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Research Article

A Residual Theorem Approach Applied to Stokes' Problems with Generally Periodic Boundary Conditions including a Pressure Gradient Term

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The differential problem given by a parabolic equation describing the purely viscous flow generated by a constant or an oscillating motion of a boundary is the well-known Stokes' problem. The one-dimensional equation is generally solved for unbounded or bounded domains; for the latter, either free slip (i.e., zero normal gradient) or no-slip (i.e., zero velocity) conditions are enforced on one boundary. Generally, the analytical strategy to solve these problems is based on finding the solutions of the Laplace-transformed (in time) equation and on inverting these solutions. In the present paper this problem is solved by making use of the residuals theorem; as it will be shown, this strategy allows achieving the solutions of First and Second Stokes' problems in both infinite and finite depth. The extension to generally periodic boundaries with the presence of a periodic pressure gradient is also presented. This approach allows getting closed form solutions in the time domain in a rather fast and simple way. An ad hoc numerical algorithm, based on a finite difference approximation of the differential equation, has been developed to check the correctness of the analytical solutions.

1. Introduction

The analysis of purely viscous unsteady flows, or Stokes' flow, consists in finding analytical solutions of parabolic equations (like the heat equation) for prescribed boundary conditions. In Batchelor [1], the fluid is supposed to occupy the half plane (x, y) for $y > 0$, with the boundary at $y = 0$ initially at the rest and then moving in its own plane with a velocity which suddenly assumes a constant (*first problem*) or sinusoidal (*second problem*) time law.

Even if considered as an old-fashioned topic, the importance in solving this class of problems for different boundary conditions lies in the wide application fields in which they are encountered. For example, both the sudden or the periodic movements of oceanic geophysical faults during an earthquake are a clear example, as well as "*the oscillatory motion which arises in the design of an oscillating half-plate flow chamber for examining the effects of fluid shear stress on cultured cell monolayers*" (see Zeng and Weinbaum [2]). Moreover, some actual and interesting problems, including the development of new materials, make the analysis of Stokes problems a fundamental topic; indeed, the flow over superhydrophobic surfaces is excellently modelled by a non-convective equation with mixed (homogeneous Dirichlet-homogeneous Neumann) boundary conditions (see, for example, Lauga and Stone [3], Rothstein [4]).

Ocean wave dynamics are another example: among others, Xu and Bowen [5] exploit simplified solutions of Stokes problems in order to estimate the wave stress over the ocean bottom. The great interest for these solutions lies in the examination of the net mass transport phenomena associated with an ocean wave (Stokes' drift) generated by the wind in a rotating frame (see Janssen [6], Hanley and Belcher [7], and Polton et al. [8]). For large-scale problems, transitional effects are normally neglected, focusing the interest on long time dynamics. However the solutions from which final conclusions are drawn contain some approximations and, obviously, anything can be said on the time scale of the problem or on fast dynamics events.

The solutions of these problems in one-dimensional case and for unbounded domains are well established in literature, in particular for heat diffusion problems (see, for example, Watson [9], Batchelor [1], Erdogan [10], and Liu and Liu [11]). The strategy for the solution, as in the present paper, is to consider the Laplace transformation in time; however, the inverse transformation is not simple to achieve (see, for example, Devakar and Iyengar [12] where the inverse transform is obtained through a numerical procedure); therefore a clear and a more general strategy, which would allow the achievement of the analytical solution for different boundary conditions, has to be found.

An example of analytical solutions in unbounded domain can be found in Watson [9], where Stokes' equation is considered for the study of the boundary layer past an infinite porous wall, assuming the pressure gradient as an arbitrary function of time. The two-side Laplace transform in time is used in order to obtain analytical solutions for homogeneous Dirichlet boundary condition. However the approach to the problem is carried out in a way which is difficult to extend for finite domains and time-dependent boundary conditions.

The analytical solutions of parabolic equations for heat conduction problems are strictly related to Stokes problems and in the book of Carslaw [13] a wide class of interesting cases are discussed and often solved with different techniques such as Fourier series or Laplace transform. In the latter case, the inversion theorem is adopted only for simple cases and no mention is given for multidimensional problems, which are considered instead in Section 15.11 but only for parallelepipeds with a prescribed condition on one face. In this way these examples are extensions of one-dimensional problems and not two- or three-dimensional real ones.

When a bounded domain is considered and two spatial dimensions are involved, the deduction of analytical solutions is a no-trivial task. In Zeng and Weinbaum [2] the two-dimensional problem is approached with an *ad hoc* definition of similar variables for unbounded domains and extended to bounded domains through an image superimposition technique. In a technical note by Liu and Liu [11], the one-dimensional *second Stokes' problem* is solved in an unbounded domain and for arbitrary initial phase; a complete analysis of *First* and *Second Stokes' problems* in one- and two-dimensional bounded and unbounded domains can be found in Liu [14]. In this paper, the author rearranges the inversion integrals in such a way that the integrands are always straight invertible, with the aid of Laplace transform tables. This has to point out that the strategy adopted by these authors makes an extension to more general boundary conditions very hard, as well as the possibility of taking into account a pressure field. In a following paper, Liu [15] used the same strategy to find the solution of Stokes' problems with porous wall in a two-dimensional unbounded domain. A similar strategy was also adopted in Khaled and Vafai [16].

In the present paper, the use of the residuals theorem is exploited; once the path along which the complex integrals of the inverse transformation should be evaluated is defined, this theorem allows a straight deduction of the time- and space-dependent solution. As it will be demonstrated, in this way the analytical treatment of more general boundary conditions becomes possible. In particular, the analytical solutions of Stokes' flows between moving walls with constant and sinusoidal velocities with different frequency and initial phase will be given.

It will be also shown that, by using the actual strategy, the introduction of a constant or sinusoidal pressure gradient offers only a slight complication which can be easily overtaken; in fact, this term is normally a function of time only, Batchelor [1], so it plays the role of a simple nonhomogeneity in the transformed equation. However, even if the pressure is a product of two functions (one of which of time only), the achievement of a complete solution is possible in some cases (see Section 6.2).

Moreover the extension of the present solving technique to two-dimensional problems avoids the onset of additional complications and will be the matter of a future work.

The extension to generally periodic boundary conditions and pressure gradient is also possible; the development in Fourier series of both the boundary conditions and the pressure gradient permits superimposing the solutions for constant and sinusoidal cases, the problem being linear. In this light, the definitions of the first and the second Stokes' problem are only particular cases of a *generalized Stokes problem*, where the boundary conditions are referred to generally periodic functions.

As it can be seen in the following, the expression of the analytical solutions could be rather complicated. In some cases, its computation involves the evaluation of integrals, which are solved by means of series expansions whose convergence could be rather slow. In order to check the correctness of the computed analytical solutions, an algorithm which resolves numerically the differential problem has been also implemented. The algorithm is based on a finite difference approximation of the Laplace operator for a generally nonuniform orthogonal distribution of the discretization points. The time integration is achieved by means of a Crank-Nicholson scheme; the resulting tridiagonal system of algebraic equations is solved by means of the classical Thomas algorithm. It is easy to demonstrate that the algorithm is unconditionally stable and globally second-order accurate (see, for example, Morton [17]). Both the algorithms for the computation of the analytical and the numerical solutions are available on request.

The paper is organized as follows: in Section 2, the methodology is explained and the classical one-dimensional solution for infinite depth is recalled. In Sections 3 and 4, the solutions of First and Second Stokes' problem for the finite depth case are obtained. In Section 3, a free slip wall is considered on the top; this will highlight how the simple solutions obtained in the previous section can be easily used in a more complex cases. In Section 4, solutions are deduced for both walls with either constant or oscillating velocities. It has to be highlighted that the linearity of the problem allows obtaining these solutions as a superposition of solutions with only one wall with a nonzero velocity. In Section 5, the presence of a constant and sinusoidal pressure gradient is considered. Some examples are given in Section 6; in 6.1, a comparison with the numerical solution is presented, whereas in 6.2 an extension of the primary wave motion

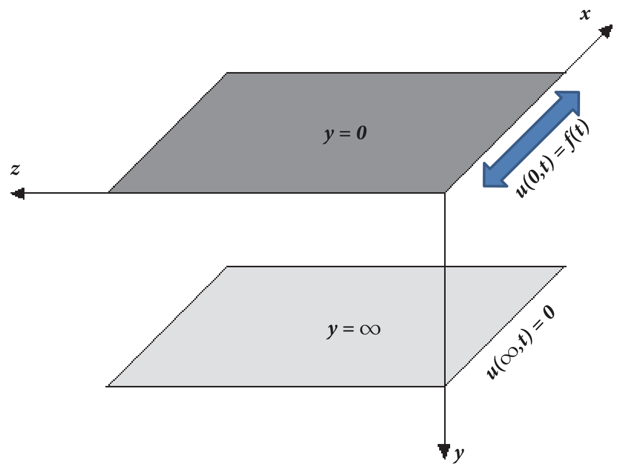

FIGURE 1: Sketch of the Stokes problem with an infinite depth.

solution for a gravity wave in a viscous fluid is addressed. The aim is to show how the solutions presented are very easy to handle and to be used for different problems, achieving at the same time the best possible accuracy when problems modelled with a purely viscous equation are considered. In Section 7, a discussion on the possibility of handling more general boundary conditions and pressure gradient is carried out. In Section 8, conclusions and perspectives will wind up the paper. The extension to the solutions of two-dimensional problems will be the matter of future works.

2. One-Dimensional Infinite Depth Flows

A Newtonian fluid with kinematical viscosity ν is contained in an unbounded half space with $y > 0$. The boundary starts to move at time $t = 0$ with a given velocity law $f(t)$, moving in its own plane ($y = 0$). The problem is clearly one-dimensional and the motion of the fluid is described by the solution of the Stokes' problem:

$$\begin{aligned} &\partial_t u = \nu \partial^2_{yy} u \\ &u(0,t) = f(t), \quad u(+\infty, t) \equiv 0 \\ &u(y,0) \equiv 0, \end{aligned} \tag{1}$$

where u is the velocity component parallel to the plane $y = 0$. The First and Second Stokes' problem are defined by the velocity law of the boundary $y = 0$ for $t > 0$:

$$\begin{array}{cc} \text{FIRST PROBLEM} & \text{SECOND PROBLEM} \\ f(t) = u_0 & f(t) = u_0 \cos(\sigma t + \theta) \end{array} \tag{2}$$

being $f(t) = 0$ at initial time. The problem is sketched in Figure 1.

