EQUIVALENT DISSIPATION POSTULATES IN CLASSICAL PLASTICITY

Internal Report C88-51

November 1988

M. Lucchesi
P. Podio Guidugli

EQUIVALENT DISSIPATION POSTULATES IN CLASSICAL PLASTICITY

by

M. Lucchesi and **P. Podio-Guidugli**

Istituto CNUCE
Consiglio Nazionale delle Ricerche
Via S.Maria, 34 - 56100 Pisa

Dip. di Ingegneria Civile Edile
Università di Roma 2
Via O.Raimondo, 00173 Roma

1. Introduction

In the classical theory of plasticity, the notions of *yield surface* and *plastic stretching* play a crucial role. The yield surface is the boundary of the region in stress space of all stresses which are attainable from the current one without any further plastic deformation; it characterizes the transitions from a purely elastic to an elastic-plastic regime. On the other hand, knowing the plastic stretching, the total stretching and the elastic response law of the material, one can calculate the stress at each instant during each admissible processes.

The two notions are tightly connected by stating some sort of *dissipation postulate*, on the form of which there has been recurrent speculation of plasticity theorists (*vid.* the discussion of Lubliner in [1], where many of the relevant references are listed). For example, a well-known dissipation condition involving the additional working of internal action has been suggested by Drucker in [2]; Drucker's Postulate implies convexity of the yield surface, as well as the so-called associated flow rule, according to which the plastic stretching agrees with the outward normal to the yield surface evaluated at the current stress. Another condition having similar consequences, this one restricting the sign of the complementary working, has been proposed for study by Martin [3]. Moreover, we have showed in [6] that properties of the yield surface and the plastic stretching completely analogous to the classical ones can be obtained by a version of Il'yushin's Postulate [4] appropriate to the theory of materials with elastic range undergoing large deformations developped in [5] [6] [7].

In this paper we prove that, within the linear theory of isotropic materials with elastic range [7], the postulates of Drucker, Martin and Il'yushin are indeed equivalent; this result, whose formal proof is rather delicate, had been previously stated by us in [8].

2. The Linear Theory of Isotropic Materials with Elastic Range

In this section we present in abridged form the elements of the title theory, in order to make our paper reasonably self-contained. A general theory of materials with elastic range undergoing large deformations has been formulated in [5] and [6]; in [7], among other things, that general theory has been linearized by choosing the norm of both the displacement gradient from a fixed reference placement and its time derivative to be small, and the resulting theory has been shown to have all key features of classical infinitesimal plasticity. Once and for all we refer the reader to our papers just quoted for a precise and detailed exposition of those concepts and results that we here only skim.

2.1 Histories

We begin by introducing the basic notion of a *history*, namely, a continuous and piecewise continuously differentiable mapping

$$\hat{E} : [0,1] \to \mathrm{Sym}, \quad E = \hat{E}(\tau) , \tag{2.1}$$

from the closed unit interval of the real line into the space of second-order symmetric tensors. The value $\hat{E}(\tau)$ of $\hat{E}$ at "instant" τ is interpreted as the infinitesimal strain tensor at a fixed material point; by $\dot{E}$ we denote the time derivative of $\hat{E}$, when it exists, or else the derivative from the right at an instant τ where $\dot{E}$ has a discontinuity.

The set $\mathfrak{H}$ of all histories is closed with respect to certain operations of section and continuation: for $\hat{E} \in \mathfrak{H}$ and $\tau \in [0,1]$, the *τ-section* of $\hat{E}$ is the history $\hat{E}_\tau$ such that

$$\tau' \mapsto \hat{E}_\tau(\tau') = \hat{E}(\tau\tau') \ ; \tag{2.2}$$

a *continuation* $\hat{G}$ of $\hat{E}$ is any history $\hat{G}$ such that $\hat{G}_\tau = \hat{E}$ for some $\tau \in]0,1]$, and a *continuation* of $\hat{E}$ *up to* a given $A \in \mathrm{Sym}$ is a continuation of $\hat{E}$ whose final value $\hat{E}(1)$ is A. For the purpose of this presentation, it is expedient to assume that all histories begin at the origin 0 of Sym:

$$\hat{E}(0) = 0 \quad \text{for all } \hat{E} \in \mathfrak{H} \ ; \tag{2.3}$$

this assumption reflects those real life situations when experiments are made only on identically prepared specimina.

2.2 The Constitutive Functional

We model the mechanical response of isotropic elastic-plastic solids by means of a *constitutive functional*

$$\tilde{T} : \mathfrak{H} \to \mathrm{Sym}, \qquad T = \tilde{T}(\hat{E}) , \tag{2.4}$$

whose value $\tilde{T}(\hat{E})$ assigns the Cauchy stress at the end of history $\hat{E}$. $\tilde{T}$ is assumed to be *frame-indifferent* and invariant under time-rescalings (briefly, *rate-independent*). As a consequence of rate-independence, $\tilde{T}(\hat{E}_\tau)$, the stress corresponding to the τ-section of $\hat{E}$, can be interpreted as the stress obtained at instant τ during the history $\hat{E}$ itself; we stress this interpretation by writing $\hat{T}_E(\tau)$ in place of $\tilde{T}(\hat{E}_\tau)$, and notice that

$$\hat{T}_E(\tau) = \hat{T}_{E_\tau}(1) . \qquad (2.5)$$

The type of mechanical response we envisage is further specified by introducing the notions of elastic range and unloaded history (see [6], Sections 3 and 5, respectively).

