# Corrigendum to "Solvability and uniqueness criteria for generalized Sylvester-type equations"*

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**Abstract**

We provide an amended version of Corollaries 7 and 9 in [De Terán, Iannazzo, Poloni, Robol, "Solvability and uniqueness criteria for generalized Sylvester-type equations"]. These results characterize the unique solvability of the matrix equation $AXB + CX^{\star}D = E$ (where the coefficients need not be square) in terms of an equivalent condition on the spectrum of certain matrix pencils of the same size as one of its coefficients.

**Keywords.** Sylvester equation, eigenvalues, matrix pencil, matrix equation

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## 1 Setting

We consider the *generalized ⋆-Sylvester equation*

$$AXB + CX^{\star}D = E \tag{1}$$

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for the unknown $X \in \mathbb{C}^{m\times n}$, with $\star$ being either the transpose ($\top$) or the conjugate transpose ($*$), and $A, B, C, D, E$ being matrices with appropriate sizes. We are interested in the most general situation, where both the coefficients and the unknown are allowed to be rectangular. The most general setting in which all the matrix products in (1) make sense is $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$.

We recall a few definitions from [3] which are necessary to state and prove our results.

Throughout the paper we denote by $I$ the identity matrix of appropriate size. A matrix pencil $\mathcal{P}(\lambda) = \lambda M + N$ is said to be *singular* if either $\mathcal{P}(\lambda)$ is rectangular or $p(\lambda) := \det\big(\mathcal{P}(\lambda)\big)$ is identically zero. If $\mathcal{P}(\lambda)$ is not singular, and so $M, N$ are $n \times n$ matrices, then it is said to be *regular* and the set of roots of $p(\lambda)$, complemented with $\infty$ if the degree of $p(\lambda)$ is less than $n$, is the *spectrum* of $\mathcal{P}$, denoted by $\Lambda(\mathcal{P})$. With $m_\lambda(\mathcal{P})$ we denote the *algebraic multiplicity* of the eigenvalue $\lambda$ in $\mathcal{P}$, namely, the multiplicity of $\lambda$ as a root of $p(\lambda)$, if $\lambda \in \mathbb{C}$, or $n - \deg p(\lambda)$ if $\lambda = \infty$. If $M$ is a square matrix, by $\Lambda(M)$ and $m_\lambda(M)$ we denote, respectively, the spectrum of $M$ and the algebraic multiplicity of $\lambda$ as an eigenvalue of $M$.

We shall deal with certain matrices and matrix pencils that always have $|m-n|$ zero or infinite eigenvalues which are *dimension-induced*, that is, they are present simply because of the sizes of the coefficient matrices they are constructed from (see [6]). Hence we define a variant of the spectrum in which these eigenvalues are omitted:

$$\widehat{\Lambda}(\mathcal{P}) := \begin{cases} \Lambda(\mathcal{P}), & \text{if } m_\infty(\mathcal{P}) > |m-n|, \\ \Lambda(\mathcal{P}) \setminus \{\infty\}, & \text{if } m_\infty(\mathcal{P}) = |m-n|, \end{cases}$$

$$\widetilde{\Lambda}(\mathcal{P}) := \begin{cases} \Lambda(\mathcal{P}), & \text{if } m_0(\mathcal{P}) > |m-n|, \\ \Lambda(\mathcal{P}) \setminus \{0\}, & \text{if } m_0(\mathcal{P}) = |m-n|. \end{cases}$$

Following [6], we refer to the eigenvalues in either $\widehat{\Lambda}(\mathcal{P})$ or $\widetilde{\Lambda}(\mathcal{P})$ as *core eigenvalues*. If $M$ is a square matrix, we use the notation $\widetilde{\Lambda}(M)$ to denote $\widetilde{\Lambda}(\lambda I - M)$.

**Definition 1.** (Reciprocal free and $*$-reciprocal free set) [1,5]. Let $\mathcal{S}$ be a subset of $\mathbb{C} \cup \{\infty\}$. We say that $\mathcal{S}$ is

(a) reciprocal free if $\lambda \neq \mu^{-1}$, for all $\lambda, \mu \in \mathcal{S}$;

(b) $*$-reciprocal free if $\lambda \neq (\overline{\mu})^{-1}$, for all $\lambda, \mu \in \mathcal{S}$.

This definition includes the values $\lambda = 0, \infty$, with the customary assumption $\lambda^{-1} = (\overline{\lambda})^{-1} = \infty, 0$, respectively.