The solution of the problem could be obtained through a Laplace transformation in time [1], according to which

$$\hat{u}(y,s) = \exp(-\beta\, y)\, \hat{f}(s). \tag{3}$$

where $\beta = \sqrt{s/\nu}$ and the transformed quantities are indicated with a hat.

The solutions of the classical Stokes' problem are briefly addressed, in order to establish the methodology. In the following, the nondimensional quantities are indicated with a tilde, where needed.

2.1. First Stokes' Problem. In this case $\hat{f}(s) = u_0/s$ and the time derivative of the nondimensional velocity $\tilde{u} = u/u_0$ is

$$\tilde{u}_t = \frac{1}{2\pi i} \lim_{M\longrightarrow\infty} \int_{\mu-iM}^{\mu+iM} ds \exp(ts - \beta y) \tag{4}$$

μ being an arbitrary positive real number. The function $\tilde{u}_t(y,t)$ is evaluated through an application of the Cauchy theorem on the path in Figure 2(a):

$$\begin{aligned} \tilde{u}_t &= \frac{1}{2\pi i} \int_{-\infty}^{0} dx \left[\exp\left(xt + i\frac{y}{\sqrt{\nu}}\sqrt{-x} \right) \right. \\ &\quad \left. - \exp\left(xt - i\frac{y}{\sqrt{\nu}}\sqrt{-x} \right) \right] = \frac{1}{\pi} \\ &\quad \cdot \frac{\tilde{\eta} e^{-\tilde{\eta}^2}}{t} \left[\int_0^{+\infty} d\xi\, e^{-(\xi - i\tilde{\eta})^2} + \int_0^{+\infty} d\xi\, e^{-(\xi + i\tilde{\eta})^2} \right], \end{aligned} \tag{5}$$

where $\xi^2 = |x|t$ and $\tilde{\eta} = y/(2\sqrt{\nu t})$. The two integrals in the above relation are evaluated along the lower and upper paths of Figure 2(b) and their sum gives $\sqrt{\pi}$. As a consequence, $\tilde{u}_t$ assumes the following form:

$$\tilde{u}_t(\tilde{\eta}, t) = \frac{\tilde{\eta}}{\sqrt{\pi t}} \exp\left(-\tilde{\eta}^2\right). \tag{6}$$

An integration in time leads to the classical solution:

$$\tilde{u}(\tilde{\eta}) = \frac{2}{\sqrt{\pi}} \int_{\tilde{\eta}}^{+\infty} d\xi\, e^{-\xi^2} = \operatorname{erfc}(\tilde{\eta}). \tag{7}$$

As it is evident, the solution depends on a similar variable $\tilde{\eta}$. The wall stress τ_w, made nondimensional as $\tilde{\tau}_w = \tau_w/(\rho u_0^2)$, ρ being the density of the fluid, follows as

$$\tilde{\tau}_w = \frac{\nu}{u_0}\ \tilde{u}_y\Big|_{y=0} = -\frac{1}{\sqrt{\pi \tilde{t}}} \tag{8}$$

defining $\tilde{t} = t\ u_0^2/\nu$.

2.2. Second Stokes' Problem. In second Stokes' problem the Laplace transform of the wall velocity is

$$\hat{f}(s) = \frac{u_0}{2} \left(\frac{e^{-i\theta}}{s + i\sigma} + \frac{e^{+i\theta}}{s - i\sigma} \right) \tag{9}$$

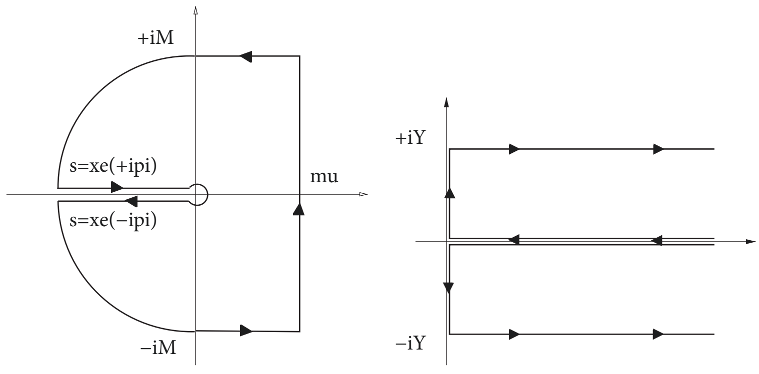

FIGURE 2: Integration paths in the plane of s: for the evaluation of the function $\tilde{u}_t$ (a) and of the integrals $I_{1,2}$.

and solution (3) becomes

$$\tilde{u}(y,s) = \frac{1}{2}\left\{ e^{-i(\sigma t+\theta)} \underbrace{\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\, \frac{\exp\left[(s+i\sigma)t-\beta y\right]}{s+i\sigma}}_{G_1} + e^{+i(\sigma t+\theta)} \underbrace{\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\, \frac{\exp\left[(s-i\sigma)t-\beta y\right]}{s-i\sigma}}_{G_2} \right\} \quad (10)$$

in which the time derivatives of the functions $G_{1,2}(y,t)$ are easily evaluated in terms of (6), so that the following is obtained:

$$G_{1,2}(y,t) = G_{1,2}(y,0) + \frac{2}{\sqrt{\pi}}\int_{\tilde{\eta}}^{\infty} d\xi \exp\left[\pm iT\frac{\tilde{\eta}^2}{\xi^2} - \xi^2\right]. \quad (11)$$

where $T = \sigma t$. An integration along the path of Figure 2(a) proves that $G_{1,2}(y,0) = 0$ (see Appendix A), so the initial condition $u(y,0) \equiv 0$ is satisfied. Solution (10) is as follows:

$$\begin{aligned}\tilde{u}(\tilde{y},T) &= \cos(T+\theta)\frac{2}{\sqrt{\pi}}\int_{\tilde{\eta}}^{+\infty} d\xi\, e^{-\xi^2}\cos\left(T\frac{\tilde{\eta}^2}{\xi^2}\right) + \sin(T+\theta)\frac{2}{\sqrt{\pi}}\int_{\tilde{\eta}}^{+\infty} d\xi\, e^{-\xi^2}\sin\left(T\frac{\tilde{\eta}^2}{\xi^2}\right) \\ &= \cos(T+\theta)\; I_c(\tilde{\eta},T) + \sin(T+\theta)\; I_s(\tilde{\eta},T).\end{aligned} \quad (12)$$

The above solution is the real form of the solution in Erdogan [10] for $\theta = 0$ and $\theta = \pi/2$ and of the solution in Liu and Liu [11]. Integrals I_c and I_s can be calculated by recursive formulas for little values of T, as discussed in Appendix B.

The asymptotic expression of I_c and I_s for large T is as follows:

$$\begin{aligned} I_c(\tilde{\eta},T) &\longrightarrow e^{-\tilde{y}}\cos(\tilde{y}) \\ I_s(\tilde{\eta},T) &\longrightarrow e^{-\tilde{y}}\sin(\tilde{y})\end{aligned} \quad (13)$$

where $\tilde{y} = y\sqrt{\sigma/(2\nu)}$. In Figure 3 integrals I_c and I_s are compared with their asymptotic expression for different times. For long times, the convergence is evident.

The wall stress is given by

$$\tilde{\tau}_w(T) = -\frac{\cos\theta}{\sqrt{\pi\tilde{t}}} + \alpha'\left[\sin(T+\theta)C_1\left(\sqrt{T}\right) - \cos(T+\theta)S_1\left(\sqrt{T}\right)\right], \quad (14)$$

where C_1 and S_1 are the Fresnel integrals (see Abramowitz and Stegun [18] page 300 equations 7.3.3 and 7.3.4, respectively), while $\alpha^2 = \sigma\nu/u_0^2$ and $\alpha' = \sqrt{2}\alpha$. Note that the first term is the wall stress of the First Stokes problem, due to the initial nonvanishing value $(\cos\theta)$ of the wall velocity.

3. One-Dimensional Finite Depth Flows

Stokes' problem is posed in the following way:

$$\begin{aligned} \partial_t u &= \nu\partial_{yy}^2 u \\ u(0,t) &= f(t), \\ \partial_y u(h,t) &\equiv 0 \\ u(y,0) &\equiv 0,\end{aligned} \quad (15)$$

h being the height of the fluid (hereafter, the corresponding nondimensional height $\tilde{y} = y/h$ will be used). Note that the second boundary condition enforces vanishing viscous stresses at the interface.

The Laplace transform (in time) of the solution is

$$\hat{u}(y,s) = \frac{\cosh\left[(h-y)\beta\right]}{\cosh(h\beta)}\hat{f}(s). \quad (16)$$

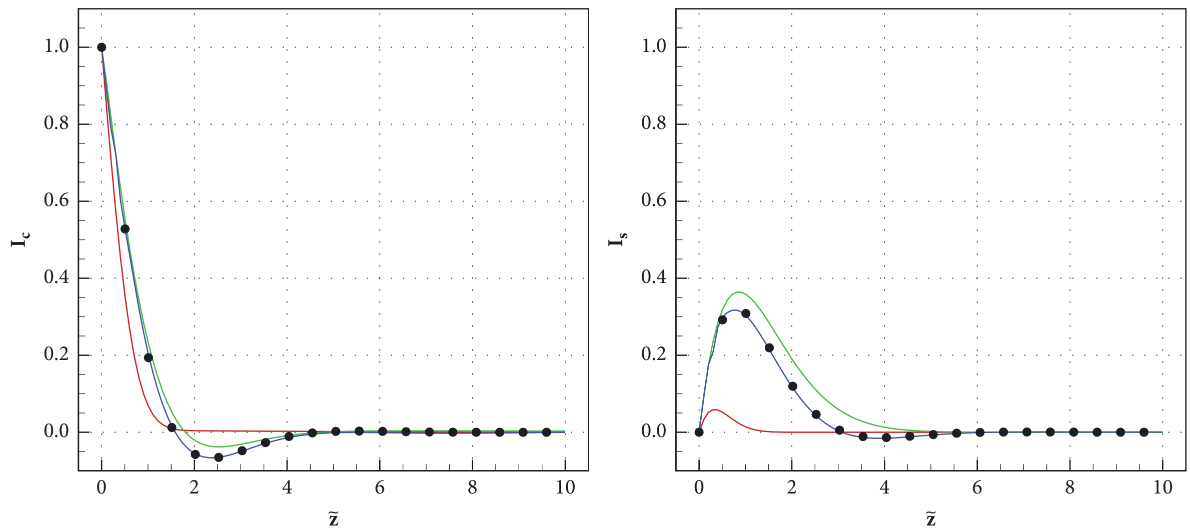

FIGURE 3: Behaviour of I_c and I_s integrals for short (red line), medium (green line), and long (blue line) times. Circular symbols represent the asymptotic solution (13).

