The conforming virtual element method for polyharmonic problems

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Abstract

In this work, we exploit the capability of virtual element methods in accommodating approximation spaces featuring high-order continuity to numerically approximate differential problems of the form $\Delta^p u = f$, $p \geq 1$. More specifically, we develop and analyze the conforming virtual element method for the numerical approximation of polyharmonic boundary value problems, and prove an abstract result that states the convergence of the method in the energy norm.

Key words: virtual element method, polytopal mesh, polyharmonic problem, high-order methods

1. Introduction

In the recent years, there has been a tremendous interest to numerical methods that approximate partial differential equations (PDEs) on computational meshes with arbitrarily-shaped polygonal/polyhedral (polytopic, for short) elements. A nonexhaustive list of such methods include the Mimetic Finite Difference method (see e.g., [22,20,4,16,8]), the Polygonal Finite Element Method (see e.g., [43]),the polygonal Discontinuous Galerkin Finite Element Methods (see e.g., [5,26,24,7,11]) the Hybridizable Discontinuous Galerkin and Hybrid High-Order Methods (see e.g., [31,32]), the Gradient Discretization method (see e.g., [34].[33]), An alternative approach that is also proved to be very successful is provided by the Virtual Element method (VEM), which was originally proposed in [14] for the numerical treatment of second-order elliptic problems [29,28], and readily extended to Cahn-Hilliard equation [3], Stokes equations [2], Laplace-Beltrami equation [35], Darcy-Brinkam equation [44], discrete topology optimization problems [6], fracture networks problems [17], eigenvalue problems [38]. The mixed virtual element formulation was proposed in [21]; the nonconforming Virtual element formulations was proposed for second-order elliptic problems in [12], and later extended to general advection-reaction-diffusion problems, Stokes equation, the biharmonic problems, the eigenvalue problems, and the Schrodinger equation in [27,30,48,37,9]. Efficient multigrid methods for the resulting linear system of equations in [10]. A posteriori error estimates can be found in [25].

In this work, we propose the conforming VEM for the numerical approximation of polyharmonic problems. A peculiar feature of VEM is the possibility of designing approximation spaces characterized by high-order continuity properties [15]. This turns out to be crucial when differential operators of order higher than two have to be considered, as, for example, in the numerical treatment of biharmonic problems (see, e.g., the plate bending problem or the Cahn-Hilliard equation) and polyharmonic problems. The numerical approximation of polyharmonic problems has been first addressed in the eighties by [18] and, more recently, in [13,40,45,42,36]. It is worth mentioning an increasing interest in the numerical approximation of models involving high-order differential operators, e.g., [46,47,41] in the context of sixth order Cahn-Hilliard equations. To the best of our knowledge, the conforming VEM proposed in this article is the first work addressing the approximation of arbitrary-order polyharmonic problems on polygonal meshes.

The outline of the paper is as follows. In Section 2, we introduce the continuous polyharmonic problem involving the differential operator Δ^p for any integer $p \geq 1$. In Section 3, we introduce the conforming VEM approximation of arbitrary order. In this case, the global VEM space is made of C^{p-1} functions. As a collateral result, we obtain a virtual element formulation that includes the VEM for the Poisson and the biharmonic equation, where the basis functions are globally C^r for $r \geq 1$. An abstract result proves the convergence of the method in the energy norm that correspond to the differential operator Δ^p. In this section, we also consider an alternative formulation with virtual element spaces of arbitrarily regular basis functions by enriching the "bulk" degrees of freedom. In Section 4, we derive the error estimates in different norms. Finally, in Section 5, we offer our final comments and conclusions.

Notation and technicalities. Throughout the paper, we consider the usual multi-index notation. In particular, if v is a sufficiently regular bivariate function and $\nu = (\nu_1, \nu_2)$ a multi-index with ν_1, ν_2 nonnegative integer numbers, the function $D^\nu v = \partial^{|\nu|} v / \partial x_1^{\nu_1} \partial x_2^{\nu_2}$ is the partial derivative of v of order $|\nu| = \nu_1 + \nu_2 > 0$. For $\nu = (0, 0)$, we adopt the convention that $D^\nu v$ coincides with v. Also, for the sake of exposition, we may use the shortcut notation $\partial_x v$, $\partial_y v$, $\partial_{xx} v$, $\partial_{xy} v$, $\partial_{yy} v$, to denote the first- and second-order partial derivatives along the coordinate directions x and y; $\partial_n v$, $\partial_t v$, $\partial_{nn} v$, $\partial_{nt} v$, $\partial_{tt} v$ to denote the first- and second-order normal and tangential derivatives of order one and two along a given mesh edge; and $\partial_n^m v$ and $\partial_t^m v$ to denote the normal and tangential derivative of v of order m along a given mesh edge. Finally, let $\mathbf{n} = (n_x, n_y)$ and $\boldsymbol{\tau} = (\tau_x, \tau_y)$ be the unit normal and tangential vectors to a given edge e of an arbitrary polygon K. We recall the following relations between the first derivatives of v:

$$\partial_n v = n_x \partial_x v + n_y \partial_y v, \quad \partial_\tau v = \tau_x \partial_x v + \tau_y \partial_y v, \tag{1}$$

and the second derivatives of v:

$$\partial_{nn} v = \mathbf{n}^T \mathsf{H}(v) \mathbf{n}, \quad \partial_{n\tau} v = \mathbf{n}^T \mathsf{H}(v) \boldsymbol{\tau}, \quad \partial_{\tau\tau} v = \boldsymbol{\tau}^T \mathsf{H}(v) \boldsymbol{\tau}, \tag{2}$$

where matrix $\mathsf{H}(v)$ is the Hessian of v, i.e., $\mathsf{H}_{11}(v) = \partial_{xx} v$, $\mathsf{H}_{12}(v) = \mathsf{H}_{21}(v) = \partial_{xy} v$, $\mathsf{H}_{22}(v) = \partial_{yy} v$.

2. The continuous polyharmonic problem

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary Γ. For an integer $p \geq 1$, we are interested in developing the conforming Virtual Element method for the numerical approximation of the following problem:

$$\Delta^p u = f \qquad \text{in } \Omega, \tag{3a}$$

$$\partial_n^j u = 0 \qquad \text{for } j = 0, \dots, p-1 \text{ on } \Gamma, \tag{3b}$$

(recall the conventional notation $\partial_n^0 u = u$). Let

$$V \equiv H_0^p(\Omega) = \left\{ v \in H^p(\Omega) : \partial_n^j v = 0 \text{ on } \Gamma,\ j = 0, \dots, p-1 \right\}.$$

We denote the duality pairing between V and its dual V^* by $\langle \cdot, \cdot \rangle$. The variational formulation of (3) reads as: *Find $u \in V$ such that*

$$a(u, v) = \langle f, v \rangle \qquad \forall v \in V, \tag{4}$$

where, for any nonnegative integer ℓ, the bilinear form on the left is given by:

$$
a(u,v)=\begin{cases}\displaystyle\int_\Omega \nabla\Delta^\ell u\cdot\nabla\Delta^\ell v\,dx & \text{for } p=2\ell+1,\\ \displaystyle\int_\Omega \Delta^\ell u\,\Delta^\ell v\,dx & \text{for } p=2\ell.\end{cases} \tag{5}
$$

Whenever $f\in L^2(\Omega)$, we may consider the duality pairing between $L^2(\Omega)$ and itself given by the L^2-inner product:

$$
\langle f,v\rangle=(f,v)=\int_\Omega fv dx. \tag{6}
$$

The existence and uniqueness of the solution to (4) follows from the Lax-Milgram lemmabecause of the continuity and coercivity of the bilinear form $a(\cdot,\cdot)$ with respect to $\|\cdot\|_V:=|\cdot|_{p,\Omega}$ which is a norm on $H_0^p(\Omega)$. Moreover, since Ω is a convex polygon, from [39] we know that $u\in H^{2p-m}(\Omega)\cap H_0^p(\Omega)$ if $f\in H^{-m}(\Omega)$, $m\leq p$ and it holds that $\|u\|_{2p-m}\leq C\,\|f\|_{-m}$. In the following, we denote the coercivity and continuity constants of $a(\cdot,\cdot)$ by α and M, respectively.

