LMI-based design of a robust model predictive controller for a class of Recurrent Neural Networks with guaranteed properties*

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Abstract— **This work proposes a novel robust nonlinear model predictive control (NMPC) algorithm for systems described by a generic class of recurrent neural networks. The algorithm enables tracking of constant setpoints in the presence of input and output constraints. The terminal set and cost are defined based on linear matrix inequalities to ensure convergence and recursive feasibility in presence of process disturbances. Simulation results on a quadruple tank nonlinear process demonstrate the effectiveness of the proposed control approach.**

I. INTRODUCTION

In recent years, the increased plant sensorization and the availability of new tools and methods for storing and extracting information from data [1] have led to increasing attention to data-based control approaches [2]. Within the framework of data-based control, recurrent neural networks (RNNs) are gaining increased popularity and are currently under investigation for their ability to model plants that display nonlinear dynamics. Unlike standard neural networks, RNNs retain memory of past data, making them particularly suitable for learning dynamical systems [3]. In view of their modelling capabilities, RNNs are widely employed for developing model-based control strategies, such as serving as system models for model predictive control (MPC) [4].
Many types of RNNs have been adopted as system models for MPC, including gated recurrent units (GRU) [5], [6], long short-term memory (LSTM) [7]–[9], echo state networks (ESN) [10], [11], and neural network autoregressive exogenous (NARX) [12] models.

Within this framework, a commonly adopted approach for MPC design consists of constraining the RNN training procedure so that the identified model fulfils some stability properties, such as input-to-state stability (ISS) [13], and incremental ISS (δISS) [14]. Leveraging these open-loop system properties enables the derivation of nonlinear MPC (NMPC) laws with guaranteed closed-loop stability, where no stabilizing auxiliary control law is basically required. In [11], for instance, conditions to train stable ESN are provided, and a model predictive controller with guaranteed stabilizing properties in the presence of input constraints is devised. In [8], LSTM stability properties are exploited to design a stabilizing NMPC with input constraints. A nominal NMPC law is derived in [5] for δISS GRU networks. Additionally, in [15], an NMPC law allowing for the control of general exponential δISS nonlinear systems is presented. However, the aforementioned NMPC formulations only allow for the inclusion of input constraints within the NMPC. This is a strong limitation since output/state constraints may be extremely useful if limitations are present on the physical system variables or if we need to limit the operational range in the one used in the model identification process. To address potential model-plant mismatch and disturbances, the recent work [6] presents a robust NMPC scheme for stable GRU, which also accommodates the inclusion of output constraints.
Despite its effectiveness, the approach adopted in the majority of the cited works, which involves imparting stability during training and subsequently deriving a stable NMPC law, suffers from several limitations. Firstly, it is not applicable when the system under control does not exhibit ISS or δISS stability properties. Secondly, the latter stability-related properties during training is done enforcing sufficient conditions on the model parameters, which may be rather conservative, leading to suboptimality and resulting in reduced performance. Finally, the terminal set is often not defined in the resulting NMPC formulation, with the exception of [5] and [6], precluding the definition of output/state constraints.

In this work, we propose a different NMPC design approach that does not require stability of the identified open-loop model. Specifically, we focus on a rather generic class of state-space neural network models which includes, e.g., ESN, shallow NARX, and nonlinear autoregressive with exogenous input echo state networks (NARXESN [16]). Leveraging the stability condition presented in [17] for this class of systems, we show how to derive a state-space control law ensuring δISS of the closed-loop system, together with a Lyapunov function, based on linear matrix inequalities (LMI) [18]. The proposed approach is employed to derive the NMPC terminal ingredients [4], namely, the auxiliary control law, terminal cost, and terminal set. Based on these components, a theoretically sound NMPC strategy is derived, ensuring both convergence and recursive feasibility, while accommodating the inclusion of input and output constraints. The developed approach is aimed to control systems subject to process disturbances. Specifically, exploiting the δISS of the closed-

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loop system, a robustly positive invariant (RPI) set is derived using the results presented in [14], and robustness is obtained employing the well-established tube-based MPC [19].

The rest of the work is organized as follows. In Section II, the considered class of state-space RNNs is presented and the control goal is stated. Section III formulates the NMPC algorithm, outlines the offline design procedure to define the terminal ingredients, and provides formal proof of recursive feasibility and convergence. In Section IV, the proposed controller is tested on a quadruple tank benchmark system in the presence of actuation disturbances. Conclusions are finally drawn in Section V.

