FOUR CAI BASES TOWARDS A COMPREHENSIVE CAI SYSTEM

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by

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S u m m a r y : we shall first present, as initial realizations of a pilot instructional project supported by I.E.I.(¶), the following CAI programmes

1 - E X E R interactive system for

a) Predicate Calculus, Set Combinatorics, and Groups

b) Geometrical Gaussian Optics

2 - E L G E graphic system for isometrical plane transforms

3 - M A G A programmes for Mathematical Games.

After the technical presentation we shall comment briefly on the motivation for, and the prospective developments of, these CAI programmes towards a 'comprehensive' CAI system.

T h e E X E R s y s t e m : It is APL-implemented with an added interpreter, namely: the student prints EXER sentences which are either 'questions' or 'proposals'

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(†††) Grantees of C.N.R. post-graduate scholarships

(¶) It is the I.E.I. share in a joint didactic research with the University of Pisa, participating with their project "Seminario didattico", director: Prof.V.Checcucci, Math Dep't.

from his viewpoint, and have been made easier to learn, more significant, than APL instructions. This simplification can be realized by delimiting the didactic content.

A potential CMI aspect: we have on record a connected set of didactic epidoses, i.e. a sizable history of EXER stimulus-response occurrences, which lends itself to statistical processing in view of EXER-2nd edition.

The EXER functions are either for interaction control or for tutorial control, as follows

interaction control: RIC shows how to select contents, CONT shows each content, MATA computes at student's request, and SUPER supervises how the three functions above intervene.

Here follows an English transcript of a RIC frame in Fig.1 (note that 'RIC' stands for the Italian equivalent of 'REQUEST'; same remark for other labels):

SENTENTIAL CALCULUS - SET COMBINATORICS

No *Have you taken this course previously?*
(4) Do you intend to take the whole course?
4 *(5) Do you intend to take the drill part only?*
Then you need only to know the basic rules of Sentential Calculus and Set Combinatorics. You may ask for one of the nine problems that follow

1-Truth tables, 2-Canonical forms, 3-Universal systems, 4-Predicates, 5&6-Operations over sets, 7÷Combinations, 8&9-Permutations

Directions on how to proceed shall be given at each step. The most important instructions are

-CONT- Go on to another problem
-SALT- Go back on problems already dealt with
-RIPE- Repeat this problem
-STOP- This session adjourned

The MATA function simulates an extendable desk calculator and helps the students with routine computations, i.e. within the computing level assumed as prerequisite at this learning stage. Let us now quote the main functions of

tutorial interaction: PROB generates a specific problem for each class of problems, either formal or numerical, in the fields 1 - a) and b), i.e. PROB specifies the problem parameters by symbols or numbers. Then ES finds the problem solution and compares it with that presented by the student, warning him about his mistakes if any. AES responds flexibly to HELP requests according whether the student request follows: either a BLANK, or a correct partial answer, or an uncorrect answer that is such either semantically or syntactically (i.e. EXER ill-formed). Finally, if AES remains unconclusive then SES intervenes and presents the ES found solution offering step-by-step comments.

EXER 1 - a), an example: PROB each time generates (pseudo)random numbers and decodes them, when the problem is formal, into symbols that stand for logical variables and connectives; otherwise, when the problem is numerical, PROB normalizes them into specific intervals. The specification of a problem may require several such steps; to consider an example in Predicate Calculus, assume by initial generation the following problem-class

"To find the extension of predicate $P(x) = Q(x) \Theta R(x)$" and assume that PROB, by operating as said above, specifies Θ as the connective Λ, $Q(x)$ as a 2nd degree inequality, $R(x)$ as a negation, and the negated predicate as a union. Then PROB, after the last (and numerical) specification, presents the following problem

"To find the values of x verifying the predicate

$$P(x) = (x^2 - x - 30 < 0) \Lambda \sim(x < 1 \cup x > 8)$$ "

One can see from this example that the system is performing linguistic analysis: infact PROB presents a problem after applying (well known) checks in order to discard ill-formed formal specifications, then ES solves a problem each time it is generated - problems are not retrieved from a 'répertoire'! - finally AES responds flexibly. For instance, if the student stumbles over the negation in the above example, and asks for HELP, then AES would first remind him of a De Morgan law and, if that fails, would suggest performing the union, ect.
Here follows a transcript of an AES frame (i.e. computer print-out) in Fig.2:

PROBLEM no 4

Which values of x verify the following predicate

$(0 > x^2 - x - 20) \wedge (x > -3 \wedge x < 5)$?

$|-3,5|$ *Boundary uncorrect*

$(-3,5|$ *Boundary uncorrect*

HELP *Then, find the extension of* $(x^2 - x - 20 < 0)$

$(-5,4)$ *Interval uncorrect, the correct answer is (-4,5).*
Then, what the final answer?

