DYNAMIC PERFECT PLASTICITY AS CONVEX MINIMIZATION

ELISA DAVOLI AND ULISSE STEFANELLI

Abstract. We present a novel variational approach to dynamic perfect plasticity. This is based on minimizing over entire trajectories parameter-dependent convex functionals of Weighted-Inertia-Dissipation-Energy (WIDE) type. Solutions to the system of dynamic perfect plasticity are recovered as limit of minimizing trajectories are the parameter goes to zero. The crucial compactness is achieved by means of a time-discretization and a variational convergence argument.

1. Introduction

Plasticity is the macroscopic, inelastic behavior of a solid resulting from the accumulation of slip defects at its microscopic, crystalline level. As a result of these dislocations, the behavior of the material remains purely elastic (and hence reversible) as far as the magnitude of the stress remains *small*, and becomes irreversible as soon as a given stress-threshold is reached. When that happens, a plastic flow is developed such that, after unloading, the material remains permanently plastically deformed [24].

Referring to [21, 32] for an overview on plasticity models, we focus here on *dynamic perfect plasticity* in the form of the classical *Prandtl-Reuss* model [15]

$$\rho\ddot{u} - \nabla\cdot\sigma = 0, \tag{1.1}$$

$$\sigma = \mathbb{C}(Eu - p), \tag{1.2}$$

$$\partial H(\dot{p}) \ni \sigma_D \tag{1.3}$$

describing the basics of plastic behavior in metals [20]. Here $u(t) : \Omega \to \mathbb{R}^3$ denotes the (time-dependent) *displacement* of a body with reference configuration $\Omega \subset \mathbb{R}^3$ and density $\rho > 0$, and $\sigma(t) : \Omega \to \mathbb{M}^{3\times3}_{\rm sym}$ is its *stress*. In particular, relation (1.1) expresses the conservation of momenta. The constitutive relation (1.2) relates the stress $\sigma(t)$ to the *linearized strain* $Eu(t) = (\nabla u(t) + \nabla u^\top(t))/2 : \Omega \to \mathbb{M}^{3\times3}_{\rm sym}$ and the *plastic strain* $p(t) : \Omega \to \mathbb{M}^{3\times3}_D$ (deviatoric tensors) via the fourth-order *elasticity tensor* $\mathbb{C}$. Finally, (1.3) expresses the plastic-flow rule: $H : \mathbb{M}^{3\times3}_D \to [0, +\infty)$ is a positively 1-homogeneous, convex *dissipation* function, σ_D stands for the deviatoric part of the stress, and the symbol ∂ is the subdifferential in the sense of Convex Analysis [9]. The system will be driven by imposing a nonhomogeneous boundary displacement. Details on notation and modeling are given in Section 2.

The focus of this paper is to recover weak solutions to the dynamic perfect plasticity system (1.1)-(1.3) by minimizing parameter-dependent convex functionals over entire trajectories, and by passing to the parameter limit. In particular, we consider the *Weighted-Inertia-Dissipation-Energy (WIDE)* functional of the form

$$I_\varepsilon(u,p) = \int_0^T\int_\Omega \exp\left(-\frac{t}{\varepsilon}\right)\left(\frac{\rho\varepsilon^2}{2}|\ddot{u}|^2 + \varepsilon H(\dot{p}) + \frac{1}{2}(Eu - p) : \mathbb{C}(Eu - p)\right) dx\, dt, \tag{1.4}$$

to be defined on suitable admissible classes of entire trajectories $t \in [0, T] \mapsto (u(t), p(t)) : \Omega \to \mathbb{R}^3 \times \mathbb{M}^{3\times3}_D$ fulfilling given boundary-displacement and initial conditions (on u and p, respectively). The functional bears its name from resulting from the sum of the inertial term $\rho|\ddot{u}|^2/2$, the dissipative term $H(\dot{p})$, and the energy term $(Eu - p) : \mathbb{C}(Eu - p)/2$, weighted by different powers of ε as well as the function $\exp(-t/\varepsilon)$.

2010 *Mathematics Subject Classification.* 70H03, 70H30, 74C10, 74G65.

Key words and phrases. Weighted-Inertia-Dissipation-Energy, dynamic perfect plasticity, elliptic regularization, time discretization, functions of bounded deformation.

For all $\varepsilon > 0$ one can prove that (a suitable relaxation of) the convex functional I_ε admits minimizers $(u_\varepsilon, p_\varepsilon)$ which indeed approximate solutions to the dynamic perfect plasticity system (1.1)-(1.3). In particular, by computing the corresponding Euler-Lagrange equations one finds that the minimizers $(u_\varepsilon, p_\varepsilon)$ weakly solve the elliptic-in-time approximating relations

$$\varepsilon^2 \rho \ddddot{u}_\varepsilon - 2\varepsilon^2 \rho \dddot{u}_\varepsilon + \rho \ddot{u}_\varepsilon - \nabla \cdot \sigma = 0, \tag{1.5}$$

$$\sigma = \mathbb{C}(Eu_\varepsilon - p_\varepsilon), \tag{1.6}$$

$$-\varepsilon(\partial H(\dot{p}_\varepsilon))^{\cdot} + \partial H(\dot{p}_\varepsilon) \ni \sigma_D, \tag{1.7}$$

along with Neumann conditions at the final time T.

The dynamic perfect plasticity system (1.1)-(1.3) is formally recovered by taking $\varepsilon \to 0$ in system (1.5)-(1.7). The main result of this paper consists in making this intuition rigorous, resulting in a new approximation theory for dynamic perfect plasticity.

The interest in this variational-approximation approach is threefold. First, the differential problem (1.1)-(1.3) is reformulated on purely variational grounds. This opens the possibility of applying the powerful tools of the Calculus of Variations to the problem, from the Direct Method, to relaxation, and Γ-convergence [14].

Secondly, by addressing a time-discrete analogue of this approach we contribute a novel numerical strategy in order to approximate dynamic perfect plasticity by means of space-time optimization methods. We believe that this might be of potential interest in combination with global constraints or non-cylindrical domains.

Eventually, The variational formulation via WIDE functionals is easily open to be generalized by including more refined material effects, especially in terms of additional internal-variable descriptions. This indeed has been one of the main motivations for advancing the WIDE method in the first place, see in particular [10, 25] for applications in Materials Science. Having illustrated the details of the method in the case of dynamic perfect plasticity could then serve as basis for developing complete theories.

As a by-product of our analysis, we obtain a new proof of existence of weak solutions to dynamic perfect plasticity. Note that existence results for (1.1)-(1.3) are indeed quite classical. In the quasistatic case $\rho = 0$ they date back to Suquet [49] and have been subsequently reformulated by Dal Maso, DeSimone, and Mora [11] and Francfort and Giacomini [17] within the theory of rate-independent processes (see the recent monograph [38]). In the dynamic case $\rho > 0$ both the first existence results due to Anzellotti and Luckhaus [6, 33] and their recent revisiting by Babadjian and Mora [7] are based on viscosity techniques. Dimension reduction has been tackled both in the quasistatic and the dynamic case, in [12, 26, 27] and [34], respectively. Finally, in [35] convergence of solutions of the dynamic problem to solutions of the quasistatic problem as the density ρ tends to 0 has been shown. With respect to the available existence theories our approach is new, for it does not rely on viscous approximation but rather on a global variational method.

Before moving on, let us review here the available literature on WIDE variational methods. At the level of Euler-Lagrange equations, elliptic regularization techniques are classical and have to be traced back to Lions [30, 31] and Oleinik [42]. Their variational version via global functionals is already mentioned in the classical textbook by Evans [16, Problem 3, p. 487] and has been used by Ilmanen [22], in the context of Brakke mean-curvature flow of varifolds, and by Hirano [19] in connection with periodic solutions to gradient flows.

The formalism has been then applied in the context of rate-independent systems ($\rho = 0$) by Mielke and Ortiz [37], see also the follow-up [39]. Viscous dynamics have been considered in many different settings, including gradient flows [40], curves of maximal slopes in metric spaces [43, 44], mean curvature flow [47], doubly-nonlinear equations [1, 2, 3, 4, 5], reaction-diffusion systems [36], and quasilinear parabolic equations [8].

The dynamic case $\rho > 0$ has been the object of a long-standing conjecture by De Giorgi on semilinear waves [13]. The conjecture was solved in the positive in [48] for finite-time intervals and then by Serra and Tilli in [45] for the whole time semiline, that is in its original formulation. De Giorgi himself pointed out in [13] the interest of extending the method to other dynamic problems. The task has been then taken up in [29] for mixed hyperbolic-parabolic equations, in [28] for Lagrangian Mechanics, and in [46] for other hyperbolic problems. The present paper delivers the first realization of De Giorgi's suggestion in the context of Continuum Mechanics.

The paper is organized as follows. We introduce notation and state our main result, namely Theorem 2.3 in Section 2. Then, we discuss in Section 3 the existence of minimizers of the WIDE functionals. In Section 4 a time discretization of the minimization problem is addressed. Its time-continuous limit is discussed in Section 5 by means of variational convergence arguments. A parameter-dependent energy inequality is derived in Section 6 and finally used in Section 7 in order to pass to the limit as $\varepsilon \to 0$ and prove Theorem 2.3.

2. Statement of the main result

We devote this section to the specification of the material model and its mathematical setting. Some notions from measure theory need to be recalled and we introduce the notation and assumptions to be used throughout the article. The specific form of the WIDE functionals is eventually introduced in Subsection 2.8 and we conclude by stating our main result, namely Theorem 2.3.

2.1. **Tensors.** In what follows, for any map $f : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ we will denote by $\dot f$ its time derivative, and by ∇f its spatial gradient. The set of 3×3 real matrices will be denoted by $\mathbb{M}^{3\times3}$. Given $M, N \in \mathbb{M}^{3\times3}$, we will denote their scalar product by $M : N := \operatorname{tr}(M^\top N)$ where tr denotes the trace and the superscript stands for transposition, and we will adopt the notation M_D to identify the deviatoric part of M, namely $M_D := M - \operatorname{tr}(M)\mathrm{Id}/3$ where Id is the identity matrix. The symbol $\mathbb{M}^{3\times3}_{\rm sym}$ will stand for the set of symmetric 3×3 matrices, whereas $\mathbb{M}^{3\times3}_D$ will be the subset of $\mathbb{M}^{3\times3}_{\rm sym}$ given by symmetric matrices having null trace.

2.2. **Measures.** Given a Borel set $B \subset \mathbb{R}^N$ the symbol $\mathcal{M}_b(B;\mathbb{R}^m)$ denotes the space of all bounded Borel measures on B with values in $\mathbb{R}^m$ ($m \in \mathbb{N}$). When $m = 1$ we will simply write $\mathcal{M}_b(B)$. We will endow $\mathcal{M}_b(B;\mathbb{R}^m)$ with the norm $\|\mu\|_{M_b} := |\mu|(B)$, where $|\mu| \in \mathcal{M}_b(B)$ is the total variation of the measure μ.

If the relative topology of B is locally compact, by the Riesz representation Theorem the space $\mathcal{M}_b(B;\mathbb{R}^m)$ can be identified with the dual of $C_0(B;\mathbb{R}^m)$, which is the space of all continuous functions $\varphi : B \to \mathbb{R}^m$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $\mathcal{M}_b(B;\mathbb{R}^m)$ is defined using this duality.

2.3. **Functions with bounded deformation.** Let U be an open set of $\mathbb{R}^3$. The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U;\mathbb{R}^3)$ whose symmetric gradient $Eu := \operatorname{sym} Du := (Du + Du^T)/2$ (in the sense of distributions) belongs to $\mathcal{M}_b(U;\mathbb{M}^{3\times3}_{\rm sym})$. It is easy to see that $BD(U)$ is a Banach space endowed with the norm

$$\|u\|_{L^1(U;\mathbb{M}^{3\times3})} + \|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{3\times3}_{\rm sym})}.$$

A sequence $\{u^k\}$ is said to converge to u weakly* in $BD(U)$ if $u^k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^3)$ and $Eu^k \rightharpoonup Eu$ weakly* in $\mathcal{M}_b(U;\mathbb{M}^{3\times3}_{\rm sym})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. If U is bounded and has a Lipschitz boundary, $BD(U)$ can be embedded into $L^{3/2}(U;\mathbb{R}^3)$ and every function $u \in BD(U)$ has a trace, still denoted by u, which belongs to $L^1(\partial U;\mathbb{R}^3)$. If Γ is a nonempty open subset of ∂U in the relative topology of ∂U, there exists a constant $C > 0$, depending on U and Γ, such that

$$\|u\|_{L^1(U;\mathbb{R}^3)} \leq C\|u\|_{L^1(\Gamma;\mathbb{R}^3)} + C\|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{3\times3}_{\rm sym})}. \tag{2.1}$$

(see [50, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space $BD(U)$ we refer to [50].

2.4. **The elasticity tensor.** Let $\mathbb{C}$ be the *elasticity tensor*, considered as a symmetric positive-definite linear operator $\mathbb{C} : \mathbb{M}^{3\times 3}_{\rm sym} \to \mathbb{M}^{3\times 3}_{\rm sym}$, and let $Q : \mathbb{M}^{3\times 3}_{\rm sym} \to [0,+\infty)$ be the quadratic form associated with $\mathbb{C}$, given by

$$Q(\xi) := \tfrac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{\rm sym}.$$

Let the two constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, with $0 < \alpha_{\mathbb{C}} \le \beta_{\mathbb{C}}$, be such that

$$\alpha_{\mathbb{C}}|\xi|^2 \le Q(\xi) \le \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{\rm sym}, \tag{2.2}$$

and

$$|\mathbb{C}\xi| \le 2\beta_{\mathbb{C}}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{\rm sym}. \tag{2.3}$$

2.5. **The dissipation potential.** Let K be a closed convex set of $\mathbb{M}^{3\times 3}_D$ such that there exist two constants r_K and R_K, with $0 < r_K \le R_K$, satisfying

$$\{\xi \in \mathbb{M}^{3\times 3}_D : \ |\xi| \le r_K\} \subset K \subset \{\xi \in \mathbb{M}^{3\times 3}_D : \ |\xi| \le R_K\}.$$

The boundary of K is interpreted as the *yield surface*. The *plastic dissipation potential* is given by the support function $H : \mathbb{M}^{3\times 3}_D \to [0,+\infty)$ of K, defined as

$$H(\xi) := \sup_{\sigma \in K} \sigma : \xi.$$

Note that $K = \partial H(0)$ is the subdifferential of H at 0 (see e.g. [9, Section 1.4]). The function H is convex and positively 1-homogeneous, with

$$r_K|\xi| \le H(\xi) \le R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_D. \tag{2.4}$$

In particular, H satisfies the triangle inequality

$$H(\xi+\zeta) \le H(\xi) + H(\zeta) \quad \text{for every } \xi, \zeta \in \mathbb{M}^{3\times 3}_D. \tag{2.5}$$

For every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$ let $d\mu/d|\mu|$ be the Radon-Nikodým derivative of μ with respect to its variation $|\mu|$.

According to the theory of convex functions of measures [18], we introduce the nonnegative Radon measure $H(\mu) \in \mathcal{M}_b(\Omega \cup \Gamma_0)$ defined by

$$H(\mu)(A) := \int_A H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $A \subset \Omega \cup \Gamma_0$. We also consider the functional $\mathcal{H} : \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D) \to [0,+\infty)$ defined by

$$\mathcal{H}(\mu) := H(\mu)(\Omega \cup \Gamma_0) = \int_{\Omega \cup \Gamma_0} H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$. Notice that $\mathcal{H}$ is lower semicontinuous on $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$ with respect to weak* convergence. The following lemma is a consequence of [18, Theorem 4] and [50, Chapter II, Lemma 5.2] (see also [11, Subsection 2.2]).

Lemma 2.1. *Setting* $\mathcal{K}_D(\Omega) := \{\tau \in L^2(\Omega; \mathbb{M}^{3\times 3}_D) : \tau(x) \in K \text{ for a.e. } x \in \Omega\}$, *there holds*

$$\mathcal{H}(\mu) = \sup\{\langle \tau, \mu \rangle : \tau \in C_0(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D) \cap \mathcal{K}_D(\Omega)\}$$

for every $\mu \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D)$.

2.6. **The $\mathcal{H}$-dissipation.** Let $s_1, s_2 \in [0,T]$ with $s_1 \le s_2$. For every function $t \mapsto \mu(t)$ of bounded variation from $[0,T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)$, we define the *$\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$D_{\mathcal{H}}(\mu; s_1, s_2) := \sup\left\{ \sum_{j=1}^n \mathcal{H}(\mu(t_j) - \mu(t_{j-1})) : \ s_1 = t_0 \le t_1 \le \cdots \le t_n = s_2, \ n \in \mathbb{N} \right\}. \tag{2.6}$$

Denoting by $V_{\rm tot}$ the pointwise variation of $t \to \mu(t)$, that is,

$$V_{\rm tot}(\mu; s_1, s_2) := \sup\left\{ \sum_{j=1}^n |\mu(t_j) - \mu(t_{j-1})| : \ s_1 = t_0 \le \cdots \le t_n = s_2, \, n \in \mathbb{N} \right\},$$

by (2.4) there holds

$$r_K V_{\rm tot}(\mu; s_1, s_2) \le D_{\mathcal{H}}(\mu; s_1, s_2) \le R_K V_{\rm tot}(\mu; s_1, s_2). \tag{2.7}$$

As in [37, Section 4.2] for every non-increasing and positive $a \in C([0,T])$ we define the *a-weighted $\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$\begin{aligned} D_{\mathcal{H}}(a; \mu; s_1, s_2) := \sup\Bigg\{ \sum_{j=1}^n a(t_j) \mathcal{H}(\mu(t_j) - \mu(t_{j-1})) : \ t_0, t_n \in [s_1, s_2], \\ t_0 \le t_1 \le \cdots \le t_n, \ n \in \mathbb{N} \Bigg\}, \end{aligned} \tag{2.8}$$

and for every $b \in C([0,T])$ we introduce the *b-weighted $\mathcal{H}$-dissipation* of $t \mapsto \mu(t)$ in $[s_1, s_2]$ as

$$\begin{aligned} \hat{D}_{\mathcal{H}}(b; \mu; s_1, s_2) := \lim_{\delta \to 0} \Bigg\{ \sup \Bigg[\sum_{j=1}^n b(t_j) \mathcal{H}(\mu(t_j) - \mu(t_{j-1})) : \ t_0, t_n \in [s_1, s_2], \\ t_0 \le t_1 \le \cdots \le t_n, \ n \in \mathbb{N}, \ \max_{i=1,\dots,n} (t_i - t_{i-1}) \le \delta \Bigg] \Bigg\}. \end{aligned} \tag{2.9}$$

Note that if b is non-increasing and positive, then

$$\hat{D}_{\mathcal{H}}(b; \mu; s_1, s_2) = D_{\mathcal{H}}(b; \mu; s_1, s_2).$$

2.7. **The equations of dynamic perfect plasticity.** Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$). On Γ_0 for every $t \in [0,T]$ we prescribe a boundary datum $w(t) \in W^{1,1/2}(\Gamma_0; \mathbb{R}^3)$.

The *set of admissible displacements and strains* for the boundary datum $w(t)$ is denoted by

$$\begin{aligned} \mathscr{A}(w(t)) := \Big\{ (u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D) : \\ Eu = e + p \text{ in } \Omega, \quad p = (w(t) - u) \odot \nu \mathcal{H}^2 \text{ on } \Gamma_0 \Big\}, \end{aligned} \tag{2.10}$$

where $\odot$ stands for the symmetrized tensor product, namely

$$a \odot b := (a \otimes b + b \otimes a)/2 \quad \forall \, a, b \in \mathbb{R}^3,$$

ν is the outer unit normal to $\partial\Omega$, and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. The function u represents the *displacement* of the body, while e and p are called the *elastic* and *plastic strain*, respectively.

A *solution to the equations of dynamic perfect plasticity* is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $(L^2(\Omega; \mathbb{R}^3) \cap BD(\Omega)) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times3}_D)$ such that for every $t \in [0,T]$ there holds $(u(t), e(t), p(t)) \in \mathscr{A}(w(t))$, and the following conditions are satisfied:

- (c1) *equilibrium:* $\rho \ddot{u}(t) - \operatorname{div} \sigma(t) = 0$ in Ω and $\sigma(t)\nu = 0$ on $\partial\Omega \setminus \Gamma_0$, where $\sigma(t) := \mathbb{C}e(t)$ is the stress tensor, and $\rho > 0$ is the constant density;
- (c2) *stress constraint:* $\sigma_D(t) \in K$;
- (c3) *flow rule:* $\dot{p}(t) = 0$ if $\sigma_D(t) \in \operatorname{int} K$, while $\dot{p}(t)$ belongs to the normal cone to K at $\sigma_D(t)$ if $\sigma_D(t) \in \partial K$.

Under suitable assumptions, when (c1) and (c2) are satisfied, condition (c3) can be equivalently reformulated as the following *energy inequality*

$$\text{(c3}'\text{)}\quad \int_\Omega Q(e(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,dx + \int_0^t \mathcal{H}(\dot p(t))\,dt \le \int_\Omega Q(e(0))\,dx + \frac{\rho}{2}\int_\Omega |\dot u(0)|^2\,dx + \int_0^t\int_\Omega \sigma(s):E\dot w(s)\,dx + \rho\ddot u(s)\cdot \dot w(s)\,dx\,ds.$$

A detailed analysis of the equivalence between (c1)–(c3), and (c1),(c2) complemented by (c3') has been performed in [11, Section 6]. An adaptation of the argument yields the analogous statements in the dynamic setting.

The following existence and uniqueness result holds true (see [34, Theorem 3.1 and Remark 3.2]).

Theorem 2.2 (Existence of the evolution)**.** *Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional, C^2 manifold.*

Let $w \in W^{3,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, and $(u^0,e^0,p^0) \in \mathscr{A}(w(0))$ be such that $\operatorname{div}\mathbb{C}e^0 = 0$ a.e. in Ω, $(\mathbb{C}e^0)\nu = 0$ a.e. on $\partial\Omega\setminus\Gamma_0$, and $(\mathbb{C}e^0)_D \in K$ a.e. in Ω. Eventually, let $(u^1,e^1,0) \in \mathscr{A}(\dot w(0))$.

