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Abstract

In this report, we compare two different frameworks for a performance analysis of the Gerchberg algorithm, considered either as an $L^2(\mathbb{R}^N)$ to $L^2(D)$ or as an $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ process. In the first case, the tool to be adopted for the analysis is the singular value decomposition (SVD), in the second case, the tool is the eigensystem analysis. We derive the relevant relationships in both cases and compare the final formulas, to show how the two approaches can be considered interchangeable.

Categories and Subject Descriptors: I.4.5 [**Image Processing and Computer Vision**]: Reconstruction - *Transform methods*

1 Introduction

The problem of reconstructing an N-dimensional object function from its image obtained through a linear, space-invariant, bandlimiting system is ill-posed, because in principle it has not a unique solution. Indeed, any function with the same spectrum within the passband would produce the same image. However, if the object has a compact support, by the Paley-Wiener theorem the object spectrum is an entire function, and can be known everywhere from its values on any contour in $\mathbb{C}^N$. Even with a unique solution, the problem is still ill-posed for lack of stability. Noisy images would thus produce solutions that are very different from the original object, and a regularization procedure is required to solve the problem.

The Gerchberg method [Gerchberg, 1974] is an iterative algorithm that regularizes the problem by enforcing alternatively the compact support constraint and the known spectral data. A square-integrable function whose support is contained in a compact $\mathbb{R}^N$ domain D can be seen either as an $L^2(D)$ function or as an $L^2(\mathbb{R}^N)$ function that has been space-limited to D through a certain space-limiting operator. The Gerchberg method can be shown to be equivalent to a filter in the singular space of an $L^2(D)$ to $L^2(\mathbb{R}^N)$ map or in the eigenspace of an $L^2(\mathbb{R}^N)$ operator. This filter has the effect of rejecting the components of the solution that are most affected by system noise, and cause instability in the inversion problem. The number of iterations performed takes the place of the regularization parameter. To stop the procedure at a given iteration is equivalent to reject the high-order components in the solution. The $L^2(D)$ and the $L^2(\mathbb{R}^N)$ approaches have been both adopted for the performance analysis of the algorithm. Among many other references, we mention here [Bertero and De Mol, 1996], for the first approach, and [Gori and Wabnitz, 1985] for the second approach.

The aim of this report is to establish a relationship between these two approaches, showing that they are perfectly equivalent. In Section 2, the two different analyses are performed separately, and a comparison is made on the basis of the final filtering formulas and the relationships between the eigenpairs in $L^2(\mathbb{R}^N)$ and the singular system of the $L^2(D)$ to $L^2(\mathbb{R}^N)$ map. As a byproduct of this analysis, in Section 3, we derive the global impulse response, that is, the impulse response due to both the bandlimiting linear system and the Gerchberg reconstruction procedure. This derivation is only made in the framework of the $L^2(\mathbb{R}^N)$ eigensystem; a similar expression can be derived for the case of the singular system.

2 Performance analysis of the Gerchberg method

2.1 Singular Value Decomposition

Let F(x) be a square-integrable function of the N-dimensional variable x, whose support is contained in the compact $\mathbb{R}^N$ subset D. F(x) is thus an $L^2(D)$ function. Let I(x) be the image of F(x) through a linear bandlimiting operator $\mathcal{B}_D$, whose passband is the N-dimensional domain B. I(x) is certainly square-integrable, but, being the output of a bandlimited system, it will assume nonzero values over the whole $\mathbb{R}^N$, and will thus be an $L^2(\mathbb{R}^N)$ function. In formulas, we have:

$$I(x) = (\mathcal{B}_D F)(x) \ , \tag{1}$$

$$I(x) \in L^2(\mathbb{R}^N) \ , \tag{2}$$

$$F(x) \in L^2(D) \ , \tag{3}$$

$$\mathcal{B}_D: L^2(D) \rightarrow L^2(\mathbb{R}^N): \qquad (\mathcal{B}_D F)(x) = \int_D F(x')\ K_B(x - x')\ dx' \ , \tag{4}$$

with

$$K_B(x) = \int_B \exp\{j\, 2\pi f \cdot x\}\ df \ . \tag{5}$$

The singular value decomposition of $\mathcal{B}_D$ consists in finding all the triples (u_k, v_k, λ_k) such that:

$$\begin{cases} (\mathcal{B}_D u_k)(x) = \lambda_k v_k(x) \\ (\mathcal{B}_D^* v_k)(x) = \lambda_k u_k(x) \end{cases} \qquad \|u_k\|_D = \|v_k\|_{\mathbb{R}^N} = 1 \ , \tag{6}$$