The *reversal pencil* of the matrix pencil $\mathcal{P}(\lambda) = \lambda M + N$ is the pencil $\operatorname{rev}\mathcal{P}(\lambda) := \lambda N + M$. The pencil $\mathcal{P}(\lambda)$ has an infinite eigenvalue if and only if $\operatorname{rev}\mathcal{P}(\lambda)$ has the zero eigenvalue. The multiplicity of the infinite eigenvalue in $\mathcal{P}(\lambda)$ is the multiplicity of the zero eigenvalue in $\operatorname{rev}\mathcal{P}(\lambda)$, thus

$$\widetilde{\Lambda}(\operatorname{rev}\mathcal{P}) = \left\{\lambda^{-1} \mid \lambda \in \widehat{\Lambda}(\mathcal{P})\right\}. \tag{2}$$

We recall now the main result from [3].

**Theorem 2.** *Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{m\times q}$ and set*

$$\mathcal{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}. \tag{3}$$

*The equation*

$$AXB + CX^\star D = E$$

*has a unique solution, for any right-hand side $E$, if and only if $\mathcal{Q}(\lambda)$ is regular and one of the following situations holds:*

(i) *$p = m \neq n = q$, either $m < n$ and $A$ is invertible or $m > n$ and $B$ is invertible, and*

- *If $\star = \top$, $\widehat{\Lambda}(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\widehat{\Lambda}(\mathcal{Q})$ is $*$-reciprocal free.*

(ii) *$p = n \neq m = q$, either $m > n$ and $C$ is invertible or $m < n$ and $D$ is invertible, and*

- *If $\star = \top$, $\widetilde{\Lambda}(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\widetilde{\Lambda}(\mathcal{Q})$ is $*$-reciprocal free.*

(iii) *$p = m = n = q$, and*

- *If $\star = \top$, $\Lambda(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{Q})$ is $*$-reciprocal free.*

# 2 Amended corollaries

In [3], we provided several corollaries that convert the conditions in Theorem 2 into conditions on pencils and matrices of smaller size. Unfortunately, some issues with the counting of dimension-induced eigenvalues were brought to our attention after the publication of that paper.

The following amended version of Corollary 7 has the exact same statement, but with the symbols $\Lambda$ replaced by $\widehat{\Lambda}$.

**Corollary 3** (Corollary 7 in [3], amended version)**.** *Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{m\times q}$. Then the equation $AXB + CX^\star D = E$ has a unique solution, for any right-hand side $E$, if and only if one of the following situations holds:*

(a) *$p = m \leqslant n = q$, $A$ is invertible, the pencil $\mathcal{P}_1(\lambda) := B^\star - \lambda D^\star A^{-1} C$ is regular and*

- *If $\star = \top$, $\widehat{\Lambda}(\mathcal{P}_1) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}_1) \leqslant 1$.*

– *If* $\star = *$, $\widehat{\Lambda}(\mathcal{P}_1)$ *is* $*$*-reciprocal free.*

(b) $p = m \geqslant n = q$, *$B$ is invertible, the pencil* $\mathcal{P}_2(\lambda) := A^\star - \lambda D B^{-1} C^\star$ *is regular and*

– *If* $\star = \top$, $\widehat{\Lambda}(\mathcal{P}_2) \setminus \{1\}$ *is reciprocal free and* $m_1(\mathcal{P}_2) \leqslant 1$.

– *If* $\star = *$, $\widehat{\Lambda}(\mathcal{P}_2)$ *is* $*$*-reciprocal free.*

(c) $p = n \leqslant m = q$, *$C$ is invertible, the pencil* $\mathcal{P}_3(\lambda) := D^\star - \lambda B^\star C^{-1} A$ *is regular and*

– *If* $\star = \top$, $\widehat{\Lambda}(\mathcal{P}_3) \setminus \{1\}$ *is reciprocal free and* $m_1(\mathcal{P}_3) \leqslant 1$.

– *If* $\star = *$, $\widehat{\Lambda}(\mathcal{P}_3)$ *is* $*$*-reciprocal free.*

(d) $p = n \geqslant m = q$, *$D$ is invertible, the pencil* $\mathcal{P}_4(\lambda) := C^\star - \lambda B D^{-1} A^\star$ *is regular and*

– *If* $\star = \top$, $\widehat{\Lambda}(\mathcal{P}_4) \setminus \{1\}$ *is reciprocal free and* $m_1(\mathcal{P}_4) \leqslant 1$.