Then there exist unique $u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap \mathrm{Lip}(0,T;BD(\Omega))$, $e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$, and $p \in \mathrm{Lip}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ solving (c1), (c2) *and* (c3') *with $(u(0),e(0),p(0)) = (u^0,e^0,p^0)$ and $\dot u(0) = u^1$.*

2.8. **The WIDE functional.** Let the boundary datum $w \in W^{3,2}([0,T];W^{1,2}(\Omega;\mathbb{R}^3))$ be given. By reformulating the expression in (1.4) for the triple (u,e,p) one would be tempted to introduce the functional

$$(u,e,p) \mapsto \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot u|^2\,dx + \varepsilon\mathcal{H}(\dot p) + \int_\Omega Q(e)\,dx\Big)\,dt,$$

to be defined on the set $\mathcal{V}$, given by

$$\begin{aligned}\mathcal{V} := \{&(u,e,p) \in W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^1(0,T;BD(\Omega))\\ &\times L^2((0,T)\times\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \times BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)) :\\ &(u(t),e(t),p(t)) \in \mathscr{A}(w(t)) \text{ for every } t\in[0,T],\\ &u(0)=u^0,\ \dot u(0)=u^1,\ e(0)=e^0,\ p(0)=p^0\},\end{aligned} \tag{2.11}$$

where $(u^0,e^0,p^0) \in \mathscr{A}(w(0))$, and $u^1 \in BD(\Omega)$ is such that there exists a pair $(e^1,p^1) \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)$ satisfying $(u^1,e^1,p^1) \in \mathscr{A}(\dot w(0))$.

On the other hand, one readily sees that the term

$$\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}(\dot p)\,dt$$

is not well defined in case p is not absolutely continuous with respect to time (see [11, Theorem 7.1]). We hence need to relax the form of the WIDE functional as

$$I_\varepsilon(u,e,p) := \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot u|^2\,dx + \int_\Omega Q(e)\,dx\Big)\,dt + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T), \tag{2.12}$$

for every $(u,e,p) \in \mathcal{V}$. We point out that an adaptation of [11, Theorem 7.1] yields

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T) = \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}(\dot p)\,dt$$

whenever p is absolutely continuous with respect to time.

2.9. **Main result.** We are now ready to state the main result of the paper.

Theorem 2.3 (Dynamic perfect plasticity as convex minimization). *Let Ω be a bounded open set in $\mathbb{R}^3$ with C^2 boundary. Let Γ_0 be a connected open subset of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\partial_{\partial\Omega}\Gamma_0$ is a connected, one-dimensional, C^2 manifold. Let $w \in W^{3,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, and $(u^0,e^0,p^0) \in \mathscr{A}(w(0))$ be such that $\operatorname{div}\mathbb{C}e^0 = 0$ a.e. in Ω, $(\mathbb{C}e^0)\nu = 0$ a.e. on $\partial\Omega\setminus\Gamma_0$, and $(\mathbb{C}e^0)_D \in K$ a.e. in Ω. Eventually, let $(u^1,e^1,0) \in \mathscr{A}(\dot w(0))$.*

For every $\varepsilon > 0$ there exists $\{(u^\varepsilon,e^\varepsilon,p^\varepsilon)\} \subset \mathcal{V}$ solving

$$I_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon) = \min_{(u,e,p)\in\mathcal{V}} I_\varepsilon(u,e,p). \tag{2.13}$$

For $\varepsilon \to 0$, and for all $t \in [0,T]$ there holds

$$\begin{aligned}
&u^\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)),\\
&e^\varepsilon \rightharpoonup e \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})),\\
&u^\varepsilon(t) \rightharpoonup^* u(t) \quad \text{weakly* in } BD(\Omega),\\
&e^\varepsilon(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}),\\
&p^\varepsilon(t) \rightharpoonup^* p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)
\end{aligned}$$

where $u \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap W^{1,\infty}(0,T;BD(\Omega))$, $e \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$, and $p \in W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))$ is the unique solution to the dynamic perfect plasticity problem (c1), (c2) *and* (c3') *with $(u(0),e(0),p(0)) = (u^0,e^0,p^0)$ and $\dot u(0) = u^1$.*

The rest of the paper is devoted to the proof of Theorem 2.3. Our argument runs as follows: we prove that minimizers $\{(u^\varepsilon,e^\varepsilon,p^\varepsilon)\}$ of Problem (2.13) exist in Section 3. Then, we devise an ε-independent a-priori estimate on $\{(u^\varepsilon,e^\varepsilon,p^\varepsilon)\}$ first in a discrete and then in a continuous setting (Section 4) by means of a Γ-convergence argument (Section 5). Then, we derive an energy inequality at level $\varepsilon > 0$ (Section 6) which allows discussing the limit $\varepsilon \to 0$ in Section 7.

We point out that the C^2 regularity of $\partial\Omega$ is needed in Theorem 2.3 in order to introduce a duality between stresses and plastic strains, along the footsteps of [23, Proposition 2.5]. Due to technical reasons it is not possible to use here the results in [17] and consider the case of a Lipschitz $\partial\Omega$. We refer to Remark 4.5 for some discussion of this point.

3. Minimizers of the WIDE functional

We start by focusing here on Problem (2.13) and show that the functional I_ε admits a minimizer in $\mathcal{V}$.

Proposition 3.1 (Existence of minimizers). *For every $\varepsilon > 0$ there exists a triple $(u^\varepsilon,e^\varepsilon,p^\varepsilon) \in \mathcal{V}$ such that*

$$I_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon) = \inf_{(u,e,p)\in\mathcal{V}} I_\varepsilon(u,e,p). \tag{3.1}$$

Proof. Fix $\varepsilon > 0$, and let $\{(u_n,e_n,p_n)\} \subset \mathcal{V}$ be a minimizing sequence for I_ε. We first observe that the triple

$$t \to (u^0 + tu^1 + w(t) - w(0) - t\dot w(0), e_0 + te_1 + Ew(t) - Ew(0) - tE\dot w(0), p_0 + tp_1)$$

belongs to $\mathcal{V}$. Hence,

$$\begin{aligned}
\lim_{n\to+\infty} I_\varepsilon(u_n,e_n,p_n) \le \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot w|^2\,dx + \varepsilon\mathcal{H}(p^1)\\
+ \int_\Omega Q(e^0 + te^1 + Ew(t) - Ew(0) - tE\dot w(0))\,dx\Big)\,dt \le C,
\end{aligned}$$

thus yielding the uniform bound

$$\sup_{n\in\mathbb{N}}\Big\{\|\ddot u_n\|_{L^2((0,T);L^2(\Omega;\mathbb{R}^3))} + D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p_n;0,T)$$

$$+ \|e_n\|_{L^2((0,T);L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}))}\Big\} \leq C. \tag{3.2}$$

Since $(u_n, e_n, p_n) \in \mathcal{V}$, there holds $p_n(0) = p^0$ for every $n \in \mathbb{N}$. In view of (2.9) and (2.7),

$$r_K \exp(-T/\varepsilon) V_{\rm tot}(p_n; 0, T) \leq \exp(-T/\varepsilon) D_{\mathcal{H}}(p_n; 0, T) \leq D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_n; 0, T).$$

Therefore we are in a position of applying the variant of Helly's theorem in [11, Lemma 7.2] and to deduce the existence of a subsequence, still denoted by $\{p_n\}$ and a map $p^\varepsilon \in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))$, such that

$$p_n(t) \rightharpoonup^* p^\varepsilon(t) \text{ weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D) \quad \text{for every } t\in[0,T], \tag{3.3}$$

and by the lower semicontinuity of the $\mathcal{H}$-dissipation,

$$D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T) \leq \liminf_{n\to+\infty} D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_n; 0, T). \tag{3.4}$$

By (3.2), there exist $e^\varepsilon \in L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}))$ and $u^\varepsilon \in W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ such that, up to the extraction of a (non relabeled) subsequence,

$$e_n \rightharpoonup e^\varepsilon \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})), \tag{3.5}$$

and

$$u_n \rightharpoonup u^\varepsilon \quad \text{weakly in } W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{3.6}$$

This implies that $u^\varepsilon(0) = u^0$ and $\dot{u}^\varepsilon(0) = u^1$. By (3.3) and (3.6) it follows that

$$e_n(t) \rightharpoonup e^\varepsilon(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) \tag{3.7}$$

for every $t \in [0,T]$, and hence $e^\varepsilon(0) = e^0$. Finally, by (2.1), (3.3), and (3.7), up to subsequences there holds

$$u_n(t) \rightharpoonup^* u^\varepsilon(t) \quad \text{weakly* in } BD(\Omega) \quad \text{for a.e. } t\in[0,T].$$

The fact that p^ε satisfies the boundary condition on Γ_0 follows arguing as in [11, Lemma 2.1]. The minimality of the limit triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ is a direct consequence of the lower semicontinuity of I_ε with respect to the convergences in (3.4), (3.5), and (3.6). ☐

We conclude this section with a conditional uniqueness result.

Proposition 3.2 (Uniqueness of minimizers given the plastic strain)**.** *Let (u_a, e_a, p_a) and (u_b, e_b, p_b) be two minimizers of I_ε in $\mathcal{V}$. Then*

$$\begin{aligned} &\varepsilon\sqrt{\rho}\|u_a - u_b\|_{W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))} + \sqrt{\alpha_{\mathbb{C}}}\|e_a - e_b\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}))} \\ &\quad\leq \varepsilon\sqrt{R_K} V_{\rm tot}(p_a - p_b; 0, T). \end{aligned} \tag{3.8}$$

Proof. Arguing as in [11, Theorem 3.8], we set $v = u_a - u_b$, $f = e_a - e_b$, and $q = p_a - p_b$. Since $(v, f, q) \in \mathscr{A}(0)$, it follows that $(u_a, e_a, p_a) + \lambda(v, f, q) \in \mathcal{V}$ for every $\lambda \in \mathbb{R}$. Thus,

$$\begin{aligned} I_\varepsilon(u_a, e_a, p_a) &\leq I_\varepsilon((u_a, e_a, p_a) + \lambda(v, f, q)) \\ &= \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}_a + \lambda\ddot{v}|^2\,dx + \int_\Omega Q(e_a + \lambda f)\,dx\Big)\,dt \\ &\quad + \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a + \lambda q; 0, T). \end{aligned}$$

By the arbitrariness of λ we deduce the inequality

$$-\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); q; 0, T) \leq \varepsilon^2\rho\int_\Omega \ddot{u}_a\ddot{v}\,dx + \int_\Omega \mathbb{C}e_a : f\,dx \leq \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); -q; 0, T). \tag{3.9}$$

Arguing analogously, the minimality of (u_b, e_b, p_b) yields

$$-\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); -q; 0, T) \leq -\varepsilon^2\rho\int_\Omega \ddot{u}_b\ddot{v}\,dx - \int_\Omega \mathbb{C}e_b : f\,dx \leq \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); q; 0, T). \tag{3.10}$$

Summing (3.9) and (3.10) we obtain

$$-\varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_a - p_b; 0, T) - \varepsilon D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p_b - p_a; 0, T)$$

$$\begin{aligned}&\le \varepsilon^2\rho\int_\Omega |\ddot u_a-\ddot u_b|^2\,dx+\int_\Omega Q(e_a-e_b)\,dx\\&\le \varepsilon D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p_a-p_b;0,T)+\varepsilon D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p_b-p_a;0,T).\end{aligned}$$

The thesis follows now by (2.2), (2.7), and (2.9). □

Remark 3.3. Let us point out that the previous proposition can alternatively be read as a Lipschitz regularity result for the solution operator associated to the reduced problem $p\mapsto \operatorname{Argmin} I_\varepsilon(\cdot,\,\cdot,\,p)$.

4. Discrete energy estimate

With the aim of establishing an a-priori estimate on $\{(u^\varepsilon,e^\varepsilon,p^\varepsilon)\}$ independent of ε we start by analyzing a time-discrete version of the problem. Fix $n\in\mathbb N$, set $\tau:=T/n$, and consider the time partition

$$0=t_0<t_1<\cdots<t_n=T,\qquad t_i:=i\tau.$$

We define $w_0:=w(0)$, $w_1:=w_0+\tau\dot w(0)$, and, for $i=2,\dots,n$, we set $w_i:=w(t_i)$. Our analysis will be set in the space

$$\begin{aligned}\mathscr U_\tau:=\Big\{&(u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\in\big(BD(\Omega)\times L^2(\Omega;\mathbb M^{3\times3}_{\rm sym})\times\mathcal M_b(\Omega\cup\Gamma_0;\mathbb M^{3\times3}_D)\big)^{n+1}:\\&(u_i,e_i,p_i)\in\mathscr A(w(t_i))\quad\text{for } i=1,\dots,n\Big\}.\end{aligned}$$

We define the *discrete energy functional* $I_{\varepsilon\tau}:\mathscr U_\tau\to[0,+\infty)$ as

$$\begin{aligned}I_{\varepsilon\tau}\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big):=&\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n\tau\eta_{\tau,i}\int_\Omega|\delta^2u_i|^2\,dx+\sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_i)\,dx\\&+\varepsilon\tau\sum_{i=1}^n\eta_{\tau,i+1}\mathcal H(\delta p_i),\end{aligned}\tag{4.1}$$

where, given a vector $v=(v_1,\dots,v_n)$, the operator δ denotes its discrete derivative,

$$\delta v_i:=\frac{v_i-v_{i-1}}{\tau},\qquad \delta^k v_i:=\frac{\delta^{k-1}v_i-\delta^{k-1}v_{i-1}}{\tau},$$

for $k\in\mathbb N$, $k>1$, and where the weights

$$\eta_{\tau,i}:=\left(\frac{\varepsilon}{\varepsilon+\tau}\right)^i,\quad i=0,\dots,n,$$

are a discretization of the map $t\to\exp\big(-t/\varepsilon\big)$. Define the set

$$\begin{aligned}\mathscr K_\tau(u^0,e^0,p^0,u^1):=\{&(u_0,e_0,p_0),\dots(u_n,e_n,p_n)\in\mathscr U_\tau:\ u_0=u^0,\ e_0=e^0,\ p_0=p^0,\\&\delta u_1=u^1\}.\end{aligned}$$

Arguing as in Proposition 3.1 we obtain the following result.

Lemma 4.1. *There exists a $(n{+}1)$-tuple of triples $(u^\varepsilon_k,e^\varepsilon_k,p^\varepsilon_k)$ such that $\big((u^\varepsilon_0,e^\varepsilon_0,p^\varepsilon_0),\dots,(u^\varepsilon_n,e^\varepsilon_n,p^\varepsilon_n)\big)\in\mathscr K_\tau(u^0,e^0,p^0,u^1)$, and*

$$\begin{aligned}&I_{\varepsilon\tau}\big((u^\varepsilon_0,e^\varepsilon_0,p^\varepsilon_0),\dots,(u^\varepsilon_n,e^\varepsilon_n,p^\varepsilon_n)\big)\\&\quad=\min_{\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big)\in\mathscr K_\tau(u^0,e^0,p^0,u^1)}I_{\varepsilon\tau}\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big).\end{aligned}\tag{4.2}$$

4.1. **Discrete Euler-Lagrange equations.** We first compute the discrete Euler-Lagrange equations satisfied by a minimizing $(n{+}1)$-tuple $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$.

Proposition 4.2 (Discrete Euler-Lagrange equations)**.** *Let $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$ be a solution to (4.2). Then*

$$\varepsilon^2 \rho \eta_{\tau,i} \int_\Omega \delta^2 u_i^\varepsilon \cdot \delta^2 \varphi \, dx + \eta_{\tau,i+2} \int_\Omega \mathbb{C} e_i^\varepsilon : E\varphi \, dx = 0 \tag{4.3}$$

for every $\varphi \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $\varphi = 0$ $\mathcal{H}^2$-a.e. on Γ_0, $i = 2, \dots, n-2$. In addition,

$$-\Big(\frac{\varepsilon}{\varepsilon+\tau}\Big)\mathcal{H}(\xi) - \mathcal{H}(-\xi) \le \Big(\frac{\tau}{\varepsilon+\tau}\Big) \int_\Omega \mathbb{C} e_i^\varepsilon : \xi \, dx \le \mathcal{H}(\xi) + \Big(\frac{\varepsilon}{\varepsilon+\tau}\Big)\mathcal{H}(-\xi), \tag{4.4}$$

for every $\xi \in L^2(\Omega;\mathbb{M}^{3\times3}_D)$, $i = 2, \dots, n-2$.

Proof. Let $(v_0, f_0, q_0), \dots, (v_n, f_n, q_n) \in \mathscr{K}_\tau(0,0,0,0)$, and consider the $(n{+}1)$-tuple

$$(u_0^\varepsilon \pm \lambda v_0, e_0^\varepsilon \pm \lambda f_0, p_0^\varepsilon \pm \lambda q_0), \dots (u_n^\varepsilon \pm \lambda v_n, e_n^\varepsilon \pm \lambda f_n, p_n^\varepsilon \pm \lambda q_n),$$

with $\lambda > 0$. By the minimality of $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$, there holds

$$\begin{aligned}&\frac{1}{\lambda} I_{\varepsilon\tau}\big((u_0^\varepsilon \pm \lambda v_0, e_0^\varepsilon \pm \lambda f_0, p_0^\varepsilon \pm \lambda q_0), \dots, (u_n^\varepsilon \pm \lambda v_n, e_n^\varepsilon \pm \lambda f_n, p_n^\varepsilon \pm \lambda q_n)\big)\\ &\quad - \frac{1}{\lambda} I_{\varepsilon\tau}\big((u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)\big) \ge 0.\end{aligned}$$

Therefore by (2.5) and (4.1) we deduce the inequality

$$\begin{aligned}-\varepsilon\tau \sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta q_i) &\le \varepsilon^2\rho \sum_{i=2}^n \tau\eta_{\tau,i} \int_\Omega \delta^2 u_i^\varepsilon \cdot \delta^2 v_i \, dx + \sum_{i=2}^{n-2} \tau \eta_{\tau,i+2} \int_\Omega \mathbb{C} e_i^\varepsilon : f_i \, dx\\ &\le \varepsilon\tau \sum_{i=1}^n \eta_{\tau,i+1} \mathcal{H}(-\delta q_i).\end{aligned} \tag{4.5}$$

For $i = 0, \dots, n$, let $\varphi_i \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, and let $\xi_i \in L^2(\Omega;\mathbb{M}^{3\times3}_D)$. Choosing $v_i = \varphi_i$, $f_i = E\varphi_i$, and $q_i = 0$, for $i = 1, \dots, n$, by (4.5) we obtain

$$\varepsilon^2\rho \sum_{i=2}^n \tau\eta_{\tau,i} \int_\Omega \delta^2 u_i^\varepsilon \cdot \delta^2 \varphi_i \, dx + \sum_{i=2}^{n-2} \tau\eta_{\tau,i+2} \int_\Omega \mathbb{C} e_i^\varepsilon : E\varphi_i \, dx = 0$$

for every $\varphi_1, \dots, \varphi_n \in W^{1,2}(\Omega;\mathbb{R}^3)$, $\varphi_i = 0$ $\mathcal{H}^2$-a.e. on Γ_0, $i = 0, \dots, n$, and hence (4.3). Choosing $v_i = 0$, $f_i = \xi_i$, and $q_i = -\xi_i$ for $i = 1, \dots, n$, estimate (4.5) yields

$$-\varepsilon\tau \sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(-\delta\xi_i) \le \sum_{i=2}^{n-2} \tau\eta_{\tau,i+2} \int_\Omega \mathbb{C} e_i^\varepsilon : \xi_i \, dx \le \varepsilon\tau \sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta\xi_i),$$

for every $\xi_1, \dots, \xi_n \in L^2(\Omega;\mathbb{M}^{3\times3}_D)$, and thus (4.4). □

We observe that it follows from (4.4) that $(\mathbb{C} e_i^\varepsilon)_D \in L^\infty(\Omega;\mathbb{M}^{3\times3}_D)$ for every i and ε, although the bound is not uniform with respect to τ nor ε. Indeed, for every B Borel subset of Ω and for every $M \in \mathbb{M}^{3\times3}_D$ we can choose $\xi = M\chi_B$ in (4.4), where χ_B denotes the characteristic function of B. We have

$$-\Big(\frac{\varepsilon}{\varepsilon+\tau}\Big)H(M) - H(-M) \le \Big(\frac{\tau}{\varepsilon+\tau}\Big)\mathbb{C} e_i^\varepsilon(x) : M \le H(M) + \Big(\frac{\varepsilon}{\varepsilon+\tau}\Big)H(-M),$$

for $i = 2, \dots, n-2$, and a.e. $x \in \Omega$, which by (2.4) imply

$$-2r_K|M| \le \Big(\frac{\tau}{\varepsilon+\tau}\Big)\mathbb{C} e_i^\varepsilon(x) : M \le 2R_K|M|,$$

for $i = 2, \dots, n-2$, and every $M \in \mathbb{M}^{3\times3}_D$, for a.e. $x \in \Omega$. Thus we get the estimate

$$\|(\mathbb{C} e_i^\varepsilon)_D\|_{L^\infty(\Omega;\mathbb{M}^{3\times3}_D)} \le 2\Big(\frac{\varepsilon+\tau}{\tau}\Big)R_K, \tag{4.6}$$

for $i = 2, \dots, n-2$.

As a consequence of inequality (4.4), the discrete stresses $\sigma_i^\varepsilon := \mathbb{C}e_i^\varepsilon$, $i = 2, \dots, n-2$, belong to the subdifferential in 0 of suitable convex and positively 1-homogeneous functions. Indeed, by (4.4) we have

$$\Big(\frac{\tau}{\varepsilon+\tau}\Big)\sigma_i^\varepsilon(x) \in \partial F_H^\varepsilon(0), \quad \text{for a.e. } x \in \Omega,\, i = 2, \dots, n-2.$$

where $F_H^\varepsilon : \mathbb{M}_D^{3\times 3} \to [0, +\infty)$ is defined as

$$F_H^\varepsilon(M) := H(M) + \left(\frac{\varepsilon}{\varepsilon+\tau}\right) H(-M),$$

for every $M \in \mathbb{M}_D^{3\times 3}$. The convexity and positive one-homogeneity of F_H^ε follow directly by the corresponding properties of H.