2.3 The Elastic Range

The *elastic range* corresponding to the history $\hat{E} \in \mathfrak{D}$ is a set $E(\hat{E}) \subset \mathrm{Sym}$, the closure of an arcwise connected open set containing $\hat{E}(1)$, whose boundary is attainable from interior points only, and whose points are interpreted as the symmetric gradients of all deformations from the reference configuration to configurations which are elastically accessible from the current configuration. A history $\hat{G}$ is called an *elastic continuation* of $\hat{E}$ if there exists $\tau \in [0,1]$ such that both $\hat{G}_\tau = \hat{E}$ and $\hat{G}(\tau') \in E(\hat{E})$ for all $\tau' \in [\tau, 1]$; we let $\mathcal{C}(\hat{E}, E(\hat{E}))$ denote the subset of $\mathfrak{D}$ consisting of all elastic continuations of $\hat{E}$, and assume that the constitutive functional $\tilde{T}$ is *path-independent* when restricted to $\mathcal{C}(\hat{E}, E(\hat{E}))$, in the sense that $\tilde{T}(\hat{G}) = \tilde{T}(\hat{H})$ for all $\hat{H}, \hat{G} \in \mathcal{C}(\hat{E}, E(\hat{E}))$ such that $\hat{G}(1) = \hat{H}(1)$. Two crucial properties of the material class we wish to study are that, for all $\hat{E} \in \mathfrak{D}$,

(i) the elastic range $E(\hat{E})$ is *invariant under elastic continuation, i.e.*, $E(\hat{E}) = E(\hat{G})$ for all $\hat{G} \in \mathcal{C}(\hat{E}, E(\hat{E}))$;

(ii) the elastic range *evolves smoothly, i.e.*, for each $\tau \in [0,1[$ and for each A belonging to the interior part $E^\circ(\hat{E}_\tau)$ of $E(\hat{E}_\tau)$ there exists $\delta > 0$ such that

$$A \in E^\circ(\hat{E}_{\tau'}) \quad \text{per } \tau' \in [\tau, \tau + \delta] .$$

2.4 Unloaded Histories

For a given $\hat{E} \in \mathfrak{D}$, a history $\hat{E}^P$ is called an *unloaded history* corresponding to $\hat{E}$ if, for all $\tau \in [0,1]$, both $\hat{E}^P(\tau) \in E(\hat{E}_\tau)$ and, for $\hat{G}$ any elastic continuation of $\hat{E}_\tau$ up to $\hat{E}^P(\tau)$, $\tilde{T}(\hat{G}) = 0$. The notion of unloaded history is a mathematical formulation of the idea that there should be a stress-free configuration attainable *via* elastic continuation from the current configuration; in particular, the initial configuration is stress-free. For all $\hat{E} \in \mathfrak{D}$, we assume here that there is exactly one unloaded history $\hat{E}^P$ corresponding to $\hat{E}$, and that there is *no plastic change of volume, i.e.*, $\hat{E}^P$ is such that

$$\mathrm{tr}\,\hat{E}^P(\tau) = 0 \quad \text{for all} \quad \tau \in [0,1] . \qquad (2.6)$$

For $\hat{E} \in \mathfrak{D}$ and for all $\tau \in [0,1]$ it is clear that

$$\hat{E}^p(\tau) = \hat{E}^p_\tau(1) \,. \tag{2.7}$$

Moreover, for $\hat{E} \in \mathfrak{D}$ and $\hat{G}$ an elastic continuation of $\hat{E}$ such that $\hat{G}_\tau = \hat{E}$, it is not difficult to show that

$$\hat{G}^p(\tau') = \hat{E}^p(1) \qquad \text{for all } \tau' \in [\tau, 1] \,, \tag{2.8}$$

so that

$$\dot{G}^p(\tau') = 0 \qquad \text{for all } \tau' \in [\tau, 1] \,. \tag{2.9}$$

We now state two further properties of histories which are in general left tacit. These are:

(i) (*existence of continuations implying plastic stretching*) for each $\hat{E} \in \mathfrak{D}$ such that $\hat{E}(1) \in \partial E(\hat{E})$, there is a continuation $\hat{G}$ such that

$$\hat{G}_\tau = \hat{E} \quad \text{and} \quad \dot{G}^p(\tau) \neq 0 \,; \tag{2.10}$$

(ii) (*insensitivity of plastic stretching to elastic cycles*) let $\hat{E}, \hat{G} \in \mathfrak{D}$ be a pair of histories such that, for some positive numbers τ, $\overline{\tau}$ and δ with $0 < \tau < \overline{\tau} < \overline{\tau} + \delta \leq 1$,

(1) $\hat{E}_\tau = \hat{G}_\tau$ and $\hat{E}(\tau) \in \partial E(\hat{E}_\tau)$;

(2) for all $\tau' \in [\tau, \overline{\tau}]$, $\hat{G}(\tau') \in E(\hat{E}_\tau)$, with $\hat{G}(\overline{\tau}) = \hat{G}(\tau)$;

(3) for all $\theta \in [0, \delta]$, $\hat{E}(\tau + \theta) = \hat{G}(\overline{\tau} + \theta)$ (*vid.* Fig. 1);

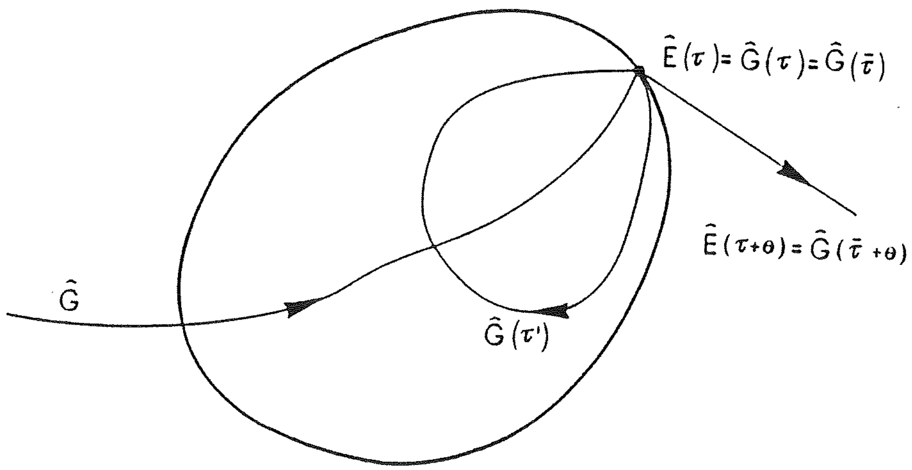

Fig. 1

then,

$$\dot{E}^{p}(\tau) = \dot{G}^{p}(\overline{\tau}) \ . \qquad (2.11)$$

Property (i) guarantees that each history attaining the boundary of its elastic range has a continuation with non-vanishing instantaneous plastic stretching; property (ii), on the other hand, << formalizes the requirement that plastic stretching should not be affected by any prior deformation cycle remaining within the elastic range >> (*cf.* [6], Axioms 8 and 9, respectively).