where, as usual, the asterisk denotes the adjoint operator, $u_k(x) \in L^2(D)$, $v_k(x) \in L^2(\mathbb{R}^N)$, and the subscripts D and $\mathbb{R}^N$ denote the Euclidean norms in $L^2(D)$ and in $L^2(\mathbb{R}^N)$, respectively. $\mathcal{B}_D$ is a compact operator, the singular values λ_k are all positive, less than one, and accumulate to zero. Since the singular functions u_k and v_k form orthogonal complete bases for $L^2(D)$ and $L^2(\mathbb{R}^N)$, respectively, we can write:

$$F(x) = \sum_{k=1}^{\infty} (F,\ u_k)_D\ u_k(x) \ , \tag{7}$$

$$I(x) = \sum_{k=1}^{\infty} (I,\ v_k)_{\mathbb{R}^N}\ v_k(x) \ , \tag{8}$$

where, as in (6), the subscripts D and $\mathbb{R}^N$ mark operations in $L^2(D)$ and in $L^2(\mathbb{R}^N)$, respectively; they will be omitted hereafter, where no ambiguity is possible. From (1) and (6)-(8):

$$\sum_{k=1}^{\infty} (I,\ v_k)\ v_k(x) = \sum_{k=1}^{\infty} \lambda_k\ (F,\ u_k)\ v_k(x) \tag{9}$$

from which

$$(F,\ u_k) = \frac{(I,\ v_k)}{\lambda_k} \tag{10}$$

and, substituting (10) in (7),

$$F(x) = \sum_{k=1}^{\infty} \frac{(I,\ v_k)}{\lambda_k}\ u_k(x)\ . \tag{11}$$

Equation (11) is a reconstruction formula for F, assuming a perfect knowledge of the singular functions and values, and the availability of a noiseless image, from which the coefficients of the summation can be calculated as scalar products in $\mathbb{R}^N$. However, since the singular values accumulate to zero faster than the noise components (see for example [Salerno, 1998]), solution (11) is unstable, and no good estimate can be obtained from a noisy data image.

Let us now consider the Gerchberg algorithm in the above setting. The procedure, which is shown here to regularize formula (11), can be written as follows [Bertero and De Mol,1996]:

$$\begin{cases} F_o(x) \equiv 0 \\ F_n(x) = (\mathcal{B}_D^* I)(x) + [(\delta - \mathcal{B}_D^* \mathcal{B}_D) F_{n-1}](x) \end{cases} , \tag{12}$$

where δ is the identity operator. From (6) and (8), we have:

$$\mathcal{B}_D^* I = \sum_{k=1}^{\infty} \lambda_k\ (I,\ v_k)\ u_k\ , \tag{13}$$

and from an expression similar to (7), written for F_{n-1}, we have:

$$\mathcal{B}_D^* \mathcal{B}_D F_{n-1} = \sum_{k=1}^{\infty} \lambda_k^2\ (F_{n-1},\ u_k)\ u_k\ . \tag{14}$$

Then, from (12)-(14) and the orthonormality of the singular functions:

$$(F_n,\ u_k) = \lambda_k\ (I,\ v_k) + (1 - \lambda_k^2)(F_{n-1},\ u_k)\ . \tag{15}$$

From (12) and (15), it is easy to prove by induction that

$$(F_n,\ u_k) = \lambda_k \left[1 + (1 - \lambda_k^2) + (1 - \lambda_k^2)^2 + \ldots + (1 - \lambda_k^2)^{n-1}\right] (I,\ v_k) =$$

$$= \left[1 - (1 - \lambda_k^2)^n\right] \frac{(I,\ v_k)}{\lambda_k}\ , \tag{16}$$

thus

$$F_n(x) = \sum_{k=1}^{\infty} \left[1 - (1 - \lambda_k^2)^n\right] \frac{(I,\ v_k)}{\lambda_k}\ u_k(x)\ . \tag{17}$$

From (11) and (17), and from the mentioned properties of the singular values, we see that the n-th estimate F_n in the noiseless case converges to the object F as n goes to infinity. It is now clear how the Gerchberg algorithm acts as a filter in the singular space of $\mathcal{B}_D$. The smaller is n, the more high-order components in the solution are attenuated, thus avoiding the strongest noise components in (11). It is also obvious that knowing the behavior of the singular functions and values it is possible to study from (17) the convergence rate of the algorithm and the maximum number of significant components in the solution for each value of the signal-to-noise ratio (some useful references can be found in [Salerno, 1998]).