Equation (4.3) can be equivalently reformulated in the following useful form.

Proposition 4.3 (Discrete Euler-Lagrange equations 2)**.** *Let $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon), \dots, (u_n^\varepsilon, e_n^\varepsilon, p_n^\varepsilon)$ be a solution to (4.2). Then*

$$\delta^2 u_n^\varepsilon = \delta^3 u_n^\varepsilon = 0, \tag{4.7}$$

$$\int_\Omega \big[\rho(\varepsilon^2\delta^4 u_{i+2}^\varepsilon - 2\varepsilon\delta^3 u_{i+1}^\varepsilon + \delta^2 u_i^\varepsilon)\cdot\varphi + \mathbb{C}e_i^\varepsilon : E\varphi\big]\,dx = 0 \tag{4.8}$$

for $i = 2, \dots, n-2$, and every $\varphi \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $\varphi = 0$ $\mathcal{H}^2$-a.e. on Γ_0.

We omit the proof of this proposition as it follows arguing exactly as in [48, Subsection 2.3]. In view of (4.8) there holds

$$\begin{cases}\operatorname{div}\mathbb{C}e_i^\varepsilon = \rho(\varepsilon^2\delta^4 u_{i+2}^\varepsilon - 2\varepsilon\delta^3 u_{i+1}^\varepsilon + \delta^2 u_i^\varepsilon) & \text{a.e. in } \Omega,\\ \mathbb{C}e_i^\varepsilon\nu = 0 & \mathcal{H}^2-\text{ a.e. on } \partial\Omega\setminus\Gamma_0,\end{cases} \tag{4.9}$$

and hence, $\operatorname{div}\mathbb{C}e_i^\varepsilon \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, $i = 2, \dots, n-2$.

4.2. **Stress-strain duality.** In order to establish a uniform discrete energy estimate we need to preliminary introduce a notion of duality for the discrete stresses σ_i^ε and the plastic strains p_i^ε.

We work along the footsteps of [23] and [11, Subsection 2.3]. Define the set

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}) : \sigma_D \in L^\infty(\Omega;\mathbb{M}_D^{3\times 3}) \text{ and } \operatorname{div}\sigma \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)\}. \tag{4.10}$$

By [23, Proposition 2.5] for every $\sigma \in \Sigma(\Omega)$ there holds

$$\sigma \in L^6(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}),$$

and

$$\|\mathrm{tr}\sigma\|_{L^6(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})} \le C\big(\|\sigma\|_{L^1(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})} + \|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}_D^{3\times 3})} + \|\operatorname{div}\sigma\|_{L^2(\Omega;\mathbb{R}^3)}\big).$$

In addition, we can introduce the trace $[\sigma\nu] \in W^{1,1/2}(\partial\Omega;\mathbb{R}^3)$ (see e.g. [50, Theorem 1.2, Chapter I]) as

$$\int_{\partial\Omega}[\sigma\nu]\cdot\psi\,d\mathcal{H}^2 := \int_\Omega \operatorname{div}\sigma\cdot\psi\,dx + \int_\Omega \sigma : E\psi\,dx$$

for every $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$. Defining the normal and the tangential part of $[\sigma\nu]$ as

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu \quad \text{and} \quad [\sigma\nu]_\nu^\perp := [\sigma\nu] - ([\sigma\nu]\cdot\nu)\nu,$$

by [23, Lemma 2.4] we have that $[\sigma\nu]_\nu^\perp \in L^\infty(\partial\Omega;\mathbb{R}^3)$, and

$$\|[\sigma\nu]_\nu^\perp\|_{L^\infty(\partial\Omega;\mathbb{R}^3)} \le \frac{1}{\sqrt{2}}\|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}_D^{3\times 3})}.$$

Let $\sigma \in \Sigma(\Omega)$ and let $u \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, with $\operatorname{div} u \in L^2(\Omega)$. We define the distribution $[\sigma_D : E_D u]$ on Ω as

$$\langle[\sigma_D : E_D u],\, \varphi\rangle := -\int_\Omega \varphi\operatorname{div}\sigma\cdot u\,dx - \frac{1}{3}\int_\Omega \varphi\,\mathrm{tr}\,\sigma\cdot\operatorname{div} u\,dx - \int_\Omega \sigma : (u\odot\nabla\varphi)\,dx \tag{4.11}$$

for every $\varphi \in C_c^\infty(\Omega)$. By [23, Theorem 3.2] it follows that $[\sigma_D : E_D u]$ is a bounded Radon measure on Ω, whose variation satisfies

$$|[\sigma_D : E_D u]| \leq \|\sigma_D\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)}|E_D u| \quad \text{in } \Omega.$$

Let $\Pi_{\Gamma_0}(\Omega)$ be the set of admissible plastic strains, namely the set of maps $p \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ such that there exist $u \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$, $e \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$, and $w \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $(u,e,p) \in \mathscr{A}(w)$. Note that the additive decomposition $Eu = e+p$ implies that $\operatorname{div} u \in L^2(\Omega)$.

It is possible to define a duality between elements of $\Sigma(\Omega)$ and $\Pi_{\Gamma_0}(\Omega)$. To be precise, given $p \in \Pi_{\Gamma_0}(\Omega)$, and $\sigma \in \Sigma(\Omega)$, we fix (u,e,w) such that $(u,e,p) \in \mathscr{A}(w)$, with $u \in L^2(\Omega;\mathbb{R}^3)$, and we define the measure $[\sigma_D : p] \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ as

$$[\sigma_D : p] := \begin{cases} [\sigma_D : E_D u] - \sigma_D : e_D & \text{in } \Omega \\ [\sigma\nu]^\perp_\nu \cdot (w-u)\mathcal{H}^2 & \text{on } \Gamma_0, \end{cases}$$

so that

$$\int_{\Omega\cup\Gamma_0} \varphi \, d[\sigma_D : p] = \int_\Omega \varphi \, d[\sigma_D : E_D u] - \int_\Omega \varphi\sigma_D : e_D \, dx + \int_{\Gamma_0} \varphi[\sigma\nu]^\perp_\nu \cdot (w-u)\, d\mathcal{H}^2$$

for every $\varphi \in C(\bar\Omega)$. Arguing as in [11, Section 2] one can prove that the definition of $[\sigma_D : p]$ is independent of the choice of (u,e,w), and that if $\sigma_D \in C(\bar\Omega;\mathbb{M}^{3\times 3}_D)$ and $\varphi \in C(\bar\Omega)$, then

$$\int_{\Omega\cup\Gamma_0} \varphi \, d[\sigma_D : p] = \int_{\Omega\cup\Gamma_0} \varphi\sigma_D : dp.$$

We finally rewrite [11, Proposition 2.2] in our framework.

Proposition 4.4. *Let $\sigma \in \Sigma(\Omega)$, $w \in W^{1,2}(\Omega;\mathbb{R}^3)$, and $(u,e,p) \in \mathscr{A}(w)$, with $u \in L^2(\Omega;\mathbb{R}^3)$. Then*

$$[\sigma_D : p](\Omega\cup\Gamma_0) + \int_\Omega \sigma : (e - Ew)\, dx = -\int_\Omega \operatorname{div}\sigma \cdot (u-w)\, dx + \int_{\partial\Omega\setminus\Gamma_0} [\sigma\nu]\cdot(u-w)\, dx.$$

Remark 4.5. We point out that the C^2 regularity of $\partial\Omega$ is needed here in order to apply [23, Proposition 2.5]. It is not possible to use here the results in [17] and extend the analysis to the case in which $\partial\Omega$ is Lipschitz, as (4.9) only implies that $\operatorname{div}\mathbb{C}e_i^\varepsilon \in L^2(\Omega;\mathbb{R}^3)$, whereas [17, Proposition 6.1] would require $\operatorname{div}\mathbb{C}e_i^\varepsilon \in L^3(\Omega;\mathbb{R}^3)$.

4.3. **Discrete energy estimate.** We preliminary establish a lower bound on the mass of the measures $[(\mathbb{C}e_i^\varepsilon)_D : q]$, $i = 2,\dots,n-2$, where $q \in \Pi_{\Gamma_0}(\Omega)$ is such that there exist $v \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$ and $f \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$ satisfying $(v,f,q) \in \mathscr{A}(0)$.

A caveat on notation: in the following we use the symbol C to indicate a generic constant, possibly depending on data and varying from line to line.

The following estimate holds true.

Proposition 4.6. *Let $q \in \Pi_{\Gamma_0}(\Omega)$, let $v \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3)$ and $f \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$ be such that $(v,f,q) \in \mathscr{A}(0)$. Then*

$$\tau[(\mathbb{C}e_i^\varepsilon)_D : q](\Omega\cup\Gamma_0) + (\varepsilon+\tau)\mathcal{H}(\delta p_i^\varepsilon - q) + \varepsilon\mathcal{H}(q) \geq (\varepsilon+\tau)\mathcal{H}(\delta p_i^\varepsilon) \tag{4.12}$$

for every $i = 2,\dots,n-2$.

Proof. Let q be as in the statement of the proposition. By (4.6) and (4.9) it follows that $\mathbb{C}e_i^\varepsilon \in \Sigma(\Omega)$, $i = 2,\dots,n-2$. In view of the triangular inequality (2.5), since $(u_0^\varepsilon,e_0^\varepsilon,p_0^\varepsilon),\dots,(u_n^\varepsilon,e_n^\varepsilon,p_n^\varepsilon)$ is a solution to (4.2) it also solves the implicit minimum problem

$$\begin{aligned} &I_{\varepsilon\tau}\big((u_0^\varepsilon,e_0^\varepsilon,p_0^\varepsilon),\dots,(u_n^\varepsilon,e_n^\varepsilon,p_n^\varepsilon)\big) \\ &\quad = \min_{(u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\in\mathscr{K}_\tau(u^0,e^0,p^0,u^1)} J_{\varepsilon\tau}\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big) \end{aligned}$$

where

$$J_{\varepsilon\tau}\big((u_0,e_0,p_0),\dots,(u_n,e_n,p_n)\big) := \frac{\varepsilon^2\rho}{2}\sum_{j=2}^{n}\tau\eta_{\tau,j}\int_\Omega|\delta^2u_j|^2\,dx$$
$$+\sum_{j=2}^{n-2}\tau\eta_{\tau,j+2}\int_\Omega Q(e_j)\,dx+\varepsilon\tau\sum_{j=1}^{n}\eta_{\tau,j+1}\left[\mathcal{H}\Big(\frac{p_j-p^\varepsilon_{j-1}}{\tau}\Big)+\mathcal{H}\Big(\frac{p^\varepsilon_{j-1}-p_{j-1}}{\tau}\Big)\right].$$

Arguing as in Proposition 4.2 we compute the Euler-Lagrange equations associated to the minimum problem above, and we perform variations $(u^\varepsilon_0\pm\lambda v_0,e^\varepsilon_0\pm\lambda f_0,p^\varepsilon_0\pm\lambda q_0),\dots(u^\varepsilon_n\pm\lambda v_n,e^\varepsilon_n\pm\lambda f_n,p^\varepsilon_n\pm\lambda q_n)$, with $\lambda>0$, and $(v_0,f_0,q_0),\dots,(v_n,f_n,q_n)\in\mathscr{K}_\tau(0,0,0,0)$. The convexity of $\mathcal{H}$ yields

$$\varepsilon^2\rho\sum_{j=2}^{n}\tau\eta_{\tau,j}\int_\Omega\delta^2u^\varepsilon_j\cdot\delta^2v_j\,dx+\sum_{j=2}^{n-2}\tau\eta_{\tau,j+2}\int_\Omega\mathbb{C}e^\varepsilon_j:f_j\,dx$$
$$+\varepsilon\tau\sum_{j=1}^{n}\eta_{\tau,j+1}\left[\mathcal{H}\Big(\delta p^\varepsilon_j+\frac{q_j}{\tau}\Big)-\mathcal{H}(\delta p^\varepsilon_j)+\mathcal{H}\Big(-\frac{q_{j-1}}{\tau}\Big)\right]\geq 0.$$

By combining Proposition 4.4 with the Euler-Lagrange equation (4.9), and performing the discrete integration by parts in [48, Subsection 2.3], we have

$$-\sum_{j=2}^{n-2}\tau\eta_{\tau,j+2}[(\mathbb{C}e^\varepsilon_j)_D:q_j](\Omega\cup\Gamma_0)$$
$$+\varepsilon\tau\sum_{j=1}^{n-1}\eta_{\tau,j+1}\left[\mathcal{H}\Big(\delta p^\varepsilon_j+\frac{q_j}{\tau}\Big)-\mathcal{H}(\delta p^\varepsilon_j)+\mathcal{H}\Big(-\frac{q_{j-1}}{\tau}\Big)\right]\geq 0.$$

The thesis follows choosing $q_j=-\tau q$ for $j=i$, and $q_j=0$ otherwise. □

Given a vector $(w_0,\dots,w_n)$ we denote by $\bar{w}_\tau$ and w_τ its backward piecewise-constant and its piecewise-affine interpolants on the partition, that is

$$\bar{w}_\tau(0)=w_\tau(0)=w_0,\quad \bar{w}_\tau(t)=w_i,\quad w_\tau(t):=\alpha_\tau(t)w_i+(1-\alpha_\tau(t))w_{i-1}\tag{4.13}$$

for $t\in((i-1)\tau,i\tau]$, $i=1,\dots,n$, where

$$\alpha_\tau(t):=\frac{(t-(i-1)\tau)}{\tau}\quad\text{for } t\in((i-1)\tau,i\tau],\quad i=1,\dots,n.$$

In particular, $\dot{w}_\tau(t)=\overline{\delta w}_\tau(t)$ for almost every $t\in(0,T)$. Analogously, we define the piecewise constant maps

$$\bar{\eta}_\tau(t):=\eta_{\tau,i}\quad\text{for } t\in((i-1)\tau,i\tau],\quad i=1,\dots,n.$$

In addition, as in [48, Subsection 2.5.1] we denote by $\tilde{w}_\tau$ the piecewise quadratic interpolants, defined via

$$\begin{aligned}&\tilde{w}_\tau(t):=w_\tau(t)\quad\text{in }[0,\tau]\\&\dot{\tilde{w}}_\tau(t)=\alpha_\tau(t)\dot{w}_\tau(t)+(1-\alpha_\tau(t))\dot{w}_\tau(t-\tau)\quad\text{in }(\tau,T].\end{aligned}\tag{4.14}$$

Notice that

$$\dot{\tilde{w}}_\tau(t)=\dot{w}_\tau(t-\tau)+\tau\alpha_\tau(t)\ddot{\tilde{w}}_\tau(t)\quad\text{for a.e. } t\in(\tau,T].$$

Theorem 4.7 (Discrete energy estimate)**.** *Let $(u^\varepsilon_0,e^\varepsilon_0,p^\varepsilon_0),\dots,(u^\varepsilon_n,e^\varepsilon_n,p^\varepsilon_n)$, be a solution to (4.2). Assume in addition that $p^1=0$. Let $(u^\varepsilon_\tau,e^\varepsilon_\tau,p^\varepsilon_\tau)$ and $(\tilde{u}^\varepsilon_\tau,\tilde{e}^\varepsilon_\tau,\tilde{p}^\varepsilon_\tau)$ be the triples of associated piecewise affine and piecewise quadratic interpolants, respectively. Then there exists a constant C (independent of ε and τ) such that*

$$\varepsilon\rho\int_{2\tau}^{T-2\tau}\int_{2\tau}^{t}\int_\Omega|\ddot{\tilde{u}}^\varepsilon_\tau|^2\,dx\,ds\,dt+\rho\int_\tau^{T-2\tau}\int_\Omega|\dot{u}^\varepsilon_\tau|^2\,dx\,dt$$
$$+\int_\tau^{T-2\tau}\int_\Omega Q(e^\varepsilon_\tau)\,dx\,dt+\int_\tau^{T-2\tau}\mathcal{H}(\dot{p}^\varepsilon_\tau)\,dt\leq C\Big(1+\frac{\tau}{\varepsilon}\Big).\tag{4.15}$$

Proof. Take the map $\varphi = \tau(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))$ as test function in (4.9). For $k = 2, \dots, n-2$ we obtain

$$\begin{aligned}
&\varepsilon^2\rho\sum_{i=2}^k \tau\int_\Omega \delta^4 u_{i+2}^\varepsilon\cdot(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx\\
&- 2\varepsilon\rho\sum_{i=2}^k \tau\int_\Omega \delta^3 u_{i+1}^\varepsilon\cdot(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx\\
&+ \rho\sum_{i=2}^k \tau\int_\Omega \delta^2 u_{i}^\varepsilon\cdot(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx\\
&- \sum_{i=2}^k \tau\int_\Omega \mathrm{div}\,\mathbb{C}e_i^\varepsilon\cdot(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx = 0. \qquad (4.16)
\end{aligned}$$

Arguing as in [48, Subsection 2.4] we estimate the first three terms in the left-hand side of (4.16) from below as

$$\begin{aligned}
&\varepsilon^2\rho\sum_{i=2}^k \tau\int_\Omega \delta^4 u_{i+2}^\varepsilon\cdot(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx \geq \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx\\
&+ \varepsilon^2\rho\int_\Omega \delta^3 u_{k+2}^\varepsilon\cdot(\delta u_k^\varepsilon - u^1 - \delta w_k + \dot{w}(0))\,dx - \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u_{k+1}^\varepsilon|^2\,dx\\
&+ \frac{\varepsilon^2\rho}{4}\sum_{i=2}^k\int_\Omega |\delta^2 u_{i+1}^\varepsilon - \delta^2 u_i^\varepsilon|^2\,dx - \varepsilon^2\rho\int_\Omega |\delta^2 w_2|^2\,dx + \varepsilon^2\rho\int_\Omega \delta^2 u_{k+1}^\varepsilon \delta^2 w_k\,dx\\
&- \varepsilon^2\rho\int_\Omega \delta^2 u_3^\varepsilon\delta^2 w_3\,dx - \frac{\varepsilon^2\rho}{2}\sum_{i=4}^k\tau\int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx - \frac{\varepsilon^2\rho}{2}\sum_{i=4}^k\tau\int_\Omega |\delta^3 w_i|^2\,dx, \qquad (4.17)
\end{aligned}$$

$$\begin{aligned}
&- 2\varepsilon\rho\sum_{i=2}^k \tau\int_\Omega \delta^3 u_{i+1}^\varepsilon\cdot(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx \geq -\varepsilon\rho\sum_{i=2}^k\tau\int_\Omega |\delta^2 w_i|^2\,dx\\
&- 2\varepsilon\rho\int_\Omega \delta^2 u_{k+1}^\varepsilon\cdot(\delta u_k^\varepsilon - u^1 - \delta w_k + \dot{w}(0))\,dx + \varepsilon\rho\sum_{i=3}^k\tau\int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx, \qquad (4.18)
\end{aligned}$$

and

$$\begin{aligned}
&\rho\sum_{i=2}^k \tau\int_\Omega \delta^2 u_{i}^\varepsilon\cdot(\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx = \frac{\rho}{2}\int_\Omega |\delta u_k^\varepsilon - u^1|^2\,dx\\
&+ \frac{\rho}{2}\sum_{i=2}^k\int_\Omega |\delta u_i^\varepsilon - \delta u_{i-1}^\varepsilon|^2\,dx - \rho\sum_{i=2}^k\int_\Omega (\delta u_i^\varepsilon - \delta u_{i-1}^\varepsilon)\cdot(\delta w_i - \dot{w}(0))\,dx\\
&\geq \frac{\rho}{2}\int_\Omega |\delta u_k^\varepsilon - u^1|^2\,dx + \frac{\rho}{2}\sum_{i=2}^k\int_\Omega |\delta u_i^\varepsilon - \delta u_{i-1}^\varepsilon|^2\,dx - \frac{\rho}{16}\sum_{i=2}^{k-1}\tau\int_\Omega |\delta u_i^\varepsilon|^2\,dx\\
&- \frac{\rho}{2}\int_\Omega |\dot{w}(0)|^2\,dx - \rho\int_\Omega \delta u_k^\varepsilon\delta w_k\,dx + \rho\int_\Omega \delta u_1^\varepsilon\delta w_2\,dx - 4\rho\sum_{i=2}^{k-1}\tau\int_\Omega |\delta^2 w_i|^2\,dx. \qquad (4.19)
\end{aligned}$$

Regarding the fourth term in the right-hand side of (4.16), by (4.6) and (4.9) there holds $\mathbb{C}e_i^\varepsilon \in \Sigma(\Omega)$ for $i = 2, \dots, n-2$ (see (4.10)). Therefore, in view of Proposition 4.4 and (4.9), we have

$$- \sum_{i=2}^k \tau\int_\Omega \mathrm{div}\,\mathbb{C}e_i^\varepsilon : (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx$$

$$= \sum_{i=2}^{k} \tau \int_\Omega \mathbb{C}e_i^\varepsilon : (\delta e_i^\varepsilon - e^1 - E\delta w_i + E\dot{w}(0))\,dx + \sum_{i=2}^{k} \tau[(\mathbb{C}e_i^\varepsilon)_D : \delta p_i^\varepsilon](\Omega \cup \Gamma_0)$$

for $k = 2, \dots, n-2$. On the one hand

$$\sum_{i=2}^{k} \tau \int_\Omega \mathbb{C}e_i^\varepsilon : (-E\delta w_i + E\dot{w}(0))\,dx \geq -\frac{1}{4}\sum_{i=2}^{k} \tau \int_\Omega Q(e_i^\varepsilon)\,dx$$
$$- 4\sum_{i=2}^{k} \tau \int_\Omega Q(E\delta w_i - E\dot{w}(0))\,dx,$$

and

$$\sum_{i=2}^{k} \tau \int_\Omega \mathbb{C}e_i^\varepsilon : (\delta e_i^\varepsilon - e^1)\,dx \geq \int_\Omega Q(e_k^\varepsilon)\,dx - \int_\Omega Q(e^1)\,dx - \sum_{i=2}^{k} \tau \int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx.$$