2.5 Stress Range and Yield Surface

Let a history $\hat{E}$ and its corresponding unloaded history $\hat{E}^{p}$ be chosen in $\mathfrak{D}$. The classic assumption that the stress response to a purely elastic strain from the unstressed configuration reached after unloading at the current instant τ is both unaffected by the past deformation process and completely determined by $\hat{E}(\tau)$ and $\hat{E}^{p}(\tau)$, together with (2.6) and the fact that we here deal exclusively with the linear theory of isotropic materials, yields the following representation formula for the linear mapping $\mathbb{T}$ delivering the stress $\hat{T}_{E}(\tau)$:

$$\hat{T}_{E}(\tau) = \mathbb{T}[\hat{E}(\tau) - \hat{E}^{p}(\tau)] = 2\mu\,(\hat{E}(\tau) - \hat{E}^{p}(\tau)) + \lambda(\mathrm{tr}\,\hat{E}(\tau))\mathrm{I} \ , \qquad (2.12)$$

where the constitutive moduli λ and μ are supposed to satisfy the usual *a priori* positivity restrictions

$$3\lambda + 2\mu > 0 \ , \quad \mu > 0 \ . \qquad (2.13)$$

Consequently, $\mathbb{T}$ is invertible, and we have

$$\hat{E}(\tau) = \mathbb{T}^{-1}[\hat{T}_{E}(\tau)] + \hat{E}^{p}(\tau) = $$
$$= (1/2\mu)\ [\hat{T}_{E}(\tau) + (\lambda/(2\mu+3\lambda))(\mathrm{tr}\,\hat{T}_{E}(\tau))\,\mathrm{I}] + \hat{E}^{p}(\tau) \ . \qquad (2.14)$$

For each $\hat{E} \in \mathfrak{D}$ and $\tau \in [0,1]$, let

$$\mathcal{T}(\hat{E}_{\tau}) := \{\mathrm{T} \in \mathrm{Sym} \mid \mathrm{T} = \mathbb{T}[\mathrm{E} - \hat{E}^{p}(\tau)],\ \mathrm{E} \in \mathcal{E}(\hat{E}_{\tau})\} \qquad (2.15)$$

be the collection of all stresses T elastically attainable starting from the current stress $\hat{T}_{E}(\tau)$. We call $\mathcal{T}(\hat{E}_{\tau})$ and its boundary $\partial\mathcal{T}(\hat{E}_{\tau})$ the *stress range* and the *yield surface*, respectively, associated to history $\hat{E}$ and instant τ. Due to the invertibility of $\mathbb{T}$, the elastic range $\mathcal{E}(\hat{E}_{\tau})$ and the stress range $\mathcal{T}(\hat{E}_{\tau})$ are in one-to-one pointwise correspondence; moreover,

$$\hat{E}(\tau) \in \partial\mathcal{E}(\hat{E}_{\tau}) \quad \Leftrightarrow \quad \hat{T}_{E}(\tau) \in \partial\mathcal{T}(\hat{E}_{\tau}) \ . \qquad (2.16)$$

3. Working Integrals, Potentials and Cyclic Powers

For $\hat{E} \in \mathfrak{D}$ and $\tau_0, \tau_1 \in [0,1]$ such that $0 \le \tau_0 < \tau_1 \le 1$, the integrals

$$\Lambda\{\hat{E}\} := \int_{\tau_0}^{\tau_1} \hat{T}_E(\tau) \cdot \dot{E}(\tau)\, d\tau \quad \text{and} \quad \Lambda_c\{\hat{E}\} := \int_{\tau_0}^{\tau_1} \dot{T}(\tau) \cdot \hat{E}(\tau)\, d\tau \qquad (3.1)$$

define, respectively, the *working* and the *complementary working* (per unit volume) done by internal actions along history $\hat{E}$ during the time interval $[\tau_0, \tau_1]$. We proceed to show that the working ⟦the complementary working⟧ is, in a sense we make precise, path-independent over the elastic range ⟦the stress range⟧.

For each $\hat{E} \in \mathfrak{D}$ and $E \in \mathcal{E}(\hat{E})$, let $\hat{G} \in \mathfrak{D}$ be any elastic continuation of $\hat{E}$ up to E, with $\hat{G}_\tau = \hat{E}$. We observe that, for all $\tau' \in [\tau, 1]$,

$$\hat{G}^P(\tau') = \hat{E}^P(1)\ , \quad \hat{G}(\tau) = \hat{E}(1)\ , \quad \hat{G}(1) =: E\ ; \qquad (3.2)$$

by (2.12) and (2.14), respectively, we then have that

$$\hat{T}_G(\tau') = \mathbb{T}[\hat{G}(\tau') - \hat{E}^P(1)] \quad , \quad \hat{G}(\tau') = \hat{E}^P(1) + \mathbb{T}^{-1}[\hat{T}_G(\tau')]\ ; \qquad (3.3)$$

in particular,

$$T := \hat{T}_G(1) = \mathbb{T}[E - \hat{E}^P(1)]\ , \quad E = \hat{G}(1) = \hat{E}^P(1) + \mathbb{T}^{-1}[T]\ . \qquad (3.4)$$

We now define the *stress* and the *strain energy* to be, respectively,

$$2\overline{\sigma}(A) := A \cdot \mathbb{T}^{-1}[A] \quad \text{and} \quad 2\underline{\sigma}(A) := A \cdot \mathbb{T}[A]\ . \qquad (3.5)$$