3. The conforming Virtual Element approximation

3.1. *Abstract framework*

Let $\{\Omega_h\}_h$ be a sequence of decompositions of Ω where each mesh Ω_h is a collection of nonoverlapping polygonal elements K with boundary ∂K, and let $\mathcal{E}_h$ be the set of edges e of Ω_h. Each mesh is labeled by h, the diameter of the mesh, defined as usual by $h=\max_{K\in\Omega_h}h_K$, where $h_K=\sup_{\mathbf{x},\mathbf{y}\in K}|\mathbf{x}-\mathbf{y}|$. We denote the set of vertices in $\mathcal{T}_h$ by $V_h=V_h^i\cup V_h^\Gamma$, where V_h^i and V_h^Γ are the subsets of interior and boundary vertices, respectively. Accordingly, V_h^K is the set of vertices of K. The symbol h_{v} denotes the average of the diameters of the polygons sharing the vertex v. For functions in $\Pi_{K\in\Omega_h}H^p(K)$, we define the seminorm $\|v\|_h^2=\sum_{K\in\Omega_h}a^K(v,v)$, being $a^K(\cdot,\cdot)$ the restriction of $a(\cdot,\cdot)$ to K.

The formulation of the Virtual Element method for solving problem (4) only requires three mathematical objects: the finite dimensional conforming Virtual Element space $V_{h,r}^p\subset V$, the bilinear form $a_h(\cdot,\cdot)$, and the linear functional $\langle f_h,\cdot\rangle$. Their definition is the topic of this section. Using such objects, we formulate the VEM as: *Find $u_h\in V_{h,r}^p$ such that*

$$
a_h(u_h,v_h)=\langle f_h,v_h\rangle\quad\forall v_h\in V_{h,r}^p. \tag{7}
$$

The well-posedness of the VEM given in (7), which implies existence and uniqueness of the solution u_h, is a consequence of the Lax-Milgram lemma. An abstract convergence result is available, which depends only on the following assumptions:

(H1) for each h and an assigned integer number $r\geq 2p-1$ we are given:

(i) the *global* Virtual Element space $V_{h,r}^p$ with the following properties:
- $V_{h,r}^p$ is a finite dimensional subspace of $H_0^p(\Omega)$;
- its restriction $V_{h,r}^p(K)$ to any element K of a given mesh Ω_h, called the *local* Virtual Element space, is a finite dimensional subspace of $H^p(K)$;
- $\mathbb{P}_r(K)\subset V_{h,r}^p(K)$ where $\mathbb{P}_r(K)$ is the space polynomials of degree up to $r\geq 1$ defined on K

(ii) the symmetric and coercive bilinear form $a_h: V_{h,r}^p\times V_{h,r}^p\to\mathbb{R}$ admitting the decomposition

$$
a_h(u_h,v_h)=\sum_{K\in\Omega_h}a_h^K(u_h,v_h)\quad\forall u_h,\,v_h\in V_{h,r}^p,
$$

where each local summation term $a_h^K(\cdot,\cdot)$ is also a symmetric and coercive bilinear form;

(iii) an element f_h of the dual space V_h^*, which allows us to define the continuous linear functional $\langle f_h,\cdot\rangle$.

(H2) for each h and each mesh element $K \in \Omega_h$, the local symmetric bilinear form $a_h^K(\cdot,\cdot)$ possesses the two following properties:

(i) r-**Consistency**: for every polynomial $q \in \mathbb{P}_r(K)$ and function $V_{h,r}^p(K)$ we have:

$$a_h^K(v_h, q) = a^K(v_h, q); \tag{8}$$

(ii) **Stability**: there exist two positive constants α_*, α^* independent of h and K such that for every $v_h \in V_{h,r}^p(K)$ it holds:

$$\alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h). \tag{9}$$

To apply the Lax-Milgram lemma we need $a_h(\cdot,\cdot)$ to be coercive and continue. The coercivity of $a_h(\cdot,\cdot)$ follows from the coercivity of $a(\cdot,\cdot)$ and the stability property **(H2)** (with coercivity constant $\alpha^*\alpha$). The continuity of $a_h(\cdot,\cdot)$ follows from its symmetry, assumption **(H2)** and the continuity of $a(\cdot,\cdot)$ (with continuity constant $\alpha^* M$). Denoting by $\mathbb{P}_r(\Omega_h)$ the space of piecewise (possibly discontinuous) polynomials defined over the mesh Ω_h, the following abstract convergence result hold.

Theorem 3.1 *Let u be the solution of the variational problem* (4)*. Then, for every Virtual Element approximation u_I in $V_{h,r}^p$ and any piecewise polynomial approximation $u_\pi \in \mathbb{P}_r(\Omega_h)$ of u we have:*

$$\|u - u_h\|_V \leq C\Big(\|u - u_I\|_V + \|u - u_\pi\|_h + \|f_h - f\|_{V_{h,r}^*} \Big), \tag{10}$$

where C is a constant independent of h that may depend on α, α_*, α^*, M, and r, and,

$$\|f - f_h\|_{V_{h,r}^*} = \sup_{v_h \in V_{h,r}^p \setminus \{0\}} \frac{\langle f - f_h, v_h \rangle}{\|v_h\|_V} \tag{11}$$

is the approximation error of the right-hand side given in the norm of the dual space $V_{h,r}^*$.

Proof. The proof of this theorem is similar to the proofs of the convergence theorem in the energy norm for the Virtual Element approximation of lower-order elliptic problems [14,23]. We briefly sketch how the proof works for completeness of exposition. First, an application of the triangular inequality implies that:

$$\|u - u_h\|_V \leq \|u - u_I\|_V + \|u_I - u_h\|_V. \tag{12}$$

Let $\delta_h = u_h - u_I$. Starting from the definition of $\|\cdot\|_V$, we find that:

$$\begin{aligned}
\alpha_* \|\delta_h\|_V^2 &= \alpha_* a(\delta_h, \delta_h) && \big[\text{use } (9)\big] \\
&\leq a_h(\delta_h, \delta_h) && \big[\text{use } \delta_h = u_h - u_I\big] \\
&\leq a_h(\delta_h, u_h) - a_h(\delta_h, u_I) && \big[\text{use } (7)\big] \\
&\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} a_h^K(\delta_h, u_I) && \big[\text{add } \pm u_\pi\big] \\
&\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} \Big(a_h^K(\delta_h, u_I - u_\pi) + a_h^K(\delta_h, u_\pi) \Big) && \big[\text{use } (8)\big] \\
&\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} \Big(a_h^K(\delta_h, u_I - u_\pi) + a^K(\delta_h, u_\pi) \Big) && \big[\text{add } \pm u\big] \\
&\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} \Big(a_h^K(\delta_h, u_I - u_\pi) + a^K(\delta_h, u_\pi - u) + a^K(\delta_h, u) \Big) && \big[\text{use } (4)\big] \\
&= \langle f_h - f, \delta_h \rangle - \sum_{K \in \Omega_h} \Big(a_h^K(\delta_h, u_I - u_\pi) + a^K(\delta_h, u_\pi - u) \Big).
\end{aligned}$$