Notation: Given a vector $v \in \mathbb{R}^n$, $v^\top$ represents its transpose and v_i denotes its i-th entry. Given a matrix $M \in \mathbb{R}^{n\times n}$, its entry in position (i,j) is denoted as m_{ij}, its maximum eigenvalue is denoted as $\lambda_{\max}(M)$, and its minimum eigenvalue is $\lambda_{\min}(M)$. The diagonal function is denoted as diag$(\cdot)$, the average function as avg$(\cdot)$, the identity function as id$(\cdot)$, the set-interior function as $\mathcal{INT}(\cdot)$, and the $n \times n$ identity matrix as I_n. The sequence from time k up to time $k+N$, $u(k),\dots,u(k+N)$, is denoted as $u([k:k+N])$. The Minkowski set addition is defined as $\mathbb{X} \oplus \mathbb{Y} = \{x+y \mid x \in \mathbb{X},\ y \in \mathbb{Y}\}$, and the Pontryagin set difference as $\mathbb{X} \ominus \mathbb{Y} = \{x \in \mathbb{X} \mid x+z \in \mathbb{X},\ \forall z \in \mathbb{Y}\}$. Finally, $\mathcal{B}_\epsilon^{(n)}(0)$ denotes a ball of radius ϵ, centered at 0, in $\mathbb{R}^n$. In the text, we make reference to the definitions of δISS provided in [14], and globally Lipschitz continuous function, available in [20].

II. PROBLEM STATEMENT

Consider the following discrete-time nonlinear system

$$\begin{cases} x(k+1) = f(Ax(k) + Bu(k) + w(k)) \\ y(k) = Cx(k) \end{cases}, \qquad (1)$$

where $x \in \mathbb{R}^n$ denotes the state vector, $u \in \mathbb{R}^m$ the input vector, $y \in \mathbb{R}^p$ the output vector, $w \in \mathbb{R}^n$ the process disturbance, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $f(\cdot) = \begin{bmatrix} f_1(\cdot) & \dots & f_n(\cdot)\end{bmatrix}^\top$ is a vector of scalar functions applied element-wise. We consider the following assumptions.

Assumption 1: The disturbance $w(k)$ satisfies, for all $k \geq 0$, $w(k) \in \mathbb{W}$, where $\mathbb{W}$ is a compact and convex set.

Assumption 2: Functions $f_i(\cdot)$ $i = 1,\dots,n$ are nonlinear globally Lipshitz continuous functions with Lipshitz constant L_{pi} or identity functions.

Thanks to Assumption 2, we can define matrix $W = \mathrm{diag}(L_{p1},\dots,L_{pn})$, where $L_{pi} = 1$ for all i such that $f_i(\cdot) = \mathrm{id}(\cdot)$. As discussed in [17], the class of systems described by (1) and satisfying assumption 2 includes, for instance, some classes of recurrent neural networks, such as echo state networks (ESN), shallow neural nonlinear autoregressive exogenous models (NARX), and nonlinear autoregressive with exogenous input echo state networks (NARXESN).

The control goal is to steer the plant's output to a piecewise constant setpoint $\bar{y} \in \mathbb{R}^p$ while fulfilling the input constraint $u \in \mathbb{U}$ and the output constraint $y \in \mathbb{Y}$. The sets $\mathbb{U}$ and $\mathbb{Y}$ are assumed to be compact and convex.

To address this control problem, we propose a state-feedback NMPC law with convergence and recursive feasibility properties. Indeed we assume that, at any time instant $k \geq 0$, an observation $x(k)$ of the system state is available. Note, however, that this assumption is made in this work to simplify the framework but may be unrealistic in many applications; in fact, especially since we are dealing with RNN models, the state variables may not be directly related to measurable physical quantities. To solve this issue, an observer of the type proposed in [Tesi William] can be used for providing an estimate of $x(k)$: the related quantifiable uncertainty can be accounted for as part of the process disturbance $w(k)$ and no ad-hoc extension of the approach proposed in this paper is needed. In any case, future work will be dedicated to provide a comprehensive output-feedback controller design procedure.

III. THE PROPOSED NONLINEAR MPC ALGORITHM

In this section we provide the details for the application of the well-known tube-based approach [19] for the design of a dedicated robust controller for the system described by the nonlinear model (1).

A. The nominal prediction model

The model used for computing the state prediction in the MPC-related optimization problem is derived from (1), i.e.,

$$\hat{x}(k+1) = f(A\hat{x}(k) + B\hat{u}(k)). \qquad (2)$$

The input $\hat{u}(k)$ will be used to compute the real plant input $u(k)$ consistently with the following equation

$$u(k) = \hat{u}(k) + K(x(k) - \hat{x}(k)). \qquad (3)$$

The gain K is defined in order to fulfil the main assumption of the following proposition.