It follows from (3.3)÷(3.5) that the stress and the strain energies are equal over $[\tau, 1]$, in the sense that

$$2\overline{\sigma}(\hat{T}_G(\tau')) = \hat{T}_G(\tau') \cdot \mathbb{T}^{-1}[\hat{T}_G(\tau')] =$$
$$= (\hat{G}(\tau') - \hat{E}^P(1)) \cdot \mathbb{T}[\hat{G}(\tau') - \hat{E}^P(1)] = 2\underline{\sigma}(\hat{G}(\tau') - \hat{E}^P(1))\ . \qquad (3.6)$$

Finally, (3.3) and (3.6) imply that

$$\hat{T}_G(\tau') \cdot \dot{G}(\tau') = \hat{T}_G(\tau') \cdot \mathbb{T}^{-1}[\dot{T}_G(\tau')] = \dot{\overline{\sigma}}(\hat{T}_G(\tau')) = \dot{\underline{\sigma}}(\hat{G}(\tau') - \hat{E}^P(1))$$

and

$$\dot{T}_G(\tau') \cdot \hat{G}(\tau') = (\mathbb{T}[\hat{G}(\tau') - \hat{E}^P(1)])^{\cdot} \cdot (\hat{G}(\tau') - \hat{E}^P(1)) + \dot{T}_G(\tau') \cdot \hat{E}^P(1) =$$
$$= [\underline{\sigma}(\hat{G}(\tau') - \hat{E}^P(1)) + \hat{T}_G(\tau') \cdot \hat{E}^P(1)]^{\cdot} = [\overline{\sigma}(\hat{T}_G(\tau')) + \hat{T}_G(\tau') \cdot \hat{E}^P(1)]^{\cdot}\ .$$

Thus, the working is *path-independent* within the elastic range, and the complementary working within the stress range, in the following sense:

$$\int_{\tau}^{1} \hat{T}_G(\tau')\cdot\dot{G}(\tau')\,d\tau' = \underline{\sigma}(E-\hat{E}^p(1)) - \underline{\sigma}(\hat{E}(1)-\hat{E}^p(1)) \quad , \tag{3.7}$$

and

$$\int_{\tau}^{1} \dot{T}_G(\tau')\cdot\hat{G}(\tau')\,d\tau' = \tilde{\sigma}(T,\hat{E}^p(1)) - \tilde{\sigma}(\hat{T}_E(1),\hat{E}^p(1)) \, , \tag{3.8}$$

where we have set

$$\tilde{\sigma}(A,B) := \overline{\sigma}(A) + A\cdot B \quad . \tag{3.9}$$

For each $\hat{E}\in\mathfrak{D}$, we are then in a position to define the *potential*

$$\varphi(\cdot;\hat{E}) : \mathcal{E}(\hat{E}) \to \mathbb{R} \, , \quad \varphi(E;\hat{E}) := \int_0^1 \hat{T}_G(\tau)\cdot\dot{G}(\tau)\,d\tau \tag{3.10}$$

and the *complementary potential*

$$\varphi_c(\cdot;\hat{E}) : \mathcal{T}(\hat{E}) \to \mathbb{R} \, , \quad \varphi_c(T;\hat{E}) := \int_0^1 \dot{T}_G(\tau)\cdot\hat{G}(\tau)\,d\tau \quad , \tag{3.11}$$

where $\hat{G}$ is an arbitrary elastic continuation of $\hat{E}$ up to $E\in\mathcal{E}(\hat{E})$, and $T=\mathbb{T}[E-\hat{E}^p(1)]$ (*cf.* $(3.4)_{1,2}$). It is easily seen that the potentials φ and φ_c are differentiable over $\mathcal{E}^\circ(\hat{E})$ and $\mathcal{T}^\circ(\hat{E})$, respectively, with derivatives

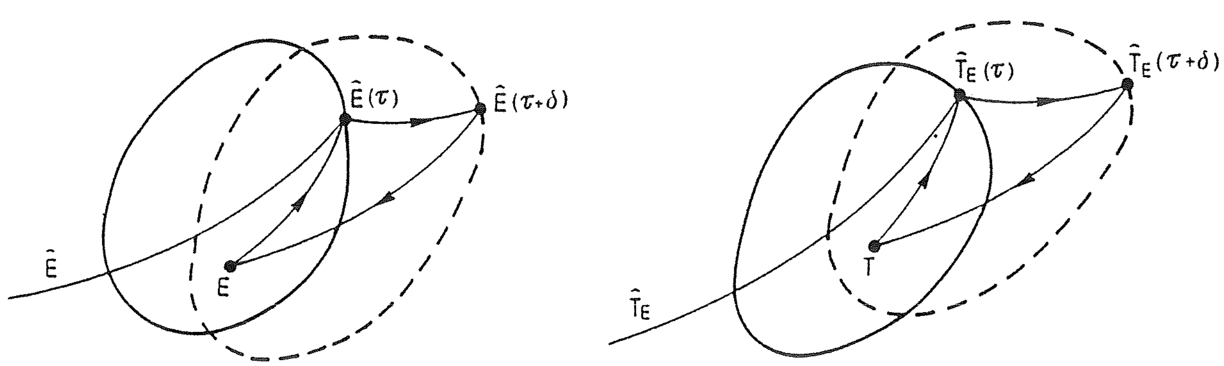

Fig. 2

$$\partial_E \varphi(E:\hat{E}) = T \quad \text{and} \quad \partial_T \varphi_c(T:\hat{E}) = E\ . \tag{3.12}$$

Now, for each $\hat{E} \in \mathfrak{D}$, choose $\delta > 0$, E and T in such a way that

$$E \in E(\hat{E}_{\tau'})\ ,\ T \in T(\hat{E}_{\tau'}) \quad \text{for all } \tau' \in [\tau,\ \tau+\delta]\ ; \tag{3.13}$$

then, $[\varphi(E:\hat{E}_{\tau+\delta}) - \varphi(E:\hat{E}_\tau)]$ and $[\varphi_c(T:\hat{E}_{\tau+\delta}) - \varphi_c(T:\hat{E}_\tau)]$ measure the working and the complementary working during the "cycles" $\{E \to \hat{E}(\tau) \to \hat{E}(\tau+\delta) \to E\}$ and $\{T \to \hat{T}_E(\tau) \to \hat{T}_E(\tau+\delta) \to T\}$, respectively (Fig. 2).