Note that the kernel $\cosh[(h-y)\beta]/\cosh(h\beta)$ possesses in the s−plane a branch cut along the negative real axis and also a countable set of real and negative poles of the first order: by accounting for that, for any integer k, $2k+1$ will be indicated with k' and $k'\pi/2$ with K', they are placed on the points $s_k = -K'^2\nu/h^2$. Kernel (16) being an even function of β results in being continuous across the above branch cut, with the exception of the poles.

3.1. First Stokes' Problem. The time derivative of the nondimensional velocity is

$$\tilde{u}_t = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\, e^{st}\frac{\cosh[(h-y)\beta]}{\cosh(h\beta)} := F \tag{17}$$

where the integral is evaluated by applying the residue theorem on the path in Figure 2(a):

$$F = 2\frac{\nu}{h^2}\sum_{k=0}^{\infty} K'\, e^{-K'^2\nu t/h^2}\sin\left(\frac{K'y}{h}\right). \tag{18}$$

By accounting for the identity (see Abramowitz and Stegun [18], pag. 1005),

$$\sum_{k=0}^{\infty}\frac{\sin(K'y/h)}{K'} \equiv \frac{1}{2}, \tag{19}$$

the solution of the problem becomes

$$\tilde{u}(\tilde{y},\tilde{t}) = 1 - 2\sum_{k=0}^{\infty}\frac{\sin(K'\tilde{y})}{K'}\exp\left(-K'^2\tilde{t}\right), \tag{20}$$

where $\tilde{t} = t\nu/h^2$. The wall stress is given by

$$\tilde{\tau}_w = -\frac{2}{Re_0}\sum_{k=0}^{\infty}\exp\left(-K'^2\tilde{t}\right). \tag{21}$$

where $Re_0 = u_0 h/\nu$ is the Reynolds number referring to the constant velocity u_0 and to the distance h between walls.

3.2. Second Stokes' Problem. By inserting the Laplace transform of the wall velocity (9) inside solution (16), its nondimensional value is as follows:

$$\tilde{u} = \frac{1}{2}\left\{e^{-i(\sigma t+\theta)}\underbrace{\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\,\frac{e^{(s+i\sigma)t}}{s+i\sigma}\frac{\cosh[(h-y)\beta]}{\cosh(h\beta)}}_{L_1}\right. \\ \left. + e^{+i(\sigma t+\theta)}\underbrace{\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\,\frac{e^{(s-i\sigma)t}}{s-i\sigma}\frac{\cosh[(h-y)\beta]}{\cosh(h\beta)}}_{L_2}\right\}. \tag{22}$$

Two functions $L_{1,2}(y,t)$ are evaluated by observing that their time derivatives are related to the function F defined in (18):

$$\partial_t L_{1,2} = e^{\pm i\sigma t}F, \tag{23}$$

which are integrated by starting from the initial values:

$$L_{1,2}(\tilde{y},0) = -2\sum_{k=0}^{\infty}\frac{K'\sin(K'\tilde{y})}{K'^2 \mp i\alpha^2} \\ + \frac{\cosh\left[\alpha\left((1\mp i)/\sqrt{2}\right)(1-\tilde{y})\right]}{\cosh\left(\alpha\left((1\mp i)/\sqrt{2}\right)\right)}, \tag{24}$$

where $\alpha = \sqrt{\sigma/\nu}h$. It follows the nondimensional solution:

$$\tilde{u}(\tilde{y},\tilde{t}) = c(\tilde{y})\cos(T+\theta) + d(\tilde{y})\sin(T+\theta) \\ + -2\sum_{k=0}^{\infty}\frac{K'\sin(K'\tilde{y})}{K'^4+\alpha^4}e^{-K'^2T/\alpha^2}\left(K'^2\cos\theta\right. \\ \left. + \alpha^2\sin\theta\right), \tag{25}$$

where the two functions $c(\tilde{y}), d(\tilde{y})$ are given in Appendix C.

In (25), the stationary part of the solution is separated from the transient one and can be conveniently written as Fourier series:

$$\begin{aligned} & c(\tilde{y})\cos(T+\theta) + d(\tilde{y})\sin(T+\theta) \\ & \quad = 2\sum_{k=0}^{\infty}\frac{K'}{K'^4+\alpha^4}\left[K'^2\cos(T+\theta)\right. \\ & \quad \left. + \alpha^2\sin(T+\theta)\right]\sin\left(K'\tilde{y}\right), \end{aligned} \tag{26}$$

allowing the straightforward evaluation of the wall stress:

$$\begin{aligned} \tilde{\tau}_w = {} & \frac{2}{Re_0}\sum_{k=0}^{\infty}\frac{K'^2}{K'^4+\alpha^4}\left[K'^2\cos(T+\theta)\right. \\ & + \alpha^2\sin(T+\theta) \\ & \left. - \left(K'^2\cos\theta + \alpha^2\sin\theta\right)e^{-K'^2T/\alpha^2}\right] \end{aligned} \tag{27}$$

4. One-Dimensional Flow between Moving Walls

The following Stokes' problem is considered:

$$\begin{aligned} & \partial_t u = \nu\partial^2_{yy}u \\ & u(0,t) = f(t), \quad u(h,t) = g(t) \\ & u(y,0) \equiv 0, \end{aligned} \tag{28}$$

where the second boundary condition represents a moving wall with a given velocity law $g(t)$.

The Laplace transform in time of the solution is

$$\hat{u}(y,s) = \hat{f}(s)\frac{\sinh[\beta(h-y)]}{\sinh[\beta h]} + \hat{g}(s)\frac{\sinh[\beta y]}{\sinh[\beta h]}. \tag{29}$$

As in the previous case, the ratios possess in the s-plane a branch cut along the negative real axis and a countable set of real and negative poles of the first order. By indicating with $K'' = k\pi$, they are placed on the points $s_k = -K''^2\nu/h^2$ for $k = 1, 2, \ldots$. The ratios $\sinh[\beta(h-y)]/\sinh[\beta h]$ and $\sinh[\beta y]/\sinh[\beta h]$ are even functions of β, so they are continuous across the branch cut with exception of the poles.

The case with two walls is more complicated, because it could lead to a new class of problems. If the walls have the same velocity law (either constant or periodical), one of the classic Stokes problems arises, but if the laws are different the resulting problem is a new one, which could be denoted as the *mixed problem*. Anyhow it is easy to show that the problem with one or both walls with constant velocity can be seen as a special case of the more general periodical velocity laws with zero frequency and initial phase.

4.1. First Stokes Problem. By assuming constant velocity laws for the walls $\hat{f} = u_0/s$ and $\hat{g} = v_0/s$ and indicating with $\gamma_0 = v_0/u_0$, the time derivative of the velocity is

$$\begin{aligned} \tilde{u}_t = {} & \underbrace{\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\, e^{st}\frac{\sinh[\beta(h-y)]}{\sinh[\beta h]}}_{F_1} \\ & + \underbrace{\frac{\gamma_0}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\, e^{st}\frac{\sinh[\beta y]}{\sinh[\beta h]}}_{F_2}. \end{aligned} \tag{30}$$

The integrals are evaluated by applying the residue theorem on the path of Figure 2(a):

$$\tilde{u}_t = 2\frac{\nu}{h^2}\sum_{k=1}^{\infty}K''\, e^{-K''^2\nu t/h^2}\sin\left(\frac{K''y}{h}\right)\left(1-\gamma_0(-1)^k\right). \tag{31}$$

By taking into account the identity (see Abramowitz and Stegun [18], pag.1005),

$$\sum_{k=1}^{\infty}\frac{\sin\left(K''y/h\right)}{K''}(-1)^k \equiv -\frac{y}{2h} \tag{32}$$

the solution is found with a simple integration in time:

$$\begin{aligned} \tilde{u}(\tilde{y},\tilde{t}) = {} & 1 + (\gamma_0 - 1)\tilde{y} \\ & - 2\sum_{k=1}^{\infty}\frac{\sin\left(K''\tilde{y}\right)}{K''}e^{-K''^2\tilde{t}}\left(1-\gamma_0(-1)^k\right) \end{aligned} \tag{33}$$

where $\tilde{t} = t\nu/h^2$ and $\tilde{y} = y/h$. As expected, the stationary part of the solution is a linear function of $\tilde{y}$ and it depends only on the ratio of the velocities on the walls.

The shear stresses on the walls can be easily computed taken the spatial derivative of the previous relation; for the bottom wall, it reads

$$\tilde{\tau}^b_w(\tilde{t}) = \frac{1}{\tilde{h}}\left[(\gamma_0 - 1) - 2\sum_{k=1}^{\infty}e^{-K''^2\tilde{t}}\left(1-\gamma_0(-1)^k\right)\right] \tag{34}$$

while the stress on the upper wall returns:

$$\tilde{\tau}^u_w(\tilde{t}) = \frac{1}{\tilde{h}}\left[(\gamma_0 - 1) - 2\sum_{k=1}^{\infty}e^{-K''^2\tilde{t}}\left((-1)^k - \gamma_0\right)\right]. \tag{35}$$

4.2. Second Stokes' Problem. By assuming periodical velocity laws for the walls,

$$\begin{aligned} \hat{f}(s) &= \frac{u_0}{2}\left(\frac{e^{-i\theta}}{s+i\sigma} + \frac{e^{i\theta}}{s-i\sigma}\right) \\ \hat{g}(s) &= \frac{v_0}{2}\left(\frac{e^{-i\phi}}{s+i\omega} + \frac{e^{i\phi}}{s-i\omega}\right), \end{aligned} \tag{36}$$

the inverse Laplace transform of the velocity is

$$\tilde{u}(y,t) = \frac{1}{2}\left\{ \underbrace{e^{-i(\sigma t+\theta)} \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds \frac{e^{(s+i\sigma)t}}{s+i\sigma}\frac{\sinh[(h-y)\beta]}{\sinh[h\beta]}}_{S_1} + \underbrace{e^{+i(\sigma t+\theta)} \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds \frac{e^{(s-i\sigma)t}}{s-i\sigma}\frac{\sinh[(h-y)\beta]}{\sinh[h\beta]}}_{S_2} + \underbrace{e^{-i(\omega t+\phi)} \frac{\gamma_0}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds \frac{e^{(s+i\omega)t}}{s+i\omega}\frac{\sinh[y\beta]}{\sinh[h\beta]}}_{S_3} + \underbrace{e^{+i(\omega t+\phi)} \frac{\gamma_0}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds \frac{e^{(s-i\omega)t}}{s-i\omega}\frac{\sinh[y\beta]}{\sinh[h\beta]}}_{S_4} \right\}. \tag{37}$$