Then, we use (9), add and subtract u, use the continuity of a^K, sum over all the elements K, divide by $\|\delta_h\|_V$, take the supremum of the right-hand side error term on $V_{h,r}^p \setminus \{0\}$, and obtain

$$\alpha_* \|\delta_h\|_V \leq \sup_{v_h \in V_{h,r}^p \setminus \{0\}} \frac{|\langle f_h - f, v_h \rangle|}{\|v_h\|_V} + M\left(\alpha^* \|u_I - u\|_V + (1 + \alpha^*) \|u - u_\pi\|_h\right). \tag{13}$$

The assertion of the theorem follows by using (13) in (12) and suitably defining constant the C. $\square$

Let $K \subset \mathbb{R}^2$ be a polygonal element and set

$$a^K(u,v) = \begin{cases} \displaystyle\int_K \nabla\Delta^\ell u \cdot \nabla\Delta^\ell v \, dx & \text{for } p = 2\ell+1, \\ \displaystyle\int_K \Delta^\ell u \, \Delta^\ell v \, dx & \text{for } p = 2\ell. \end{cases}$$

For an odd p, i.e., $p = 2\ell+1$, a repeated application of the integration by parts formula yields

$$\begin{aligned} a^K(u,v) = &- \int_K \Delta^p u \, v \, dx + \int_{\partial K} \partial_n(\Delta^\ell u) \, \Delta^\ell v \, ds \\ &+ \sum_{i=1}^{\ell} \left(\int_{\partial K} \partial_n(\Delta^{p-i} u) \, \Delta^{i-1} v \, ds - \int_{\partial K} \Delta^{p-i} u \, \partial_n(\Delta^{i-1} v) \, ds \right), \end{aligned} \tag{14}$$

while, for an even p, i.e., $p = 2\ell$, we have

$$a^K(u,v) = - \int_K \Delta^p u \, v \, dx + \sum_{i=1}^{\ell} \left(\int_{\partial K} \partial_n(\Delta^{p-i} u) \, \Delta^{i-1} v \, ds - \int_{\partial K} \Delta^{p-i} u \, \partial_n(\Delta^{i-1} v) \, ds \right). \tag{15}$$

3.2. *Virtual element spaces*

For $p \geq 1$ and $r \geq 2p-1$, the local Virtual Element space on element K is defined by

$$\begin{aligned} V^p_{h,r}(K) = \Big\{ v_h \in H^p(K) \, : \Delta^p v_h \in \mathbb{P}_{r-2p}(K), \, D^\nu v_h \in C^0(\partial K), |\nu| \leq p-1, \\ v_h \in \mathbb{P}_r(e), \, \partial^i_n v_h \in \mathbb{P}_{r-i}(e), \, i = 1, \ldots, p-1 \; \forall e \in \partial K \Big\}, \end{aligned} \tag{16}$$

with the conventional notation that $\mathbb{P}_{-1}(K) = \{0\}$. The Virtual Element space $V^p_{h,r}(K)$ contains the space of polynomials $\mathbb{P}_r(K)$, for $r \geq 2p-1$. Moreover, for $p = 1$, it coincides with the conforming Virtual Element space for the Poisson equation [14]; for $p = 2$, it coincides with the conforming Virtual Element space for the biharmonic equation [23].

We characterize the functions in $V^p_{h,r}(K)$ through the following *degrees of freedom*:

$(D1)$ $h_{\mathsf{v}}^{|\nu|} D^\nu v_h(\mathsf{v})$, $|\nu| \leq p-1$ for any vertex v of K;

$(D2)$ $h_e^{-1} \displaystyle\int_e q v_h \, ds$ for any $q \in \mathbb{P}_{r-2p}(e)$ and any edge e of ∂K;

$(D3)$ $h_e^{-1+j} \displaystyle\int_e q \partial^j_n v_h \, ds$ for any $q \in \mathbb{P}_{r-2p+j}(e)$, $j = 1, \ldots, p-1$ and any edge e of ∂K;

$(D4)$ $h_K^{-2} \displaystyle\int_K q v_h \, ds$ for any $q \in \mathbb{P}_{r-2p}(K)$.

Here, as usual, we assume that $\mathbb{P}_{-n}(\cdot) = \{0\}$ for $n \geq 1$. In $(D3)$, the index j starts from 1 instead of 0 since for $j = 0$ we would find the degrees of freedom that are already listed in $(D2)$ (recall that $\partial^j_n v_h = v_h$ for $j = 0$). We note that for any sufficiently regular two-dimensional domain Ω we have the embedding $C^m(\Omega) \subset H^p(\Omega)$ if $m \leq p-1$. This regularity is reflected by the previous choice of the degrees of freedom, which allows us to reconstruct the trace of v_h and the derivatives $\partial^j_n v_h \in \mathbb{P}_{r-j}(e)$ on each edge of ∂K. Since these polynomial traces on a given edge only depend on the edge degrees of freedom, the traces are the same from inside the two mesh elements sharing that edge. To interpolate $v_h \in \mathbb{P}_r(e)$ we need $r+1$ conditions for each edge e. Let v_A and v_B denote the vertices of edge e and use the shortcut notation: $v_A = v_h(\mathsf{v}_A)$, $\partial_n v_A = \partial_n v_h(\mathsf{v}_A)$, etc. Then,

Fig. 1. Triharmonic problem: edge degrees of freedom of the Virtual Element space $V_{h,r}(K)$. Here, p is the order of the partial differential operator and $p = 3$ corresponds to the triharmonic case; $r = 5, 6$ are the integer parameters that specify the maximum degree of the polynomial subspace $\mathbb{P}_r(K)$ of the VEM space $V_{h,r}(K)$. The (green) dots at the vertices represent the vertex values and each (red) vertex circle represents an order of derivation. The (black) dot on the edge represents the moment of $v_{h|e}$; the arrows represent the moments of $\partial_n v_{h|e}$; the double arrows represent the moments of $\partial_{nn} v_{h|e}$.

- the degrees of freedom $(D1)$ provides v_A, v_B, $\partial_\tau^k v_A$ and $\partial_\tau^k v_B$ for $k = 1, \ldots, p-1$, i.e., $2p$ degrees of freedom, which are enough to interpolate $v_{h|e}$ in $\mathbb{P}_r(e)$ if $r = 2p-1$. When $r > 2p-1$ the remaining $(r+1) - 2p$ conditions required to interpolate $v_{h|e}$ in $\mathbb{P}_r(e)$ are provided by the degrees of freedom $(D2)$. The tangential derivatives $\partial_\tau^k v_{h|e}$ in $\mathbb{P}_{r-k}(e)$ for $k = 1, \ldots, r-1$ can be obtained by deriving k times the interpolated polynomial $v_{h|e}$ along e;
- similarly, for each $j = 1, \ldots, p-1$, the degrees of freedom $(D1)$ provides $2(p-j)$ conditions, i.e., $\partial_\tau^k \partial_n^j v_A$ and $\partial_\tau^k \partial_n^j v_B$, for $k = j, \ldots, p-1$. The remaining $r + 1 - 2(p-j)$ conditions to interpolate $\partial_n^j v_h$ in $\mathbb{P}_{r-j}(e)$ are provided by the $(r - 2p + j) + 1$ degrees of freedom $(D3)$. The tangential derivatives $\partial_\tau^k \partial_n^j v_{h|e}$ in $\mathbb{P}_{r-j-k}(e)$ for $k = 1, \ldots, r-j-1$ can be obtained by deriving k times the interpolated polynomial $\partial_n^j v_h$ along e.

Figure 1 illustrates the degrees of freedom on a given edge e for $p = 3$ (triharmonic case) and $r = 5, 6$. Finally, we note that the internal degrees of freedom $(D4)$ make it possible to define the orthogonal polynomial projection of v_h onto the space of polynomial of degree $r - 2p$.