Remark. It is important to realize that the "cycles" considered above are not parts of any history: to see how to interpret them in the case, say, of the complementary working, let $T \in T(\hat{E}_{\tau'})$ for all $\tau' \in [\tau,\ \tau+\delta]$, and let $\underline{\hat{G}}$ be an elastic continuation of $\hat{E}_\tau$ up to $E = \hat{E}^P(\tau) + \mathbb{T}^{-1}[T]$; moreover, for

$$0 < \tau < \tau_0 < \overline{\tau} < \overline{\tau} + \delta < \tau_1 \leq 1$$

let $\hat{G}$ be a continuation of $\underline{\hat{G}}$, such that $\hat{G}_{\tau_0} = \underline{\hat{G}}$ and :
(1) $\hat{G}(\tau') \in E(\hat{E}_\tau)$ for all $\tau' \in [\tau_0,\ \overline{\tau}]$, with $\hat{T}_G(\tau_0) = T$ and $\hat{T}_G(\overline{\tau}) = \hat{T}_E(\tau)$);
(2) $\hat{G}(\overline{\tau} + \theta) = \hat{E}(\tau + \theta)$ for all $\theta \in [0, \delta]$;
(3) $\hat{G}(\tau') \in E(\hat{E}_{\tau+\delta})$ for all $\tau' \in]\overline{\tau}+\delta, \tau_1]$, and $\hat{T}_G(\tau_1) = T$.
Suppose now, as is common experience in applications, that one were willing to strenghten our former assumption on the insensitivity of plastic stretching to elastic cycles slightly , in such a way that

$$\varphi_c(T:\hat{E}_{\tau+\delta}) = \varphi_c(T:\hat{G}_{\overline{\tau}+\delta})$$

would follow. Then, by definition of complementary working, and as the constitutive functional is rate-independent and the complementary working is path-independent within the elastic range, one would have that

$$\varphi_c(T:\hat{E}_{\tau+\delta}) - \varphi_c(T:\hat{E}_\tau) = \int_0^{\tau_1} \dot{T}_G(\tau')\cdot\hat{G}(\tau')\,d\tau' - \int_0^1 \dot{T}_{\underline{G}}(\tau')\cdot\underline{\hat{G}}(\tau')\,d\tau' =$$

$$= \int_{\tau_0}^{\tau_1} \dot{T}_G(\tau')\cdot\hat{G}(\tau')\,d\tau' \tag{3.14}$$

and a physically convincing interpretation of $[\varphi_c(T:\hat{E}_{\tau+\delta}) - \varphi_c(T:\hat{E}_\tau)]$ as the complementary working done on the cyclic part of a history would ensue. □

As the elastic range evolves smoothly, for each $\hat{E} \in \mathfrak{D}$ and $\tau \in [0, 1[$ fixed, and for each $E \in E^\circ(\hat{E}_\tau)$ and $T \in T^\circ(\hat{E}_\tau)$, there is $\delta > 0$ such that (3.13) is obeyed. Consequently, whenever a finite limit exists, we may set

$$\dot{\varphi}(E:\hat{E}_\tau) = \lim_{\delta \to 0+} \frac{\varphi(E:\hat{E}_{\tau+\delta}) - \varphi(E:\hat{E}_\tau)}{\delta} \tag{3.15$_1$}$$

and likewise

$$\dot{\varphi}_c(T;\hat{E}_\tau) = \lim_{\delta \to 0+} \frac{\varphi_c(T;\hat{E}_{\tau+\delta}) - \varphi_c(T;\hat{E}_\tau)}{\delta} \quad . \tag{3.15$_2$}$$

We call $\dot{\varphi}$ the *cyclic power* and $\dot{\varphi}_c$ the *complementary cyclic power.* The main properties of cyclic powers are the contents of our first proposition.

Proposition 1. *For each* $\hat{E} \in \mathfrak{D}$ *and* $\tau \in [0,1[$, $\dot{\varphi}_c$ $[\![\dot{\varphi}]\!]$ *is well-defined over* $T(\hat{E}_\tau)$ $[\![E(\hat{E}_\tau)]\!]$, *and*

$$\dot{\varphi}_c(T;\hat{E}_\tau) = -[\hat{T}_E(\tau) - T]\cdot \dot{E}^p(\tau)$$
$$[\![\ \dot{\varphi}(E;\hat{E}_\tau) = (\hat{T}_E(\tau) - \mathbb{T}[E - \hat{E}^p(\tau)])\cdot \dot{E}^p(\tau)\ ]\!] \ . \tag{3.16}$$

Moreover, if, for $0 \le \tau_0 < \tau_1 \le 1$, $T \in T(\hat{E}_\tau)$ $[\![E \in E(\hat{E}_\tau)]\!]$ *for all* $\tau \in [\tau_0, \tau_1]$, *then the function* $\tau \mapsto \dot{\varphi}_c(T;\hat{E}_\tau)$ $[\![\ \tau \mapsto \dot{\varphi}(E;\hat{E}_\tau)\ ]\!]$ *is piecewise continuous on* $[\tau_0, \tau_1]$.

Corollary. *For each* $\hat{E} \in \mathfrak{D}$ *and* $\tau \in [0,1[$,

$$\dot{\varphi}(E;\hat{E}_\tau) + \dot{\varphi}_c(T;\hat{E}_\tau) = 0 \ , \tag{3.17}$$

provided $T = \mathbb{T}[E - \hat{E}^p(\tau)]$.