Proposition 1: Define matrices $\tilde{A} = WA$ and $\tilde{B} = WB$. Assume that there exists a symmetric and positive definite matrix $P = P^\top \succ 0$

- structured in such a way that its off-diagonal entries $p_{i,j}$ are zero for all i where $f_i(\cdot) \neq \mathrm{id}(\cdot)$ and $j \neq i$;
- is a solution to the inequality

$$(\tilde{A} + \tilde{B}K)^\top P(\tilde{A} + \tilde{B}K) - P \prec 0 \qquad (4)$$

then the system

$$x(k+1) = f((A+BK)x(k) + w(k)) \qquad (5)$$

is δISS [14] with respect to the exogenous variable $w(k)$. Also, denote the set $\delta\mathbb{X}$, relative to the dynamics of the error $\delta x(k) = x(k) - \hat{x}(k)$ as

$$\delta\mathbb{X} = \{\delta x \in \mathbb{R}^n \mid \delta x^\top P \delta x \leq b\},$$

where

$$b = \frac{\lambda_{max}(P)\lambda_w}{\lambda_{min}(A^\star)}\bar{v}\max_{w\in\mathbb{W}} \|w\|^2, \qquad (6)$$

and where $0 < \bar{v} < 1$, $A^* = P - (1+\tau^2)(\tilde{A} + \tilde{B}K)^\top P(\tilde{A} + \tilde{B}K)$, with τ such that $0 < \tau^2 < \frac{\lambda_{min}(P-(\tilde{A}+\tilde{B}K)^\top P(\tilde{A}+\tilde{B}K))}{\lambda_{max}((\tilde{A}+\tilde{B}K)^\top P(\tilde{A}+\tilde{B}K))}$, and $\lambda_w > \lambda_{max}(B^*)$, with $B^* = \left(1 + \frac{1}{\tau^2}\right) P$. If $\delta x(k) \in \delta\mathbb{X}$, then $\delta x(k+1) \in \delta\mathbb{X}$ for all $w(k) \in \mathbb{W}$.

Proof: The first claim relies entirely on Theorem 2 in [17]. More specifically, the δISS Lyapunov function related to system (5) is $V(x_1, x_2) = \|x_1 - x_2\|_P^2$. More specifically, in [17] it is shown that, if we denote $x_1^+ = f((A + BK)x_1 + w_1)$ and $x_2^+ = f((A + BK)x_2 + w_2)$, then $V(x_1^+, x_2^+) - V(x_1 - x_2) \leq -\lambda_{min}(A^*)\|x_1 - x_2\|^2 + \lambda_w\|w_1 - w_2\|^2$ for any $\lambda_w > \lambda_{max}(B^*)$, where $A^* := P - (1+\tau^2)(\tilde{A} + \tilde{B}K)^\top P(\tilde{A} + \tilde{B}K)$ for all τ such that $0 < \tau^2 < \frac{\lambda_{min}(P-(\tilde{A}+\tilde{B}K)^\top P(\tilde{A}+\tilde{B}K))}{\lambda_{max}((\tilde{A}+\tilde{B}K)^\top P(\tilde{A}+\tilde{B}K))}$ and $B^* := \left(1 + \frac{1}{\tau^2}\right) P$. The second claim, related to the robust invariance of $\delta\mathbb{X}$, is derived by Lemma 2 in [14]. Specifically, for $x_1 = x$ and $x_2 = \hat{x}$, the δISS Lyapunov function $V(x, \hat{x}) = \delta x^\top P \delta x$. The expression to compute b in (6) can be derived by noting that $\lambda_{min}(P)\|\delta x\|^2 \leq V(x, \hat{x}) \leq \lambda_{max}(P)\|\delta x\|^2$, and $V(x^+, \hat{x}^+) - V(x, \hat{x}) \leq -\lambda_{min}(A^*)\|\delta x\|^2 + \lambda_w\|w\|^2$. This concludes the proof. ■

For making the proposed control design problem sound, the set $\delta\mathbb{X}$ must be compliant with the following assumption.

Assumption 3: There exist $\epsilon_u, \epsilon_y > 0$ such that $C\delta\mathbb{X} \oplus \mathcal{B}_{\epsilon_y}^{(p)}(0) \subseteq \mathbb{Y}$ and $K\delta\mathbb{X} \oplus \mathcal{B}_{\epsilon_u}^{(m)}(0) \subseteq \mathbb{U}$.

Also, the setpoint $\bar{y}$ must be properly defined, i..e, in line with the following assumption. Before stating it, we need to define, thanks to Assumption 3, the tightened sets $\hat{\mathbb{U}} := \mathbb{U} \ominus K\delta\mathbb{X}$ and $\hat{\mathbb{Y}} := \mathbb{Y} \ominus C\delta\mathbb{X}$.