Two functions $S_{1,2}(y,t)$ are evaluated by observing that their time derivatives are related to the function F_1 defined in (30):

$$\partial_t S_{1,2} = e^{\pm i\sigma t} F_1, \tag{38}$$

which are integrated by starting from the initial values:

$$S_{1,2}(\tilde{y},0) = -2\sum_{k=1}^{\infty}\frac{K'' \sin(K''\tilde{y})}{K''^2 \mp i\alpha^2} + \frac{\sinh\left[\alpha\left((1\mp i)/\sqrt{2}\right)(1-\tilde{y})\right]}{\sinh\left(\alpha\left((1\mp i)/\sqrt{2}\right)\right)}. \tag{39}$$

where $\alpha = \sqrt{\sigma/\nu}h$. In the same way, the time derivatives of the functions $S_{3,4}(y,t)$ are related to function F_2 defined in (30),

$$\partial_t S_{3,4} = e^{\pm i\omega t} F_2, \tag{40}$$

and are integrated by starting from the initial values:

$$S_{3,4}(\tilde{y},0) = 2\sum_{k=1}^{\infty}(-1)^k \frac{K'' \sin(K''\tilde{y})}{K''^2 \mp i\alpha_2^2} + \gamma_0\frac{\sinh\left(\alpha_2\left((1\mp i)/\sqrt{2}\right)\tilde{y}\right)}{\sinh\left(\alpha_2\left((1\mp i)/\sqrt{2}\right)\right)}, \tag{41}$$

where $\alpha_2 = \sqrt{\omega/\nu}h$. The solution may be rewritten in terms of $S_{1,2,3,4}$ as

$$\tilde{u} = \frac{1}{2}\left[\overbrace{(S_1+S_2)\cos(T+\theta) + i(S_2-S_1)\sin(T+\theta)}^{AA} + \underbrace{(S_3+S_4)\cos(T_2+\phi) + i(S_4-S_3)\sin(T_2+\phi)}_{BB}\right] \tag{42}$$

where $T_2 = \omega t$ is the nondimensional time with the frequency of the top wall. In order to simplify the expression of the solution, avoiding the presence of complex terms, the following quantities are evaluated:

$$AA = -4\sum_{k=1}^{\infty}\frac{K''\sin(K''\tilde{y})}{K''^4+\alpha^4} e^{-T/\alpha^2 K''^2}\left[K''^2\cos\theta + \alpha^2\sin\theta\right] + 2c_1(\tilde{y})\cos(T+\theta) + 2d_1(\tilde{y})\sin(T+\theta)$$
$$BB = +4\gamma_0\sum_{k=1}^{\infty}\frac{K''\sin(K''\tilde{y})}{K''^4+\alpha_2^4} e^{-T_2/\alpha_2^2 K''^2}\left[K''^2\cos\phi + \alpha_2^2\sin\phi\right](-1)^k + 2\gamma_0 c_2(\tilde{y})\cos(T\psi+\phi) + 2\gamma_0 d_2(\tilde{y})\sin(T\psi+\phi) \tag{43}$$

where the complete expressions of the stationary terms c_1, d_1, c_2, d_2 are given in Appendix C. It follows the nondimensional solution:

$$\tilde{u}(\tilde{y},T,T_2) = \tilde{u}_{st}(\tilde{y},T,T_2) + \tilde{u}_{tr}(\tilde{y},T,T_2) \tag{44}$$

where $\tilde{u}_{st}$ is the steady-state part of the solution:

$$\tilde{u}_{st}(\tilde{y},T,T_2) = c_1(\tilde{y})\cos(T+\theta) + d_1(\tilde{y})\sin(T+\theta) + \gamma_0\left(c_2(\tilde{y})\cos(T_2+\phi) + d_2(\tilde{y})\sin(T_2+\phi)\right) \tag{45}$$

while $\tilde{u}_{tr}$ is the transient part of the solution:

$$\tilde{u}_{tr}(\tilde{y},T,T_2) = -2\sum_{k=1}^{\infty}K''\sin(K''\tilde{y})e^{-T/\alpha^2 K''^2} \cdot \frac{K''^2\cos\theta + \alpha^2\sin\theta}{K''^4+\alpha^4} + 2\sum_{k=1}^{\infty}K''\sin(K''\tilde{y}) \cdot e^{-T_2/\alpha_2^2 K''^2}\frac{K''^2\cos\phi + \alpha_2^2\sin\phi}{K''^4+\alpha_2^4}\gamma_0(-1)^k. \tag{46}$$

The shear stresses on the walls can be computed by deriving expression (44) with respect to the normal direction; their value for $\tilde{y} = 0$ gives the stress on the bottom wall:

$$\tilde{\tau}_w^b(T,T_2) = c_1'(0)\cos(T+\theta) + d_1'(0)\sin(T+\theta) + +\gamma_0\left(c_2'(0)\cos(T_2+\phi) + d_2'(0)\sin(T_2+\phi)\right) + -2\sum_{k=1}^{\infty}\frac{K''^2\left(K''^2\cos\theta+\alpha^2\sin\theta\right)}{K''^4+\alpha^4}e^{-T/\alpha^2 K''^2} + +2\sum_{k=1}^{\infty}\frac{K''^2\left(K''^2\cos\phi+\alpha_2^2\sin\phi\right)}{K''^4+\alpha_2^4}e^{-T_2/\alpha_2^2 K''^2}\gamma_0(-1)^k \tag{47}$$

while the value of the derivative at $\tilde{y} = 1$ gives the shear stress on the upper wall:

$$\tilde{\tau}_w^u(T, T_2) = c_1'(1)\cos(T+\theta) + d_1'(1)\sin(T+\theta) + +\gamma_0\left(c_2'(1)\cos(T_2+\phi) + d_2'(1)\sin(T_2+\phi)\right) + -2\sum_{k=1}^{\infty}\frac{K''^2\left(K''^2\cos\theta + \alpha^2\sin\theta\right)}{K''^4+\alpha^4}e^{-T/\alpha^2 K''^2}(-1)^k + +2\sum_{k=1}^{\infty}\frac{K''^2\left(K''^2\cos\phi + \alpha_2^2\sin\phi\right)}{K''^4+\alpha_2^4}e^{-T_2/\alpha_2^2 K''^2}\gamma_0 \quad (48)$$

The expression of the derivatives c_1', d_1', c_2', d_2' can be found in Appendix C.

As it has been anticipated, the solution for the *First Stokes'* problem can be recovered from the solution of the problem with two moving boundaries (44). As a matter of the fact, by considering that $c_1(\tilde{y}) \longrightarrow (1-\tilde{y})$ for $\alpha \longrightarrow 0$ and $c_2(\tilde{y}) \longrightarrow \tilde{y}$ for $\alpha_2 \longrightarrow 0$, the steady-state solution (45) goes to the stationary part of the (33), whereas, for the transient part it can be easily demonstrated that, by taking θ, ϕ, α, and α_2 in (46) equal to zero, the transient part of (33) is recovered.

5. One-Dimensional Flow between Moving Walls with Pressure Gradient

A useful extension of the Stokes' problems can be achieved by considering the presence of a pressure gradient in the motion equation (see Xu and Bowen [5]; Lauga and Stone [3]); this term must be a function of time only [1], so that the Laplace transform of the solution is very similar to the previous case, allowing a direct inversion with the same procedure of Section 4. Stokes' problem considered is as follows:

$$\partial_t u = -\partial_x P + \nu\partial_{yy}^2 u$$
$$u(0,t) = f(t), \quad u(h,t) = g(t) \quad (49)$$
$$u(y,0) \equiv 0,$$

where $P = p/\rho$ and the boundary conditions refer to oscillatory moving walls, since the case with constant velocity can be seen as a particular case (see Section 4). A constant $-\partial_x P = \lambda_0$ and sinusoidal time law $-\partial_x P = \lambda_0\cos(\chi t + \Theta)$ are considered for the new term; in this way, as it will be shown in Section 7, the solution for the case with a generally periodical function of time for the pressure gradient can be easily obtained by exploiting the linearity of the problem and taking the Fourier series of the pressure term. The Laplace transform in time of the solution (49) is

$$\hat{u}(y,s) = \hat{f}(s)\frac{\sinh[\beta(h-y)]}{\sinh[\beta h]} + \hat{g}(s)\frac{\sinh[\beta y]}{\sinh[\beta h]} - \frac{\hat{\lambda}(s)}{s}\left\{\frac{\sinh[\beta(h-y)]}{\sinh[\beta h]} + \frac{\sinh[\beta y]}{\sinh[\beta h]} - 1\right\}. \quad (50)$$

The first two quantities in the right-hand side are the solution of the associated homogeneous equation (the same just discussed in Section 4); similarly the pressure term is composed by two terms with the same poles $s_k = -K''^2\nu/h^2$ plus a pole on the origin and a last term with a single pole on the origin.

The inverse Laplace transform of the velocity is a linear combination of the solutions deduced in Section 4 (the first or second Stokes' problem according to the expressions of $f(t)$ and $g(t)$), so only the inversion of pressure term, indicated with $\hat{u}^p$, will be here discussed:

$$u^p(y,t) = \frac{1}{2\pi i}\cdot\int_{\mu-i\infty}^{\mu+i\infty} ds\, e^{st}\frac{\hat{\lambda}}{s}\left[\frac{\sinh[(h-y)\beta]}{\sinh[h\beta]} + \frac{\sinh[y\beta]}{\sinh[h\beta]}\right] - \int_0^t dt\,\lambda(t) \quad (51)$$

5.1. Constant Pressure Gradient. By assuming a constant pressure gradient, the expression of u_t^p is as follows:

$$u_t^p = \frac{\lambda_0}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds\,\frac{e^{st}}{s}\left[\frac{\sinh[(h-y)\beta]}{\sinh[h\beta]} + \frac{\sinh[y\beta]}{\sinh[h\beta]}\right] - \lambda_0. \quad (52)$$

A simple integration in time, starting by the initial value,

$$u_0^p = \frac{\lambda_0 h^2}{2\nu}\left(\tilde{y}^2 - \tilde{y}\right) + 4\frac{\lambda_0 h^2}{\nu}\sum_{k=1}^{\infty}\frac{\sin(K_o\tilde{y})}{K_o^3}, \quad (53)$$

which can be derived from the general expression of the inverse transform of $\hat{u}^p$ for $t = 0$, leads to the particular solution:

$$\tilde{u}^p(\tilde{y},\tilde{t}) = \frac{Hg}{2\,Re_0}\left(\tilde{y}^2 - \tilde{y}\right) + 4\frac{Hg}{Re_0}\sum_{k=1}^{\infty}\frac{\sin(K_o\tilde{y})}{K_o^3}e^{-K_o^2\tilde{t}} \quad (54)$$

where the stationary part of the solution is the well-known Poiseuille solution between planar solid walls. In the previous relations $K_o = (2k-1)\pi$ and $Hg = \lambda_0 h^3/\nu^2$ is the Hagen number. It is worth noticing that, from the identity

$$\sum_{k=1}^{\infty}\frac{\sin(K_o\tilde{y})}{K_o^3} \equiv \frac{1}{8}\left(\tilde{y} - \tilde{y}^2\right) \quad (55)$$

(see Abramowitz and Stegun [18] pag. 1005) the initial condition $\tilde{u}^p(\tilde{y},0) = 0$ is satisfied.