By Proposition 4.6 we infer that

$$\sum_{i=2}^{k} \tau[(\mathbb{C}e_i^\varepsilon)_D : \delta p_i^\varepsilon](\Omega \cup \Gamma_0) \geq \sum_{i=2}^{k} \tau \mathcal{H}(\delta p_i^\varepsilon).$$

Therefore

$$\begin{aligned} &- \sum_{i=2}^{k} \tau \int_\Omega \operatorname{div} \mathbb{C}e_i^\varepsilon :\, (\delta u_i^\varepsilon - u^1 - \delta w_i + \dot{w}(0))\,dx \\ &\geq \int_\Omega Q(e_k^\varepsilon)\,dx - \int_\Omega Q(e^1)\,dx - \sum_{i=2}^{k} \tau \int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx \\ &- \frac{1}{4}\sum_{i=2}^{k} \tau \int_\Omega Q(e_i^\varepsilon)\,dx - 4\sum_{i=2}^{k} \tau \int_\Omega Q(E\delta w_i - E\dot{w}(0))\,dx + \sum_{i=2}^{k} \tau \mathcal{H}(\delta p_i^\varepsilon). \end{aligned} \tag{4.20}$$

By combining (4.17)–(4.20), equality (4.16) yields

$$\begin{aligned} &\varepsilon^2\rho \int_\Omega \delta^3 u_{k+2}^\varepsilon \cdot (\delta u_k^\varepsilon - u^1 - \delta w_k + \dot{w}(0))\,dx - \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u_{k+1}^\varepsilon|^2\,dx \\ &+ \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx + \frac{\varepsilon^2\rho}{4}\sum_{i=2}^{k}\int_\Omega |\delta^2 u_{i+1}^\varepsilon - \delta^2 u_i^\varepsilon|^2\,dx \\ &+ \varepsilon^2\rho \int_\Omega \delta^2 u_{k+1}^\varepsilon \delta^2 w_k\,dx + \frac{\rho}{2}\int_\Omega |\delta u_k^\varepsilon - u^1|^2\,dx - \varepsilon^2\rho \int_\Omega \delta^2 u_3^\varepsilon \delta^2 w_3\,dx \\ &- 2\varepsilon\rho \int_\Omega \delta^2 u_{k+1}^\varepsilon \cdot (\delta u_k^\varepsilon - u^1 - \delta w_k + \dot{w}(0))\,dx + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho \sum_{i=3}^{k} \tau \int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx \\ &+ \frac{\rho}{2}\sum_{i=2}^{k}\int_\Omega |\delta u_i^\varepsilon - \delta u_{i-1}^\varepsilon|^2\,dx - \rho \int_\Omega \delta u_k^\varepsilon \delta w_k\,dx + \rho \int_\Omega \delta u_1^\varepsilon \delta w_2\,dx \\ &- \frac{\rho}{16}\sum_{i=2}^{k-1} \tau \int_\Omega |\delta u_i^\varepsilon|^2\,dx + \int_\Omega Q(e_k^\varepsilon)\,dx + \sum_{i=2}^{k} \tau \mathcal{H}(\delta p_i^\varepsilon) \\ &\leq \varepsilon^2\rho \int_\Omega |\delta^2 w_2|^2\,dx + \frac{\varepsilon^2\rho}{2}\sum_{i=4}^{k} \tau \int_\Omega |\delta^3 w_i|^2\,dx + \varepsilon\rho \sum_{i=2}^{k} \tau \int_\Omega |\delta^2 w_i|^2\,dx \\ &+ \frac{\rho}{2}\int_\Omega |\dot{w}(0)|^2\,dx + 4\rho \sum_{i=2}^{k-1} \tau \int_\Omega |\delta^2 w_i|^2\,dx + 4\sum_{i=2}^{k} \tau \int_\Omega Q(E\delta w_i - E\dot{w}(0))\,dx \end{aligned}$$

$$+\int_\Omega Q(e^1)\,dx+\sum_{i=2}^k\tau\int_\Omega \mathbb{C}e_i^\varepsilon : e^1\,dx+\frac14\sum_{i=2}^k\tau\int_\Omega Q(e_i^\varepsilon)\,dx. \tag{4.21}$$

Since $w\in W^{3,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, by Hölder's inequality there holds

$$\begin{aligned}\tau\int_\Omega|\delta w_2|^2\,dx&=\tau\int_\Omega\Big|\frac{w(2\tau)-w(0)-\tau\dot w(0)}{\tau}\Big|^2\,dx\\&\le 2\tau\int_\Omega\Big|\frac{w(2\tau)-w(\tau)}{\tau}\Big|^2\,dx+2\tau\int_\Omega\Big|\frac{w(\tau)-w(0)-\tau\dot w(0)}{\tau}\Big|^2\,dx\\&\le 2\tau\int_\Omega\Big|\frac{w(2\tau)-w(\tau)}{\tau}\Big|^2\,dx+\frac{2}{\tau}\int_\Omega\Big|\int_0^\tau(\dot w(t)-\dot w(0))\,dt\Big|^2\,dx\\&\le 2\tau\int_\Omega\Big|\frac{w(2\tau)-w(\tau)}{\tau}\Big|^2\,dx+2\tau^2\int_0^T\int_\Omega|\ddot w|^2\,dx\,dt.\end{aligned}$$

Thus, we have found that

$$\begin{aligned}\sum_{i=2}^k\tau\int_\Omega|\delta w_i|^2\,dx&\le 2\tau^2\int_0^T\int_\Omega|\ddot w|^2\,dx\,dt+2\sum_{i=2}^k\tau\int_\Omega\Big|\frac{w(t_i)-w(t_{i-1})}{\tau}\Big|^2\,dx\\&\le 2\tau^2\int_0^T\int_\Omega|\ddot w|^2\,dx\,dt+\frac{2}{\tau}\sum_{i=2}^k\int_\Omega\Big|\int_{(i-1)\tau}^{i\tau}\dot w(t)\,dt\Big|^2\,dx\\&\le 2\tau^2\int_0^T\int_\Omega|\ddot w|^2\,dx\,dt+2\int_0^T\int_\Omega|\dot w|^2\,dx\,dt.\end{aligned}\tag{4.22}$$

Analogously, one checks that

$$\begin{aligned}\varepsilon^2\rho\int_\Omega|\delta^2 w_2|^2\,dx&=\varepsilon^2\rho\int_\Omega\Big|\frac{w(t_2)-2\tau\dot w(0)-w(0)}{\tau^2}\Big|^2\,dx\\&=\varepsilon^2\rho\int_\Omega\Big|\frac{1}{\tau^2}\int_0^{2\tau}\int_0^\xi\ddot w(\lambda)\,d\lambda\,d\xi\Big|^2\,dx\\&\le C\varepsilon^2\rho\int_\Omega|\ddot w(0)|^2\,dx+2\varepsilon^2\rho\tau\int_0^T\int_\Omega|\dddot w|^2\,dx\,dt,\end{aligned}\tag{4.23}$$

as well as the following

$$\begin{aligned}\sum_{i=2}^{k-1}\tau\int_\Omega|\delta^2 w_i|^2\,dx&=\sum_{i=3}^{k-1}\tau\int_\Omega\Big|\int_{(i-1)\tau}^{i\tau}\frac{\dot w(t)-\dot w(t-\tau)}{\tau^2}\,dt\Big|^2\,dx+C\tau\\&\le\frac1\tau\sum_{i=3}^{k-1}\int_\Omega\int_{(i-1)\tau}^{i\tau}\int_{t-\tau}^t|\ddot w(\xi)|^2\,d\xi\,dt\,dx+C\tau\le\int_0^T\int_\Omega|\ddot w|^2\,dx\,dt+C\tau.\end{aligned}\tag{4.24}$$

In addition, we have that

$$\begin{aligned}\sum_{i=4}^k\tau\int_\Omega|\delta^3 w_i|^2\,dx&=\frac{1}{\tau^5}\sum_{i=4}^k\int_\Omega\Big|\int_{(i-1)\tau}^{i\tau}\int_{\xi-\tau}^\xi(\ddot w(s)-\ddot w(s-\tau))\,ds\,d\xi\Big|^2\,dx\\&\le C\int_\Omega\int_0^T|\dddot w|^2\,dt\,dx.\end{aligned}\tag{4.25}$$

Finally, in view of Jensen's inequality, we compute

$$\begin{aligned}&4\sum_{i=2}^k\tau\int_\Omega Q(E\delta w_i-E\dot w(0))\,dx\le 4\tau(k-2)\int_\Omega Q(E\dot w(0))\,dx\\&+8\sum_{i=2}^k\tau\int_\Omega Q\Big(\frac1\tau\int_{(i-1)\tau}^{i\tau}Ew(\xi)\,d\xi\Big)+8\tau\int_\Omega Q\Big(\frac1\tau\int_0^\tau(E\dot w(\xi)-E\dot w(0))\,d\xi\Big)\,dx\end{aligned}$$

$$\leq 4\tau n \int_\Omega Q(E\dot{w}(0))\,dx + 8\int_\Omega \int_0^T Q(Ew)\,dt\,dx + 8\int_\Omega \int_0^\tau Q(E\dot{w}(t) - E\dot{w}(0))\,dt\,dx. \tag{4.26}$$

By (4.22)–(4.26), the first two rows of the right-hand side of (4.21) are uniformly bounded in terms of the boundary datum w, independently of τ and ε. Therefore we obtain the estimate

$$\begin{aligned}
&\varepsilon^2\rho\int_\Omega \delta^2 u^\varepsilon_{k+1}\delta^2 w_k\,dx - \varepsilon^2\rho\int_\Omega \delta^2 u^\varepsilon_3\delta^2 w_3\,dx \\
&\quad + \varepsilon^2\rho\int_\Omega \delta^3 u^\varepsilon_{k+2}\cdot(\delta u^\varepsilon_k - u^1 - \delta w_k + \dot{w}(0))\,dx - \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u^\varepsilon_{k+1}|^2\,dx \\
&\quad + \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u^\varepsilon_2|^2\,dx - 2\varepsilon\rho\int_\Omega \delta^2 u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k - u^1 - \delta w_k + \dot{w}(0))\,dx \\
&\quad + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho\sum_{i=3}^k \tau\int_\Omega |\delta^2 u^\varepsilon_i|^2\,dx + \int_\Omega Q(e^\varepsilon_k)\,dx + \tau\sum_{i=2}^k \mathcal{H}(\delta p^\varepsilon_i) - \rho\int_\Omega \delta u^\varepsilon_k\delta w_k\,dx \\
&\quad + \rho\int_\Omega \delta u^\varepsilon_1\delta w_2\,dx - \frac{\rho}{16}\sum_{i=2}^{k-1}\tau\int_\Omega |\delta u^\varepsilon_i|^2\,dx \leq C + \int_\Omega Q(e^1)\,dx \\
&\quad + \sum_{i=2}^k \tau\int_\Omega \mathbb{C}e^\varepsilon_i : e^1\,dx + \frac{1}{4}\sum_{i=2}^k \tau\int_\Omega Q(e^\varepsilon_i)\,dx.
\end{aligned} \tag{4.27}$$

Multiplying the previous inequality by τ and summing for $k = 2, \dots, n-2$, one obtains

$$\begin{aligned}
&\varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^2 u^\varepsilon_{k+1}\delta^2 w_k\,dx - \varepsilon^2\rho\tau(n-3)\int_\Omega \delta^2 u^\varepsilon_3\delta^2 w_3\,dx + \frac{\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega |\delta u^\varepsilon_k - u^1|^2\,dx \\
&\quad + \varepsilon^2\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^3 u^\varepsilon_{k+2}\cdot(\delta u^\varepsilon_k - u^1 - \delta w_k + \dot{w}(0))\,dx - \frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau\int_\Omega |\delta^2 u^\varepsilon_{k+1}|^2\,dx \\
&\quad + \frac{\varepsilon^2\rho}{2}\tau(n-3)\int_\Omega |\delta^2 u^\varepsilon_2|^2\,dx - 2\varepsilon\rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta^2 u^\varepsilon_{k+1}\cdot(\delta u^\varepsilon_k - u^1 - \delta w_k + \dot{w}(0))\,dx \\
&\quad + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho\sum_{k=3}^{n-2}\sum_{i=3}^k \tau^2\int_\Omega |\delta^2 u^\varepsilon_i|^2\,dx + \sum_{k=2}^{n-2}\tau\int_\Omega Q(e^\varepsilon_k)\,dx + \sum_{k=2}^{n-2}\sum_{i=2}^k \tau^2\mathcal{H}(\delta p^\varepsilon_i) \\
&\quad - \rho\sum_{k=2}^{n-2}\tau\int_\Omega \delta u^\varepsilon_k\cdot\delta w_k\,dx + \rho\tau(n-3)\int_\Omega \delta u^\varepsilon_1\cdot\delta w_2\,dx - \frac{\rho}{16}\sum_{k=2}^{n-2}\sum_{i=2}^{k-1}\tau^2\int_\Omega |\delta u^\varepsilon_i|^2\,dx \\
&\quad \leq C + \tau(n-3)\int_\Omega Q(e^1)\,dx + \sum_{k=2}^{n-2}\sum_{i=2}^k \tau^2\int_\Omega \mathbb{C}e^\varepsilon_i : e^1\,dx \\
&\quad + \frac{(n-3)}{4}\sum_{i=2}^{n-2}\tau^2\int_\Omega Q(e^\varepsilon_i)\,dx.
\end{aligned} \tag{4.28}$$

By choosing $k = n-2$ in (4.27) and by (4.7), we have

$$\begin{aligned}
&-\varepsilon^2\rho\int_\Omega \delta^2 u^\varepsilon_3\delta^2 w_3\,dx + \frac{\varepsilon^2\rho}{2}\int_\Omega |\delta^2 u^\varepsilon_2|^2\,dx + \frac{\rho}{2}\int_\Omega |\delta u^\varepsilon_{n-2} - u^1|^2\,dx \\
&\quad + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho\sum_{i=3}^{n-2}\tau\int_\Omega |\delta^2 u^\varepsilon_i|^2\,dx + \int_\Omega Q(e^\varepsilon_{n-2})\,dx + \sum_{i=2}^{n-2}\tau\mathcal{H}(\delta p^\varepsilon_i) \\
&\quad - \rho\int_\Omega \delta u^\varepsilon_{n-2}\cdot\delta w_{n-2}\,dx + \rho\int_\Omega \delta u^\varepsilon_1\cdot\delta w_2\,dx - \frac{\rho}{16}\sum_{i=2}^{n-3}\tau\int_\Omega |\delta u^\varepsilon_i|^2\,dx
\end{aligned}$$

$$\leq C + \int_\Omega Q(e^1)\,dx + \sum_{i=2}^{n-2} \tau \int_\Omega \mathbb{C} e_i^\varepsilon : e^1\,dx + \frac{1}{4}\sum_{i=2}^{n-2} \tau \int_\Omega Q(e_i^\varepsilon)\,dx. \tag{4.29}$$

In view of (4.7) and (4.24) we deduce the lower bounds

$$\begin{aligned}
&\varepsilon^2\rho \sum_{k=2}^{n-2} \tau \int_\Omega \delta^3 u_{k+2}^\varepsilon \cdot (\delta u_k^\varepsilon - u^1 - \delta w_k + \dot w(0))\,dx \\
&\quad = -\varepsilon^2\rho \sum_{k=2}^{n-2} \tau \int_\Omega \delta^2 u_{k+1}^\varepsilon \cdot (\delta^2 u_k^\varepsilon - \delta^2 w_k)\,dx \geq -\frac{3\varepsilon^2\rho}{2} \sum_{k=2}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx \\
&\quad - \frac{\varepsilon^2\rho}{2} \sum_{k=2}^{n-2} \tau \int_\Omega |\delta^2 w_k|^2\,dx \geq -\frac{3\varepsilon^2\rho}{2} \sum_{k=2}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx - C,
\end{aligned} \tag{4.30}$$

and, analogously,

$$\begin{aligned}
&\varepsilon^2\rho \sum_{k=2}^{n-2} \tau \int_\Omega \delta^2 u_{k+1}^\varepsilon \cdot \delta^2 w_k\,dx \geq -\frac{\varepsilon^2\rho}{2} \sum_{k=3}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx - \frac{\varepsilon^2\rho}{2} \sum_{k=2}^{n-2} \tau \int_\Omega |\delta^2 w_k|^2\,dx \\
&\geq -\frac{\varepsilon^2\rho}{2} \sum_{k=3}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx - C.
\end{aligned} \tag{4.31}$$

In addition, arguing as in [48, Subsection 2.4],

$$\begin{aligned}
&-2\varepsilon\rho \sum_{k=2}^{n-2} \tau \int_\Omega \delta^2 u_{k+1}^\varepsilon \cdot (\delta u_k^\varepsilon - u^1 - \delta w_k + \dot w(0))\,dx = \varepsilon\rho \sum_{k=2}^{n-2} \int_\Omega |\delta u_{k+1}^\varepsilon - \delta u_k^\varepsilon|^2\,dx \\
&\quad - \varepsilon\rho \int_\Omega |\delta u_{n-1}^\varepsilon|^2\,dx + \varepsilon\rho \int_\Omega |\delta u_2^\varepsilon|^2\,dx - 2\varepsilon\rho \int_\Omega (\delta u_{n-1}^\varepsilon - \delta u_2^\varepsilon) \cdot (-u^1 + \dot w(0))\,dx \\
&\quad + 2\varepsilon\rho \sum_{k=2}^{n-2} \int_\Omega (\delta u_{k+1}^\varepsilon - \delta u_k^\varepsilon) \cdot \delta w_k\,dx \geq -3\varepsilon\rho \int_\Omega |\delta u_{n-1}^\varepsilon|^2\,dx - 2\varepsilon\rho \int_\Omega |u^1|^2\,dx \\
&\quad - \varepsilon\rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon|^2\,dx - C,
\end{aligned} \tag{4.32}$$

where we used (4.7) and (4.24). By collecting the terms in (4.28)–(4.32) involving second-order differences, we have

$$\begin{aligned}
&(\varepsilon - 3\varepsilon^2)\rho \sum_{k=3}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx + \frac{\varepsilon^2\rho}{2}(1 + \tau(n-6)) \int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx \\
&\quad + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho \sum_{k=3}^{n-2} \sum_{i=3}^{k} \tau^2 \int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx - \varepsilon^2\rho(1 + \tau(n-3)) \int_\Omega \delta^2 u_3^\varepsilon \cdot \delta^2 w_3\,dx \\
&\quad \geq \Big(\varepsilon - 3\varepsilon^2 - \frac{\varepsilon^2(1 + \tau(n-3))}{2}\Big)\rho \sum_{k=3}^{n-2} \tau \int_\Omega |\delta^2 u_k^\varepsilon|^2\,dx \\
&\quad + \Big(\varepsilon - \frac{\varepsilon^2}{2}\Big)\rho \sum_{k=2}^{n-2} \sum_{i=3}^{k} \tau^2 \int_\Omega |\delta^2 u_i^\varepsilon|^2\,dx + \frac{\varepsilon^2\rho(1 + \tau(n-6))}{2} \int_\Omega |\delta^2 u_2^\varepsilon|^2\,dx - C.
\end{aligned} \tag{4.33}$$

Finally, using the elementary inequality

$$|\delta u_i^\varepsilon|^2 \leq 2|\delta u_i^\varepsilon - u^1|^2 + 2|u^1|^2 \quad \text{a.e. in } \Omega, \quad \text{for every } i,$$

we deduce that

$$-3\varepsilon\rho \int_\Omega |\delta u_{n-1}^\varepsilon|^2\,dx - 2\varepsilon\rho \int_\Omega |u^1|^2\,dx - \varepsilon\rho \sum_{k=2}^{n-2} \tau \int_\Omega |\delta u_k^\varepsilon|^2\,dx - \rho \sum_{k=2}^{n-2} \tau \int_\Omega \delta u_k^\varepsilon \cdot \delta w_k\,dx$$

$$
\begin{aligned}
&+\rho\tau(n-3)\int_\Omega \delta u_1^\varepsilon\cdot\delta w_2\,dx-\frac{\rho}{16}\sum_{k=2}^{n-2}\sum_{i=2}^{k-1}\tau^2\int_\Omega|\delta u_i^\varepsilon|^2\,dx-\rho\int_\Omega\delta u_{n-2}^\varepsilon\cdot\delta w_{n-2}\,dx\\
&+\rho\int_\Omega\delta u_1^\varepsilon\cdot\delta w_2\,dx-\frac{\rho}{16}\sum_{i=2}^{n-3}\tau\int_\Omega|\delta u_i^\varepsilon|^2\,dx\\
&\geq-3\varepsilon\rho\int_\Omega|\delta u_{n-1}^\varepsilon|^2\,dx-2\varepsilon\rho\int_\Omega|u^1|^2\,dx\\
&-\Big(\varepsilon+\frac{1}{16}\Big)\rho\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u_k^\varepsilon|^2\,dx-\frac{\rho}{16}\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega|\delta u_i^\varepsilon|^2\,dx-C\\
&\geq-6\varepsilon\rho\int_\Omega|\delta u_{n-1}^\varepsilon-u^1|^2\,dx-\Big(2\varepsilon+\frac{1}{8}\Big)\rho\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u_k^\varepsilon-u^1|^2\,dx\\
&-\frac{\rho}{8}\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega|\delta u_i^\varepsilon-u^1|^2\,dx-C\int_\Omega|u^1|^2\,dx-C. \qquad (4.34)
\end{aligned}
$$

Summing (4.28) with (4.29), in view of (4.7), estimates (4.30)–(4.34) yield the inequality

$$
\begin{aligned}
&\Big(\frac14-2\varepsilon\Big)\rho\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u_k^\varepsilon-u^1|^2\,dx+\Big(\frac12-6\varepsilon\Big)\rho\int_\Omega|\delta u_{n-1}^\varepsilon-u^1|^2\,dx\\
&\quad+\frac{\varepsilon^2\rho(1+\tau(n-6))}{2}\int_\Omega|\delta u_2^\varepsilon|^2\,dx+\Big(\varepsilon-\frac{\varepsilon^2}{2}\Big)\rho\sum_{k=3}^{n-2}\sum_{i=3}^{k}\tau^2\int_\Omega|\delta^2u_i^\varepsilon|^2\,dx\\
&\quad+\sum_{k=2}^{n-2}\tau\int_\Omega Q(e_k^\varepsilon)\,dx+\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\mathcal{H}(\delta p_i^\varepsilon)+\int_\Omega Q(e_{n-2}^\varepsilon)\,dx+\tau\sum_{i=2}^{n-2}\mathcal{H}(\delta p_i^\varepsilon)\\
&\quad\leq(1+\tau(n-3))\int_\Omega Q(e^1)\,dx+\sum_{k=2}^{n-2}\sum_{i=2}^{k}\tau^2\int_\Omega\mathbb{C}e_i^\varepsilon:e^1\,dx+\frac14\sum_{i=2}^{n-2}\tau\int_\Omega Q(e_i^\varepsilon)\,dx\\
&\quad+\frac{(n-3)}{4}\sum_{i=2}^{n-2}\tau^2\int_\Omega Q(e_i^\varepsilon)\,dx+\sum_{i=2}^{n-2}\tau\int_\Omega\mathbb{C}e_i^\varepsilon:e^1\,dx+C\int_\Omega|u^1|^2\,dx+C. \qquad (4.35)
\end{aligned}
$$

By the definition of τ, for ε small enough we eventually obtain

$$
\begin{aligned}
&\varepsilon\rho\sum_{k=3}^{n-2}\sum_{i=3}^{k}\tau^2\int_\Omega|\delta^2u_i^\varepsilon|^2\,dx+\rho\sum_{k=2}^{n-2}\tau\int_\Omega|\delta u_k^\varepsilon-u^1|^2\,dx\\
&\quad+\sum_{k=2}^{n-2}\tau\int_\Omega Q(e_k^\varepsilon)\,dx+\sum_{k=2}^{n-2}\tau\mathcal{H}(\delta p_k^\varepsilon)\leq C \qquad (4.36)
\end{aligned}
$$

and the assertion follows. □

5. Γ-Convergence from discrete to continuous

In this section we prove that for fixed $\varepsilon>0$ the sequence of discrete energy functionals $\{I_{\varepsilon\tau}\}$ (see (4.1)) converges, as the time step τ tends to zero, to the functional I_ε. This will allow us to pass to the limit $\tau\to0$ in the discrete energy estimate (4.15) in order to obtain its continuous analogue, see (5.37) below.