The dimension of $V_{h,r}^p(K)$ is

$$\begin{aligned} \dim V_{h,r}^p(K) &= \frac{p(p+1)}{2}\mathcal{N}^K + \mathcal{N}^K \sum_{j=0}^{p-1} \mathbb{P}_{r-2p+j}(e) + \dim \mathbb{P}_{r-2p}(K) \\ &= \frac{p(p+1)}{2}\mathcal{N}^K + \mathcal{N}^K \sum_{j=0}^{p-1} (r - 2p + j + 1) + \frac{(r-2p+1)(r-2p+2)}{2}, \end{aligned}$$

where $\mathcal{N}^K$ is the number of vertices, which equals the number of edges, of K.

The following lemma ensures that the above choice of degrees of freedom is unisolvent in $V_{h,r}^p(K)$.

Lemma 3.2 *The degrees of freedom (D1)-(D4) are unisolvent for* $V_{h,r}^p(K)$.

Proof. To ease the presentation, we first consider the lowest order space ($r = 5$) for the triharmonic problem ($p = 3$). A counting argument implies that the cardinality of the set of degrees of freedom $(D1) - (D3)$ is equal to the dimension of $V_{h,5}^3$ (note that in this specific case $(D4)$ is empty as there is no volumetric integral in the right-hand side of (14)). Then, we are left to prove that a function v_h in $V_{h,5}^3$ is zero if its degrees of freedom are zero. From the previous discussion on the degrees of freedom, we know that the edge polynomial interpolation of the traces of v_h, $\partial_n v_h$, $\partial_{nn} v_h$ and $\partial_{\tau\tau} v_h$, and, hence, $\Delta v_h = \partial_{tt} v_h + \partial_{nn} v_h$, must be zero if the degrees of freedom of v_h are zero. Hence, from (14) with $\ell = 1$, we find that $\|\nabla(\Delta v_h)\|_{L^2(K)}^2 = 0$ inside K, which implies that Δv_h is constant in K. Using the Divergence Theorem we find that:

$$\Delta v_h |K| = \int_K \Delta v_h \, dx = \int_{\partial K} \partial_n v_h \, ds = 0.$$

Therefore, v_h is the solution of the boundary value problem $\Delta v_h = 0$ in K with boundary conditions $v_h = 0$ on ∂K, and, thus, $v_h = 0$ in K.

The case of a generic p can be treated analogously by properly employing relations (14) (odd p) and (15) (even p) in combination with the following observations:

(a) the polynomial trace $\Delta^\nu v_{h|e} = \partial_n^\alpha \partial_\tau^\beta v_h$ for every integers α, β, and ν (with $\nu \geq 1$) such that $\alpha+\beta = 2\nu$ must be zero if the degrees of freedom $(D1)-(D3)$ of v_h are zero;

(b) the volumetric integrals in (14) and (15) are zero if the degrees of freedom $(D4)$ are zero;

(c) $a^K(v_h, v_h)$ is a norm on $H_0^p(K)$.

□

To define the elliptic projection $\Pi_r^{\nabla,K} : V_{h,r}^p(K) \to \mathbb{P}_r(K)$, we first need to introduce the *vertex average projector* $\widehat{\Pi}^K : V_{h,r}^p(K) \to \mathbb{P}_0(K)$, which projects any smooth enough function defined on K onto the space of constant polynomials. Let ψ be a continuous function defined on K. The *vertex average projection* of ψ onto the constant polynomial space is defined as:

$$\widehat{\Pi}^K \psi = \frac{1}{\mathcal{N}^K} \sum_{\mathsf{v}\in\partial K} \psi(\mathbf{x}_{\mathsf{v}}), \tag{17}$$

where $\mathbf{x}_{\mathsf{v}}$ is the position of vertex v. Finally, we define the elliptic projection $\Pi_r^{\nabla,K} : V_{h,r}^p(K) \to \mathbb{P}_r(K)$ as the solution of the finite dimensional variational problem

$$a^K(\Pi_r^{\nabla,K} v_h, q) = a^K(v_h, q) \qquad \forall q \in \mathbb{P}_r(K), \tag{18}$$

$$\widehat{\Pi}^K D^\nu \Pi_r^{\nabla,K} v_h = \widehat{\Pi}^K D^\nu v_h \qquad |\nu| \leq p-1. \tag{19}$$

Such operator has two important properties:

(i) it is a polynomial-preserving operator in the sense that $\Pi_r^{\nabla,K} q = q$ for every $q \in \mathbb{P}_r(K)$.

(ii) $\Pi_r^{\nabla,K} v_h$ is *computable* using only the degrees of freedom of v_h. In fact, in view of the integration by parts formulas (14) and (15), the right-hand side of (18) takes the form (depending on the parity of p):

$$a^K(v_h, q) = -\int_K \Delta^p q\, v_h\, dx + \int_{\partial K} \partial_n(\Delta^\ell q)\, \Delta^\ell v_h\, ds + \sum_{i=1}^{\ell} \left\{ \int_{\partial K} \partial_n(\Delta^{p-i} q)\, \Delta^{i-1} v_h\, ds - \int_{\partial K} \Delta^{p-i} q\, \partial_n(\Delta^{i-1} v_h)\, ds \right\}, \tag{20}$$

or

$$a^K(v_h, q) = -\int_K \Delta^p q\, v_h\, dx + \sum_{i=1}^{\ell} \left\{ \int_{\partial K} \partial_n(\Delta^{p-i} q)\, \Delta^{i-1} v_h\, ds - \int_{\partial K} \Delta^{p-i} q\, \partial_n(\Delta^{i-1} v_h)\, ds \right\}. \tag{21}$$

In (20) and (21), $\Delta^{p-i}q$, and $\Delta^p q$ are easily computable from q. The volumetric integral on K can be expressed using the degrees of freedom $(D5)$ since it is the moment of v_h against $\Delta^p q$, which is a polynomial of degree $r-2p$. The edge traces of $\Delta^\ell v_h$, $\partial_n(\Delta^{i-1} v_h)$ and $\Delta^{i-1} v_h$ are computable from the degrees of freedom $(D1)-(D4)$ of v_h by solving suitable polynomial interpolation problems.

Building upon the local spaces $V_{h,r}^p(K)$ for all $K \in \Omega_h$, the global conforming Virtual Element space $V_{h,r}^p$ is defined on Ω as

$$V_{h,r}^p = \left\{ v_h \in H_0^p(\Omega) \,:\, v_{h|K} \in V_{h,r}^p(K)\ \forall K \in \Omega_h \right\}. \tag{22}$$

We remark that the associated global space is made of C^{p-1} functions. Indeed, the restriction of a Virtual Element function v_h to each element K belongs to $H^p(K)$ and glues with C^{p-1}-regularity across the internal mesh faces. The global degrees of freedom induced by the local degrees of freedom are listed as follows:

- $h_{\mathsf{v}}^{|\nu|} D^\nu v_h(\mathsf{v})$, $|\nu| \leq p-1$ for every interior vertex v of Ω_h;

- $h_e^{-1}\displaystyle\int_e q v_h\, ds$ for any $q \in \mathbb{P}_{r-2p}(e)$ and every interior edge $e \in \mathcal{E}_h$;
- $\displaystyle\int_e q\partial_n v_h\, ds$ for every $q \in \mathbb{P}_{r-5}(e)$ and every interior edge $e \in \mathcal{E}_h$;
- $h_e^{-1+j}\displaystyle\int_e q\partial_n^j v_h\, ds$ for any $q \in \mathbb{P}_{r-2p+j}(e)$ $i = 1, \ldots, p-1$ and every interior edge $e \in \mathcal{E}_h$;
- $h_K^{-2}\displaystyle\int_K q v_h\, ds$ for any $q \in \mathbb{P}_{r-2p}(K)$ and every $K \in \Omega_h$.