Proof. We prove the assertion on $\dot{\varphi}_c$: the proof of the corresponding assertion on $\dot{\varphi}$, which is completely analogous, has been given in [6]; the proof of the corollary is a straightforward consequence of $(3.16)_{1,2}$.

Let $\tau \in [0,1[$ and $T \in T^o(\hat{E}_\tau)$ be arbitrarily chosen. In view of (3.11), (3.8), (3.9) and the assumption that the elastic range evolves smoothly, and recalling also (2.5), (2.7), we can write:

$$\varphi_c(T;\hat{E}_{\tau+\delta}) - \varphi_c(T;\hat{E}_\tau) = [\tilde{\sigma}(T;\hat{E}^p(\tau+\delta)) - \tilde{\sigma}(T;\hat{E}^p(\tau))] +$$
$$-[\tilde{\sigma}(\hat{T}_E(\tau+\delta);\hat{E}^p(\tau+\delta)) - \tilde{\sigma}(\hat{T}_E(\tau);\hat{E}^p(\tau))] +$$
$$+\int_\tau^{\tau+\delta} \dot{T}_E(\tau')\cdot \hat{E}(\tau)\, d\tau' \ .$$

But, in view again of (3.9),

$$\tilde{\sigma}(T;\hat{E}^p(\tau+\delta)) - \tilde{\sigma}(T;\hat{E}^p(\tau)) = T\cdot[\hat{E}^p(\tau+\delta) - E^p(\tau)]$$

and

$$\tilde{\sigma}(\hat{T}_E(\tau+\delta);\hat{E}^p(\tau+\delta)) - \tilde{\sigma}(\hat{T}_E(\tau);\hat{E}^p(\tau)) = [\overline{\sigma}(\hat{T}_E(\tau+\delta)) - \overline{\sigma}(\hat{T}_E(\tau))] +$$

$$+[\hat{T}_E(\tau+\delta)\cdot\hat{E}^p(\tau+\delta)-\hat{T}_E(\tau)\cdot\hat{E}^p(\tau)]\ .$$

Dividing by δ and passing to the limit for $\delta\to 0+$, and taking into account $(3.5)_1$, we then have $(3.16)_1$ for all $T\in T^\circ(\hat{E}_\tau)$; as the right hand-side of $(3.16)_1$ is well-defined for all $T\in \mathrm{Sym}$, $(3.16)_1$ holds on the boundary $\partial T(\hat{E}_\tau)$ of $T^\circ(\hat{E}_\tau)$ as well. Finally, the stated smoothness properties of function $\tau\mapsto\dot{\varphi}_c(T;\hat{E}_\tau)$ follow directly from the corresponding properties of $\hat{E}^p$ and $(3.16)_1$. □

4. Equivalent Dissipation Postulates

Recall from the former section that, for $\hat{E}\in\mathfrak{D}$ and $\tau_0,\tau_1\in[0,1]$ such that $0\le\tau_0<\tau_1\le 1$, the integrals

$$\Lambda\{\hat{E}\}=\int_{\tau_0}^{\tau_1}\hat{T}_E(\tau)\cdot\dot{E}(\tau)\,d\tau \quad\text{and}\quad \Lambda_c\{\hat{E}\}=\int_{\tau_0}^{\tau_1}\dot{T}_E(\tau)\cdot\hat{E}(\tau)\,d\tau \tag{4.1}$$

define, respectively, the *working* and the *complementary working* done by internal actions along history $\hat{E}$ during the time interval $[\tau_0,\tau_1]$. The integral

$$\Lambda_a\{\hat{E}\}:=\int_{\tau_0}^{\tau_1}[\hat{T}_E(\tau)-\hat{T}_E(\tau_0)]\cdot\dot{E}(\tau)\,d\tau \tag{4.2}$$

defines the *additional working* along $\hat{E}$ during $[\tau_0,\tau_1]$.

A history $\hat{E}\in\mathfrak{D}$ is *strain-closed* ⟦*stress-closed*⟧ *in the time interval* $[\tau_0,\tau_1]$ if $\hat{E}(\tau_0)=\hat{E}(\tau_1)=E$ for some $E\in E(\hat{E}_\tau)$ for all $\tau\in[\tau_0,\tau_1]$ ⟦ $\hat{T}_E(\tau_0)=\hat{T}_E(\tau_1)=T$ for some $T\in T(\hat{E}_\tau)$ for all $\tau\in[\tau_0,\tau_1]$ ⟧; we briefly say that $\hat{E}$ is *strain-closed* ⟦*stress-closed*⟧ if it is strain-closed in some time interval.

With the above notions of working integrals and closed histories, we are in a position to formulate the dissipation postulates of Il'yushin [4], Martin [3] and Drucker [2].

Il'yushin's Postulate. *Let a history $\hat{E}\in\mathfrak{D}$ be strain-closed in the time interval $[\tau_0,\tau_1]$. Then, the working along $\hat{E}$ during $[\tau_0,\tau_1]$ is nonnegative:*

$$\Lambda\{\hat{E}\}\ge 0\ . \tag{4.3}$$

Martin's Postulate. *Let a history $\hat{E}\in\mathfrak{D}$ be stress-closed in the time interval $[\tau_0,\tau_1]$. Then, the complementary working along $\hat{E}$ during $[\tau_0,\tau_1]$ is nonpositive:*

$$\Lambda_c\{\hat{E}\}\le 0\ . \tag{4.4}$$

Drucker's Postulate. *Let a history* $\hat{E} \in \mathfrak{D}$ *be stress-closed in the time interval* $[\tau_0, \tau_1]$. *Then, the additional working along* $\hat{E}$ *during* $[\tau_0, \tau_1]$ *is nonnegative*:

$$\Lambda_a\{\hat{E}\} \geq 0 \ . \tag{4.5}$$

As $\mathbb{T}$ is invertible, $\hat{E}$ is stress-closed in $[\tau_0, \tau_1]$ if and only if

$$\hat{E}(\tau_0) - \hat{E}^P(\tau_0) = \hat{E}(\tau_1) - \hat{E}^P(\tau_1) \ ; \tag{4.6}$$

consequently, as

$$[\hat{T}_E(\tau) - T] \cdot \dot{\hat{E}}(\tau) = [(\hat{T}_E(\tau) - T) \cdot (\hat{E}(\tau) - \hat{E}^P(\tau))]^{\cdot} + [\hat{T}_E(\tau) - T] \cdot \dot{\hat{E}}^P(\tau) \ ,$$