Assumption 4: The output reference $\bar{y} \in \mathcal{INT}(\hat{\mathbb{Y}})$ must be selected in such a way that there exist $\bar{x}$ and $\bar{u} \in \mathcal{INT}(\hat{\mathbb{U}})$ where $\bar{x} = f(A\bar{x} + B\bar{u})$, $\bar{y} = C\bar{x}$.

B. Online Optimal Control Problem

Consistently with the tube-based control paradigm, the optimization variables, at each time instant k, are the nominal system (2) inputs $\hat{u}(k), \ldots, \hat{u}(k + N - 1)$ on a given prediction horizon of length N and the nominal state $\hat{x}(k)$. The finite-horizon optimal control problem (FHOCP) is formulated as follows

$$\min_{\hat{x}(k), \hat{u}([k:k+N-1])} J(\hat{x}([k : k + N]), \hat{u}([k : k + N - 1])$$

subject to:

$$x(k) - \hat{x}(k) \in \delta\mathbb{X} \tag{7a}$$

$$\forall \tau = 0, \ldots, \; N - 1 :$$

$$\hat{x}(k + \tau + 1) = f(A\hat{x}(k + \tau) + B\hat{u}(k + \tau)) \tag{7b}$$

$$\hat{u}(k + \tau) \in \hat{\mathbb{U}} \tag{7c}$$

$$C\hat{x}(k + \tau) \in \hat{\mathbb{Y}} \tag{7d}$$

$$\hat{x}(k + N) \in \mathbb{X}_f \tag{7e}$$

Note that constraint (7b) embeds the dynamics of the predictive model, which is initialized by constraint (7a) in the neighborhood of the state measurement $x(k)$. Input and output constraints are enforced via (7c) and (7d), respectively. The cost function, which penalizes the deviation of the nominal input and state from the target equilibrium $(\bar{x},\ \bar{u})$, is

$$J = \sum_{\tau=0}^{N-1} \left(\|\hat{x}(k + \tau) - \bar{x}\|_Q^2 + \|\hat{u}(k + \tau) - \bar{u}\|_R^2\right) + V_f(\hat{x}(k + N)), \tag{8}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are positive definite matrices, and V_f is the terminal cost that will be specified below.

The solution to the FHOCP (7) at time k is $\hat{x}(k), \hat{u}([k : N - 1])$. The control action $u(k) = \hat{u}(k) + K(x(k) - \hat{x}(k))$ is applied to system (1). This process is iterated over time according to a receding horizon procedure.

C. Terminal ingredients

Convergence and recursive feasibility of the control system are guaranteed by proper design of the terminal ingredients, namely the terminal set $\mathbb{X}_f$ in (7e), where the nominal state is constrained to lie at the end of the prediction horizon, and the terminal cost $V_f(\hat{x}(k + N))$. As customary, their definition requires the preliminary definition of the auxiliary control law; the latter, in this work, is

$$\hat{u}(k) = K_f(\hat{x}(k) - \bar{x}) + \bar{u}, \tag{9}$$

where K_f is defined in such a way that there exists a matrix S, structured (similarly to matrix P in (4)) in such a way that its off-diagonal entries $s_{i,j}$ are zero for all i where $f_i(\cdot) \neq \mathrm{id}(\cdot)$ and $j \neq i$ and verifying the following Lyapunov Inequality

$$(\tilde{A} + \tilde{B}K_f)^\top S(\tilde{A} + \tilde{B}K_f) - S \preceq -(Q + K_f^\top RK_f). \tag{10}$$

The terminal set can now be defined as

$$\mathbb{X}_f = \{\hat{x} \in \mathbb{R}^n : (\hat{x} - \bar{x})^\top S(\hat{x} - \bar{x}) \leq \sigma\}, \tag{11}$$

where $\sigma \in \mathbb{R}$ is set in such a way that:

$$\sigma = \max_{\hat{x}\in\mathbb{R}^n} \{(\hat{x} - \bar{x})^\top S(\hat{x} - \bar{x}) \mid C\hat{x} \in \hat{\mathbb{Y}},\ K_f\hat{x} \in \hat{\mathbb{U}}\}. \tag{12}$$

The terminal cost is eventually defined as

$$V_f(\hat{x}) = \|\hat{x} - \bar{x}\|_S^2. \tag{13}$$

D. Main result

The theoretical properties of the resulting control system are stated in the following result.

Theorem 2: Suppose that Assumptions 1, 2, 3, and 4 are verified. Then if the FHOCP (7) admits a solution at time $k = 0$, then it admits solution for all $k > 0$. Moreover, the output y asymptotically converges to $\bar{y} \oplus C\delta\mathbb{X}$ for $k \to \infty$.