Remark 3.3 *For $p = 3$ (triharmonic case) we can also consider the following modified lowest order space*

$$\widetilde{V}_{h,5}(K) = \Big\{ v_h \in H^3(K) \,:\Delta^3 v_h = 0, v_h,\, \partial_n v_h,\, \partial_{nn} v_h \in C^0(\partial K), \\ v_h \in \mathbb{P}_5(e),\, \partial_n v_h \in \mathbb{P}_3(e),\, \partial_{nn} v_h \in \mathbb{P}_2(e)\ \forall e \in \partial K \Big\}$$

with associated dofs
$(D1')$ $h_{\mathsf{v}}^{|\nu|} D^\nu v_h(\mathsf{v})$, $|\nu| \le 2$ for any vertex v of ∂K;
$(D2')$ $h_e \displaystyle\int_e \partial_{nn} v_h\, ds$ for any edge e of ∂K.
Using the same argument of the proof of Lemma 3.2, we can still prove that (i) the degrees of freedom $(D1')$ and $(D2')$ are unisolvent in $\widetilde{V}_{h,5}(K)$; (ii) the space of polynomials of degree 4 are a subspace of $\widetilde{V}_{h,5}(K)$; (iii) the elliptic projection of v_h is still computable from this choice of degrees of freedom; (iv) the associated global space

$$\widetilde{V}_{h,5} = \Big\{ v_h \in H_0^3(\Omega) \,:\, v_h|_K \in \widetilde{V}_{h,5}(K)\ \forall K \in \Omega_h \Big\}, \tag{23}$$

which is obtained by gluing together all the elemental spaces $\widetilde{V}_{h,5}(K)$, is still made of C^2 functions. Analogously, in the general case one can build the following modified lowest order spaces (containing the space of polynomials of degree $2p-2$)

$$\tilde{V}^p_{h,2p-1}(K) = \Big\{ v_h \in H^p(K) \,:\Delta^p v_h = 0,\, D^\nu v_h \in C^0(\partial K), |\nu| \le p-1, \\ v_h \in \mathbb{P}_{2p-1}(e),\, \partial_n^i v_h \in \mathbb{P}_{2p-2-i}(e),\, i = 1, \ldots, p-1\ \forall e \in \partial K \Big\}, \tag{24}$$

with associated dofs
$(D1')$ $h_{\mathsf{v}}^{|\nu|} D^\nu v_h(\mathsf{v})$, $|\nu| \le p-1$ for any vertex v of ∂K;
$(D2')$ $h_e^{-1+j} \displaystyle\int_e q\partial_n^i v_h\, ds$ for any $q \in \mathbb{P}_{i-2}(e)$ and any edge e of ∂K, $i = 1, \ldots, p-1$.

3.3. *Construction of the bilinear form*

We write the symmetric bilinear form $a_h : V^p_{h,r} \times V^p_{h,r} \to \mathbb{R}$ as the sum of local terms

$$a_h(u_h, v_h) = \sum_{K \in \Omega_h} a_h^K(u_h, v_h), \tag{25}$$

where each local term $a_h^K : V^p_{h,r}(K) \times V^p_{h,r}(K) \to \mathbb{R}$ is a symmetric bilinear form. We set

$$a_h^K(u_h, v_h) = a^K(\Pi_r^{\nabla,K} u_h, \Pi_r^{\nabla,K} v_h) + S^K(u_h - \Pi_r^{\nabla,K} u_h, v_h - \Pi_r^{\nabla,K} v_h), \tag{26}$$

where $S^K : V^p_{h,r}(K) \times V^p_{h,r}(K) \to \mathbb{R}$ is a symmetric positive definite bilinear form such that

$$\sigma_* a^K(v_h, v_h) \le S^K(v_h, v_h) \le \sigma^* a^K(v_h, v_h) \qquad \forall v_h \in V^p_{h,r}(K) \text{ with } \Pi_r^{\nabla,K} v_h = 0, \tag{27}$$

for two some positive constants σ_*, σ^* independent of h and K. The bilinear form $a_h^K(\cdot,\cdot)$ has the two fundamental properties of *consistency* and *stability* stated by the following lemma.

Lemma 3.4 *The bilinear form $a_h^K(\cdot,\cdot)$ defined in* (26) *possesses both* (*i*) *r-stability and* (*ii*) *consistency properties stated in* (8) *and* (9), *respectively, as required by assumption **(H2)**.*

Proof. The r-consistency property follows by noting that the stability term in (26) is zero when one of its entries is a polynomial of degree r as $\Pi_r^{\nabla,K}$ is a polynomial-preserving operator. The stability property is easily established by applying (9) to definition (26) and setting $\alpha_* = \min(\sigma_*, 1)$ and $\alpha^* = \max(\sigma^*, 1)$, where σ_* and σ^* are the constants defined in (27). □

Furthermore, $a_h^K(\cdot,\cdot)$ is V-elliptic and continuous for every K, and so is the global bilinear form $a_h(\cdot,\cdot)$. The V-ellipticity of $a_h^K(\cdot,\cdot)$ is indeed a consequence of the left inequality in (9). Since $a_h^K(\cdot,\cdot)$ is symmetric and coercive, it is a scalar product on $V_{h,r}(K)$ and satisfies the Cauchy-Schwarz inequality. Using the right inequality in (9) we (easily) prove the continuity of $a_h^K(\cdot,\cdot)$ with respect to norm $\|\cdot\|_{V,K}$:

$$
\begin{aligned}
a_h^K(u_h,v_h) &\leq \left(a_h^K(u_h,u_h)\right)^{\frac{1}{2}} \left(a_h^K(v_h,v_h)\right)^{\frac{1}{2}} \leq \alpha^*\left(a^K(u_h,u_h)\right)^{\frac{1}{2}} \left(a^K(v_h,v_h)\right)^{\frac{1}{2}} \\
&\leq \alpha^* M \; \|u_h\|_{V,K} \; \|v_h\|_{V,K} \qquad \forall u_h,\, v_h \in V_{h,r}^p(K). \qquad (28)
\end{aligned}
$$

Collecting together the local terms, we can formulate the global V-ellipticity and continuity properties as follows:

$$
\alpha_* a(v_h,v_h) \leq a_h(v_h,v_h) \leq \alpha^* a^K(v_h,v_h) \qquad \forall v_h \in V_{h,r}^p \qquad (29)
$$

$$
a_h(u_h,v_h) \leq \alpha^* M \; \|u_h\|_V \; \|v_h\|_{V,K} \qquad \forall u_h,\, v_h \in V_{h,r}^p. \qquad (30)
$$

3.4. *Construction of the load term*

We denote by f_h the piecewise polynomial approximation of f on Ω_h given by

$$
f_{h|K} = \Pi_{r-p}^{0,K} f, \qquad (31)
$$

for $r \geq 2p-1$ and $K \in \Omega_h$. Then, we set

$$
\langle f_h, v_h\rangle = \sum_{K\in\Omega_h} \int_K f_h v_h \, dx. \qquad (32)
$$

Using the definition of the L^2-orthogonal projection we find that

$$
\langle f_h, v_h\rangle = \sum_{K\in\mathcal{T}_h} \int_K \Pi_{r-p}^{0,K} f \, v_h \, dx = \sum_{K\in\mathcal{T}_h} \int_K \Pi_{r-p}^{0,K} f \Pi_{r-p}^{0,K} v_h \, dx = \sum_{K\in\mathcal{T}_h} \int_K f \, \Pi_{r-p}^{0,K} v_h \, dx. \qquad (33)
$$

The right-hand side of (33) is computable by using the degrees of freedom $(D1)-(D5)$ and the enhanced approach [1].

