28)
64	0.76 (22)	0.73 (28)	0.75 (30)	0.76 (31)	0.77 (32)
256	0.77 (21)	0.72 (25)	0.74 (30)	0.76 (31)	0.78 (31)
1,024	0.77 (19)	0.72 (23)	0.71 (25)	0.72 (25)	0.73 (29)
4,096	0.71 (17)	0.72 (21)	0.71 (22)	0.72 (23)	0.72 (24)

Similar behavior is obtained with nonconforming grids that are not embedded, these results are presented in [30].

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