– *If* $\star = *$, $\widehat{\Lambda}(\mathcal{P}_4)$ *is* $*$*-reciprocal free.*

*Proof.* Let us assume first that (1) has a unique solution, for any right-hand side $E$. Then Theorem 2 implies that at least one of the following situations holds: (C1) $p = m < n = q$ and $A$ is invertible, (C2) $p = m > n = q$ and $B$ is invertible, (C3) $p = n < m = q$ and $C$ is invertible, (C4) $p = n > m = q$ and $D$ is invertible, or (C5) $p = m = n = q$. Let us first assume that case (C1) holds. We can perform the following unimodular equivalence on $\mathcal{Q}(\lambda)$:

$$\begin{bmatrix} I & -\lambda D^\star A^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} = \begin{bmatrix} 0 & B^\star - \lambda^2 D^\star A^{-1} C \\ A & \lambda C \end{bmatrix}. \tag{4}$$

Taking determinants in (4) we arrive at

$$\det(\mathcal{Q}(\lambda)) = \pm \det(A) \det(\mathcal{P}_1(\lambda^2)). \tag{5}$$

This shows that $\mathcal{P}_1$ is regular. Note that $D^\star A^{-1} C$ has rank at most $m < n$, hence $\det(\mathcal{P}_1(\lambda))$ has degree at most $m$ and $|n-m|$ dimension-induced infinite eigenvalues are present in $\Lambda(\mathcal{P}_1)$. Similarly, $\mathcal{Q}(\lambda)$ has $|n-m|$ dimension-induced infinite eigenvalues. The left- and right-hand sides of Equation (5) are nonzero polynomials in $\lambda$ with degree at most $2m$; therefore we have $\widehat{\Lambda}(\mathcal{Q}) = \sqrt{\widehat{\Lambda}(\mathcal{P}_1)} := \{\mu : \ \mu^2 \in \widehat{\Lambda}(\mathcal{P}_1)\}$, including multiplicities and core infinite eigenvalues. Then Theorem 2 implies that part (a) in the statement holds.

If case (C2) holds, then we apply the $\star$ operator in (1) and apply the previous arguments to the new equation and its corresponding pencil.

If case (C3) holds, then after introducing the change of variables $Y = X^\star$, the roles of $A, B$ and $C, D$ are exchanged, so we apply the same arguments as in case (C1) to the corresponding pencil, $\mathcal{P}_3(\lambda)$.

In case (C4), we apply the $\star$ operator in (1) and introduce the change of variables $Y = X^\star$. Then we apply the same arguments as for case (C1) to the pencil corresponding to this new equation.

Finally, if we are in case (C5), Corollary 12 of [2] guarantees that at least one of $A, B, C, D$ is invertible and thus at least one of (a)–(d) in the statement holds, and we are done.

To prove the converse, let us assume that any of (a)–(d) in the statement holds. Then, reversing the previous arguments, we can conclude that at least one of the situations (i)–(iii) in the statement of Theorem 2 occurs, and Theorem 2 implies that (1) has a unique solution, for any right-hand side. □

The statement and proof of Corollary 8 in [3] are true without need for corrections. The statement of Corollary 9 still holds, but its proof needs a correction.

**Corollary 4** (Corollary 9 in [3])**.** *Let $A, B \in \mathbb{C}^{n\times m}$. Then the equation $AXB + X^\star = E$ has a unique solution, for any right-hand side $E$, if and only if the following conditions hold:*

- *If $\star = \top$, $\Lambda(AB^\top) \setminus \{1\}$ is reciprocal free and $m_1(AB^\top) \leqslant 1$.*
- *If $\star = *$, $\Lambda(AB^*)$ is $*$-reciprocal free.*

*Proof.* It is sufficient to observe that the condition in Corollary 3 (taking $C = I$, $D = I$) is equivalent to the condition stated on the spectrum of $AB^\star$, for each of the cases in Corollary 3. If $m > n$, we are in case (c), with $\mathcal{P}_3 = I - \lambda B^\star A$. The eigenvalues of $\mathcal{P}_3$ are the reciprocals of the eigenvalues of $B^\star A$. Note that $B^\star A$ has $m - n$ dimension-induced zero eigenvalues and $\widetilde{\Lambda}(B^\star A) = \Lambda(AB^\star)$ (this equality follows from [4, Theorem 1.3.20]). Hence the set $\Lambda(AB^\star)$ is the reciprocal of $\widehat{\Lambda}(\mathcal{P}_3)$, so one of the two is ($*$-)reciprocal-free if and only if the other is, while the multiplicity of 1 is the same in both spectra.

Similarly, if $m < n$, we can take $\mathcal{P}_4(\lambda) = I - \lambda BA^\star$; then $\widehat{\Lambda}(\mathcal{P}_4)$ is the reciprocal of $\widetilde{\Lambda}(BA^\star)$, or, applying the $\star$ operator, the ($*$-)reciprocal of $\widetilde{\Lambda}(AB^\star)$. Since a matrix never has $\infty$ as an eigenvalue, $\widetilde{\Lambda}(AB^\star)$ is a ($*$-)reciprocal-free set if and only if $\Lambda(AB^\star)$ is so, regardless of the additional zero eigenvalues.

The cases with $m = n$ can be proved in a similar way. □

A version of [3] which incorporates these corrections is available at `https://arxiv.org/abs/1608.01183`.

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