2.2 *Eigensystem analysis*

Let us now note that our object $F(x)$ is also an $L^2(\mathbb{R}^N)$ function vanishing outside D. The image $I(x)$ can thus be thought of as the output of two cascaded $L^2(\mathbb{R}^N)$ operators: a band-pass operator $\mathcal{B}$

$$\mathcal{B}: L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N): \qquad (\mathcal{B}F)(x) = \int_{\mathbb{R}^N} F(x')\ K_B(x - x')\ dx' \quad , \tag{18}$$

where K_B is the same as in (5), and a space-limiting operator $\mathcal{T}_D$

$$\mathcal{T}_D: L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N): \qquad (\mathcal{T}_D F)(x) = \begin{cases} F(x) & \forall\ x \in D \\ 0 & \text{elsewhere} \end{cases} \quad , \tag{19}$$

that is, we have

$$I(x) = (\mathcal{B}\mathcal{T}_D F)(x) \quad . \tag{20}$$

The search space of the iteration is $L^2(\mathbb{R}^N)$, but our procedure converges on F, which vanishes outside D. The Gerchberg iteration in this case can be formalized as follows [Gori and Wabnitz, 1985]:

$$\begin{cases} F_o(x) \equiv 0 \\ F_n(x) = I(x) + [(\delta - \mathcal{B}\mathcal{T}_D)F_{n-1}](x) \end{cases} \quad . \tag{21}$$

$\mathcal{B}\mathcal{T}_D$ is a compact operator; its eigenfunctions thus form a complete base for $L^2(\mathbb{R}^N)$. The eigenpairs (ϕ_k, μ_k) are all the $L^2(\mathbb{R}^N)$ functions, ϕ_k, and all (generally complex) numbers, μ_k, satisfying the following equation:

$$(\mathcal{B}\mathcal{T}_D \phi_k)(x) = \mu_k\ \phi_k(x) \quad . \tag{22}$$

In a way similar to Equations (7)-(10), we can write

$$F(x) = \sum_{k=1}^{\infty} (F,\ \phi_k)\ \phi_k(x) \quad , \tag{23}$$

$$I(x) = \sum_{k=1}^{\infty} (I,\ \phi_k)\ \phi_k(x) \quad , \tag{24}$$

$$\sum_{k=1}^{\infty} (I, \phi_k)\, \phi_k(x) = \sum_{k=1}^{\infty} \mu_k\, (F, \phi_k)\, \phi_k(x) \ , \tag{25}$$

$$(F, \phi_k) = \frac{(I, \phi_k)}{\mu_k} \ , \tag{26}$$

thus, analogously to (11),

$$F(x) = \sum_{k=1}^{\infty} \frac{(I, \phi_k)}{\mu_k}\, \phi_k(x) \ . \tag{27}$$

The same considerations of Equation (11) hold true for this last reconstruction formula. From (21), we can now derive a noniterative expression for the n-th iteration, as we did for Equation (17):

$$F_n(x) = \sum_{k=1}^{\infty} \left[1 - (1 - \mu_k)^n \right] \frac{(I, \phi_k)}{\mu_k}\, \phi_k(x) \ . \tag{28}$$

The same observations can be made for both (17) and (28). Let us now show how these formulas are indeed the same.

2.3 *Comparison between formulas (17) and (28)*

Note that if we operate the following substitutions (see also [Bertero, 1992]):

$$\lambda_k = \sqrt{\mu_K}; \quad u_k(x) = \frac{1}{\sqrt{\mu_K}}\, \phi_k(x); \qquad v_k(x) = \phi_k(x) \tag{29}$$

Equations (17) and (28) become the same, and, as far as the performance analysis of the Gerchberg method is concerned, they are perfectly equivalent. This equivalence does not hold anymore if (17) and (28), or, respectively, (11) and (27), are used as noniterative reconstruction formulas for F. Indeed, in [Bertero and Pike, 1982] it is proved that Equations (11) and (17) have some advantages over Equations (27) and (28) in order to obtain a noniterative estimate of F.

3 Global impulse response

Let us consider the eigenspace approach. It is easy to see that the impulse response of the bandlimiting system $\mathcal{BT}_D$ is:

$$h(x, \xi) = \begin{cases} K_B(x - \xi) & \forall\ (x, \xi) \in \mathbb{R}^N \times D \\ 0 & \text{elsewhere} \end{cases} \ , \tag{30}$$

where K_B is given by Equation (5).

Cascading the Gerchberg algorithm to operator $\mathcal{BT}_D$ will result in a global linear space-variant system, whose impulse response will be, as is known, the output of the Gerchberg procedure for a Dirac impulse input. We are able to derive all the expressions in the eigenspace of the operator. Let us assume $F(x) = \delta(x - \xi)$. In this case, we have

$$(F, \phi_k) = \int \delta(x - \xi)\, \phi_k(x)\, dx = \phi_k(\xi) \tag{31}$$

and, from (28) and (26)

$$h_n(x, \xi) = \sum_{k=1}^{\infty} \left[1 - (1 - \mu_k)^n \right] \phi_k(\xi)\, \phi_k(x) \quad . \tag{32}$$

$h_n(x, \xi)$ is the global impulse response of the system at the n-th iteration. The above equations are valid if $\xi \in D$, otherwise the output of the system is zero everywhere. Note that for n infinite we have $h_\infty(x, \xi) = \delta(x - \xi)$ (perfect reconstruction), and for n=1 we have $h_1(x, \xi) = h(x, \xi)$ as in equation (30) (PSF of the bandlimiting system).

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