5.2. Sinusoidal Pressure Gradient. By assuming a sinusoidal law for the pressure gradient, the expression of u^p is as follows:

$$u^p(y,t) = \frac{\lambda_0}{2}(Z_1 + Z_2 + Z_3 + Z_4) + \frac{\lambda_0}{\chi}\left(\sin\Theta - \sin(\chi t + \Theta)\right) \quad (56)$$

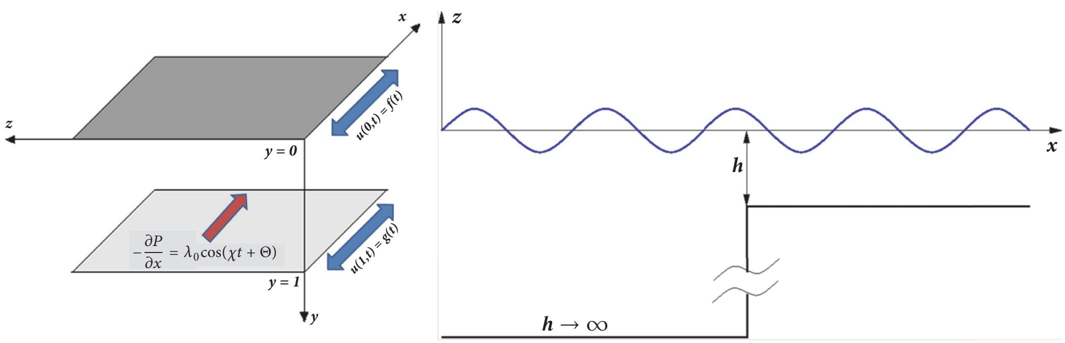

FIGURE 4: Sketches of the example problems presented in Section 6.1 (left) and in Section 6.2 (right).

where

$$Z_{1,2} = e^{\mp i(\chi t+\Theta)} \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} ds \frac{e^{(s\pm i\chi)t}}{s(s\pm i\chi)} \frac{\sinh[(h-y)\beta]}{\sinh[h\beta]} \quad (57)$$

$$Z_{3,4} = e^{\mp i(\chi t+\Theta)} \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} ds \frac{e^{(s\pm i\chi)t}}{s(s\pm i\chi)} \frac{\sinh[y\beta]}{\sinh[h\beta]}. \quad (58)$$

By observing that the time derivatives of $Z_{1,2}$ are similar to $S_{1,2}$ and the derivatives of $Z_{3,4}$ are similar to $S_{3,4}$ (the sums $Z_1 + Z_2$ and $Z_3 + Z_4$ are given in Appendix D), a simple integration in time gives back the solution:

$$\tilde{u}^p(\tilde{y}, T_3) = \frac{Hg}{\alpha_3 Re_0} \left(\tilde{u}^p_{st} + \tilde{u}^p_{tr}\right) \quad (59)$$

where the definition of α_3 is given in Appendix D. The stationary part of the solution is

$$\begin{aligned} \tilde{u}^p_{st}(\tilde{y}, T_3) = & -\sin(T_3 + \Theta) \\ & + [c_1(\tilde{y}) + c_2(\tilde{y})] \sin(T_3 + \Theta) \\ & - [d_1(\tilde{y}) + d_2(\tilde{y})] \cos(T_3 + \Theta) \end{aligned} \quad (60)$$

while the transient part is

$$\begin{aligned} \tilde{u}^p_{tr}(\tilde{y}, T_3) = & 4\alpha_3^2 \sum_{k=1}^{\infty} \frac{\sin(K_o \tilde{y})}{K_o(K_o^4 + \alpha_3^4)} e^{-K_o^2 T_3/\alpha_3^2} \left[K_o^2 \cos\Theta \right. \\ & \left. + \alpha_3^2 \sin\Theta\right] \end{aligned} \quad (61)$$

whereas the definition of T_3 is given in Appendix D.

It is useful to notice that

$$\begin{aligned} & \lim_{\chi\to 0} [c_1(\tilde{y}) + c_2(\tilde{y})] = 1 \\ & \lim_{\chi\to 0} [d_1(\tilde{y}) + d_2(\tilde{y})] \frac{Hg}{\alpha_3 Re_0} = \frac{Hg}{2Re_0}\left(\tilde{y} - \tilde{y}^2\right) \end{aligned} \quad (62)$$

so that, for $\Theta = 0$, the limit for $\chi \longrightarrow 0$ of $\tilde{u}^p_{st}$ goes to the stationary part of the solution (54), while for the transient part $\tilde{u}^p_{tr}$ the limit is obvious.

6. Examples

In this section, two examples of solutions for the extended Stokes's problem are given; namely, a general case of Stokes' flow within two moving walls without and with a pressure gradient (see also Durante and Broglia [19]) and a more physical application in which the complete primary wave motion for gravity waves will be given as an extension of the solutions presented in Xu and Bowen [5]. The problems are outlined in Figure 4 for the sake of clearness.

6.1. Flow within Two Moving Walls. In this section, the cases of the Stokes' flow within two moving walls without (Section 4) or with (Section 5) a pressure gradient are presented. For both cases the two walls have an oscillatory movement, in particular, the problem is defined by the following periodical boundary conditions:

$$\begin{aligned} u(0,t) &= f(t) = u_0 \cos(\sigma t + \theta) \\ u(1,t) &= g(t) = v_0 \cos(\omega t + \phi) \end{aligned} \quad (63)$$

with $v_0 = 0.5\, u_0$, $\sigma = 10\pi$, $\omega = 20\pi$, $\theta = \pi/4$ and $\phi = 0$. For the problem with nonzero pressure gradient, the following law has been set:

$$-\partial_x P = \lambda_0 \cos(\chi t + \Theta) \quad (64)$$

with $\lambda_0 = 10$, $\chi = 3/10\pi$ and $\Theta = \pi/2$. In Figure 5, the solutions at six time instants are given; for the sake of completeness, the transient and the steady-state components of the solutions are individually plotted as well. The time instants shown are chosen between an early stage ($t = 0.01$) (where the transient part is still of the same order of the stationary part), passing through an intermediate stage ($t = 0.07$) (the energy associated with the transient part is only a small fraction of that associated with the steady-state part) and finally to a late stage ($t = 1.84$) (the transient part is negligible).

In order to give a comparison with the case where a pressure gradient is enforced, Figure 6 is sketched. In an initial stage the contributions coming from stationary and transient parts are higher respect to the former case. The transient part becomes practically zero at the late stage, as before. The correctness of the analytical solutions and of

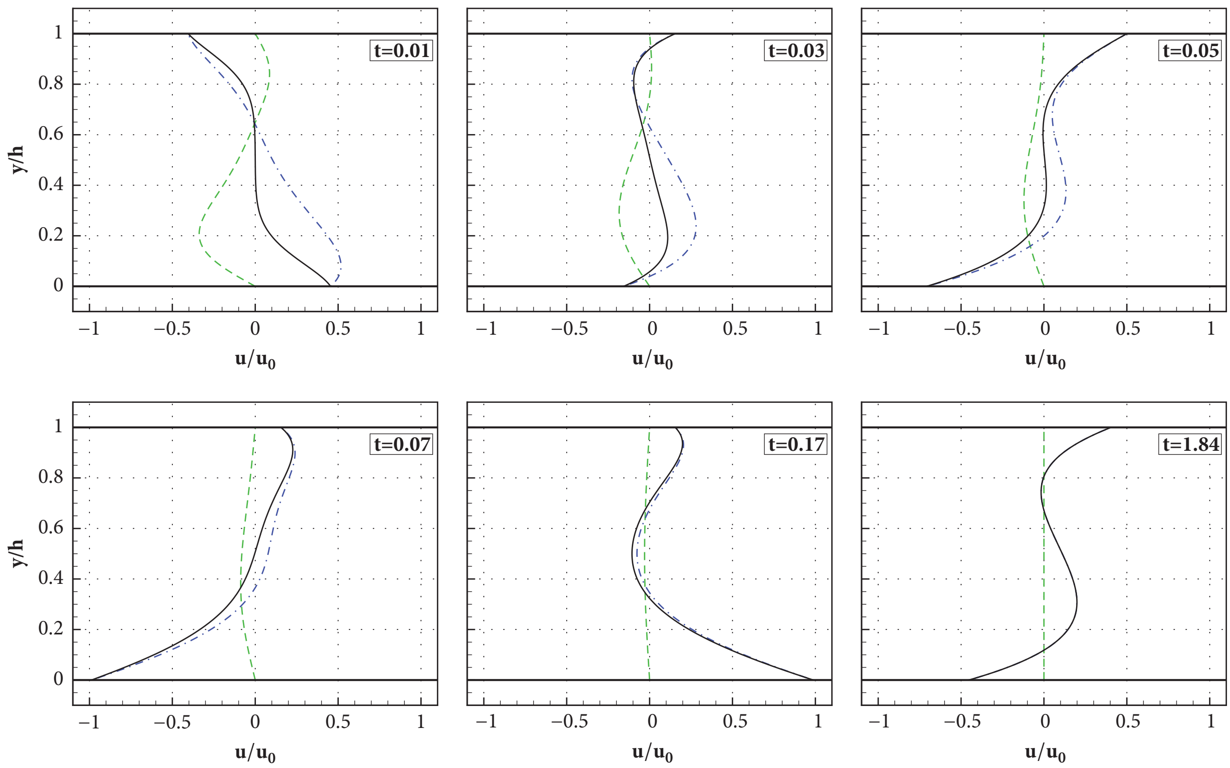

FIGURE 5: One-dimensional flow between two walls with periodical motion. Solid line: global solution; dashed line: transient part; dashed-dotted line, stationary part.