In order to state the convergence result we need to introduce a few auxiliary spaces and to extend the energy functionals I_ε and $I_{\varepsilon\tau}$. Let

$$
\begin{aligned}
\mathscr{U}:=\{(u,e,p)\in\,& W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))\cap L^1(0,T;BD(\Omega))\\
&\times L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))\times L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D))\}
\end{aligned}
$$

and

$$\begin{aligned}\mathscr{U}_\tau^{\rm affine} :=\{&(u,e,p):[0,T]\to (BD(\Omega)\cap L^2(\Omega;\mathbb{R}^3))\times L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})\times \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)\\ &\text{piecewise affine on the time partition of step } \tau \text{ on } [0,T],\\ &\text{and such that } (u(0),e(0),p(0)),\ (u(\tau),e(\tau),p(\tau)),\dots,\\ &(u(T),e(T),p(T))\in \mathscr{K}_\tau(u^0,e^0,p^0,u^1)\}.\end{aligned}$$

We set

$$G_\varepsilon(u,e,p):=\begin{cases} I_\varepsilon(u,e,p) & \text{if } (u,e,p)\in\mathcal{V},\\ +\infty & \text{otherwise in } \mathscr{U},\end{cases}$$

(where $\mathcal{V}$ is the space defined in (2.11)), and

$$G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau):=\begin{cases} I_{\varepsilon\tau}\big((u_\tau(0),e_\tau(0),p_\tau(0)),\ (u_\tau(\tau),e_\tau(\tau),p_\tau(\tau)),\dots,(u_\tau(T),e_\tau(T),p_\tau(T))\big)\\ \qquad \text{if } (u_\tau,e_\tau,p_\tau)\in \mathscr{U}_\tau^{\rm affine},\\ +\infty \qquad \text{otherwise in } \mathscr{U}.\end{cases}$$

We now show that the sequence of energies $\{G_{\varepsilon\tau}\}$ converges to G_ε in the sense of Γ-convergence in $\mathscr{U}$ as $\tau\to 0$.

Theorem 5.1 (Liminf inequality)**.** *Let $\{(u_\tau,e_\tau,p_\tau)\}\subset\mathscr{U}$ and $(u,e,p)\in\mathscr{U}$ be such that*

$$u_\tau \rightharpoonup u \quad \text{weakly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.1}$$

$$p_\tau(t)\overset{*}{\rightharpoonup} p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D) \text{ for every } t\in[0,T], \tag{5.2}$$

$$e_\tau\rightharpoonup e \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})). \tag{5.3}$$

Then, we have that

$$G_\varepsilon(u,e,p)\le \liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau).$$

Proof. Let $\{(u_\tau,e_\tau,p_\tau)\}$ and (u,e,p) be as in the statement of the theorem. If $\liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau)=+\infty$ there is nothing to prove, therefore without loss of generality we can assume that

$$\begin{aligned}\liminf_{\tau\to 0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau)&=\lim_{\tau\to 0}\Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n \tau\eta_{\tau,i}\int_\Omega |\delta^2 u_\tau(i\tau)|^2\,dx\\ &+\sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx+\varepsilon\tau\sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big]<+\infty,\end{aligned}\tag{5.4}$$

In view of (5.1) and (5.2) it follows that $u(0)=u^0$ and $p(0)=p^0$. Denoting by $\bar{u}_\tau$ and $\tilde{u}_\tau$ the piecewise constant and piecewise quadratic interpolants associated to u_τ (see (4.13) and (4.14)), respectively, by (5.4), up to the extraction of a (not relabeled) subsequence, we have

$$\begin{aligned}\liminf_{\tau\to 0}\Big[&\frac{\varepsilon^2\rho}{2}\int_\tau^T \bar{\eta}_\tau\int_\Omega|\ddot{\tilde{u}}_\tau|^2\,dx\,dt+\int_\tau^{T-2\tau}\bar{\eta}_\tau(\cdot+2\tau)\int_\Omega Q(\bar{e}_\tau)\,dx\,dt\\ &+\varepsilon\int_0^T \bar{\eta}_\tau(\cdot+\tau)\mathcal{H}(\dot{p}_\tau)\,dt\Big]<+\infty.\end{aligned}\tag{5.5}$$

In view of (5.5) for τ small there holds

$$\begin{aligned}\liminf_{\tau\to 0}\Big[&\frac{\varepsilon^2\rho}{2}\int_\tau^T\int_\Omega(|\ddot{\tilde{u}}_\tau|^2+|\dot{u}_\tau|^2)\,dx\,dt+\int_\tau^{T-2\tau}\int_\Omega Q(\bar{e}_\tau)\,dx\,dt\\ &+\varepsilon\int_0^T\mathcal{H}(\dot{p}_\tau)\,dt\Big]<+\infty.\end{aligned}\tag{5.6}$$

Therefore, there exists a map $v\in W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ such that

$$\tilde{u}_\tau\rightharpoonup v \quad \text{weakly in } W^{2,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)). \tag{5.7}$$

Arguing as in [48, Subsection 2.5.1], we obtain that $u=v$, and $\dot{u}(0)=u^1$.

By (5.4) we deduce the upper bound

$$\liminf_{\tau\to 0} D_{\mathcal{H}}(\bar{p}_\tau;0,T) \le C. \tag{5.8}$$

Since $\bar{p}_\tau(0) = p^0$ for every τ, by [11, Lemma 7.2] there exists a map $q \in BV([0,T];\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))$ such that

$$\bar{p}_\tau(t) \rightharpoonup^* q(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D) \quad \text{for every } t\in[0,T] \tag{5.9}$$

and

$$D_{\mathcal{H}}(q;0,T) \le \liminf_{\tau} D_{\mathcal{H}}(\bar{p}_\tau;0,T).$$

By (5.5) for a.e. $t \in [0,T]$ there exists $f^t \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$, and a t-dependent subsequence τ_t such that

$$\bar{e}_{\tau_t}(t) \rightharpoonup f^t \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}). \tag{5.10}$$

By (5.9) and (5.10), for a.e. $t\in[0,T]$, the sequence $\{E\bar{u}_{\tau_t}(t)\}$ is bounded in $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)$ (see [11, Theorem 3.3]). This implies that for a.e. $t\in[0,T]$ there exists a map $v^t \in BD(\Omega)$ such that

$$\bar{u}_\tau(t) \rightharpoonup^* v^t \quad \text{weakly* in } BD(\Omega), \tag{5.11}$$

$$Ev^t = f^t + q(t), \tag{5.12}$$

$$q(t) = (w(t) - v^t)\odot \nu\mathcal{H}^1 \quad \text{on } \Gamma_0. \tag{5.13}$$

In view of (5.1) there holds

$$u_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3) \quad \text{for every } t\in[0,T]. \tag{5.14}$$

In addition, for fixed $i\in\mathbb{N}$, and $t\in((i-1)\tau, i\tau]$, we have

$$\bar{u}_\tau(t) - u_\tau(t) = (i\tau - t)\dot{u}_\tau(t).$$

Thus by (5.6) we obtain the estimate

$$\|\bar{u}_\tau - u_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} = \frac{\tau}{\sqrt{3}}\|\dot{u}_\tau(t)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le C\tau,$$

which in turn by (5.14) implies that

$$\bar{u}_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3) \quad \text{for a.e. } t\in[0,T]. \tag{5.15}$$

By (5.9)–(5.11) we conclude that

$$v^t = u(t), \tag{5.16}$$

and the convergence in (5.11) holds for the entire sequence $\bar{u}_\tau(t)$.

Fix $i\in\mathbb{N}$ and $t\in((i-1)\tau, i\tau]$, then

$$\|\bar{p}_\tau(t) - p_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)} = \|(t-i\tau)\dot{p}_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)}. \tag{5.17}$$

Therefore by (2.4)

$$\begin{aligned}\|\bar{p}_\tau - p_\tau\|_{L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))} &= \frac{\tau}{2}\|\dot{p}_\tau\|_{L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))}\\ &\le \frac{\tau}{2r_K}\int_0^T \mathcal{H}(\dot{p}_\tau)\,dt \le C\tau,\end{aligned} \tag{5.18}$$

where the last inequality is due to (5.6). In view of (5.18),

$$\|\bar{p}_\tau(t) - p_\tau(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)} \to 0 \quad \text{for a.e. } t\in[0,T].$$

Thus, by (5.2) and (5.9) we deduce that

$$p(t) = q(t) \quad \text{for a.e. } t\in[0,T]. \tag{5.19}$$

Finally, by (5.11)–(5.13), (5.16), and (5.19), there holds

$$\bar{e}_\tau(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) \quad \text{for a.e. } t\in[0,T]. \tag{5.20}$$

By Fatou's lemma, (5.7), and (5.20), one gets that

$$\begin{aligned}
&\frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega |\ddot u(t)|^2\,dx\,dt + \frac12\int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q(e(t))\,dx\,dt\\
&\le \frac12\int_0^T \liminf_{\tau\to0}\Big[\varepsilon^2\rho\bar\eta_\tau(t)\chi_{[\tau,T-\tau]}(t)\int_\Omega |\ddot{\bar u}_\tau(t)|^2\,dx\\
&\qquad + \bar\eta_\tau(t+2\tau)\chi_{[\tau,T-2\tau]}(t)\int_\Omega Q(\bar e_\tau(t))\,dx\Big]\,dt\\
&\le \liminf_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\int_\tau^T \bar\eta_\tau\int_\Omega |\ddot{\bar u}_\tau|^2\,dx\,dt + \frac12\int_\tau^{T-2\tau}\bar\eta_\tau(\cdot+2\tau)\int_\Omega Q(\bar e_\tau)\,dx\,dt\Big]\\
&= \liminf_{\tau\to0}\Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n \tau\eta_{\tau,i}\int_\Omega |\delta^2 u_\tau(i\tau)|^2\,dx + \sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx\Big],
\end{aligned}\tag{5.21}$$

where $\chi_{[\tau,T-\tau]}$ and $\chi_{[\tau,T-2\tau]}$ are the characteristic functions of the sets $[\tau,T-\tau]$ and $[\tau,T-2\tau]$, respectively.

To conclude we need to prove a liminf inequality for the plastic dissipation. To this purpose, let $0\le t_0<t_1<\dots<t_m\le T$. In view of (5.8) and (5.19), and since $\bar p_\tau$ only jumps in the points $i\tau$, $i=1,\dots,N$, we have

$$\begin{aligned}
&\sum_{i=1}^m \exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(p(t_i)-p(t_{i-1})) \le \liminf_{\tau\to0}\Big[\sum_{i=1}^m \exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(\bar p_\tau(t_i)-\bar p_\tau(t_{i-1}))\Big]\\
&\le \liminf_{\tau\to0}\Big[\sum_{i=1}^n \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(\bar p_\tau(i\tau)-\bar p_\tau((i-1)\tau)) + \frac{C\tau}{\varepsilon}D_{\mathcal{H}}(\bar p_\tau;0,T)\Big]\\
&= \liminf_{\tau\to0}\Big[\tau\sum_{i=1}^n \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big] \le \liminf_{\tau\to0}\Big[\tau\sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big]\\
&+\lim_{\tau\to0}\tau\Big|\sum_{i=1}^n\Big(\exp\Big(-\frac{i\tau}{\varepsilon}\Big)-\eta_{\tau,i+1}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big|.
\end{aligned}$$

Since $\bar\eta_\tau(\cdot+\tau)\to\exp(-t/\varepsilon)$ strongly in $L^\infty(0,T)$ as $\tau\to0$, by (5.8) we deduce

$$\begin{aligned}
&\lim_{\tau\to0}\tau\Big|\sum_{i=1}^n\Big(\exp\Big(-\frac{i\tau}{\varepsilon}\Big)-\eta_{\tau,i+1}\Big)\mathcal{H}(\delta p_\tau(i\tau))\Big|\\
&\le \lim_{\tau\to0}\Big\|\exp\Big(-\frac{t}{\varepsilon}\Big)-\bar\eta_\tau(t+\tau)\Big\|_{L^\infty(0,T)}\sum_{i=1}^n\mathcal{H}(\delta p_\tau(i\tau))\\
&\le \lim_{\tau\to0}\Big\|\exp\Big(-\frac{t}{\varepsilon}\Big)-\bar\eta_\tau(t+\tau)\Big\|_{L^\infty(0,T)}D_{\mathcal{H}}(\bar p_\tau;0,T)\\
&\le \lim_{\tau\to0}C\Big\|\exp\Big(-\frac{t}{\varepsilon}\Big)-\bar\eta_\tau(t+\tau)\Big\|_{L^\infty(0,T)}=0.
\end{aligned}$$

Thus, we have checked that

$$\begin{aligned}
&\sum_{i=1}^m \exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(p(t_i)-p(t_{i-1})) \le \liminf_{\tau\to0}\Big[\sum_{i=1}^m \exp\Big(-\frac{t_i}{\varepsilon}\Big)\mathcal{H}(\bar p_\tau(t_i)-\bar p_\tau(t_{i-1}))\Big]\\
&\le \liminf_{\tau\to0}\Big[\tau\sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big].
\end{aligned}$$

The arbitrariness of the time partition $\{t_j\}_{j=0,\dots,m}$ yields that

$$D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p;0,T)\le\liminf_\tau\Big[\tau\sum_{i=1}^{n-1}\eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big].\tag{5.22}$$

The thesis follows now by combining (5.21) and (5.22). □

We now prove that the lower bound identified in Theorem 5.1 is optimal.

Theorem 5.2 (Limsup inequality)**.** *Let $(u,e,p) \in \mathcal{V}$. There exists a sequence of triples $(u_\tau, e_\tau, p_\tau) \in \mathscr{U}_\tau^{\mathrm{affine}}$ such that*

$$u_\tau \to u \quad \textit{strongly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.23}$$

$$p_\tau(t) \to p(t) \quad \textit{strongly in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D) \textit{ for every } t\in[0,T], \tag{5.24}$$

$$e_\tau \to e \quad \textit{strongly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})), \tag{5.25}$$

and

$$\limsup_{\tau\to0} G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau) \le G_\varepsilon(u,e,p). \tag{5.26}$$

Proof. Let u_τ be defined as the affine-in-time interpolant of the following values

$$\begin{cases} u_\tau(0) = u^0, \\ u_\tau(\tau) = u^0 + \tau u^1, \\ u_\tau(i\tau) = M_\tau(u)(i\tau), \quad \text{for every } i = 2,\dots,n, \end{cases}$$

where M_τ is the backward mean operator,

$$M_\tau(u)(t) := \frac{1}{\tau}\int_{t-\tau}^t u(s)\,ds \quad \text{for every } t>\tau.$$

Define e_τ accordingly, and let p_τ be the measure satisfying

$$\begin{cases} p_\tau(0) = p^0, \\ p_\tau(\tau) = p^0 + \tau p^1, \\ p_\tau(i\tau) = M_\tau(p)(i\tau), \quad \text{for every } i = 2,\dots,n, \end{cases}$$

where

$$\langle \varphi,\, M_\tau(p)(i\tau)\rangle := \frac{1}{\tau}\int_{t-\tau}^t \int_{\Omega\cup\Gamma_0} \varphi\,dp(s)\,ds \quad \text{for every } \varphi\in C_0(\Omega\cup\Gamma_0).$$

The triple (u_τ,e_τ,p_τ) satisfies $(u_\tau,e_\tau,p_\tau)\in\mathscr{U}_\tau^{\mathrm{affine}}$, (5.25) follows by the definition, and (5.23) is obtained arguing as in [48, Subsection 2.5.2].

Regarding the plastic strains, fix $t\in(0,T]$. For τ small enough, there holds $t\in((i-1)\tau, i\tau]$, $i\ge2$. Thus, for every $\varphi\in C_0(\Omega;\mathbb{M}^{3\times3})$, there holds

$$\begin{aligned} &\Big|\int_{\Omega\cup\Gamma_0}\varphi\,dp_\tau(t) - \int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big| \\ &\quad= \frac{1}{\tau}\Big|\Big(\frac{t-(i-1)\tau}{\tau}\Big)\int_{(i-1)\tau}^{i\tau}\Big(\int_{\Omega\cup\Gamma_0}\varphi\,dp(s) - \int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big)\,ds \\ &\qquad+\Big(1-\Big(\frac{t-(i-1)\tau}{\tau}\Big)\Big)\int_{(i-2)\tau}^{(i-1)\tau}\Big(\int_{\Omega\cup\Gamma_0}\varphi\,dp(s) - \int_{\Omega\cup\Gamma_0}\varphi\,dp(t)\Big)\,ds\Big| \\ &\quad\le \frac{\|\varphi\|_{L^\infty(\Omega\cup\Gamma_0)}}{\tau}\int_{t-\tau}^{t+\tau}\|p(s)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,ds. \end{aligned} \tag{5.27}$$

In particular, for τ small enough we have

$$\|p_\tau(t)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)} \le \frac{1}{\tau}\int_{t-\tau}^{t+\tau}\|p(s)-p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)}\,ds.$$

Since $t \mapsto \|p(t)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)}$ is $L^1(0,T)$, in view of Lebesgue differentiation theorem we obtain (5.24). In addition, by the definition of p_τ there holds

$$\begin{aligned}
D_{\mathcal{H}}(p_\tau;0,T) &= D_{\mathcal{H}}(p;0,T) + \tau\|p^1\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)} + \sum_{i=2}^n \int_{(i-1)\tau}^{i\tau} \|\dot{p}_\tau\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)}\,dt \\
&= D_{\mathcal{H}}(p;0,T) + \tau\|p^1\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)} + \sum_{i=2}^n \|M_\tau(i\tau) - M_\tau((i-1)\tau)\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)} \\
&\leq D_{\mathcal{H}}(p;0,T) + \tau\|p^1\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)} + 2\int_0^T \|p\|_{\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)}\,dt \leq C.
\end{aligned} \tag{5.28}$$

Arguing as in [48, Subsection 2.5.2] we obtain the inequality

$$\begin{aligned}
&\limsup_{\tau\to 0} \Big[\frac{\varepsilon^2\rho}{2}\sum_{i=2}^n \tau\eta_{\tau,i}\int_\Omega |\delta^2 u_\tau(i\tau)|^2\,dx + \sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e_\tau(i\tau))\,dx\Big] \\
&\leq \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |\ddot{u}|^2\,dx + \int_\Omega Q(e)\,dx\Big)\,dt.
\end{aligned}$$

To prove (5.26) it remains only to show that

$$\limsup_{\tau\to 0}\Big[\tau\sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau))\Big] \leq D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T). \tag{5.29}$$

We first observe that

$$\begin{aligned}
\tau\sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}(\delta p_\tau(i\tau)) &= \sum_{i=1}^n \eta_{\tau,i+1}\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big) \\
&= \sum_{i=1}^n \Big(\eta_{\tau,i+1} - \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\Big)\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big) \\
&\quad + \sum_{i=1}^n \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big).
\end{aligned} \tag{5.30}$$

By (5.28) the first term in the right-hand side of (5.30) can be bounded from above as follows

$$\begin{aligned}
&\Big|\sum_{i=1}^n \Big(\eta_{\tau,i+1} - \exp\Big(-\frac{i\tau}{\varepsilon}\Big)\Big)\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big)\Big| \\
&\leq \sum_{i=1}^n \mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big)\|\bar{\eta}_\tau(\cdot+\tau) - \exp(-\,\cdot\,/\varepsilon)\|_{L^\infty(0,T)} \\
&\leq D_{\mathcal{H}}(p_\tau;0,T)\|\bar{\eta}_\tau(\cdot+\tau) - \exp(-\,\cdot\,/\varepsilon)\|_{L^\infty(0,T)} \\
&\leq C\|\bar{\eta}_\tau(\cdot+\tau) - \exp(-\,\cdot\,/\varepsilon)\|_{L^\infty(0,T)}
\end{aligned} \tag{5.31}$$

and converges to zero as $\tau \to 0$.