(4.5) can be written in the well-known equivalent form

$$\Lambda_a\{\hat{E}\} = \int_{\tau_0}^{\tau_1} [\hat{T}_E(\tau) - T] \cdot \dot{\hat{E}}^P(\tau) d\tau \geq 0 \ . \tag{4.7}$$

Recalling $(3.16)_1$ in Proposition 1, we may also write

$$\Lambda_a\{\hat{E}\} = - \int_{\tau_0}^{\tau_1} \dot{\varphi}_c(T; \hat{E}_\tau) d\tau \ ; \tag{4.8}$$

thus, the assumption that, for each stress-closed history $\hat{E}$ and for each $T \in T(\hat{E}_\tau)$ for all instants τ belonging to the time interval during which $\hat{E}$ is stress-closed,

$$\dot{\varphi}_c(T; \hat{E}_\tau) \leq 0 \tag{4.9}$$

appears as a natural differential formulation of Drucker's Postulate, and of course implies it. Actually, as stated in our next proposition, under our present hypotheses Drucker's Postulate implies that (4.9) holds for all histories $\hat{E} \in \mathfrak{D}$, for all $\tau \in [0,1[$ and for all $T \in T(\hat{E}_\tau)$: thus, stipulating (4.9) only for stress-closed histories has an illusory greater generality.

Proposition 2. *The additional working along stress-closed histories is nonnegative if and only if the complementary cyclic power is nonpositive along all histories at each instant* $\tau \in [0,1[$.

Proof. It suffices to prove that the complementary cyclic power is nonpositive along all histories at each instant $\tau \in [0,1[$ if the additional working along stress-closed histories is nonnegative.

For all $\hat{E} \in \mathfrak{D}$ and $\tau \in [0,1[$ such that $\hat{E}(\tau) \in \partial E(\hat{E}_\tau)$, let $\hat{G}$ be a continuation of $\hat{E}_\tau$ of the type considered already in stating the insensitivity of plastic stretching to elastic cycles, namely, such that, for some positive numbers $\overline{\tau}$ and δ with $0 < \tau < \overline{\tau} < \overline{\tau} + \delta < 1$, both $\hat{G}(\tau') \in E(\hat{E}_\tau)$ for all $\tau' \in [\tau, \overline{\tau}]$, with $\hat{G}(\overline{\tau}) = \hat{G}(\tau)$ (so that also $\hat{T}_G(\overline{\tau}) = \hat{T}_G(\tau)$), and $\hat{E}(\tau + \theta) = \hat{G}(\overline{\tau} + \theta)$ for all $\theta \in [0, \delta]$. For each fixed T belonging to $T(\hat{E}_{\tau'})$ for all $\tau' \in [\tau, \overline{\tau} + \delta[$, assume in addition that $\hat{G}$ has been chosen in such a way that $\hat{T}_G(\tau_0) = T$ for some $\tau_0 \in]\tau, \overline{\tau}[$ and, finally, that, for some τ_1 such that $\overline{\tau} + \delta < \tau_1 \le 1$, $\hat{G}(\tau') \in E(\hat{E}_{\tau + \delta})$ for all $\tau' \in [\overline{\tau} + \delta, \tau_1]$ and $\hat{T}_G(\tau_1) = T$ (*vid.* Fig. 3).

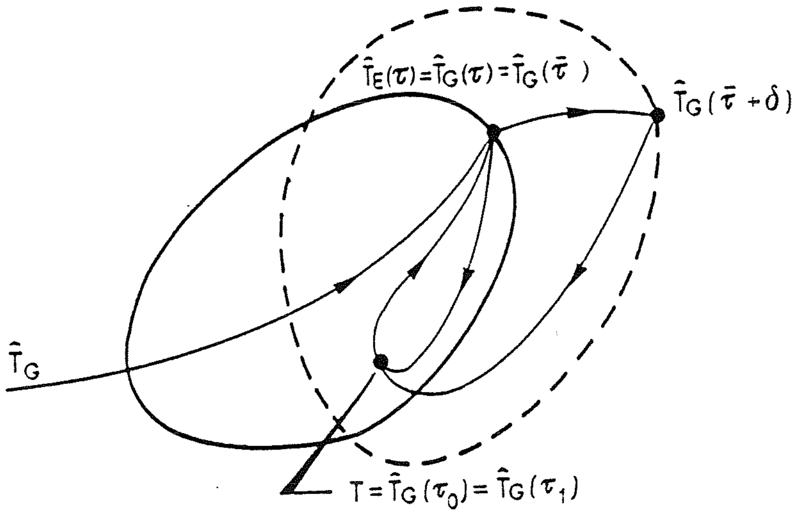

Fig. 3

By construction, history $\hat{G}$ is stress-closed in the time interval $[\tau_0, \tau_1]$. We apply Drucker's Postulate (4.5) to $\hat{G}$, and get

$$\int_{\tau_0}^{\tau_1} [\hat{T}_G(\tau') - T] \cdot \dot{G}^p(\tau')\, d\tau' = \int_{\overline{\tau}}^{\overline{\tau}+\delta} [\hat{T}_G(\tau') \cdot \dot{G}^p(\tau')\, d\tau' \;;$$

the arbitrariness in the choice of δ, together with (2.11) and the fact that, again by construction, $\hat{T}_G(\overline{\tau}) = \hat{T}_E(\tau)$, then implies that

$$0 \le (\hat{T}_G(\overline{\tau}) - T) \cdot \dot{G}^p(\overline{\tau}) = (\hat{T}_E(\tau) - T) \cdot \dot{E}^p(\tau) \;; \qquad (4.10)$$

finally, (4.10) yields the desired conclusion. □

Thus, the local formulation of Drucker's Postulate is that the complementary cyclic power $\dot{\varphi}_c(\cdot\,; \hat{E}_\tau)$ is nonpositive along all histories at each instant $\tau \in [0, 1[$. We now give equivalent local formulations of both Martin's and Il'yushin's Postulate: remarkably, the local formulation of Martin's Postulate turns out to coincide with Drucker's.