Proof: The proof uses standard arguments. For instance, the proof of Theorem 3 in [14] can be used, with two slight differences, i.e., the notation and the fact that here a setpoint triple $(\bar{x}, \bar{u}, \bar{y})$ is considered, while in [14] a regulation problem is addressed. For this reason we will not write the steps of the demonstration but, on the other hand, we will just verify that the three main assumptions of Theorem 3 in

[14] (using the notation introduced in this work) are fulfilled, i.e.,

A1: $C\mathbb{X}_f \in \hat{\mathbb{Y}}$, $\mathbb{X}_f$ is closed, $\bar{x} \in \mathbb{X}_f$;

A2: $\exists \kappa_f(\hat{x})$ s.t. $\kappa_f(\hat{x}) \in \hat{U}$ and $f(A\hat{x} + B\kappa_f(\hat{x})) \in \mathbb{X}_f$ $\forall \hat{x} \in \mathbb{X}_f$;

A3: $V_f(f(A\hat{x}+B\kappa_f(\hat{x}))) - V_f(\hat{x}) \leq -(\|\hat{x}-\bar{x}\|_Q^2 + \|\kappa_f(\hat{x}) - \bar{u}\|_R^2)$.

Assumption A1 is fulfilled in view of (11) and (12). Also, the first claim in Assumption A2 follows from (11) and (12). The second claim of A2 can be proved by resorting to Proposition 1. In fact, since S is structured in the same way as P and (10) is verified by the pair (K_f, S), it follows that (4) is verified by setting $K = K_f$ and $P = S$. In view of Proposition 1, then the system $\hat{x}(k+1) = f((A + BK_f)\hat{x}(k) + \hat{w}(k))$ is δISS with respect to $w(k)$; consider the equilibrium motion defined by $\bar{x} = f(A\bar{x} + B\bar{u} \pm BK_f\bar{x}) = f((A + BK_f)\bar{x} + B\bar{u} - BK_f\bar{x})$ and the nominal system (2) controlled by the auxiliary control law (9) $\hat{x}(k+1) = f(A\hat{x}(k) + B(\bar{u} + K_f(\hat{x}(k) - \bar{x}))) = f((A+BK_f)\hat{x}(k) + B\bar{u} - BK_f\bar{x})$; from Proposition 1, $\delta\mathbb{X} = \{\delta x \in \mathbb{R}^n | \delta x^\top S \delta x \leq \sigma\}$ (in case σ is selected as in b in (6)) is robust positively invariant for $\delta x(k) = \hat{x}(k) - \bar{x}$ in case $\hat{w}(k) \in \mathbb{W}$; here, however, the equivalent differential disturbance is $\hat{w}(k) = 0$, making $\mathbb{X}_f$ positively invariant for all $\sigma > 0$. This proves that also A2 is fulfilled.

The proof of A3 follows similar lines of the proof of Theorem 2 in [17], but we report it here for completeness. Consider again the dynamics of the equilibrium and of (2) under (9). The latter can be rewritten as

$$\begin{aligned}
&\hat{x}(k+1) = f(A\hat{x}(k) + BK_f(\hat{x}(k) - \bar{x}) + B\bar{u}) \\
&= (\tilde{A} + \tilde{B}K_f)\hat{x}(k) - \tilde{B}K_f\bar{x} + \tilde{B}\bar{u} + f((A + BK_f)\hat{x}(k) \\
&- BK_f\bar{x} + B\bar{u}) - (\tilde{A} + \tilde{B}K_f)\hat{x}(k) + \tilde{B}K_f\bar{x} - \tilde{B}\bar{u}.
\end{aligned}$$

Defining variables $v(k) = (A + BK_f)\hat{x}(k) - BK_f\bar{x} + B\bar{u}$, $\bar{v} = A\bar{x} + B\bar{u}$ and $\Delta(k) = f(v(k)) - f(\bar{v}) + W(\bar{v} - v(k))$, and noting that $v(k) - \bar{v} = (A + BK_f)\delta x(k)$, we can write

$$\delta x(k+1) = W(v(k) - \bar{v}) + \Delta(k). \tag{14}$$

Using (14) in the final cost, we can write

$$\begin{aligned}
&V_f(\hat{x}(k+1)) - V_f(\hat{x}(k)) \\
&= \|\delta x(k+1)\|_S^2 - \|\delta x(k)\|_S^2 \\
&= \|W(v(k) - \bar{v}) + \Delta(k)\|_S^2 - \|\delta x(k)\|_S^2 \\
&= 2(v(k) - \bar{v})^\top W^\top S\Delta(k) + \Delta(k)^\top S\Delta(k) \\
&+ (W(v(k) - \bar{v}))^\top S(W(v(k) - \bar{v})) - \|\delta x(k)\|_S^2.
\end{aligned}$$