Remark 3.5 *An alternative formulation that does not require the enhancement is given by taking $\overline{r} = \max(0, r-2p)$ for $r \geq 2p-1$ and $f_h = \Pi_{\overline{r}}^{0,K} f$. The resulting approximation is suboptimal.*

3.5. *VEM approximation of polyharmonic problems with basis functions of arbitrary degree of continuity*

In this last section we briefly sketch the construction of global Virtual Element spaces with higher order of continuity. More precisely, let us consider the local Virtual Element space defined as before, for $r \geq 2p-1$:

$$
\begin{aligned}
V_{h,r}^p(K) = \Big\{ & v_h \in H^p(K) : \Delta^p v_h \in \mathbb{P}_{r-2p}(K), D^\nu v_h \in C^0(\partial K), |\nu| \leq p-1, \\
& v_h \in \mathbb{P}_r(e),\, \partial_n^j v_h \in \mathbb{P}_{r-j}(e),\, j = 1,\dots,p-1 \; \forall e \in \partial K \Big\}. \qquad (34)
\end{aligned}
$$

Differently from the previous section, we make the degrees of freedom depend on a given parameter t with $0 \leq t \leq p-1$. For a given value of t we choose the *degrees of freedom* of $V_h^{K,r}$ as follows

$(D1)$ $h^{|\nu|}D^\nu v_h(\mathsf{v})$, $|\nu| \leq p-1$ for any vertex v of K;

$(D2)$ $h_e^{-1}\displaystyle\int_e v_h q\, ds$ for any $q \in \mathbb{P}_{r-2p}(e)$, for any edge e of ∂K;

$(D3)$ $h_e^{-1+j}\displaystyle\int_e \partial_n^j v_h q\, ds$ for any $q \in \mathbb{P}_{r-2p+j}(e), j = 1, \ldots, p-1$ for any edge e of ∂K;

$(D4')$ $h_K^{-2}\displaystyle\int_K q v_h\, ds$ for any $q \in \mathbb{P}_{r-2(p-t)}(K)$;

where as usual we assume $\mathbb{P}_{-n}(\cdot) = \{0\}$ for $n = 1, 2, 3, \ldots$.

In view of the above choice of the degrees of freedom, the following properties hold true:

(i) the dofs are unisolvent. Indeed, proceeding as before, it is enough to use (14) or (15) and observe that $\Delta^i v_h|_e = \partial_n^\alpha \partial_\tau^\beta v_h$ with $\alpha + \beta = 2i$ is a polynomial uniquely identified by the values of the dofs;

(ii) $\mathbb{P}_r(K) \subset V_h^{K,r}$, for $r \geq 2p-1$;

(iii) the choice $(D4')$ instead of $(D4)$ still guarantees that the associated global space is made of C^{p-1} functions, but now $(D1)$-$(D4')$ can be employed to solve a differential problem involving the Δ^{p-t} operator by employing $C^{p-1}(\Omega)$ basis functions. For instance:

  (a) Choosing p and t such that $p - t = 1$ we obtain C^{p-1} conforming VEM for the solution of the Laplacian problem. For example, for $p = 3$, $t = 2$ and $r = 5$, the local space $V_{h,5}^3(K)$ endowed with the corresponding degrees of freedom $(D1) - (D4')$ can be employed to build a global space made of C^2 functions for the approximation of the Laplace problem. It is worth mentioning that the new choice $(D4')$ (differently from the original choice $(D4)$) is essential for the computability of the elliptic projection (see (18)-(19)) with respect to the bilinear form $a^K(\cdot,\cdot) = \int_K \nabla(\cdot)\nabla(\cdot)dx$.

  (b) Choosing p and t such that $p - t = 2$ we get C^{p-1} conforming VEM for the solution of the Bilaplacian problem. For example, for $p = 3$, $t = 1$ and $r = 5$, similarly to the previous case, the space $V_{h,5}^3(K)$ together with $(D1) - (D4')$ gives rise to a global space made of C^2 functions that can be employed for the solution of the biharmonic problem. Again, the particular choice $(D4')$ makes possible the computability of the ellliptic projection with respect to the bilinear form $a^K(\cdot,\cdot) = \int_K \Delta(\cdot)\Delta(\cdot)dx$.

4. Convergence analysis

4.1. *Mesh regularity and polynomial interpolation error estimates*

We consider the following mesh regularity assumptions:

(M) There exists a positive constant γ independent of h (and K) such that $\{\Omega_h\}$:

  (i) K is star-shaped with respect to every point of a ball of radius γh_K, where h_K is the diameter of K;

  (ii) for every edge e of the cell boundary ∂K of every cell K of Ω_h, it holds that $h_e \leq \gamma h_K$, where h_e denotes the length of e.

We refer to γ as the *mesh regularity constant*.

In view of assumptions $\mathbf{M}(i)$-$\mathbf{M}(ii)$ on Ω_h, we define, for every smooth enough function w the Virtual Element interpolant w^I, which is the function in $V_{h,r}$ uniquely identified by the same degrees of freedom of w. More precisely, if $\chi_i(w)$ denotes the i-th global degree of freedom of w, there exists a unique Virtual Element function $w^I \in V_{h,r}$ such that $\chi_i(w - w^I) = 0$. Combining the Bramble-Hilbert Lemma and scaling arguments as in the finite element framework (see, e.g., [14]and [19]) we can prove that for every $K \in \Omega_h$ and every function $w \in H^\beta(K)$, it holds

$$\|w - w^I\|_{s,K} \leq C h_K^{\min(\beta,r+1)-s}|w|_{\beta,K} \qquad s = 0, 1, \ldots, p \tag{35}$$

for some positive constant C independent of h.

Under the same assumptions and using similar techniques, we can also prove that the existence of a piecewise polynomial approximation $w_\pi \in \mathbb{P}_\ell(\Omega_h)$ such that the local estimate holds

$$\|w - w_\pi\|_{s,K} \le C h_K^{\min(\beta,\ell+1)-s} |w|_{\beta,K} \qquad s = 0, 1, \ldots, p, \quad 1 \le \beta \le \ell + 1, \tag{36}$$

for some positive constant C independent of h and every mesh element K. The elliptic projection $\Pi_\ell^{\nabla,K} w$ and the L^2-ortogonal projection $\Pi_\ell^{0,K} w$ of w are both instances of w_π for which estimate (36) holds.

4.2. *Convergence in the energy norm*

By using standard estimates for the interpolation error, we can derive the convergence rate of the approximation error in the energy norm. First, we need a technical lemma that estimates the approximation error of the load term.

Lemma 4.1 *Consider a function $f \in H^{r-(p-1)}(\Omega)$ and its L^2-orthogonal projection on the space of polynomials of degree $r-p$, denoted by $f_h = \Pi_{r-p}^{0,K}$. Then, there exists a positive constant C, which is independent of h, such that*

$$\langle f - f_h, v_h \rangle \le C h^{r+1} |f|_{r-(p-1)} \, |v_h|_p \qquad \forall v_h \in V_{h,r}^p. \tag{37}$$

Proof. First, we note that $\big(I - \Pi_{r-p}^{0,K}\big) f$ is orthogonal to the polynomials of degree $r-p$ (recall that $r \ge 2p-1$) and that v_h belongs to $H_0^p(\Omega)$. We employ the Cauchy-Schwarz inequality (twice) and use (35) (with $w = f$ and $w_\pi = \Pi_{r-p}^{0,K} f$) to obtain the estimate:

$$\begin{aligned}
\langle f - f_h, v_h \rangle &= \sum_{K \in \Omega_h} \int_K \big(I - \Pi_{r-p}^{0,K}\big) f \ \big(I - \Pi_{p-1}^{0,K}\big) v_h \, dx \le \sum_{K\in\Omega_h} \Big\| \big(I - \Pi_{r-p}^{0,K}\big) f \Big\|_{0,K} \Big\| \big(I - \Pi_{p-1}^{0,K}\big) v_h \Big\|_{0,K} \\
&\le C \sum_{K\in\Omega_h} h_K^{r-(p-1)} |f|_{r-(p-1),K} \ h_K^p |v_h|_{p,K} \le C h^{r+1} |f|_{r-(p-1)} |v_h|_p \,,
\end{aligned}$$

where C denotes a positive constant independent of h. □

Theorem 4.2 *Let $f \in H^{r-p+1}(\Omega)$ and let u be the solution of the variational problem (4) and $u_h \in V_{h,r}^p$ be the solution of the Virtual Element problem (7). Under the mesh regularity assumption **(M)**, we find that*

$$\|u - u_h\|_V \le C h^{r-(p-1)} \big(|u|_{r+p+1} + |f|_{r-p+1} \big). \tag{38}$$

Proof. The assertion of the theorem follows by estimating each term of the right-hand side of (10) separately and using interpolation estimate (35) and Lemma 4.1, cf. inequality (37). □

4.3. *Convergence in lower order norms*

We first prove a technical lemma that will be useful in the error analysis of the next subsections.