$|-3,5)$ *Boundary uncorrect*

$(-3,5)$ *Correct*

EXER 1 - b), a formalization: the system first shows the student how the empirical context of light phenomena can be formalized by Geometrical Optics in the simpler case of the Gaussian Optics, i.e. when the light rays are so near the axis that one may neglect higher order terms in the Taylor series for the components of the characteristic function. Namely, the system presents refraction and reflexion together with the concept of DIOPTRE, French for "surface optique séparant deux milieux transparents inégalement réfringents" (we ignore an English word for the unbound system consisting of two adjacent optical media).

The student then learns that in a narrow region around the axis he may represent 1) a light ray by two numbers: ordinate of intersection point and axis-to-ray angle, and 2) the DIOPTRE refraction by a 2-by-2 linear transform between the incident and the refracted ray. Therefore he sees this simple Gaussian system as a 'black box' whose input-output relation is

$$\begin{pmatrix} Y' \\ U' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} Y \\ U \end{pmatrix}, \quad \text{where} \quad \alpha\,\delta - \beta\,\gamma = 1$$

Once given this formalization, the system 1 - b) operates according to the general EXER scheme, though now the function PROB has to follow some specific criteria for generating its problems in Gaussian Optics. (We should note, at this point, that one research aim for EXER-2nd edition is - infact - homogeneity of problem-structure generation through heterogeneity of models). Here follows a transcript of the computer frame about formalization in Fig.3

==

READ CAREFULLY BEFORE PRESSING THE RETURN KEY

A light ray can be thought of as an oriented straight line uniquely definable by two parameters:

1- The ordinate of the intersection point, Y, if the DIOPTRE axis is the X-axis;

2- The angle of incidence U, i.e. the angle between the X-axis and the light ray.

Therefore you have a one-to-one correspondence between light rays and vectors of a bi-dimensional space

$$\{light\ ray\} \leftrightarrow \{Y, U\}$$

Refraction of light induces a transform in this space

$$\begin{cases} Y' = Y \\ U' = U + F \times Y \end{cases}$$

where primed letters refer to the second optical medium.

Which, in matricial form, is written as

$$\begin{pmatrix} Y' \\ U' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \times \begin{pmatrix} Y \\ U \end{pmatrix}$$

Thus the DIOPTRE is represented by a matrix with DET=1 and separatedly from the light ray, which is represented by a vector. Actually the latter implies the index N through U and the two could be represented independently from each other, but less simply.

In EXER 1 - b) the student will carry over his routine computations by means of the function MATA, as said, and he learns to solve many compound Gaussian (i.e. centered) systems by applying matrix algebra in classical exercises of Geo^metrical Optics.

To conclude, we believe that EXER 1-b) would help the student to appreciate the power and beauty of linear transforms in one empirical context of immediate interest.

T h e E L G E s y s t e m : The display of this graphic system is an IBM-2250 with light pen. The student is able to

1 - generate any number of segments on the screen by pointing the light pen along the lower and the left side in order to assign the Cartesian coordinates of end-points; he may want his segments randomly placed or forming a known figure, convex or concave; and

2 - assign an isometrical transform of these segments by pointing at one of the instruction labels lit on the right side; and, at last, specify this transform by generating, Cartesian-like as before, a line or a point for a reflexion - if the transform is a reflexion to be executed about that line or that point - an applied vector for a translation along that vector, or, finally, a point plus an angle for a rotation around that point and by that angle.

The student may iterate step 2 and perform several transforms in any order, each time holding on the screen the patterns already formed. That can be done in two distinct ways: after each transform, either the new operand remains unmodified, or it will include the previous result. Thus we have composition of transforms such that the output of one becomes the input of the next one.

ELGE at present would help the student to familiarize with the notion of a group (see on this some display photographs in Fig.4). In order to test this potentiality we have on file some lengthy video-tape recordings with a fair variety of segment generations,transform assignments, and transform specifications.

ELGE leads naturally to developments of interactive type, as soon as we tackle with EXER-2nd edition.

T h e M A G A s y s t e m : As for ELGE, these computer-run games at present are just a base of learning processes to be guided by the teacher. Speaking of the competitive games, the student

1) first of all, learns to code his moves on the keyboard, thus widening - through coding awareness - his concept of 'representation' (any fact = a sequence of symbols or operations);
2) then he is led to analyze sequences of moves and to infer: either the existence of a winning strategy that depends on who starts the game (as in the games FILETTO, NIM, WYTHOFF, and PILES), or the absence of such a strategy (as in SIM);
3) finally, he would be led to abstract this concept of a winning strategy out of a variety of games. The teacher would help him to perform syntactical analysis from semantical contexts, namely (keeping for himself such technical terms) to represent a strategy formally, as a binary coded rule for NIM, as star-symmetries and right angle rotations for FILETTO, as a sequence of upward-and-

rightward steps in a square grid for WYTHOFF.