Expanding the first two terms, we obtain

$$\begin{aligned}
&2(v(k) - \bar{v})^\top W^\top S\Delta(k) + \Delta(k)^\top S\Delta(k) \\
&= (2W(v(k) - \bar{v}) + \Delta(k))^\top S\Delta(k) \\
&= (W(v(k) - \bar{v}) + f(v(k)) - f(\bar{v}))^\top S(f(v(k)) - f(\bar{v}) + W(\bar{v} - v(k))) \\
&= q(k)^\top S r(k).
\end{aligned}$$

Define the following set

$$\mathcal{W} = \{i \in [1, n] \ \mid \ f_i(\cdot) \neq \mathrm{id}(\cdot)\}.$$

Note that $r_i(k) = 0$ for all $i \notin \mathcal{W}$. Since matrix S is selected in such a way that its off-diagonal entries $s_{i,j} = 0$ for all $i \in \mathcal{W}$, $j \neq i$, we can write

$$q(k)^\top S r(k) = \sum_{i \in \mathcal{W}} s_{i,i}((f_i(v_i(k)) - f_i(\bar{v}_i))^2 - L_{pi}(v_i(k) - \bar{v}_i)^2) \leq 0, \tag{15}$$

due to the positive definiteness of S and Assumption 2. Exploiting (15) and choosing matrix S satisfying the LMI (4), it follows that

$$\begin{aligned}
&V_f(\hat{x}(k+1)) - V_f(\hat{x}(k)) \\
&\leq \|W(v(k) - \bar{v})\|_S^2 - \|\delta x(k)\|_S^2 \\
&= \|\delta x(k)\|^2_{((\tilde{A}+\tilde{B}K)^\top S(\tilde{A}+\tilde{B}K) - S)} \\
&\leq -\|\delta x(k)\|^2_{Q + K_f^\top R K_f}.
\end{aligned}$$

Since $\|\delta x\|^2_{Q+K_f^\top RK_f} = \|\hat{x} - \bar{x}\|_Q^2 + \|K_f(\hat{x} - \bar{x})\|_R^2$ and $K_f(\hat{x} - \bar{x}) = \kappa_f(\hat{x}) - \bar{u}$, this completes the proof. ■

E. Offline design procedure

In this section we now discuss how the design parameters are defined. Note that we basically need K, P fulfilling (4) and K_f, S fulfilling (10). To do so, the following procedure is proposed.

Step 1: Define a symmetric and positive definite matrix $Y \in \mathbb{R}^{n\times n}$ structured like P, and a matrix $L \in \mathbb{R}^{m \times n}$. Solve the following LMI with unknowns Y and L:

$$\begin{bmatrix} Y & Y\tilde{A}^\top + L^\top \tilde{B}^\top \\ \tilde{A}Y + \tilde{B}L & Y \end{bmatrix} \succ 0. \tag{16}$$

If a solution exists, set $P = Y^{-1}$ and $K = LP$. Note that *Step 1* guarantees the fulfilment of (4) in view of the Schur complement [18].

Step 2: Set $K_f = K$ and compute $S = \alpha P$ where $\alpha > 0$ is defined such that

$$\begin{aligned}
\alpha = \arg\min_{\alpha} \{ &(\tilde{A} + \tilde{B}K_f)^\top \alpha P(\tilde{A} + \tilde{B}K_f) \\
&- \alpha P \preceq -(Q + K_f^\top R K_f)\}. \quad (17)
\end{aligned}$$

IV. Simulation Example

In this section, the proposed control approach is tested to control the model of a quadruple tank, firstly presented in [21], and then used as a benchmark, e.g., in [3] [22].

The system consists of four tanks containing water. The tank levels are denoted as h_1, h_2, h_3 and h_4. Water is delivered to the system by two pumps with flow rates q_a and q_b, respectively, and it is split in the four tanks according to two triplet valves. The dynamics of the system is given by

$$\begin{aligned}
\dot{h}_1 &= -\frac{a_1}{S}\sqrt{2gh_1} + \frac{a_3}{S}\sqrt{2gh_3} + \frac{\gamma_a}{S}q_a \\
\dot{h}_2 &= -\frac{a_2}{S}\sqrt{2gh_2} + \frac{a_4}{S}\sqrt{2gh_4} + \frac{\gamma_b}{S}q_b \\
\dot{h}_3 &= -\frac{a_3}{S}\sqrt{2gh_3} + \frac{1-\gamma_b}{S}q_b \\
\dot{h}_4 &= -\frac{a_4}{S}\sqrt{2gh_4} + \frac{1-\gamma_a}{S}q_a
\end{aligned} \tag{18}$$

where the system parameters are summarized in Table I.