Lemma 4.3 *Let $f \in H^{r-p+1}(\Omega)$ and let u be the solution of the variational problem (4) and $u_h \in V_{h,r}^p$ be the solution of the Virtual Element problem (7). Then, for any function $\psi \in H^\beta(\Omega) \cap H_0^p(\Omega)$ ($\beta > p$) it holds that*

$$a(\psi, u - u_h) \le C h^{(r-(p-1)) + \min(\beta, r+1) - p} \Big(|u|_{r+p+1} + |f|_{r-p+1} \Big) \|\psi\|_\beta \,, \tag{39}$$

for some positive constant C independent of h.

Proof. To derive (39), we add and substract the Virtual Element interpolant of ψ denoted by ψ_I to the left-hand side of (39), and, then, use (3) and (7), and obtain:

$$\begin{aligned}
a(\psi, u - u_h) &= a(u - u_h, \psi - \psi_I) + a(u - u_h, \psi_I) \\
&= a(u - u_h, \psi - \psi_I) + \langle f - f_h, \psi_I \rangle + a_h(u_h, \psi_I) - a(u_h, \psi_I) \\
&= T_1 + T_2 + T_3.
\end{aligned} \tag{40}$$

To estimate term T_1, we use the continuity of $a(\cdot,\cdot)$ with respect to the norm $\|\cdot\|_V = |\cdot|_p$, the estimate in the energy norm (38), and interpolation error estimate (35) ($s=p$)

$$T_1 \le \|u-u_h\|_V\ \|\psi-\psi_I\|_V \le Ch^{r-(p-1)}\Big(\,|u|_{r+p+1} + |f|_{r-(p-1)}\Big)\ |\psi-\psi_I|_p \tag{41}$$

$$\le Ch^{(r-(p-1))+(\min(\beta,r+1)-p)}\Big(\,|u|_{r+p+1} + |f|_{r-(p-1)}\Big)\,|\psi|_\beta\,. \tag{42}$$

To estimate term T_2, we first note that $\big(I-\Pi^{0,K}_{r-p}\big)f$ is orthogonal to the polynomials of global degree up to $r-p$ defined on K and that $\psi_I \in H^p(\Omega)$. Then, we apply the Cauchy-Schwarz inequality (twice) and obtain:

$$\langle f-f_h,\psi_I\rangle = \sum_{K\in\Omega_h}\int_K \big(I-\Pi^{0,K}_{r-p}\big)f\ \big(I-\Pi^{0,K}_{p-1}\big)\psi_I\,dx \le \sum_{K\in\Omega_h}\Big\|\big(I-\Pi^{0,K}_{r-p}\big)f\Big\|_{0,K}\ \Big\|\big(I-\Pi^{0,K}_{p-1}\big)\psi_I\Big\|_{0,K}$$

$$\le \left(\sum_{K\in\Omega_h}\Big\|\big(I-\Pi^{0,K}_{r-p}\big)f\Big\|^2_{0,K}\right)^{\frac12}\ \left(\sum_{K\in\Omega_h}\Big\|\big(I-\Pi^{0,K}_{p-1}\big)\psi_I\Big\|^2_{0,K}\right)^{\frac12}.$$

The first term on the right is bounded by using local estimate (36):

$$\Big\|\big(I-\Pi^{0,K}_{r-p}\big)f\Big\|_{0,K} \le Ch_K^{r-(p-1)}\,|f|_{r-(p-1),K}\,.$$

The second term on the right is transformed by applying estimate (36) (with $w=\psi_I$ and $w_\pi = \Pi^{0,K}_{p-1}\psi_I$), adding and subtracting ψ and applying estimate (35):

$$\Big\|\big(I-\Pi^{0,K}_{p-1}\big)\psi_I\Big\|_{0,K} \le Ch^p\,|\psi_I|_{p,K} \le Ch^p\big(\,|\psi|_{p,K} + |\psi_I-\psi|_{p,K}\,\big) \le Ch^p\big(\,|\psi|_{p,K} + h^{\min(\beta,r+1)-p}\,|\psi|_{\beta,K}\,\big).$$

Collecting all the local terms, using the Cauchy-Schwarz inequality, and the assumption that $\beta \ge p$ (so $h^p \ge h^{\min(\beta,r+1)}$) yields:

$$T_2 \le Ch^{r-(p-1)}\,|f|_{r-(p-1)}\ \Big(h^p\,|\psi|_p + h^{p+(\min(\beta,r+1)-p)}\,|\psi|_\beta\Big) \le Ch^{r+1}\,|f|_{r-(p-1)}\ |\psi|_\beta\,. \tag{43}$$

To bound T_3, we first split it in the summation of local terms. Then, we use the r-consistency and stability property of a_h, and the continuity property of a and a_h, and we obtain

$$\begin{aligned}
T_3 &= \sum_{K\in\Omega_h}\Big(a_h^K(u_h,\psi_I) - a^K(u_h,\psi_I)\Big)\\
&= \sum_{K\in\Omega_h}\Big(a_h^K(u_h-u_\pi,\psi_I) - a^K(u_h-u_\pi,\psi_I)\Big)\\
&= \sum_{K\in\Omega_h}\Big(a_h^K(u_h-u_\pi,\psi_I-\psi_\pi) - a^K(u_h-u_\pi,\psi_I-\psi_\pi)\Big)\\
&\le C\,\|u_h-u_\pi\|_V\ \|\psi_I-\psi_\pi\|_V\,.
\end{aligned} \tag{44}$$

Adding and subtracting u and using the estimate in the energy norm (38) and the estimate for the polynomial interpolation (36), we find that

$$\|u_h-u_\pi\|_V \le \|u_h-u\|_V + \|u-u_\pi\|_V \le Ch^{r-(p-1)}. \tag{45}$$

Adding and subtracting ψ, and, then, using estimates (35) and (36) we find that

$$\|\psi_I-\psi_\pi\|_V \le \|\psi_I-\psi\|_V + \|\psi-\psi_\pi\|_V \le Ch^{\min(\beta,r+1)-p}\,\|\psi\|_\beta\,. \tag{46}$$

The bound on T_3 following by using (45) and (46) in (44):

$$T_3 \le Ch^{(r-(p-1))+\min(\beta,r+1)-p}\,\|\psi\|_\beta\,. \tag{47}$$

The assertion of the lemma follows by subtituting (42), (43), and (47), in (40). □

In view of this lemma, we can readily state and prove the convergence theorems for the four possible combinations of even and odd p and even and odd norm indices.

Theorem 4.4 (Even p, even norms) *Let $f \in H^{r-p+1}(\Omega)$ and let u be the solution of the variational problem* (4) *with $p = 2\ell$ and u_h be the solution of the Virtual Element method* (7)*. Then, there exists a positive constant C independent of h such that*

$$\left|u - u_h\right|_{2i} \leq Ch^{r+1-2i}\Big(\left|u\right|_{r+p+1} + \left|f\right|_{r-(p-1)} \Big), \tag{48}$$

for every integer $i = 0, \dots, \ell - 1$.