To study the second term in the right-hand side of (5.30) we remark that

$$\mathcal{H}\big(p_\tau(i\tau) - p_\tau((i-1)\tau)\big) \leq \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau} \mathcal{H}(p(t) - p(s))\,ds\,dt. \tag{5.32}$$

Indeed, for every $\varphi \in C_0(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D) \cap \mathcal{K}_D(\Omega)$ by Lemma 2.1 there holds

$$\begin{aligned}
\langle\varphi,\, p_\tau(i\tau) - p_\tau((i-1)\tau)\rangle &= \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}\int_{\Omega\cup\Gamma_0}\varphi\cdot\,dp(t)\,dt - \frac{1}{\tau}\int_{(i-2)\tau}^{(i-1)\tau}\int_{\Omega\cup\Gamma_0}\varphi\cdot\,dp(s)\,ds \\
&= \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau}\int_{\Omega\cup\Gamma_0}\varphi\cdot\,d(p(t) - p(s))\,ds\,dt
\end{aligned}$$

$$\leq \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau}\mathcal{H}(p(t)-p(s))\,ds\,dt.$$

A further application of Lemma 2.1 indeed yields (5.32).

In view of (5.32) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}\big(p_\tau(i\tau)-p_\tau((i-1)\tau)\big)\\
&\quad\leq\sum_{i=1}^{n}\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\int_{(i-2)\tau}^{(i-1)\tau}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\mathcal{H}(p(t)-p(s))\,ds\,dt\\
&\quad\leq\sum_{i=1}^{n}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)D_{\mathcal{H}}(p;0,i\tau)\\
&\quad\leq\sum_{i=1}^{n}\exp\Big(-\frac{i\tau}{\varepsilon}\Big)\sup\Big\{\sum_{j=1}^{m}\mathcal{H}(p(s_j)-p(s_{j-1})):\,0\leq s_1<\dots<s_m\leq i\tau\Big\}\\
&\quad\leq D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon);p;0,T).
\end{aligned}\tag{5.33}$$

Estimate (5.29) follows now by combining (5.30)–(5.33). □

As a corollary of Theorems 5.1 and 5.2, we obtain a uniform energy estimate for minimizers of G_ε.

Corollary 5.3 (Uniform energy estimate)**.** *Let $p^1=0$. For every $\tau>0$, let $(u_\tau,e_\tau,p_\tau)\in\mathscr{U}_\tau^{\rm affine}$ be a minimizer of $G_{\varepsilon\tau}$. Then, there exists a a minimizer $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ of G_ε in $\mathcal{V}$, such that*

$$\tilde u_\tau\rightharpoonup u^\varepsilon\quad\text{weakly in } W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3)),\tag{5.34}$$

$$p_\tau(t)\rightharpoonup^* p^\varepsilon(t)\quad\text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times3}_D)\text{ for every } t\in[0,T],\tag{5.35}$$

$$\bar e_\tau\rightharpoonup e^\varepsilon\quad\text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})),\tag{5.36}$$

where $\tilde u_\tau$ and $\bar e_\tau$ are the piecewise quadratic and piecewise constant interpolants of u_τ and e_τ, respectively (see (4.13) and (4.14)). In addition, there exists a constant C, independent of ε, and such that

$$\begin{aligned}
&\varepsilon\rho\int_0^T\int_0^t\int_\Omega|\ddot u^\varepsilon|^2\,dx\,ds\,dt+\frac{\varepsilon^2\rho}{2}\int_0^T\int_\Omega|\ddot u^\varepsilon|^2\,dx\,dt\\
&\quad+\rho\int_0^T\int_\Omega|\dot u^\varepsilon|^2\,dx\,dt+\int_0^T\int_\Omega Q(e^\varepsilon)\,dx\,dt+D_{\mathcal{H}}(p^\varepsilon;0,T)\leq C.
\end{aligned}\tag{5.37}$$

Proof. Let $\{(u_\tau,e_\tau,p_\tau)\}$ be as in the statement of the theorem. Since $(u^0+tu^1,e^0+te^1,p^0)\in\mathscr{U}_\tau^{\rm affine}$ for every $\tau>0$, there holds

$$\begin{aligned}
G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau)\leq G_{\varepsilon\tau}(u^0+tu^1,e^0+te^1,p^0)&=\sum_{i=2}^{n-2}\tau\eta_{\tau,i+2}\int_\Omega Q(e^0+i\tau e^1)\,dx\\
&\leq 2\int_\Omega Q(e^0)\,dx+2T\int_\Omega Q(e^1)\,dx
\end{aligned}\tag{5.38}$$

for every $\tau>0$. Arguing as in the proof of Theorem 5.1, in view of (5.38) there exists $(u^\varepsilon,e^\varepsilon,p^\varepsilon)\in\mathcal{V}$ such that (5.34)–(5.36) hold true, and

$$G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)\leq\liminf_{\tau\to0}G_{\varepsilon\tau}(u_\tau,e_\tau,p_\tau).\tag{5.39}$$

Let now $(v,f,q)\in\mathcal{V}$. By Theorem 5.2 there exist maps $(v_\tau,f_\tau,q_\tau)\in\mathscr{U}_\tau^{\rm affine}$ such that

$$\limsup_{\tau\to0}G_{\varepsilon\tau}(v_\tau,f_\tau,q_\tau)\leq G_\varepsilon(v,f,q).\tag{5.40}$$

The minimality of $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ follows then by the minimality of (u_τ,e_τ,p_τ), and by combining (5.39) with (5.40).

In view of Theorem 4.7, by (5.34) and (5.36) we have

$$\varepsilon\rho\int_0^T\int_0^t\int_\Omega|\ddot{u}^\varepsilon|^2\,dx\,ds\,dt+\varepsilon^2\rho\int_0^T\int_\Omega|\dot{u}^\varepsilon|^2\,dx\,dt+\rho\int_0^T\int_\Omega|\dot{u}^\varepsilon|^2\,dx\,dt$$
$$+\int_0^T\int_\Omega Q(e^\varepsilon)\,dx\,dt\le C. \tag{5.41}$$

In addition, by (5.35), the lower semicontinuity of $\mathcal{H}$, and Theorem 4.7,

$$\sup_{a>0}D_{\mathcal{H}}(p^\varepsilon;a,T-a)\le\sup_{a>0}\liminf_{\tau\to 0}D_{\mathcal{H}}(p_\tau;a,T-a)\le C. \tag{5.42}$$

The thesis follows by combining (5.41) and (5.42). □

6. Energy inequality at level ε

The central result of this section is Proposition (6.2) delivering an ε-dependent energy inequality fulfilled by minimizers, namely (6.6). We start by proving a somehow technical lemma.

Lemma 6.1. *Let $\mu\in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D))$ and let $\varphi\in C^\infty_c(0,T)$. Then*

$$\hat{D}_{\mathcal{H}}(\varphi;\mu;0,T)\le-\int_0^T\dot{\varphi}(t)D_{\mathcal{H}}(\mu;0,t)\,dt.$$

Proof. We subdivide the proof into two steps.

Step 1: Since the function $t\to D_{\mathcal{H}}(\mu;0,t)$ is nondecreasing, it has only a countable number of jumps. Thus, for every $\lambda>0$ there exists a time partition

$$0=t_0^\lambda<t_1^\lambda<\dots<t^\lambda_{m_\lambda}=T$$

such that, setting

$$\phi^\lambda(t):=D_{\mathcal{H}}(\mu;0,t_j^\lambda)\quad\text{for every }t\in[t_j^\lambda,t_{j+1}^\lambda),\ j=0,\dots,m_\lambda-1,$$

ϕ^λ satisfies

$$\max_{t\in[0,T]}\{D_{\mathcal{H}}(\mu;0,t)-\phi^\lambda(t)\}\le\frac{\lambda}{2}. \tag{6.1}$$

Let now $0=s_0^\delta<s_1^\delta<\dots<s^\delta_{N_\delta}=T$ be a time partition such that $\max_{i=1,\dots,N_\delta}(s_i^\delta-s_{i-1}^\delta)\le\delta$, and

$$D_{\mathcal{H}}(\mu;0,T)\le\sum_{i=1}^{N_\delta}\mathcal{H}(\mu(s_i^\delta)-\mu(s_{i-1}^\delta))+\lambda.$$

Up to taking a further refinement of $\{s_i^\delta\}$, we can assume that, setting

$$f^\delta(t):=\sum_{i=1}^{j}\mathcal{H}(\mu(s_i^\delta)-\mu(s_{i-1}^\delta))\quad\text{for every }t\in[s_j^\delta,s_{j+1}^\delta),\ j=1,\dots,N_\delta-1,$$

there holds

$$\phi^\lambda(t)=D_{\mathcal{H}}(\mu;0,t_i^\lambda)\le f^\delta(t_i^\lambda)+\frac{\lambda}{2}=f^\delta(t)+\frac{\lambda}{2}, \tag{6.2}$$

for every $t\in[t_i^\lambda,t_{i+1}^\lambda)$, for every $i=1,\dots,m_\lambda-1$. By combining (6.1) and (6.2) we deduce the inequality

$$\max_{t\in[0,T]}\{D_{\mathcal{H}}(\mu;0,t)-f^\delta(t)\}\le\lambda. \tag{6.3}$$

Step 2: consider a time partition $0=r_0^\delta<r_1^\delta<\dots<r^\delta_{M_\delta}=T$ such that $\{r_i^\delta\}$ is a refinement of the time partition $\{s_i^\delta\}$ constructed in Step 1, and

$$\sup\Big\{\sum_{i=1}^m\varphi(t_i)\mathcal{H}(\mu(t_i)-\mu(t_{i-1})):\ 0=t_0\le t_1\le\dots\le t_m=T,\ m\in\mathbb{N},\ \max_{i=1,\dots,m}(t_i-t_{i-1})\le\delta\Big\}$$

$$\le \sum_{i=1}^{M_\delta} \varphi(r_i^\delta)\mathcal{H}(\mu(r_i^\delta) - \mu(r_{i-1}^\delta)) + \lambda.$$

An integration by parts yields

$$\begin{aligned}\sum_{i=1}^{M_\delta} \varphi(r_i^\delta)\mathcal{H}(\mu(r_i^\delta) - \mu(r_{i-1}^\delta)) &= -\sum_{i=1}^{M_\delta} \int_{r_i^\delta}^T \dot\varphi(t)\mathcal{H}(\mu(r_i^\delta) - \mu(r_{i-1}^\delta))\,dt \\ &= -\sum_{i=1}^{M_\delta-1} \int_{r_i^\delta}^{r_{i+1}^\delta} \dot\varphi(t) \sum_{j=1}^{i} \mathcal{H}(\mu(r_j^\delta) - \mu(r_{j-1}^\delta))dt \le -\sum_{i=1}^{M_\delta-1} \int_{r_i^\delta}^{r_{i+1}^\delta} \dot\varphi(t) f^\delta(t)\,dt, \end{aligned} \tag{6.4}$$

where in the last inequality we used the fact that $\{r_i^\delta\}$ is a refinement of $\{s_i^\delta\}$. Hence, by (6.3) we deduce

$$\begin{aligned}&\sup\Big\{\sum_{i=1}^{m} \varphi(t_i)\mathcal{H}(\mu(t_i) - \mu(t_{i-1})) :\ 0 = t_0 \le t_1 \le \cdots \le t_m = T,\ m \in \mathbb{N},\ \max_{i=1,\dots,m}(t_i - t_{i-1}) \le \delta\Big\} \\ &\le -\int_0^T \dot\varphi(t) D_{\mathcal{H}}(\mu; 0, t)\,dt + \lambda \int_0^T |\dot\varphi(t)|\,dt. \end{aligned} \tag{6.5}$$

The thesis follows by taking the limit in (6.5) as δ tends to zero, and by the arbitrariness of λ. □

We are now in a position to presenting the ε-dependent energy inequality.

Proposition 6.2 (Energy inequality)**.** *Let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε. Then*

$$\begin{aligned}&\int_0^T \dot\varphi(t)\Big[\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx + 2\varepsilon\rho \int_0^t \int_\Omega |\ddot u^\varepsilon(s)|^2\,dx\,ds + \frac{\rho}{2}\int_\Omega |\dot u^\varepsilon(t)|^2\,dx \\ &+ D_{\mathcal{H}}(p^\varepsilon; 0, t)\Big]\,dt \le \int_0^T \dot\varphi(t) \int_0^t \int_\Omega \mathbb{C}e^\varepsilon(s) : E\dot w(s)\,dx\,ds\,dt \\ &- \frac{3\varepsilon^2\rho}{2}\int_0^T \ddot\varphi(t)\int_0^t\int_\Omega |\ddot u^\varepsilon(s)|^2\,dx\,ds\,dt - \varepsilon^2\rho\int_0^T \ddot\varphi(t)\int_\Omega \ddot u^\varepsilon(t)\cdot(\dot w(t) - \dot u^\varepsilon(t))\,dx\,dt \\ &+ \rho\int_0^T\int_\Omega \dot u^\varepsilon(t)\cdot\partial_t[\dot w(t)(\varphi(t) + 2\varepsilon\dot\varphi(t))]\,dx\,dt - \varepsilon^2\rho\int_0^T\int_\Omega \ddot u^\varepsilon(t)\cdot\ddot w(t)\varphi(t)\,dx\,dt \\ &- \varepsilon\rho\int_0^T\int_\Omega \ddot\varphi(t)|\dot u^\varepsilon(t)|^2\,dx\,dt + 2\varepsilon\rho\int_0^T\int_\Omega \dot u^\varepsilon(t)\cdot\partial_t[\ddot w(t)(\varphi(t) + \varepsilon\dot\varphi(t))]\,dx\,dt \end{aligned} \tag{6.6}$$

for every $\varphi \in C_c^\infty(0,T)$.

Proof. We argue as in [37, Proposition 4.1] by comparing the energy associated to $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ with that of a rescaled triple $(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon)$. Consider an increasing diffeomorphism

$$\beta : [0,T] \to [0,T]$$

such that $\beta \in C^2([0,T])$, $\beta(0) = 0$, $\beta(T) = T$, and $\dot\beta(0) = 1$, and set

$$\tilde u^\varepsilon(s) := u^\varepsilon(\beta^{-1}(s)) - w(\beta^{-1}(s)) + w(s), \quad \tilde e^\varepsilon(s) := e^\varepsilon(\beta^{-1}(s)) - Ew(\beta^{-1}(s)) + Ew(s),$$

and

$$\tilde p^\varepsilon(s) := p^\varepsilon(\beta^{-1}(s))$$

for every $s \in [0,T]$. It is easy to check that $(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon) \in \mathcal{V}$. Hence, by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ there holds

$$G_\varepsilon(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon) - G(u^\varepsilon, e^\varepsilon, p^\varepsilon) \ge 0. \tag{6.7}$$

Using the definition of $(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon)$, we can rewrite its associated energy as

$$\begin{aligned}G_\varepsilon(\tilde u^\varepsilon, \tilde e^\varepsilon, \tilde p^\varepsilon) &= \int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(e^\varepsilon(t) - Ew(t) + Ew(\beta(t))\big)\,dx\,dt \\ &+ \frac{\varepsilon^2\rho}{2}\int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\Big(\int_\Omega\Big|\frac{\ddot u^\varepsilon(t)}{(\dot\beta(t))^2} - \frac{\dot u^\varepsilon(t)\ddot\beta(t)}{(\dot\beta(t))^3} - \frac{\ddot w(t)}{(\dot\beta(t))^2}\end{aligned}$$

$$+\frac{\dot w(t)\ddot\beta(t)}{(\dot\beta(t))^3}+\ddot w(\beta(t))\Big|^2\Big)\,dt+\varepsilon D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\tilde p^\varepsilon;0,T).$$

Along the footsteps of [37, Proposition 4.1], we fix $\varphi\in C_c^\infty(0,T)$. Let $\delta\in(0,1)$ be such that $\varepsilon\delta\dot\varphi(t)<\exp(-t/\varepsilon)$ for every $t\in[0,T]$, and define β as the solution to

$$\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)-\exp\Big(-\frac{t}{\varepsilon}\Big)=\delta\varphi(t). \tag{6.8}$$

It is immediate to see that $\beta(0)=0$ and $\beta(T)=T$. In addition, deriving (6.8) with respect to time, we have

$$\dot\beta(t)=\exp\Big(\frac{\beta(t)}{\varepsilon}\Big)\Big(\exp\Big(-\frac{t}{\varepsilon}\Big)-\varepsilon\delta\dot\varphi(t)\Big) \tag{6.9}$$

for every $t\in[0,T]$, yielding $\dot\beta(t)>0$ for every $t\in(0,T)$ and $\dot\beta(0)=1$. As already observed in [37, Proposition 4.1],

$$\beta(t)=t-\varepsilon\delta\varphi(t)\exp\Big(\frac{t}{\varepsilon}\Big)+\mathrm{O}(\delta^2). \tag{6.10}$$

In addition, by (6.8) and (6.9),

$$\dot\beta(t)=1-\delta(\varphi(t)+\varepsilon\dot\varphi(t))\exp\Big(\frac{t}{\varepsilon}\Big)+\mathrm{O}(\delta^2), \tag{6.11}$$

and by performing a further derivation in time of (6.9),

$$\ddot\beta(t)=-\delta\Big(\frac{\varphi(t)}{\varepsilon}+2\dot\varphi(t)+\varepsilon\ddot\varphi(t)\Big)\exp\Big(\frac{t}{\varepsilon}\Big)+\mathrm{O}(\delta^2). \tag{6.12}$$

Let us firstly observe that

$$\begin{aligned}&\lim_{\delta\to0}\frac1\delta\Big\{\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(e^\varepsilon(t)-Ew(t)+Ew(\beta(t))\big)\,dx\,dt\\&-\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dt\Big\}\\&=\lim_{\delta\to0}\frac1\delta\Big\{\int_0^T\Big(\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\&+\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(Ew(t)-Ew(\beta(t))\big)\,dx\,dt\\&-\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(Ew(t)-Ew(\beta(t))\Big)\,dx\,dt\Big\}.\end{aligned} \tag{6.13}$$

In view of (6.8) and (6.11), and by the Dominated Convergence Theorem, the first term in the right-hand side of (6.13) becomes

$$\begin{aligned}&\lim_{\delta\to0}\frac1\delta\int_0^T\Big(\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\&=\lim_{\delta\to0}\frac1\delta\int_0^T\Big(\Big(\delta\varphi(t)+\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\dot\beta(t)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt\\&=-\varepsilon\int_0^T\dot\varphi(t)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt.\end{aligned} \tag{6.14}$$

By the regularity of w and by (6.10) there holds

$$|Ew(t)-Ew(\beta(t))|=\Big|\int_t^{\beta(t)}E\dot w(\xi)\,d\xi\Big|\le\delta\|w\|_{W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}.$$

Hence, by (6.8) and (6.11) one obtains

$$\lim_{\delta\to0}\frac1\delta\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(Ew(t)-Ew(\beta(t))\big)\,dx\,dt=0. \tag{6.15}$$

Finally, by (6.8), (6.11), and the mean value theorem we get

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(Ew(t)-Ew(\beta(t))\Big)\,dx\,dt\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \Big(\exp\Big(-\frac{t}{\varepsilon}\Big)+\delta\varepsilon\dot\varphi(t)\Big)\int_\Omega \mathbb{C}e^\varepsilon(t):\Big(\int_t^{\beta(t)} E\dot w(\xi)\,d\xi\Big)\,dx\,dt\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_0^T \Big(\exp\Big(-\frac{t}{\varepsilon}\Big)+\delta\varepsilon\dot\varphi(t)\Big)\int_t^{\beta(t)}\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(\xi)\,dx\,d\xi\,dt\\
&=-\varepsilon\lim_{\delta\to 0}\int_0^T\varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(\xi^t)\,dx\,dt=-\varepsilon\int_0^T\varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(t)\,dx\,dt, \qquad (6.16)
\end{aligned}$$

where, in the second-last line, for every $t\in[0,T]$, ξ^t is an intermediate value between t and $\beta(t)$. By combining (6.13)–(6.16) we obtain

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\int_0^T \exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega Q\big(e^\varepsilon(\beta(t))-Ew(t)+Ew(\beta(t))\big)\,dx\,dt\\
&\quad-\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dt\Big\}\\
&=-\varepsilon\int_0^T\dot\varphi(t)\int_\Omega Q\big(e^\varepsilon(t)\big)\,dx\,dt+\varepsilon\int_0^T\varphi(t)\int_\Omega \mathbb{C}e^\varepsilon(t):E\dot w(t)\,dx\,dt. \qquad (6.17)
\end{aligned}$$

We proceed by performing the analogous computation for the inertial term. We seek to estimate

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\Big(\int_\Omega\Big|\frac{\ddot u^\varepsilon(t)}{(\dot\beta(t))^2}-\frac{\dot u^\varepsilon(t)\ddot\beta(t)}{(\dot\beta(t))^3}-\frac{\ddot w(t)}{(\dot\beta(t))^2}\\
&\quad+\frac{\dot w(t)\ddot\beta(t)}{(\dot\beta(t))^3}+\ddot w(\beta(t))\Big|^2\Big)\,dt-\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}. \qquad (6.18)
\end{aligned}$$

By (6.8) and (6.11) we have

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\frac{\varepsilon^2\rho}{2}\int_0^T\Big(\frac{1}{(\dot\beta(t))^3}\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)-\exp\Big(-\frac{t}{\varepsilon}\Big)\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\\
&=\frac{3\varepsilon^3\rho}{2}\int_0^T\dot\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt+2\varepsilon^2\rho\int_0^T\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt. \qquad (6.19)
\end{aligned}$$