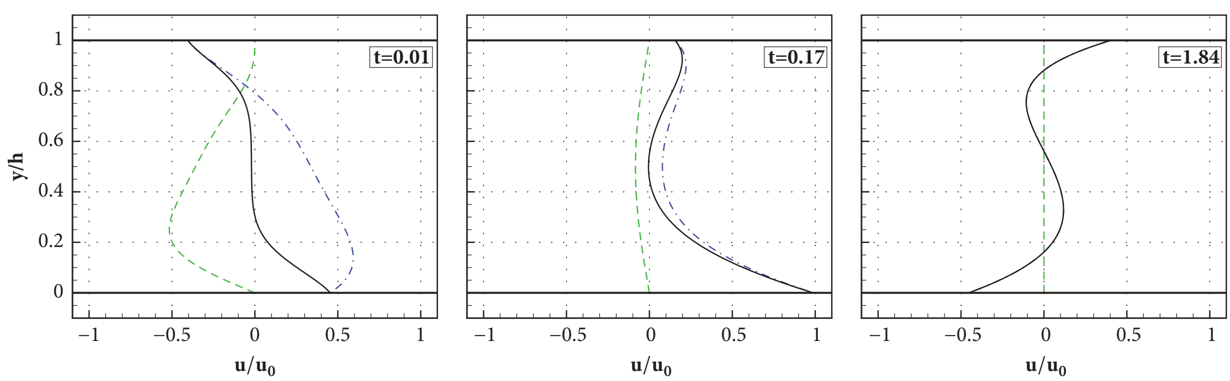

FIGURE 6: One-dimensional flow between two walls with periodical motion and pressure gradient. Solid line: global solution; dashed line: transient part; dashed-dotted line, stationary part.

their evaluation has been checked by a comparison with a numerical solution of the related Stokes' problem. The solutions and the comparison errors are given in Figure 7; the comparison error, which is substantially the truncation error of the numerical scheme, is always less than 2.5 10^{-5}. It has been tested that this error goes to zero accordingly with the accuracy of the numerical scheme.

6.2. The Complete Primary Wave Motion for Gravity Waves. The motion field of a gravity wave has been largely investigated since Stokes in 1847 found that a net material transport (called Stokes' drift) is associated with a periodic wave field, being in this way responsible for an advection process in the dispersion of passive scalars over a fluid surface. The equations of motion of a gravity wave (described as $f_s = a\cos(kx - \sigma t)$) in a viscous and incompressible fluid are normally written in terms of Stokes problems:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$
$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2} \qquad (65)$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

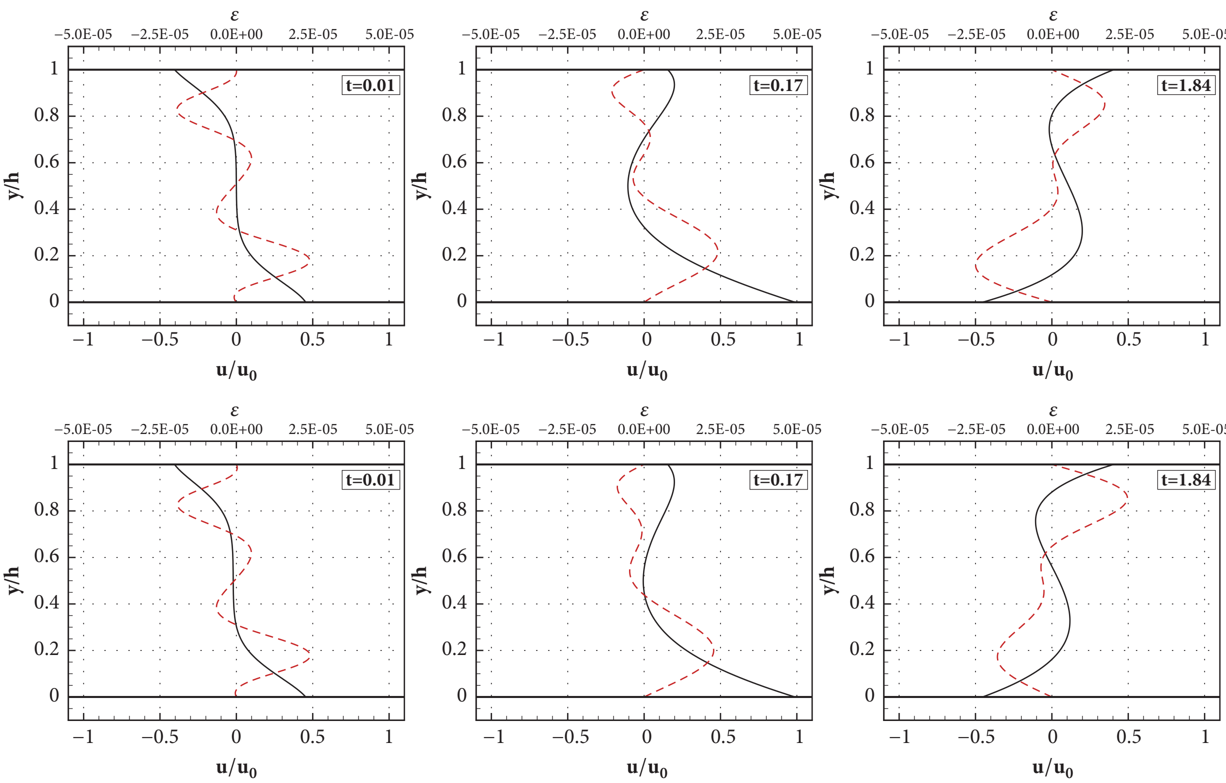

FIGURE 7: One-dimensional flow between two walls with periodical motion. The analytical solution is drawn with a solid line, whereas the error between numerical and analytical solutions is in dashed line (the reference axis is plotted on top). Top row, zero pressure gradient; bottom row, periodical pressure gradient.

where $p = p_d + \rho g z$ is the total pressure, i.e., the sum of the hydrostatic and the dynamical contributions. For a finite depth flow, the inviscid solution is achieved by supposing the velocity field to be irrotational and the amplitude a of the surface wave small enough that the kinematic condition $p = 0$, to be satisfied on the surface $z = f_s$, is instead considered on $z = 0$. In this way the pressure field is found to be a periodical function of time:

$$p = \rho g a \frac{\cosh\left(kz'\right)}{\cosh(kh)} \cos(kx - \sigma t) \tag{66}$$

where $z' = z + h$, h is the distance of mean free surface ($z = 0$) from the bottom and k is the wave-number (see Xu and Bowen [5]). The u and w components of the velocity field are found through direct integration in time of x and z pressure derivatives, respectively, giving

$$\begin{aligned} u_{nv} &= a\sigma \frac{\cosh\left(kz'\right)}{\sinh(kh)} \cos(kx - \sigma t) \\ w_{nv} &= a\sigma \frac{\sinh\left(kz'\right)}{\sinh(kh)} \sin(kx - \sigma t) \end{aligned} \tag{67}$$

where $\sigma^2 = gk\tanh(kh)$ and subscript nv stands for *nonviscous* solutions. Normally in earth sciences the transient phenomena are not interesting on large-scale problems, often adopting simplified approaches to achieve analytical solutions. This is evident in the paper Xu and Bowen [5], where the perturbed potential approach has been exploited for resolving the problem (65). A more detailed solution can be derived without any further simplification, by a simple application of the procedure discussed in the previous sections.

By considering a wave travelling along x direction in deep water (where the effect of viscosity is not taken into account) which encounters an increasing bottom, it is clear that the solution passes from a nonviscous to a viscous one. In the classical approach of earth sciences the unsteady effect which comes from this event is normally neglected. Nonetheless, it will be pointed out that once the complete sets of the Stokes problems solutions have been deduced, a solution, which considers transient effects, is simple enough to be achieved.

The boundary conditions

$$\begin{aligned} u\left(0, x, z'\right) &= u_{nv}\left(0, x, z'\right) \\ u(t, x, 0) &= 0 \\ u - u_{nv} &\longrightarrow 0 \quad \text{for large } z' \end{aligned} \tag{68}$$

for u hold, the homogeneity at $z' = 0$ being enforced for w.

The solution u will be extensively discussed, while w will be deduced by a direct integration of the continuity equation.

The viscous solution is associated with a different and unknown pressure field, which, in general, is a function of t, x, z. By considering the nonviscous solution, the pressure field must be the product of a function of z only and a function of x and t, as it will be clear later.

The Laplace transform of the solution is

$$\hat{u}_v = Ae^{-\beta z'} + Be^{+\beta z'} + \hat{f} \quad (69)$$

where $\hat{f}$ is the particular integral. By considering that for large z' the viscous solution must match the nonviscous one, the following condition holds:

$$Be^{\beta z'} + \hat{f} = \hat{u}_{nv} \quad (70)$$

where a term of kind $Be^{\beta z'}$ is not present in $\hat{u}_{nv}$. Obviously, B must be zero in order to allow this identity and $\hat{f}$ is equal to the Laplace transform of the nonviscous solution. The other constant A is found by the no-slip boundary condition, then the antitransform is as follows:

$$u = u_{nv} - \frac{a\sigma}{\sinh(kh)} \frac{1}{2\pi i} \cdot \int_{\mu-i\infty}^{\mu+i\infty} ds \frac{\exp[st - \beta z']}{2} \left(\frac{e^{-ikx}}{s - i\sigma} + \frac{e^{+ikx}}{s + i\sigma} \right) \quad (71)$$

where the integral can be directly evaluated, having the expressions of $G_{1,2}$ (see Section 2.2). The final expression of the solution can be finally found as

$$\tilde{u} = \tilde{u}_{nv} - \frac{1}{\sinh(kh)} [\cos(kx - T)\ I_c(\tilde{\eta}, T) - \sin(kx - T)\ I_s(\tilde{\eta}, T)] \quad (72)$$

where $\tilde{u} = u/(a\sigma)$, $T = \sigma t$ and I_c, I_s are the integral expressions described in Section 2.2 and numerically investigated in Appendix (Appendix B), with $\tilde{\eta} = \tilde{z}/\sqrt{2T}$ ($\tilde{z} = z'\sqrt{\sigma/(2\nu)}$). Solution (72) must match the nonviscous solution for $\tilde{z} \longrightarrow \infty$ and this is guaranteed by the behaviour of the (12) for $\tilde{y}$ diverging.