TABLE I
QUADRUPLE TANK SYSTEM PARAMETERS.

a_1	$1.31 \cdot 10^{-4}$ m^2	S	0.006 m
a_2	$1.51 \cdot 10^{-4}$ m^2	γ_a	0.3
a_3	$9.27 \cdot 10^{-4}$ m^2	γ_b	0.4
a_4	$8.82 \cdot 10^{-4}$ m^2	g	9.81 m/s^2

The water levels and the flow rates are subject to saturation limits, i.e.

$$\begin{aligned} h_i &\in [0,\ 1.36] \text{ m } i = 1,2 \\ h_i &\in [0,\ 1.3] \text{ m } i = 3,4 \\ q_a &\in [0,\ 9 \cdot 10^{-4}] \text{ m}^3/\text{s} \\ q_b &\in [0,\ 1.3 \cdot 10^{-3}] \text{ m}^3/\text{s} \end{aligned}$$

The control input vector is $u = [q_a\ q_b]^\top$, and the output vector is $y = [h_1\ h_2]^\top$.

The control procedure comprises a training phase to identify a suitable RNN model of the form (1), capable of describing the system dynamics, followed by a control design phase where the proposed NMPC algorithm is applied, leveraging the identified model.

A. Model identification

To control the system, the following NARXESN RNN architecture with $n_\chi = 8$ internal states has been considered:

$$\begin{cases} \chi(k+1) = \zeta(W_\chi \chi(k) + W_\phi \phi(k) + W_y y(k+1)) \\ y(k) = W_{o,1}\chi(k) + W_{o,2}\phi(k) \end{cases} \tag{19}$$

where $\chi \in \mathbb{R}^{n_\chi}$, and $\phi = [u(k)^\top\ u(k-1)^\top\ \dots\ u(k-n_u)^\top\ y(k)^\top \dots\ y(k-n_y)^\top]^\top \in \mathbb{R}^{n_u+n_y}$, with $n_u = n_y = 4$. The function $\zeta_i(\cdot) = \tanh(\cdot)$ for $i = 1, \dots, 5$ and $\zeta_i(\cdot) = \mathrm{id}(\cdot)$ for $i = 6, \dots, n_\chi$, and weights $W_\chi \in \mathbb{R}^{n_\chi \times n_\chi}$, $W_\phi \in \mathbb{R}^{n_\chi \times (n_y+n_u)}$, $W_y \in \mathbb{R}^{n_\chi \times p}$, $W_{o,1} \in \mathbb{R}^{m \times n_\chi}$ and $W_{o,2} \in \mathbb{R}^{m \times (n_u+n_y)}$. Note that, in (19), only $W_{o,1}$ and $W_{o,2}$ are free identification parameters, while the other weights are a priori defined hyperparameters.

The training dataset is used to identify the NARXESN (19) parameters, according to the training algorithm described in [10]. Basically, matrices W_χ, W_ϕ, and W_y are randomly generated, while $W_{o,1}$ and $W_{o,2}$ are obtained by solving a least-square problem defined on the training dataset.

To identify the identification parameters of model (19), the Quadruple Tank system (18) has been implemented in MATLAB. With a sampling time of 15 seconds, training and test datasets have been created. These datasets consist of two input/output data sequences, independently extracted by feeding the system with a multilevel pseudo-random signal (MPRS), exciting the system in different operating regions. The two input/output sequences' lengths are 10000 and 3000 samples, respectively, for the two datasets. A coloured flicker [23] noise is added to the output sequences to simulate process disturbance. Finally, data is suitably normalized so that input/output constraints translate into $y_i \in [0,1]$ and $u_i \in [0,1]$, for $i = 1,2$.

The modelling performances of the model are tested on the test dataset. To assess the identification performance, the following fitting index is calculated over the test dataset:

$$FIT_{\%} = 100 \cdot \left(1 - \frac{\|y^\star - y_s\|}{\|y^\star - \mathrm{avg}\,(y^\star)\|}\right) \in (-\infty, 100]$$

where $y^\star$ represents the real system output sequence, and y_s denotes the output sequence obtained by feeding (19) with the test dataset input sequence. A fitting $FIT_{\%} = 90.86\%$ is obtained, indicating satisfactory identification performance.