By (6.8), (6.11), and (6.12), there holds

$$\lim_{\delta\to 0}\frac{1}{\delta}\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\int_\Omega\Big[\frac{(\ddot\beta(t))^2}{(\dot\beta(t))^5}(|\dot u^\varepsilon(t)|^2+|\dot w(t)|^2-2\dot u^\varepsilon(t)\cdot\dot w(t))\Big]\,dx\,dt=0, \qquad (6.20)$$

as well as

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\varepsilon^2\rho\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\int_\Omega\frac{\ddot\beta(t)}{(\dot\beta(t))^4}\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt\\
&=-\varepsilon^3\rho\int_0^T\ddot\varphi(t)\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt-\varepsilon\rho\int_0^T(\varphi(t)+2\varepsilon\dot\varphi(t))\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt. \qquad (6.21)
\end{aligned}$$

To estimate the remaining term, we observe that by (6.11) and in view of the regularity of the boundary datum,

$$\begin{aligned}
-\frac{\ddot w(t)}{(\dot\beta(t))^2}+\ddot w(\beta(t))&=-\frac{\ddot w(t)}{(\dot\beta(t))^2}(1-(\dot\beta(t))^2)+\int_t^{\beta(t)}\dddot w(\xi)\,d\xi\\
&=-\frac{2\delta\ddot w(t)}{(\dot\beta(t))^2}(\varphi(t)+\varepsilon\dot\varphi(t))\exp\Big(\frac{t}{\varepsilon}\Big)+\int_t^{\beta(t)}\dddot w(\xi)\,d\xi+\mathrm{O}(\delta^2).
\end{aligned}$$

By the regularity of w, by (6.10), and by Lebesgue's Theorem,

$$\lim_{\delta\to 0}\Big\|\frac{1}{\delta}\int_t^{\beta(t)}\dddot{w}(\xi)\,d\xi+\varepsilon\dddot{w}(t)\varphi(t)\exp\Big(\frac{t}{\varepsilon}\Big)\Big\|_{L^2(0,T)}$$
$$=\lim_{\delta\to 0}\Big\|\frac{1}{\delta}\int_t^{\beta(t)}(\dddot{w}(\xi)-\dddot{w}(t))\,d\xi\Big\|_{L^2(0,T)}$$
$$\leq \lim_{\delta\to 0}\Big\|\frac{1}{\delta}\int_{t-\delta\varepsilon\|\varphi\|_{L^\infty(0,T)}\exp(T/\varepsilon)}^{t+\delta\varepsilon\|\varphi\|_{L^\infty(0,T)}\exp(T/\varepsilon)}|\dddot{w}(\xi)-\dddot{w}(t)|\,d\xi\Big\|_{L^2(0,T)}=0.$$

Therefore, by (6.11) and (6.12),

$$\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega\Big[\Big|-\frac{\ddot w(t)}{(\dot\beta(t))^2}+\ddot w(\beta(t))\Big|^2$$
$$+2\Big(-\frac{\ddot w(t)}{(\dot\beta(t))^2}+\ddot w(\beta(t))\Big)\cdot\frac{(\dot w(t)-\dot u(t))\ddot\beta(t)}{(\dot\beta(t))^3}\Big]\,dx\,dt\Big\}=0, \tag{6.22}$$

and

$$\lim_{\delta\to 0}\frac{1}{\delta}\varepsilon^2\rho\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega\frac{\ddot u^\varepsilon(t)}{(\dot\beta(t))^2}\cdot\Big(-\frac{\ddot w(t)}{(\dot\beta(t))^2}+\ddot w(\beta(t))\Big)\,dx\,dt$$
$$=-2\varepsilon^2\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))\,dx\,dt$$
$$-\varepsilon^3\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\dddot w(t)\varphi(t)\,dx\,dt. \tag{6.23}$$

By combining (6.18)–(6.23), we obtain

$$\lim_{\delta\to 0}\frac{1}{\delta}\Big\{\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{\beta(t)}{\varepsilon}\Big)\dot\beta(t)\int_\Omega\Big|\frac{\ddot u^\varepsilon(t)}{(\dot\beta(t))^2}-\frac{\dot u^\varepsilon(t)\ddot\beta(t)}{(\dot\beta(t))^3}-\frac{\ddot w(t)}{(\dot\beta(t))^2}$$
$$+\frac{\dot w(t)\ddot\beta(t)}{(\dot\beta(t))^3}+\ddot w(\beta(t))\Big|^2\,dx\,dt-\frac{\varepsilon^2\rho}{2}\int_0^T\exp\Big(-\frac{t}{\varepsilon}\Big)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt\Big\}$$
$$=\frac{3\varepsilon^3\rho}{2}\int_0^T\dot\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt+2\varepsilon^2\rho\int_0^T\varphi(t)\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,dt$$
$$-\varepsilon^3\rho\int_0^T\ddot\varphi(t)\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt$$
$$-\varepsilon\rho\int_0^T(\varphi(t)+2\varepsilon\dot\varphi(t))\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt$$
$$-2\varepsilon^2\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))\,dx\,dt$$
$$-\varepsilon^3\rho\int_0^T\int_\Omega\ddot u^\varepsilon(t)\cdot\dddot w(t)\varphi(t)\,dx\,dt. \tag{6.24}$$

To complete the proof of the ε-energy inequality it remains to estimate from above the quantity

$$\limsup_{\delta\to 0}\frac{1}{\delta}\left(D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);\tilde p^\varepsilon;0,T)-D_{\mathcal H}(\exp(-\,\cdot\,/\varepsilon);p^\varepsilon;0,T)\right). \tag{6.25}$$

To this aim, fix $t\in[0,T]$, and let $s\in[0,T]$ be such that $t=\beta(s)$. Let $0\le t_0<t_1<\cdots<t_m\le T$, and for $i=0,\dots,m$, let $s_i\in[0,T]$ be such that $\beta(s_i)=t_i$. By the properties of β, it follows that $0\le s_0<s_1<\cdots<s_n\le T$. In view of (6.8), we have

$$\sum_{i=1}^m\exp\left(-\frac{t_i}{\varepsilon}\right)\mathcal H(\tilde p^\varepsilon(t_i)-\tilde p^\varepsilon(t_{i-1}))=\sum_{i=1}^m\exp\left(-\frac{\beta(s_i)}{\varepsilon}\right)\mathcal H(p^\varepsilon(s_i)-p^\varepsilon(s_{i-1}))$$

$$
\begin{aligned}
&= \sum_{i=1}^m \exp\left(-\frac{s_i}{\varepsilon}\right) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1})) \\
&+ \sum_{i=1}^m \Big(\exp\left(-\frac{\beta(s_i)}{\varepsilon}\right) - \exp\left(-\frac{s_i}{\varepsilon}\right) \Big) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1})) \\
&\le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T) + \delta \sum_{i=1}^m \varphi(s_i) \mathcal{H}(p^\varepsilon(s_i) - p^\varepsilon(s_{i-1})) \\
&\le D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T) + \delta \hat{D}_{\mathcal{H}}(\varphi; p^\varepsilon; 0, T).
\end{aligned}
$$

Thus we can bound (6.25) from above as

$$
\limsup_{\delta\to 0} \frac{1}{\delta} \left(D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); \tilde{p}^\varepsilon; 0, T) - D_{\mathcal{H}}(\exp(-\,\cdot\,/\varepsilon); p^\varepsilon; 0, T) \right) \le \hat{D}_{\mathcal{H}}(\varphi; p^\varepsilon; 0, T), \tag{6.26}
$$

where $\hat{D}_{\mathcal{H}}$ is the quantity defined in (2.9). Combining (6.7), (6.17), (6.24), (6.26) and Lemma 6.1 we finally obtain the inequality

$$
\begin{aligned}
0 &\le \limsup_{\delta\to 0} \frac{1}{\varepsilon\delta} \left(G_\varepsilon(\tilde{u}^\varepsilon, \tilde{e}^\varepsilon, \tilde{p}^\varepsilon) - G(u^\varepsilon, e^\varepsilon, p^\varepsilon) \right) \\
&\le -\int_0^T \dot{\varphi}(t) \int_\Omega Q\big(e^\varepsilon(t)\big)\, dx\, dt - \int_0^T \varphi(t) \int_\Omega \mathbb{C} e^\varepsilon(t) : E\dot{w}(t)\, dx\, dt \\
&\quad + \frac{3\varepsilon^2\rho}{2} \int_0^T \dot{\varphi}(t) \int_\Omega |\ddot{u}^\varepsilon(t)|^2\, dx\, dt + 2\varepsilon\rho \int_0^T \varphi(t) \int_\Omega |\ddot{u}^\varepsilon(t)|^2\, dx\, dt \\
&\quad - \varepsilon^2\rho \int_0^T \int_\Omega \ddot{\varphi}(t) \ddot{u}^\varepsilon(t) \cdot (\dot{w}(t) - \dot{u}^\varepsilon(t))\, dx\, dt - \rho \int_0^T \int_\Omega (\varphi(t) + 2\varepsilon\dot{\varphi}(t)) \ddot{u}^\varepsilon(t) \cdot (\dot{w}(t) - \dot{u}^\varepsilon(t))\, dx\, dt \\
&\quad - 2\varepsilon\rho \int_0^T \int_\Omega \ddot{u}^\varepsilon(t) \cdot \ddot{w}(t) (\varphi(t) + \varepsilon\dot{\varphi}(t))\, dx\, dt - \varepsilon^2\rho \int_0^T \int_\Omega \ddot{u}^\varepsilon(t) \cdot \dddot{w}(t) \varphi(t)\, dx\, dt \\
&\quad - \int_0^T \dot{\varphi}(t) D_{\mathcal{H}}(p^\varepsilon; 0, t)\, dt
\end{aligned} \tag{6.27}
$$

for every $\varphi \in C_c^\infty(0, T)$. The energy inequality (6.6) follows now by performing an integration by parts. $\square$

7. Proof of Theorem 2.3

Having established the uniform estimate (5.37) we are now ready to prove Theorem 2.3. For every $\varepsilon > 0$, let $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ be a minimizer of G_ε satisfying (5.37). Since $p^\varepsilon(0) = p^0$ for every $\varepsilon > 0$, by a generalization of Helly's Theorem [11, Theorem 7.2] there exists $p \in BV(0, T; \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D))$ such that

$$
p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{3\times 3}_D) \quad \text{for every } t \in [0, T], \tag{7.1}
$$

$$
D_{\mathcal{H}}(p; 0, T) \le \liminf_{\varepsilon\to 0} D_{\mathcal{H}}(p^\varepsilon; 0, T). \tag{7.2}
$$

In addition, (5.37) yields the existence of maps $u \in W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^3))$ and $e \in L^2(0, T; L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}}))$ such that, up to subsequences,

$$
u^\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^3)), \tag{7.3}
$$

$$
e^\varepsilon \rightharpoonup e \quad \text{weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}})). \tag{7.4}
$$

In particular by (7.3), and by the embedding of $W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^3))$ into $C_w([0, T]; L^2(\Omega; \mathbb{R}^3))$ there holds

$$
u^\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^3) \quad \text{for every } t \in [0, T], \tag{7.5}
$$

and $u(0) = u^0$.

By (7.1), (7.4), and (7.5) up to subsequences there holds

$$u^\varepsilon(t) \rightharpoonup^* u(t) \quad \text{weakly* in } BD(\Omega) \quad \text{for every } t \in [0,T], \tag{7.6}$$

$$e^\varepsilon(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) \quad \text{for every } t \in [0,T]. \tag{7.7}$$

The fact that p satisfies the boundary condition on Γ_0 for every $t \in [0,T]$ follows arguing as in [11, Lemma 2.1].

Let $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$. For $\lambda > 0$, we have that

$$\Big(u^\varepsilon + \lambda \exp\Big(\frac{t}{\varepsilon}\Big)v,\, e^\varepsilon + \lambda\exp\Big(\frac{t}{\varepsilon}\Big)Ev,\, p^\varepsilon\Big) \in \mathcal{V},$$

thus by the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$,

$$\frac{1}{\lambda}\left(G_\varepsilon\left(u^\varepsilon + \lambda\exp\left(\frac{t}{\varepsilon}\right)v,\, e^\varepsilon + \lambda\exp\left(\frac{t}{\varepsilon}\right)Ev,\, p^\varepsilon\right) - G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)\right) \geq 0,$$

namely

$$\rho\int_0^T\int_\Omega \ddot{u}^\varepsilon\cdot(v + \varepsilon\dot{v} + \varepsilon^2\ddot{v})\,dx\,dt + \int_0^T\int_\Omega \mathbb{C}e^\varepsilon : Ev\,dx\,dt = 0 \tag{7.8}$$

for every $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$. Integrating by parts with respect to time, (7.3) and (7.4) yield

$$-\rho\int_0^T\int_\Omega \dot{u}\cdot\dot{v}\,dx\,dt + \int_0^T\int_\Omega \mathbb{C}e : Ev\,dx\,dt = 0$$

for every $v \in C^\infty_c((0,T)\times\Omega;\mathbb{R}^3)$, that is

$$\rho\ddot{u}(t) - \operatorname{div}\mathbb{C}e(t) = 0 \tag{7.9}$$

in the sense of distributions. Since the same procedure applies to every $v \in C^\infty_c(0,T;C^\infty(\bar{\Omega};\mathbb{R}^3))$ with $v = 0$ on Γ_0 for every $t\in[0,T]$, we also obtain

$$\mathbb{C}e(t)\nu = 0 \quad \text{on } \partial\Omega\setminus\Gamma_0. \tag{7.10}$$

Let now $q \in C^\infty_c(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_D))$, $\lambda > 0$, and consider the test triple

$$\Big(u^\varepsilon,\, e^\varepsilon - \lambda\exp\Big(\frac{t}{\varepsilon}\Big)q, p^\varepsilon + \lambda\exp\Big(\frac{t}{\varepsilon}\Big)q\Big).$$

By the minimality of $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$,

$$\frac{1}{\lambda}\left(G_\varepsilon\left(u^\varepsilon,\, e^\varepsilon - \lambda\exp\left(\frac{t}{\varepsilon}\right)q, p^\varepsilon + \lambda\exp\left(\frac{t}{\varepsilon}\right)q\right) - G_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)\right) \geq 0. \tag{7.11}$$

On the other hand,

$$\begin{aligned}&\frac{1}{\lambda}\left(D_{\mathcal{H}}(\exp(-\cdot/\varepsilon); p^\varepsilon + \lambda\exp(\cdot/\varepsilon)q; 0,T) - D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);p^\varepsilon;0,T)\right)\\ &\leq D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);\exp(\cdot/\varepsilon)q;0,T),\end{aligned}$$

and by the in-time regularity of q,

$$\begin{aligned}D_{\mathcal{H}}(\exp(-\cdot/\varepsilon);\exp(\cdot/\varepsilon)q;0,T) &= \int_0^T \exp\Big(-\frac{t}{\varepsilon}\Big)\mathcal{H}\Big(\frac{1}{\varepsilon}\exp\Big(\frac{t}{\varepsilon}\Big)q(t) + \exp\Big(\frac{t}{\varepsilon}\Big)\dot{q}(t)\Big)\\ &\leq \frac{1}{\varepsilon}\int_0^T \mathcal{H}(q(t))\,dt + \int_0^T\mathcal{H}(\dot{q}(t))\,dt.\end{aligned}$$

Thus (7.11) can be rewritten as

$$-\int_0^T\int_\Omega \mathbb{C}e^\varepsilon : q\,dx\,dt + \int_0^T\mathcal{H}(q(t))\,dt + \varepsilon\int_0^T\mathcal{H}(\dot{q}(t))\,dt \geq 0.$$

for every $q \in C^\infty_c(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_D))$, and by (7.4),

$$\int_0^T\int_\Omega \mathbb{C}e : q\,dx\,dt \leq \int_0^T\mathcal{H}(q(t))\,dt$$

for every $q \in C_c^\infty(0,T;L^2(\Omega;\mathbb{M}_D^{3\times3}))$. By approximation, the previous inequality holds in particular by choosing $q = M\chi_I\chi_B$ with $M \in \mathbb{M}^{3\times3}$, I and B Borel subsets of $(0,T)$ and $\Omega\cup\Gamma_0$, respectively. Hence, we deduce that

$$[\mathbb{C}e(t)]_D \in \partial H(0) \tag{7.12}$$

for every $t \in [0,T]$ and for a.e. $x \in \Omega$.

In order to complete the proof of the theorem it remains only to show that the limit triple satisfies the energy inequality (c5'). We argue by passing to the limit in (6.6). In view of (7.4),

$$\lim_{\varepsilon\to0}\int_0^T \varphi(t)\int_0^t\int_\Omega \mathbb{C}e^\varepsilon(s) : E\dot w(s)\,dx\,ds\,dt = \int_0^T \varphi(t)\int_0^t\int_\Omega \mathbb{C}e(s) : E\dot w(s)\,dx\,ds\,dt, \tag{7.13}$$

whereas (5.37) yields

$$\begin{aligned}\lim_{\varepsilon\to0}\Big\{&\frac{3\varepsilon^2\rho}{2}\int_0^T \ddot\varphi(t)\int_0^t\int_\Omega |\ddot u^\varepsilon(s)|^2\,dx\,ds\,dt + \varepsilon^2\rho\int_0^T\int_\Omega \ddot\varphi(t)\ddot u^\varepsilon(t)\cdot(\dot w(t)-\dot u^\varepsilon(t))\,dx\,dt\\ &+\varepsilon^2\rho\int_0^T\int_\Omega \ddot u^\varepsilon(t)\cdot\dddot w(t)\varphi(t)\,dx\,dt + \varepsilon\rho\int_0^T\int_\Omega \ddot\varphi(t)|\dot u^\varepsilon(t)|^2\,dx\,dt\Big\} = 0\end{aligned} \tag{7.14}$$

for every $\varphi \in C_c^\infty(0,T)$. In addition, by (7.3) there holds

$$\begin{aligned}&\lim_{\varepsilon\to0}\rho\int_0^T\int_\Omega \dot u^\varepsilon(t)\cdot\partial_t[\dot w(t)(\varphi(t)+2\varepsilon\dot\varphi(t))]\,dx\,dt\\ &=\lim_{\varepsilon\to0}\rho\int_0^T\int_\Omega \dot u^\varepsilon(t)\cdot\partial_t(\dot w(t)\varphi(t))\,dx\,dt\\ &=\lim_{\varepsilon\to0}\Big[-\rho\int_0^T\dot\varphi(t)\int_0^t\int_\Omega \dot u^\varepsilon(s)\cdot\ddot w(s)\,dx\,ds + \rho\int_0^T\dot\varphi(t)\int_\Omega \dot u^\varepsilon(t)\cdot\dot w(t)\,dx\,dt\Big]\\ &=\lim_{\varepsilon\to0}\rho\int_0^T\dot\varphi(t)\int_\Omega \dot u^\varepsilon(t)\cdot\dot w(t)\,dx\,dt - \rho\int_0^T\dot\varphi(t)\int_0^t\int_\Omega \dot u(s)\cdot\ddot w(s)\,dx\,ds,\end{aligned} \tag{7.15}$$

and

$$\lim_{\varepsilon\to0}2\varepsilon\rho\int_0^T\int_\Omega \dot u^\varepsilon(t)\cdot\partial_t[\ddot w(t)(\varphi(t)+\varepsilon\dot\varphi(t))]\,dx\,dt = 0 \tag{7.16}$$

for every $\varphi \in C_c^\infty(0,T)$. By collecting (7.13)–(7.16) we deduce the inequality

$$\begin{aligned}\limsup_{\varepsilon\to0}&\int_0^T\dot\varphi(t)\Big[\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega|\dot u^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,ds\\ &+ \mathcal{D}_{\mathcal H}(p^\varepsilon;0,t) - \rho\int_\Omega \dot u^\varepsilon(t)\cdot\dot w(t)\,dx\Big]\,dt\\ &\le \int_0^T\dot\varphi(t)\int_0^t\int_\Omega(\mathbb{C}e(s):E\dot w(s) - \rho\dot u(s)\cdot\ddot w(s))\,dx\,ds.\end{aligned}$$

By the Hölder inequality, and by the regularity of w, there exists a constant C independent of ε, and such that

$$\begin{aligned}\dot\varphi(t)\Big[&\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega|\dot u^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,ds + \mathcal{D}_{\mathcal H}(p^\varepsilon;0,t)\\ &-\rho\int_\Omega\dot u^\varepsilon(t)\cdot\dot w(t)\,dx\Big] \ge -C\|\varphi'\|_{L^\infty(0,T)}.\end{aligned}$$

Hence Fatou's Lemma yields

$$\begin{aligned}\int_0^T\dot\varphi(t)\limsup_{\varepsilon\to0}&\Big[\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega|\dot u^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega|\ddot u^\varepsilon(t)|^2\,dx\,ds\\ &+ \mathcal{D}_{\mathcal H}(p^\varepsilon;0,t) - \rho\int_\Omega\dot u^\varepsilon(t)\cdot\dot w(t)\,dx\Big]\,dt\end{aligned}$$

$$\leq \int_0^T \dot{\varphi}(t) \int_0^t \int_\Omega (\mathbb{C}e(s) : E\dot{w}(s) - \rho\dot{u}(s)\cdot\ddot{w}(s))\,dx\,ds\,dt \tag{7.17}$$

for every $\varphi \in C_c^\infty(0,T)$. We observe that by (2.3)

$$\begin{aligned}&\Big\| \int_\Omega (\mathbb{C}e(s) : E\dot{w}(s) - \rho\dot{u}(s)\cdot\ddot{w}(s))\,dx\Big\|_{L^2(0,T)}\\ &\quad\leq C\|e\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}))}\|E\dot{w}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}))}\\ &\quad+\rho\|\dot{u}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\|\ddot{w}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))},\end{aligned}$$

since $w \in W^{3,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, and by the continuous embedding

$$W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \hookrightarrow C([0,T];L^2(\Omega;\mathbb{R}^3)).$$