Proposition 3. *The complementary working* ⟦*working*⟧ *along stress-closed*

⟦strain-closed⟧ *histories is nonpositive* ⟦*nonnegative*⟧ *if and only if the complementary cyclic power* ⟦*cyclic power*⟧ *is nonpositive* ⟦*nonnegative*⟧ *along all histories at each instant* $\tau \in [0,1[$.

Proof. We prove the stated equivalence only for Martin's Postulate: the proof for Il'yushin's, which is completely analogous, has been given in [6] in the general case when large deformations are allowed.

We firstly show that (4.4) implies $(4.10)_2$. With reference to Fig. 3 again, let $\hat{E}$ and $\hat{G}$ be as in the proof of Proposition 3, and consider the following cyclic portion of $\hat{T}_G(\cdot)$:

$$\{ T = \hat{T}_G(\tau_0) \to \hat{T}_G(\overline{\tau}) \to \hat{T}_G(\overline{\tau}+\delta) \to \hat{T}_G(\tau_1) = T \} .$$

In view of (4.4), we have that

$$0 \le \int_{\tau_0}^{\tau_1} \dot{T}_G(\tau)\cdot\hat{G}(\tau)\,d\tau = -[\varphi_c(T;\hat{G}_{\overline{\tau}+\delta}) - \varphi_c(T;\hat{G}_{\overline{\tau}})] .$$

Therefore, by $(3.15)_2$, $(3.16)_1$, (2.11) and the fact that, by construction, $\hat{T}_G(\overline{\tau}) = \hat{T}_E(\tau)$, we conclude that

$$0 \le -\dot{\varphi}_c(T;\hat{G}_{\overline{\tau}}) = [\hat{T}_G(\overline{\tau}) - T]\cdot\dot{G}^p(\overline{\tau}) = [\hat{T}_E(\tau) - T]\cdot\dot{E}^p(\tau) , \tag{4.11}$$

which establishes (4.10).

Secondly, if (4.10) holds and $\hat{E}$ is stress-closed in $[\tau_0, \tau_1]$, (4.11) implies that

$$0 \le \int_{\tau_0}^{\tau_1} \dot{\varphi}_c(T;\hat{E}_\tau)\,d\tau = \varphi_c(T;\hat{E}_{\tau_1}) - \varphi_c(T;\hat{E}_{\tau_0}) \tag{4.12}$$

for each $T \in \mathcal{T}(\hat{E}_\tau)$ for all $\tau \in [\tau_0, \tau_1]$. Choosing now $T = \hat{T}_E(\tau_0) = \hat{T}_E(\tau_1)$ in (4.12), and recalling (3.11) and $(3.1)_2$, we arrive at

$$0 \ge \int_{\tau_0}^{\tau_1} \dot{T}_E(\tau)\cdot\hat{E}(\tau)\,d\tau = \Lambda_c\{\hat{E}\} . \quad \square$$

We are now in a position to give a straightforward proof of the main result of this paper.

Proposition 4. *Within the linear theory of isotropic materials with elastic range each one of the postulates of Il'yushin, Martin and Drucker is equivalent to each other.*

Proof. Drucker's and Martin's Postulates are trivially equivalent because they share their equivalent local formulations. Furthermore, we have from Proposition 3 that Il'yushin's Postulate holds if and only if the cyclic power is nonnegative along all histories at each instant $\tau \in [0,1[$, a condition which, in view of $(3.16)_2$ and the fact that the elastic range $E(\hat{E}_\tau)$ and the stress range

$T(\hat{E}_\tau)$ are in one-to-one correspondence (*cf.* (2.13)), is equivalent to the validity of Martin's Postulate. □

We remark in closure that the implication (4.3) ⇒ (4.10) has been noted by Pipkin & Rivlin in their seminal paper [9].

References

[1] J. Lubliner, A maximum-dissipation principle in generalized plasticity, *Acta Mechanica* **52**, 225-237 (1984).

[2] D.C.Drucker, A more fundamental approach to plastic stress-strain relations. Pp. 487-491 of Proc. First U.S. Nat. Congr. Appl. Mech., Chicago 1951, E.Sternberg Ed.. Ann Harbor: Edwards Bro.s Inc. (1952).

[3] J.B. Martin, Plasticity: Fundamental and General Results. M.I.T. Press U.S. (1975).

[4] A.A. Il'yushin, On the relation between stresses and small deformations in the mechanics of continuous media (in russo). *Prikl. Math. Mech.* **18**, 641-666 (1954).

[5] Lucchesi, M. & P. Podio-Guidugli, Materials with elastic range: A theory with a view toward applications. Part I. *Arch. Rational Mech. Anal.* **102**, 23-43 (1988).

[6] Lucchesi, M. & P. Podio-Guidugli, Materials with elastic range: A theory with a view toward applications. Part II. Nota interna CNUCE C-88 20 (1988), submitted.

[7] Lucchesi, M. & P. Podio-Guidugli, Materials with elastic range: A theory with a view toward applications. Part III. In preparation (1988).

[8] Lucchesi, M. & P. Podio-Guidugli, Sull'equivalenza di alcuni postulati di dissipazione in teoria classica della plasticità. Pp. 121-123, Vol. I, of Atti IX Congr. Naz. Mecc. Teor. Appl. , Bari, 1988.

[9] J.C. Pipkin & R.S. Rivlin, Mechanics of rate-independent materials. *Z.A.M.P.* **16**, 313-326 (1965).