Note that (19) can be expressed in the form (1) by setting $x(k) = [\chi(k)^T\ u(k-1)^T\ \dots\ u(k-n_u)^T\ y(k)^T\ \dots\ y(k-n_y)^T]^T$, $f(\cdot) = [\zeta(\cdot)\ \mathrm{id}(\cdot)]$, and matrices A, B and C as described in [17].

B. Control design and results

Based on the identified model, the NMPC algorithm proposed in Section III has been implemented in the MATLAB environment and tested for control of the simulated quadruple tank system to track a piecewise constant setpoint that satisfies assumption (4). In the simulation, we have assumed the presence of a bounded unknown actuation disturbance $d = [d_1\ d_2]$, with $d_1 \in [-0.09,\ 0.09] \cdot 10^{-4}$ m^3/s and $d_2 \in [-0.13,\ 0.13] \cdot 10^{-4}$ m^3/s. This disturbance enters system (2) through the process disturbance $w = Bd$. Matrix $S = \mathrm{diag}\,(I_5, S_f)$, with $S_f \in \mathbb{R}^{(n_x-5)\times(n_x-5)}$ being a full matrix, and parameters σ and b have been selected according to the procedure proposed in Section III. For completess, the observer proposed in [Tesi William] is adopted to estimate the RNN model state necessary to initialize the FHOCP (7) at each time step.

Figures 1-2 display the closed-loop simulation results. The figures show that the controller achieves satisfactory tracking accuracy while fulfilling the constraints on the input variable u and output variable y. Also, it can be noted that despite the presence of the disturbance, for $k > 0$ the closed-loop output trajectories and the input trajectories lie within the tubes $\hat{y}(k) \oplus C\delta\mathbb{X}$ and $\hat{u}(k) \oplus K\delta\mathbb{X}$, respectively.

V. CONCLUSIONS

In this paper, a rather novel robust NMPC for a generic class of RNNs has been proposed. The algorithm enables the tracking of constant setpoints in the presence of input and output constraints. The NMPC terminal ingredients are derived based on LMIs, leveraging stability results recently derived for the considered class of systems. Notably, no initial assumptions on the model stability are made. Proofs of convergence and recursive feasibility of the resulting NMPC law have been provided. Furthermore, robustness against process disturbances is achieved by employing the tube-based MPC formulation and exploiting the δISS of the closed-loop system when defining the RPI set.

Future work will investigate the use of conditions for local stability to increase the range of application of the approach. Also, we will extend this approach for tracking of piecewise

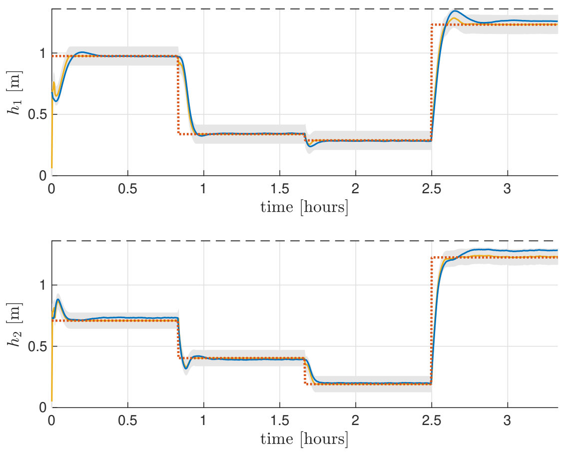

Fig. 1. Closed-loop output tracking performances of the nominal system (yellow continuous line), of the real system (blue solid line), compared to the reference (red dashed-dotted line). Output constraints are represented by the black dashed lines, and the shaded areas are the tubes around the nominal trajectories.

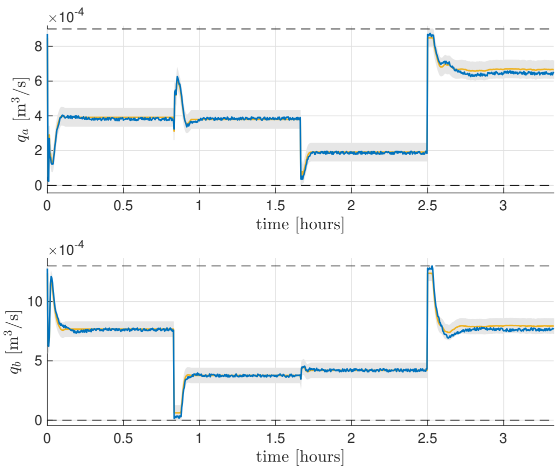

Fig. 2. Evolution of the nominal control input (yellow continuous line) and real control input (blue solid line). Input constraints are represented by the black dashed lines, and the shaded areas are the tubes around the nominal trajectories.

constant reference using the approach proposed in [24]. The application in the distributed control framework will also be subject of future research.

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