As a result, the map

$$t \to \int_0^t \int_\Omega (\mathbb{C}e(s) : E\dot{w}(s) - \rho\dot{u}(s)\cdot\ddot{w}(s))\,dx\,ds$$

is $W^{1,2}(0,T)$ and hence continuous on $[0,T]$. The arbitrariness of φ, (7.17) and the Du-Bois Raymond Lemma imply the equality

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\Big\{\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot{u}^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega |\ddot{u}^\varepsilon(s)|^2\,dx\,ds\\ &\qquad + \mathcal{D}_{\mathcal{H}}(p^\varepsilon;0,t) - \rho\int_\Omega \dot{u}^\varepsilon(t)\cdot\dot{w}(t)\,dx\Big\}\\ &=\int_0^t\int_\Omega (\mathbb{C}e(s) : E\dot{w}(s) - \rho\dot{u}(s)\cdot\ddot{w}(s))\,dx\,ds + C_0\end{aligned} \tag{7.18}$$

for every $t \in [0,T]$ and for some constant C_0. In particular, (7.18) holds for $t = 0$. The initial conditions satisfied by the triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ imply

$$C_0 = \int_\Omega Q(e^0)\,dx + \frac{\rho}{2}\int_\Omega |u^1|^2\,dx - \rho\int_\Omega u^1\cdot\dot{w}(0)\,dx. \tag{7.19}$$

Integrating (7.18) with respect to time we deduce the equality

$$\begin{aligned}&\int_B \limsup_{\varepsilon\to 0}\Big\{\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot{u}^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega |\ddot{u}^\varepsilon(s)|^2\,dx\,ds\\ &\quad + \mathcal{D}_{\mathcal{H}}(p^\varepsilon;0,t) - \rho\int_\Omega \dot{u}^\varepsilon(t)\cdot\dot{w}(t)\,dx\Big\}\,dt\\ &\quad=\int_B\int_0^t\int_\Omega (\mathbb{C}e(s) : E\dot{w}(s) - \rho\dot{u}(s)\cdot\ddot{w}(s))\,dx\,ds\,dt\\ &\quad+\int_B\Big[\int_\Omega Q(e^0)\,dx + \frac{\rho}{2}\int_\Omega |u^1|^2\,dx - \rho\int_\Omega u^1\cdot\dot{w}(0)\,dx\Big]\,dt,\end{aligned} \tag{7.20}$$

for every Borel subset $B \subset [0,T]$. In view of (7.18), an application first of the Reverse Fatou's Lemma, and then of Fatou's Lemma yields

$$\begin{aligned}&\int_B \limsup_{\varepsilon\to 0}\Big\{\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot{u}^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega |\ddot{u}^\varepsilon(s)|^2\,dx\,ds\\ &+ \mathcal{D}_{\mathcal{H}}(p^\varepsilon;0,t) - \rho\int_\Omega \dot{u}^\varepsilon(t)\cdot\dot{w}(t)\,dx\Big\}\,dt\\ &\geq \limsup_{\varepsilon\to 0}\int_B\Big\{\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot{u}^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega |\ddot{u}^\varepsilon(s)|^2\,dx\,ds\\ &+ \mathcal{D}_{\mathcal{H}}(p^\varepsilon;0,t) - \rho\int_\Omega \dot{u}^\varepsilon(t)\cdot\dot{w}(t)\,dx\Big\}\,dt\\ &\geq \liminf_{\varepsilon\to 0}\int_B\Big\{\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot{u}^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega |\ddot{u}^\varepsilon(s)|^2\,dx\,ds\end{aligned}$$

$$
\begin{aligned}
&+ D_{\mathcal{H}}(p^\varepsilon;0,t) - \rho\int_\Omega \dot u^\varepsilon(t)\cdot \dot w(t)\,dx\Big\}\,dt\\
&\geq \liminf_{\varepsilon\to 0}\int_B \Big\{\int_\Omega Q(e^\varepsilon(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot u^\varepsilon(t)|^2\,dx + 2\varepsilon\rho\int_0^t\int_\Omega |\ddot u^\varepsilon(s)|^2\,dx\,ds\\
&- \rho\int_\Omega \dot u^\varepsilon(t)\cdot \dot w(t)\,dx\Big\}\,dt + \int_B \liminf_{\varepsilon\to 0} D_{\mathcal{H}}(p^\varepsilon;0,t)\,dt
\end{aligned}
\tag{7.21}
$$

for every Borel subset $B \subset [0,T]$. By combining (7.2)–(7.4), and (7.21), we deduce the energy inequality

$$
\begin{aligned}
&\int_\Omega Q(e(t))\,dx + \frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,dx + D_{\mathcal{H}}(p;0,t) - \rho\int_\Omega \dot u(t)\cdot\dot w(t)\,dx\\
&\quad\leq \int_0^t\int_\Omega (\mathbb{C}e(s):E\dot w(s) - \rho\dot u(s)\cdot\ddot w(s))\,dx\,ds + \int_\Omega Q(e^0)\,dx\\
&\quad - \rho\int_\Omega u^1\cdot\dot w(0)\,dx + \frac{\rho}{2}\int_\Omega |u^1|^2\,dx
\end{aligned}
\tag{7.22}
$$

for a.e. $t\in[0,T]$. In view of (7.18), we obtain the uniform estimate

$$
\|e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}))} + \rho\|\dot u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C,
\tag{7.23}
$$

where the constant C is independent of ε.

In order to prove that u satisfies the first-order initial condition $\dot u(0) = u^1$ we argue as in [45, Theorem 4.2]. The minimality of the triple $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$, yields the Euler-Lagrange equation

$$
\varepsilon^2\rho\int_0^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon}\Big)\ddot u^\varepsilon(t)\cdot\ddot\phi(t)\,dx\,dt + \int_0^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon}\Big)\mathbb{C}e^\varepsilon(t):E\phi(t)\,dx\,dt = 0
\tag{7.24}
$$

for every $\phi\in W^{2,2}(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))$ satisfying $\phi(0)=\phi'(0)=0$. Let $\varepsilon_n\to 0$, and let S be a countable dense subset of $W^{1,2}_0(\Omega;\mathbb{R}^3)$. Let $I\subset(0,T)$ be defined as the set of points $t_0\in(0,T)$ such that

$$
\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\exp\Big(-\frac{t}{\varepsilon_n}\Big)\int_\Omega \ddot u^{\varepsilon_n}(t)\cdot h(x)\,dx\,dt = \exp\Big(-\frac{t_0}{\varepsilon_n}\Big)\int_\Omega \ddot u^{\varepsilon_n}(t_0)\cdot h(x)\,dx\,dt,
\tag{7.25}
$$

for every $n\in\mathbb{N}$, and for every $h\in S$. Note that by Lebesgue's theorem the set $[0,T]\setminus I$ is negligible.

Fix $t_0\in(0,T)$, and let $\varphi_{\delta n}\in C^{1,1}(\mathbb{R})$ be defined as

$$
\varphi_{\delta n}(t) := \begin{cases} 0 & t\leq t_0\\ \dfrac{(t-t_0)^2}{\delta\varepsilon_n^2} & t\in(t_0,t_0+\delta)\\ 2\dfrac{(t-t_0)}{\varepsilon_n^2} - \dfrac{\delta}{\varepsilon_n^2} & t\geq t_0+\delta.\end{cases}
$$

We observe that

$$
\varphi''_{\delta n}(t) = \frac{2}{\delta\varepsilon_n^2}\chi_{(t_0,t_0+\delta)}(t),
$$

where $\chi_{(t_0,t_0+\delta)}$ is the characteristic function of $(t_0,t_0+\delta)$. In addition,

$$
|\varphi_{\delta n}(t)| \leq \frac{2}{\varepsilon_n^2}(t-t_0)^+ \quad\text{and}\quad \varphi_{\delta n}(t)\to\frac{2}{\varepsilon_n^2}(t-t_0)^+
$$

as $\delta\to 0$ for all $t\in(0,T)$. Choosing $\phi(t,x)=\varphi_{\delta n}(t)h(x)$, with $h\in S$, by (7.24) we obtain

$$
\begin{aligned}
&\frac{2\rho}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \exp\Big(-\frac{t}{\varepsilon_n}\Big)\ddot u^{\varepsilon_n}(t)\cdot h(x)\,dx\,dt\\
&\quad + \int_{t_0}^T\int_\Omega \exp\Big(-\frac{t}{\varepsilon_n}\Big)\varphi_{\delta n}(t)\mathbb{C}e^{\varepsilon_n}(t):Eh(x)\,dx\,dt = 0.
\end{aligned}
$$

Letting $\delta \to 0$, (7.25) and the Dominated Convergence Theorem yield

$$\rho \int_\Omega \ddot{u}^{\varepsilon_n}(t_0) \cdot h(x)\, dx + \frac{1}{\varepsilon_n^2} \int_{t_0}^T \int_\Omega \exp\Big(\frac{t_0 - t}{\varepsilon_n}\Big)(t - t_0)\mathbb{C}e^{\varepsilon_n}(t) : Eh(x)\, dx\, dt = 0.$$

By (7.23), there holds

$$\begin{aligned}
&\Big|\frac{1}{\varepsilon_n} \int_{t_0}^T \int_\Omega \exp\Big(\frac{t_0 - t}{\varepsilon_n}\Big)(t - t_0)\mathbb{C}e^{\varepsilon_n}(t) : Eh(x)\, dx\, dt\Big| \\
&\leq \frac{C}{\varepsilon_n^2}\|e^{\varepsilon_n}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}))}\|Eh\|_{L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})} \int_{t_0}^T \exp\Big(\frac{t_0 - t}{\varepsilon_n}\Big)(t - t_0)\, dt \\
&\leq C\|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)} \int_0^{\frac{(T-t_0)}{\varepsilon_n}} t\exp(-t)\, dt \leq C\|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)}.
\end{aligned}$$

Thus

$$\rho\Big|\int_\Omega \ddot{u}^{\varepsilon_n}(t_0)h(x)\, dx\Big| \leq C\|h\|_{W^{1,2}_0(\Omega;\mathbb{R}^3)},$$

where the constant C is independent of ε_n and t_0. In particular, we obtain the uniform estimate

$$\rho\|\ddot{u}^{\varepsilon_n}\|_{L^\infty(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))} \leq C. \tag{7.26}$$

By combining (7.3), (7.23), and (7.26), we deduce that

$$\|\dot{u}^{\varepsilon_n}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))\cap W^{1,\infty}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))} \leq C.$$

Thus, up to the extraction of a (non-relabeled) subsequence, there holds

$$\dot{u}^{\varepsilon_n}(t) \to \dot{u}(t) \quad \text{strongly in } C([0,T]; W^{-1,2}(\Omega;\mathbb{R}^3)),$$

which in turn yields $\dot{u}(0) = u^1$.

The thesis follows now by the uniqueness of solutions for the dynamic plasticity problem (see Theorem 2.2). □

We point out that the assertion of Theorem 2.3 still holds if we generalize the minimum problem (2.13) by imposing ε-dependent initial data satisfying suitable compatibility assumptions. To be precise, for every ε, define the set

$$\begin{aligned}
\mathcal{V}_\varepsilon := \{&(u, e, p) \in W^{2,2}(0,T; L^2(\Omega;\mathbb{R}^3)) \cap L^1(0,T; BD(\Omega)) \\
&\times L^2((0,T)\times\Omega;\mathbb{M}^{3\times 3}_{\rm sym}) \times BV([0,T]; \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D)) : \\
&(u(t), e(t), p(t)) \in \mathscr{A}(w(t)) \text{ for every } t \in [0,T], \\
&u(0) = u^0_\varepsilon,\ \dot{u}(0) = u^1_\varepsilon,\ e(0) = e^0_\varepsilon,\ p(0) = p^0_\varepsilon\},
\end{aligned}$$

with $(u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon) \in \mathscr{A}(w(0))$, and $u^1_\varepsilon \in BD(\Omega)$ such that there exists $e^1_\varepsilon \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$ satisfying $(u^1_\varepsilon, e^1_\varepsilon, 0) \in \mathscr{A}(\dot{w}(0))$. Assuming that the initial data are *well-prepared*, namely

$$\begin{aligned}
&u^0_\varepsilon \rightharpoonup^* u^0 \quad \text{weakly* in } BD(\Omega), \\
&e^0_\varepsilon \rightharpoonup e^0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}), \\
&p^0_\varepsilon \rightharpoonup^* p^0 \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{3\times 3}_D), \\
&u^1_\varepsilon \to u^1 \quad \text{strongly in } W^{-1,2}(\Omega;\mathbb{R}^3),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big[\int_\Omega Q(e^0_\varepsilon)\, dx + \frac{\rho}{2}\int_\Omega |u^1_\varepsilon|^2\, dx - \rho\int_\Omega u^1_\varepsilon \cdot \dot{w}(0)\, dx\Big] \\
&= \int_\Omega Q(e^0)\, dx + \frac{\rho}{2}\int_\Omega |u^1|^2\, dx - \rho\int_\Omega u^1 \cdot \dot{w}(0)\, dx,
\end{aligned}$$

one can again prove that there exists a sequence of triples $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$, with $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \subset \mathcal{V}_\varepsilon$ for every ε, such that

$$I_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) = \min_{(v,f,q)\in\mathcal{V}_\varepsilon} I_\varepsilon(v, f, q),$$

such that $\{(u^\varepsilon, e^\varepsilon, p^\varepsilon)\}$ converges to the solution (u, e, p) of dynamic perfect plasticity, namely (c1), (c2) and (c3'), in the sense of Theorem 2.3.

Acknowledgements

We acknowledge the support by the Austrian Science Fund (FWF) projects P 27052 and I 2375. This work has been funded by the Vienna Science and Technology Fund (WWTF) through Project MA14-009.

References

[1] G. Akagi, U. Stefanelli. A variational principle for doubly nonlinear evolution. *Appl. Math. Lett.* **23**(9) (2010), 1120–1124.

[2] G. Akagi, U. Stefanelli. Periodic solutions for doubly nonlinear evolution equations. *J. Differential Equations*, **251**(7) (2011), 1790–1812.

[3] G. Akagi, U. Stefanelli. Weighted energy-dissipation functionals for doubly nonlinear evolution. *J. Funct. Anal.* **260**(9) (2011), 2541–2578.

[4] G. Akagi, U. Stefanelli. Doubly nonlinear equations as convex minimization. *SIAM J. Math. Anal.* **46**(3) (2014), 1922–1945.

[5] G. Akagi, S. Melchionna. U. Stefanelli. Weighted energy-dissipation approach to doubly-nonlinear problems on the half line. Submitted, 2016.

[6] G. Anzellotti, S. Luckhaus. Dynamical evolution of elasto-perfectly plastic bodies. *Appl. Math. Optim.* **15**(2) (1987), 121–140.

[7] J.-F. Babadjian, M.G. Mora. Approximation of dynamic and quasi-static evolution problems in plasticity by cap models. *Quart. Appl. Math.* **73**(2) (2015), 265–316.

[8] V. Bögelein, F. Duzaar, P. Marcellini. Existence of evolutionary variational solutions via the calculus of variations. *J. Differential Equations*, **256** (2014), 3912–3942.

[9] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, Springer, 2011.

[10] S. Conti, M. Ortiz. Minimum principles for the trajectories of systems governed by rate problems. *J. Mech. Phys. Solids*, **56**(5) (2008), 1885–1904.

[11] G. Dal Maso, A. DeSimone, M.G. Mora. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Ration. Mech. Anal.* **180**(2) (2006), 237–291.

[12] E. Davoli, M.G. Mora. A quasistatic evolution model for perfectly plastic plates derived by Γ-convergence. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **30**(4) (2013), 615–660.

[13] E. De Giorgi. Conjectures concerning some evolution problems. *Duke Math. J.* **81**(2) (1996), 255–268.

[14] E. De Giorgi, T. Franzoni. On a type of variational convergence. In *Proceedings of the Brescia Mathematical Seminar, Vol. 3 (Italian)*, pages 63–101, Milan, 1979. Univ. Cattolica Sacro Cuore.

[15] G. Duvaut, J.-L. Lions. *Inequalities in Mechanics and Physics.* Grundlehren der mathematischen Wissenschaften 219, Springer-Verlag, 1976.

[16] L.C. Evans. *Partial differential equations*, volume 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1998.

[17] G.A. Francfort, A. Giacomini. On periodic homogenization in perfect plasticity. *J. Eur. Math. Soc. (JEMS)*, **16**(3) (2014), 409–461.

[18] C. Goffman, J. Serrin. Sublinear functions of measures and variational integrals. *Duke Math. J.* **31** (1964), 159–178.

[19] N. Hirano. Existence of periodic solutions for nonlinear evolution equations in Hilbert spaces. *Proc. Amer. Math. Soc.* **120** (1994), 185–192.

[20] W. Han, B.D. Reddy. *Plasticity, Mathematical theory and numerical analysis*. Springer-Verlag, New York, 1999.

[21] R. Hill. *The Mathematical Theory of Plasticity.* Oxford, Clarendon Press, 1950.

[22] T. Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.* **108**(520) (1994), x+90.

[23] R. Kohn, R. Temam. Dual spaces of stresses and strains, with applications to Hencky plasticity. *Appl. Math. Optim.* **10**(1) (1983), 1–35.

[24] J. Lemaitre, J.-L. Chaboche. *Mechanics of solid materials*. Cambridge University Press, 1990.

[25] C.J. Larsen, M. Ortiz, C.L. Richardson. Fracture paths from front kinetics: relaxation and rate independence. *Arch. Ration. Mech. Anal.* **193**(3) (2009), 539–583.

[26] M. Liero, A. Mielke. An evolutionary elastoplastic plate model derived via Γ-convergence. *Math. Models Methods Appl. Sci.* **21**(9) (2011), 1961–1986.

[27] M. Liero, T. Roche. Rigorous derivation of a plate theory in linear elastoplasticity via Γ-convergence. *NoDEA Nonlinear Differential Equations Appl.* **19**(4) (2012), 437–457.

[28] M. Liero, U. Stefanelli. A new minimum principle for Lagrangian mechanics. *J. Nonlinear Sci.* **23**(2) (2013), 179–204.
[29] M. Liero, U. Stefanelli. Weighted inertia-dissipation-energy functionals for semilinear equations. *Boll. Unione Mat. Ital. (9)*, **6**(1) (2013), 1–27.
[30] J.-L. Lions. Singular perturbations and some non linear boundary value problems. Technical Report n. 421, Mathematics Research Center, University of Wisconsin Madison, 1963.
[31] J.-L. Lions. Sur certaines équations paraboliques non linéaires. *Bull. Soc. Math. France*, **93** (1965), 155–175.
[32] J. Lubliner. *Plasticity Theory.* New York, Macmillan Publishing, 1990.
[33] S. Luckhaus. Elastisch-plastische Materialien mit Viskosität. Preprint no. 65 of Heidelberg University, SFB 123.
[34] G.B. Maggiani, M.G. Mora. A dynamic evolution model for perfectly plastic plates. *Math. Models Methods Appl. Sci.* **26** (2016), 1825–1864.
[35] G. Dal Maso, R. Scala. Quasistatic evolution in perfect plasticity as limit of dynamic processes. *J. Dynam. Differential Equations*, **26**(4) (2014), 915–954.
[36] S. Melchionna. A variational principle for nonpotential perturbations of gradient flows of nonconvex energies. Preprint arXiv:1605.00801.
[37] A. Mielke, M. Ortiz. A class of minimum principles for characterizing the trajectories and the relaxation of dissipative systems. *ESAIM Control Optim. Calc. Var.* **14**(3) (2008), 494–516.
[38] A. Mielke, T. Roubíček. *Rate-independent systems*, volume 193 of *Applied Mathematical Sciences.* New York, Springer, 2015.
[39] A. Mielke, U. Stefanelli. A discrete variational principle for rate-independent evolution. *Adv. Calc. Var.* **1**(4) (2008), 399–431.
[40] A. Mielke, U. Stefanelli. Weighted energy-dissipation functionals for gradient flows. *ESAIM Control Optim. Calc. Var.* **17**(1) (2011), 52–85.
[41] M.G. Mora. Relaxation of the Hencky model in perfect plasticity. *J. Math. Pures Appl.* **106** (2016), 725–743.
[42] O.A. Oleinik. On a problem of G. Fichera. *Dokl. Akad. Nauk SSSR*, **157** (1964), 1297–1300.
[43] R. Rossi, G. Savaré, A. Segatti, U. Stefanelli. A variational principle for gradient flows in metric spaces. *C. R. Math. Acad. Sci. Paris*, **349**(23-24) (2011), 1225–1228.
[44] A. Segatti. A variational approach to gradient flows in metric spaces. *Boll. Unione Mat. Ital. (9)*, **6**(3) (2013), 765–780.
[45] E. Serra, P. Tilli. Nonlinear wave equations as limits of convex minimization problems: proof of a conjecture by De Giorgi. *Ann. of Math. (2)*, **175**(3) (2012), 1551–1574.
[46] E. Serra, P. Tilli. A minimization approach to hyperbolic Cauchy problems. *J. Eur. Math. Soc.* **18** (2016), 2019–2044.
[47] E. Spadaro, U. Stefanelli. A variational view at the time-dependent minimal surface equation. *J. Evol. Equ.* **11**(4) (2011), 793–809.
[48] U. Stefanelli. The De Giorgi conjecture on elliptic regularization. *Math. Models Methods Appl. Sci.* **21**(6) (2011), 1377–1394.
[49] P. Suquet. *Plasticité et homogénéisation.* PhD thesis, Université Pierre et Marie Curie, 1982.
[50] R. Temam. *Problèmes mathématiques en plasticité*, volume 12 of Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science]. Montrouge, Gauthier-Villars, 1983.

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

E-mail address: elisa.davoli@univie.ac.at

Faculty of Mathematics, University of Vienna & Istituto di Matematica Applicata e Tecnologie Informatiche *E. Magenes*, via Ferrata 1, I-27100 Pavia, Italy, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

E-mail address: ulisse.stefanelli@univie.ac.at