As it can be clearly noticed, the solution is composed by a term u_{nv} which resolves the nonviscous equation with a prescribed pressure gradient and by a second term u_v which resolves a viscous equation without a pressure gradient but with a prescribed oscillating boundary condition on the bottom and a homogeneous boundary condition at infinite.

For large T, solution (72) recovers the simplified one:

$$\tilde{u} = \tilde{u}_{nv} - \frac{e^{-\tilde{z}}}{\sinh(kh)} \cos(kx - T + \tilde{z}) \quad (73)$$

presented in Xu and Bowen [5], as it can be noticed from the behaviour of integrals I_c and I_s sketched in Figure 3.

By exploiting the continuity, the z-component of velocity w is found to be straightforward:

$$\tilde{w} = \tilde{w}_{nv} - \frac{1}{\sinh(kh)} \frac{k}{\chi} \left[\sin(kx - T) \int_0^{\tilde{z}} dz\ I_c + \cos(kx - T) \int_0^{\tilde{z}} dz\ I_s \right] \quad (74)$$

where $\chi^2 = \sigma/(2\nu)$. The expression of Xu and Bowen [5] can be recovered, by considering (13).

In order to establish the behaviour of the complete solution with respect to the approximate one, the velocity field may be considered as the superposition of a nonviscous and a viscous field:

$$u = u_{nv} + u_v$$
$$w = w_{nv} + w_v \quad (75)$$

and the viscous kinetic energy

$$\frac{E_c}{\rho a^3 \sigma^2} = \tilde{E}_c = \frac{1}{2} \int_0^{\tilde{z}_m} dz \left(u_v^2 + w_v^2 \right) \quad (76)$$

($\tilde{z}_m$ is the nondimensional distance from the bottom) can be evaluated. In Figure 8 the kinetic energy associated with the viscous part of the velocity fields (72) and (74) is shown in comparison with the same fields of approximate solutions. The difference is initially of about 10% and it reduces to 3% for $T = 40$ and to less than 1% for $T = 80$ (not shown). This means that, for a gravity wave with a wavelength of 62.8 meters ($\sigma \approx 1s^{-1}$), which passes from a deep to a 50-meter bottom, the kinetic energy associated with the viscous part is described with a satisfactory agreement (error less than 1%) with the approximate solutions after 80 seconds.

To get the pressure field, it is worth noticing that $\hat{u}_{nv}$ satisfies the nonhomogeneous transformed equation, so it can be inserted into together with the transformed pressure gradient, arriving to the complete expression of the pressure field:

$$p = \rho g a \cdot \frac{\cosh(kz')}{\cosh(kh)} \left[\cos(kx - \sigma t) + \nu \frac{k^2}{\sigma} \sin(kx - \sigma t) \right] \quad (77)$$

which returns (66) for $\nu \longrightarrow 0$. The pressure field possess two components: one is in phase with the velocity field u_{nv} along the propagation direction, while the other component (the viscous) is in phase with the vertical component of velocity w_{nv}. As the decomposition of the velocity field (75) is valid, it similarly holds for the pressure. In particular, the gradient of the viscous component equals the Laplacian of the nonviscous velocity.

7. Discussion

The classical approach to Stokes' problems consists in solving a parabolic linear differential problem with two different kind of boundary conditions: constant or sinusoidal. These problems may be solved for bounded or unbounded domains and for Dirichlet or Neumann boundary conditions.

When a wall (see Section 2) or two walls (Section 4) move with a generally periodical time law, the problem may be approached developing in Fourier series both the functions describing the movement of the boundaries and the periodical pressure gradient.

In order to underline this point, consider the differential problem (49) and assume $\lambda(t)$ to be a periodical function with

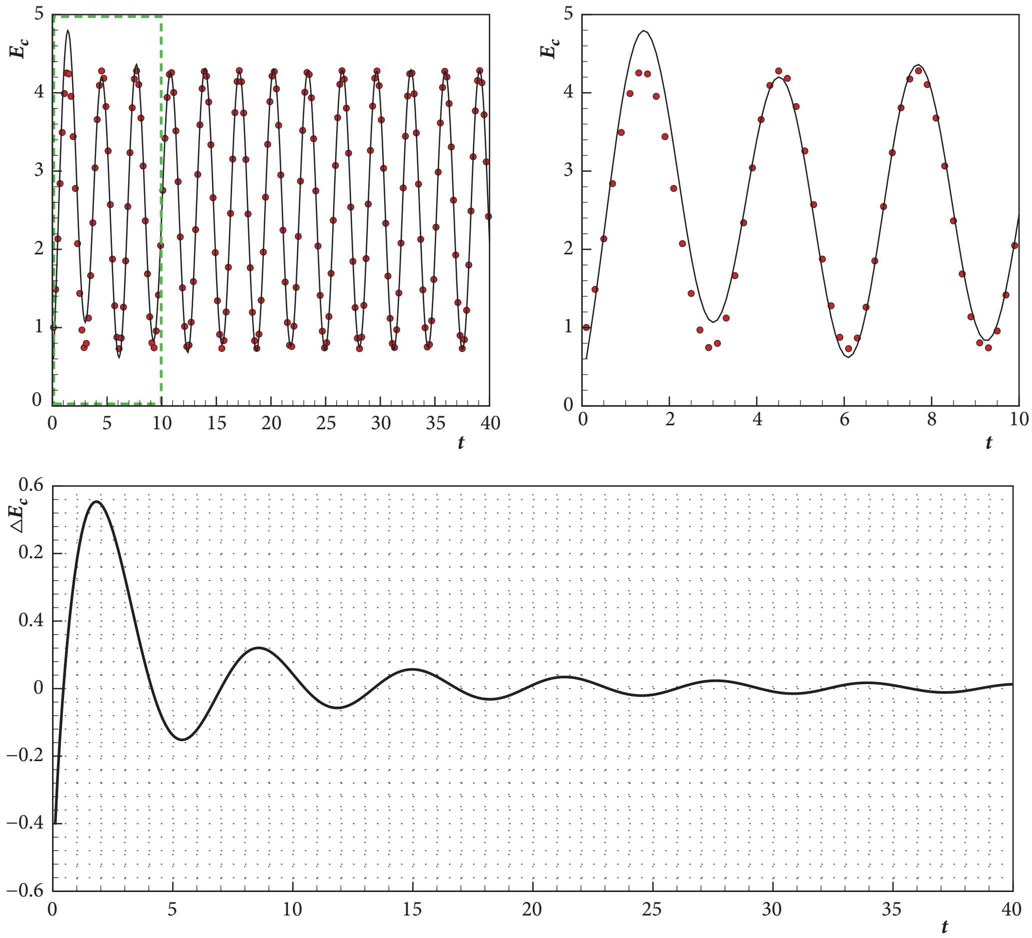

FIGURE 8: On the top, the viscous kinetic energy calculated with the approximate solutions of Xu and Bowen [5] is plotted with symbols and compared with the analytical solution, represented with solid line. On the top right, the magnifying of the comparison for short times. On the bottom, the difference between the two solutions. In abscissa, the time is nondimensional with frequency σ. The values of energy are multiplied for 10^5.

a period τ; the pressure term in (50) may be developed in Fourier series and recast as

$$\begin{aligned}\widehat{u}^p(s)+\frac{\widehat{\lambda}}{s}=&\frac{\lambda_0}{s^2}\left(\frac{\sinh[\beta(h-y)]}{\sinh[\beta h]}+\frac{\sinh[\beta y]}{\sinh[\beta h]}\right)\\&+\sum_{l=1}^{\infty}\frac{\lambda_l^c}{2s}\left(\frac{1}{s+iL}+\frac{1}{s-iL}\right)\\&\cdot\left(\frac{\sinh[\beta(h-y)]}{\sinh[\beta h]}+\frac{\sinh[\beta y]}{\sinh[\beta h]}\right)\\&+\sum_{l=1}^{\infty}\frac{\lambda_l^s}{2s}\left(\frac{i}{s+iL}-\frac{i}{s-iL}\right)\\&\cdot\left(\frac{\sinh[\beta(h-y)]}{\sinh[\beta h]}+\frac{\sinh[\beta y]}{\sinh[\beta h]}\right)\end{aligned}\tag{78}$$

where the inversion of the first row leads to a solution of kind (54), while the second and the third rows lead to a linear superposition of solutions of kind (59). The term $\widehat{\lambda}/s$ has been moved to the left-hand side and is not discussed, because the inversion is a trivial integration in time.

8. Conclusions and Future Perspectives

In this paper a general methodology for achieving the analytical solution of the one-dimensional Stokes' problem has been illustrated; solutions for constant and periodical velocity laws of the boundaries have been deduced. The present strategy has allowed taking into account the presence of a time-dependent pressure gradient. Moreover, differently from other approaches, the residual theorem allows with a reasonable mathematical effort getting time solutions in closed forms.

Solutions for constant and periodical forcing term have been derived and the complete primary motion field for a gravity wave has been achieved. The generalization to generally periodic boundary conditions and pressure gradient has been also discussed.

Numerical solutions of the Stokes' problem, provided by a second-order finite difference scheme, have been employed in order to check the correctness of the analytical solutions, as well as of their evaluations.

Thanks to the present approach the extension to two-dimensional problems with nonzero time-dependent pressure gradient is possible and will be the matter of future works. The extension of this strategy to different geometries is also under investigation.

Appendix

A. Calculation of $G_{1,2}(y,0)$

The goal is the demonstration of the nullity of $G_{1,2}$ which consists in calculating the integral:

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} ds \frac{\exp(-\beta y)}{s \pm i\sigma} \tag{A.1}$$

where the integral function possesses one pole on $\mp i\sigma$ and a phase discontinuity through the negative real axis. A double application of residuals theorem will be performed.