Proof. For $i = 0, \dots, \ell-1$, let $\psi \in H^{2(p-i)}(\Omega) \cap H_0^{p-i}(\Omega)$ be the solution of the problem

$$\Delta^{p-i}\psi = \Delta^i(u - u_h) \in L^2(\Omega), \tag{49}$$

with the stability property

$$\left\|\psi\right\|_{2(p-i)} \leq C\left|u - u_h\right|_{2i}. \tag{50}$$

We use (49) and integrate by parts to obtain:

$$\begin{aligned}\left|u - u_h\right|_{2i}^2 &= \left\|\Delta^i(u - u_h)\right\|_0^2 = \big(\Delta^i(u - u_h),\, \Delta^i(u - u_h)\big) = \big(\Delta^{p-i}\psi,\, \Delta^i(u - u_h)\big) \\ &= \big(\Delta^\ell\psi,\, \Delta^\ell(u - u_h)\big) = a(\psi, u - u_h)\end{aligned}$$

where we employed the fact that $\left|v\right|_{2i} = \left\|\Delta^i v\right\|_0$ for any $v \in H_0^p(\Omega)$. The assertion of the theorem follows from an application of Lemma 4.3 (use $\beta = 2(p-i)$ together with $r \geq 2p-1$) and the stability property (50). □

Theorem 4.5 (Even p, odd norms) *Let $f \in H^{r-p+1}(\Omega)$ and let u be the solution of the variational problem* (4) *with $p = 2\ell$ and u_h be the solution of the Virtual Element method* (7)*. Then, there exists a positive constant C independent of h such that*

$$\left|u - u_h\right|_{2i+1} \leq Ch^{(r+1)-(2i+1)}\Big(\left|u\right|_{r+p+1} + \left|f\right|_{r-(p-1)} \Big), \tag{51}$$

for every integer $i = 0, \dots, \ell - 1$.

Proof. For $i = 0, \dots, \ell-1$, let $\psi \in H^{2(p-i)-1}(\Omega) \cap H_0^{p-i}(\Omega)$ be the solution of the problem:

$$-\Delta^{p-i}\psi = \Delta^{i+1}(u - u_h) \in H^{-1}(\Omega), \tag{52}$$

with the stability property

$$\left\|\psi\right\|_{2(p-i)-1} \leq C\left|u - u_h\right|_{2i+1}. \tag{53}$$

We use (52) and integrate by parts to obtain:

$$\begin{aligned}\left|u - u_h\right|_{2i+1}^2 &= \left\|\nabla\Delta^i(u - u_h)\right\|_0^2 = \big(\nabla\Delta^i(u - u_h),\, \nabla\Delta^i(u - u_h)\big) \\ &= \big(\nabla\Delta^{i+1}(u - u_h),\, \nabla\Delta^i(u - u_h)\big) = \big(\Delta^{p-i}\psi,\, \Delta^i(u - u_h)\big) = \big(\Delta^\ell\psi,\, \Delta^\ell(u - u_h)\big) \\ &= a(\psi, u - u_h)\end{aligned}$$

where we employed the fact that $\left|v\right|_{2i+1} = \left\|\nabla\Delta^i v\right\|_0$ for any $v \in H_0^p(\Omega)$.

The assertion of the theorem follows from an application of Lemma 4.3 (use $\beta = 2(p-i)$ together with $r \geq 2p-1$) and the stability property (53). □

Theorem 4.6 (Odd p, even norms) *Let u be the solution of the variational problem* (4) *and let u_h be the solution of the Virtual Element method* (7)*. Then, there exists a positive constant C independent of h such that*

$$\left|u - u_h\right|_{2i} \leq Ch^{(r+1)-2i}\Big(\left|u\right|_{r+p+1} + \left|f\right|_{r-(p-1)} \Big), \tag{54}$$

for every integer $i = 0, \dots, \ell - 1$.

Proof. For $i = 0, \ldots, \ell$, let $\psi \in H^{2(p-i)}(\Omega) \cap H_0^{p-i}(\Omega)$ be the solution of the problem

$$-\Delta^{p-i}\psi = \Delta^i(u-u_h) \in L^2(\Omega) \tag{55}$$

with the stability property

$$\|\psi\|_{2(p-i)} \le C\,|u-u_h|_{2i}\,. \tag{56}$$

We use (58) and integrate by parts to obtain:

$$\begin{aligned} |u-u_h|_{2i}^2 &= \big\|\Delta^i(u-u_h)\big\|_0^2 = \big(\Delta^i(u-u_h),\, \Delta^i(u-u_h)\big) \\ &= \big(-\Delta^{p-i}\psi,\, \Delta^i(u-u_h)\big) = \big(-\Delta^{\ell+1}\psi,\, \Delta^\ell(u-u_h)\big) \\ &= \big(\nabla\Delta^\ell\psi,\, \nabla\Delta^\ell(u-u_h)\big) = a(\psi, u-u_h). \end{aligned}$$

The assertion of the theorem follows from an application of Lemma 4.3 (use $\beta = 2(p-i)$ together with $r \ge 2p-1$) and the stability property (56). □

Theorem 4.7 (Odd p, odd norms) *Let u be the solution of the variational problem* (4) *and let u_h be the solution of the Virtual Element method* (7)*. Then, there exists a positive constant C independent of h such that*

$$|u-u_h|_{2i+1} \le Ch^{(r+1)-(2i+1)}\Big(\,|u|_{r+p+1} + |f|_{r-(p-1)}\Big), \tag{57}$$

for every integer $i = 0, \ldots, \ell-1$.

Proof. For $i = 0, \ldots, \ell$, let $\psi \in H^{2(p-i)-1}(\Omega) \cap H_0^p(\Omega)$ be the solution of the problem:

$$-\Delta^{p-i}\psi = \Delta^{i+1}(u-u_h) \in H^{-1}(\Omega), \tag{58}$$

with the stability property

$$\|\psi\|_{2(p-i)-1} \le C\,|u-u_h|_{2i+1}\,. \tag{59}$$

We use (58) and integrate by parts to obtain:

$$\begin{aligned} |u-u_h|_{2i+1}^2 &= \big\|\nabla\Delta^i(u-u_h)\big\|_0^2 = \big(\nabla\Delta^i(u-u_h),\, \nabla\Delta^i(u-u_h)\big) \\ &= \big(\nabla\Delta^{i+1}(u-u_h),\, \nabla\Delta^i(u-u_h)\big) = \big(-\Delta^{p-i}\psi,\, \Delta^i(u-u_h)\big) = \big(-\Delta^{\ell+1}\psi,\, \Delta^\ell(u-u_h)\big) \\ &= \big(\nabla\Delta^\ell\psi,\, \nabla\Delta^\ell(u-u_h)\big) = a(\psi, u-u_h) \end{aligned}$$

where again we employed the fact that $|v|_{2i+1} = \big\|\nabla\Delta^i v\big\|_0$ for any $v \in H_0^p(\Omega)$. The assertion of the theorem follows from an application of Lemma 4.3 (use $\beta = 2(p-i)-1$ together with $r \ge 2p-1$) and the stability property (59). □

5. Conclusions

In this paper, we developed the conforming Virtual Element discretization of arbitrary order for polyharmonic problems, which requires the discretization of operator like $\Delta^p u$ for integer $p \ge 1$. To this end, we introduced local and global Virtual Element approximation spaces together with suitable discrete bilinear forms for odd and even p. The convergence of the method has been proved and optimal error estimates derived in suitable norms. The numerical implementation of the current method deserves a careful study because of the severe ill-conditioning of the polyharmonic differential operator and the need of high order polynomials, whose degree should be at least 5 in the simplest case $p = 3$. For this reason, the implementation of this conforming VEM is under investigation and will be addressed in a forthcoming publication.

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