By considering Figure 2(a), the integral can be written as the limit for $M \longrightarrow \infty$ of the difference among the integral on the whole path (called $\mathscr{C}$) and the integrals calculated on the following curves:

(1) the segment $(\mu, iM) - (0, iM)$
(2) the segment $(0, -iM) - (\mu, -iM)$
(3) the upper quarter of circle
(4) the lower quarter of circle
(5) the segment $(-iM, 0) - (0, 0)$
(6) the segment $(0, 0) - (-iM, 0)$,

where integrals (3) and (4) go to zero, while integrals (1) and (2) are opposite and their sum is identically null. By taking into account only G_1 (for G_2 the following considerations are identical), the integral can be expressed as

$$\frac{1}{2\pi i}\int_{\mathscr{C}} ds \frac{\exp(-\beta y)}{s + i\sigma} + \frac{1}{2\pi i} \cdot \int_{-\infty}^{0} dx \frac{\exp\left(iy\sqrt{-x/\nu}\right) - \exp\left(-iy\sqrt{-x/\nu}\right)}{x + i\sigma} \tag{A.2}$$

where the first integral can be evaluated directly by mean of the residuals theorem as

$$\frac{1}{2\pi i}\int_{\mathscr{C}} ds \frac{\exp(-\beta y)}{s + i\sigma} = \exp\left[-\sqrt{\frac{\sigma}{\nu}}y\frac{\sqrt{2}}{2}(1-i)\right] \tag{A.3}$$

with a consistent choice of the argument of the root $\sqrt{-i} = \sqrt{2}/2(1-i)$. By performing the change $\xi^2 = -x$, the integrals on the real axis are expressed as

$$-\frac{1}{2\pi i}\int_{0}^{+\infty} d\xi \frac{2\xi}{\xi^2 - i\sigma}\left[\exp(i\alpha\xi) - \exp(-i\alpha\xi)\right] \tag{A.4}$$

where $\alpha = y/\sqrt{\nu}$. The calculation of the latter integral needs a further application of the residuals theorem. By considering the fact that the integrand is an even function, once written in complex form, the integral is recasted as

$$-\frac{1}{\pi i}\int_{\mathscr{C}^+} dz \frac{2z\exp(i\alpha z)}{(z - z_1)(z - z_2)} \tag{A.5}$$

where $z_1 = \sqrt{2\sigma}/2(1+i)$ and $z_2 = -z_1$ are the poles of the integrands and $\mathscr{C}^+$ is the path of integration (the half of circle in the upper half plane). An application of the residuals theorem shows that

$$-\frac{1}{\pi i}\int_{\mathscr{C}^+} dz \frac{2z\exp(i\alpha z)}{(z - z_1)(z - z_2)} = -\exp(i\alpha z_1) \tag{A.6}$$

being this last term opposite to (A.3), an their sum is identically null.

B. Evaluation of the Integrals by Series

In Appendix, the integrals in the different solutions are analytically evaluated by series. Routines which calculate these integrals are available on request.

The integral

$$I_c(y,t) = \frac{2}{\sqrt{\pi}}\int_{y}^{+\infty} d\eta\, e^{-\eta^2}\cos\left(t\frac{y^2}{\eta^2}\right) \tag{B.1}$$

and the analogous one (I_s) with the sine in place of the cosine, which are present in solution (12), are calculated through the introduction of the integrals:

$$g_m(x) = x^{2m}\frac{2}{\sqrt{\pi}}\int_{x}^{+\infty} d\xi\, \frac{e^{-\xi^2}}{\xi^{2m}} \tag{B.2}$$

that follow by recurrence

$$g_0 = \operatorname{erfc}(x), \quad g_m = \frac{2x}{2m-1}\left(\frac{1}{\sqrt{\pi}}e^{-x^2} - x g_{m-1}\right) \quad \text{for } m \geq 1. \tag{B.3}$$

The integral (B.1) is evaluated as

$$I_c(y,t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k}g_{2k}(y) \tag{B.4}$$

and an analogous formula is used for I_s.

C. Stationary Solutions of the Finite Depth-Second Stokes Problem

In Appendix, the complete expressions of the constants $c(\tilde{y})$, $d(\tilde{y})$, $c_1(\tilde{y})$, $d_1(\tilde{y})$, $c_2(\tilde{y})$, $d_2(\tilde{y})$ are given:

$$
\begin{aligned}
c(\tilde{y}) &= \frac{\cosh\left[\alpha'(1-\tilde{y}/2)\right]\cos\left(\tilde{y}\alpha'/2\right)+\cos\left[\alpha'(1-\tilde{y}/2)\right]\cosh\left(\tilde{y}\alpha'/2\right)}{\cosh(\alpha')+\cos(\alpha')}\\
d(\tilde{y}) &= \frac{\sinh\left[\alpha'(1-\tilde{y}/2)\right]\sin\left(\tilde{y}\alpha'/2\right)+\sin\left[\alpha'(1-\tilde{y}/2)\right]\sinh\left(\tilde{y}\alpha'/2\right)}{\cosh(\alpha')+\cos(\alpha')}\\
c_1(\tilde{y}) &= \frac{\cosh\left[\alpha'(1-\tilde{y}/2)\right]\cos\left(\tilde{y}\alpha'/2\right)-\cos\left[\alpha'(1-\tilde{y}/2)\right]\cosh\left(\tilde{y}\alpha'/2\right)}{\cosh(\alpha')-\cos(\alpha')}\\
d_1(\tilde{y}) &= \frac{\sinh\left[\alpha'(1-\tilde{y}/2)\right]\sin\left(\tilde{y}\alpha'/2\right)-\sin\left[\alpha'(1-\tilde{y}/2)\right]\sinh\left(\tilde{y}\alpha'/2\right)}{\cosh(\alpha')-\cos(\alpha')},\\
c_2(\tilde{y}) &= \frac{\cosh\left[\alpha_2'/2(1+\tilde{y})\right]\cos\left[\alpha_2'/2(1-\tilde{y})\right]}{\cosh(\alpha_2')-\cos(\alpha_2')}+-\frac{\cos\left[\alpha_2'/2(1+\tilde{y})\right]\cosh\left[\alpha_2'/2(1-\tilde{y})\right]}{\cosh(\alpha_2')-\cos(\alpha_2')}\\
d_2(\tilde{y}) &= \frac{\sinh\left[\alpha_2'/2(1+\tilde{y})\right]\sin\left[\alpha_2'/2(1-\tilde{y})\right]}{\cosh(\alpha_2')-\cos(\alpha_2')}+-\frac{\sin\left[\alpha_2'/2(1+\tilde{y})\right]\sinh\left[\alpha_2'/2(1-\tilde{y})\right]}{\cosh(\alpha_2')-\cos(\alpha_2')}
\end{aligned}
\tag{C.1}
$$

with $\alpha' = \sqrt{2}\alpha$, $\alpha_2' = \sqrt{2}\alpha_2$. Similarly it follows the evaluation of the derivatives for $\tilde{y}=0$ and $\tilde{y}=1$:

$$
\begin{aligned}
c_1'(0) &= -\frac{\alpha'}{2}\frac{\sinh\alpha'+\sin\alpha'}{\cosh\alpha'-\cos\alpha'}\\
d_1'(0) &= +\frac{\alpha'}{2}\frac{\sinh\alpha'-\sin\alpha'}{\cosh\alpha'-\cos\alpha'}\\
c_2'(0) &= +\alpha_2'\\
&\quad\cdot\frac{\sinh\left(\alpha_2'/2\right)\cos\left(\alpha_2'/2\right)+\cosh\left(\alpha_2'/2\right)\sin\left(\alpha_2'/2\right)}{\cosh\alpha_2'-\cos\alpha_2'}\\
d_2'(0) &= +\alpha_2'\\
&\quad\cdot\frac{\cosh\left(\alpha_2'/2\right)\sin\left(\alpha_2'/2\right)-\sinh\left(\alpha_2'/2\right)\cos\left(\alpha_2'/2\right)}{\cosh\alpha_2'-\cos\alpha_2'}\\
c_1'(1) &= -\alpha'\\
&\quad\cdot\frac{\sinh\left(\alpha'/2\right)\cos\left(\alpha'/2\right)+\sin\left(\alpha'/2\right)\cosh\left(\alpha'/2\right)}{\cosh\alpha'-\cos\alpha'}\\
d_1'(1) &= -\alpha'\\
&\quad\cdot\frac{\cosh\left(\alpha'/2\right)\sin\left(\alpha'/2\right)-\sinh\left(\alpha'/2\right)\cos\left(\alpha'/2\right)}{\cosh\alpha'-\cos\alpha'}\\
c_2'(1) &= +\frac{\alpha_2'}{2}\frac{\sinh\alpha_2'+\sin\alpha_2'}{\cosh\alpha_2'-\cos\alpha_2'}\\
d_2'(1) &= -\frac{\alpha_2'}{2}\frac{\sinh\alpha_2'-\sin\alpha_2'}{\cosh\alpha_2'-\cos\alpha_2'}.
\end{aligned}
\tag{C.2}
$$

D. Second Stokes' Problem with Pressure Gradient: Some Useful Terms

In Appendix, the terms $Z_1 + Z_3$ and $Z_2 + Z_4$ in (52) are extensively shown. Starting by a calculation of $Z_{1,2}$ and $Z_{3,4}$, we have

$$
\begin{aligned}
Z_{1,2} &= e^{\mp i\theta}\left[\frac{2h^2}{\nu}\sum_{k=1}^{\infty}\frac{\sin\left(K''\tilde{y}\right)}{K''\left(K''^2\mp i\alpha_3^2\right)}e^{-T_3/\alpha_3^2K''^2}\right.\\
&\quad\pm\frac{i}{\chi}e^{\mp iT_3}\frac{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)(1-\tilde{y})\right]}{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)\right]}\\
&\quad\left.\mp\frac{i}{\chi}(1-\tilde{y})\right]\\
Z_{3,4} &= e^{\mp i\theta}\left[-\frac{2h^2}{\nu}\sum_{k=1}^{\infty}\frac{(-1)^k\sin\left(K''\tilde{y}\right)}{K''\left(K''^2\mp i\alpha_3^2\right)}e^{-T_3/\alpha_3^2K''^2}\right.\\
&\quad\left.\pm\frac{i}{\chi}e^{\mp iT_3}\frac{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)\tilde{y}\right]}{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)\right]}\mp\frac{i}{\chi}\tilde{y}\right]
\end{aligned}
\tag{D.1}
$$

where i is the imaginary unit, $\alpha_3 = \sqrt{\chi/\nu}h$ and $T_3 = \chi t$. Then sums $Z_1 + Z_3$ (indicated by Z_{1+3}) and $Z_2 + Z_4$ (Z_{2+4}) follow:

$$
\begin{aligned}
Z_{1+3,2+4} &= e^{\mp i\theta}\left[\frac{4h^2}{\nu}\sum_{k=1}^{\infty}\frac{\sin\left(K_o\tilde{y}\right)}{K_o\left(K_o^2\mp i\alpha_3^2\right)}e^{-K_o^2T_3/\alpha_3^2}\right.\\
&\quad\pm\frac{i}{\chi}e^{\mp iT_3}\left(\frac{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)(1-\tilde{y})\right]}{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)\right]}\right.\\
&\quad\left.\left.+\frac{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)\tilde{y}\right]}{\sinh\left[\alpha_3\left((1\mp i)/\sqrt{2}\right)\right]}\right)\mp\frac{i}{\chi}\right]
\end{aligned}
\tag{D.2}
$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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