We believe that programmes like MAGA are more for teenagers than for children, ...if points 1), 2), and 3) above are kept in mind; otherwise children would have more fun - not to speak of money saved - by playing these games among themselves with nuts, nails, or matches rather than against a computer.

Besides, some of the programmes are far from being a programming novelty: for instance, FILETTO is the well known 'Tic Tac Toe' which in 1959 was played at M.I.T., Mass.USA, by the TX-0 computer on a video-display.

But our game PILES deserves some attention, its programme finds itself a winning path as follows. It contains a 'urn' where each possible move is numerically coded, each number present with some replicas. At its turn, it picks up a move at random from the urn by non-destructive read-out and each time the game ends, it either inserts three replicas ('reward') for each move that turns out to be favourable, or cancels one replica ('punishment') for each move that does not.

Here follows a transcript (abridged) of a PILES computer frame in Fig.5:

THE PILES GAME

If you do not know the rules, here they are:

We play with any number of tokens, which shall be between one and nine in our programme, all of them in one row at the start. With each move any number of them can be displaced upward, and only upward, to form first one then two parallel rows above the initial one, provided that neither of them contains more tokens than a lower row (obvious rule if you play with real tokens - let's say toy cubes - instead of writing symbols on paper).

Now, there is a last move, after which the three-row array *cannot be modified any further, and the one who makes the last move is the winner. If playing with six tokens,*

```
0    }                              00   }
00   } this array is correct,       0    } this one is uncorrect
000  }                              000  }

CODE: 123                           CODE: 213
```

Let us play: how many tokens?

```
  6
    Then we have seven possible arrays:
    006    015    024    033    114    123    222
    BEGIN      006:  000000   YOU MOVE
 MY MOVE IS    024: {00
                    {0000     YOU MOVE
               222: {00
                    {00
                    {00
    YOU WIN
```

==

The didactic value of MAGA as an autonomous system begins to appear with the computer frame in Fig.5 (see there what else is not shown in the English transcript), which is meant as a first step towards realizing points 2) and 3) of page 7 without teacher help.

Namely, the arrays above (in italics) are coded from 1 through 7 in that order, which is that of decreasing array 'stability', i.e. you can't modify any array into anyother whose code number is lower. Then Fig.5 shows every admissible path from the initial "1" to the final "7", and you can see that the minimal path is "1→3→7". Finally it shows two distinct distributions (with replicas) of the internal 'urn', precisely those corresponding to the two moves quoted above in italics, computed by the PILES programme after each move according to the 'reward-and-punishment' rule already said.

Therefore the student, while playing a (reasonably) long

series of games, should be able to realize that he might win some of them initially but becomes a dead loser from a certain game on, because the computer 'learns' how to win by altering the a priori probability of its random move selections.
The student may also make a right guess: it is possible to retard the computer 'learning rate' by playing the fool in very first games and suddenly improving afterwards, in order to induce self-complacent undeserved rewards in his opponent. Of course, a zero reset is provided for, to start anew an 'ignorant' computer against another player.
To conclude on PILES: understanding the 'urn' trick would give the student a 'Darwinian' feeling of how Necessity results from Chance.
To conclude on these (competitive) games: we omit a description of the other games quoted at point 2), page 7, for the PILES game is fairly representative of them all; some details of interest may be given during the oral presentation.
Actually a variety of (similar) games is required for the didactic aim outlined at point 3), page 7; with teacher help now, or without it in the future when the MAGA system is amplified.
Simulation: We present aside the LIFEG programme, formally included in MAGA though it is not a game (it can be made such, i.e. competitive, by added rules in several ways. Incidentally, "game" and "play" both become in Italian the word "gioco" not necessarily connoting competition).
LIFEG is the well known 'GAME OF LIFE' divulged by "Scientific American": it is not *per se* a game and does not *per se* simulate anything. It is a deterministic sequence of configurations of *1*.s in a bi-dimensional array whose elements are either *1*.s or *0*.s;

i.e. a square grid where each square either contains a dot or is blank, the dotted pattern changing at each step of the sequence unless it is or becomes stable.

We said 'deterministic': there is indeed the initial configuration which is arbitrary but any other results from the previous one by generating and cancelling some *1*.s according to given'neighbourhood' rules such as, for instance,

0 → *1* if a blank square has exactly three neighbouring dotted squares,

1 → *0* if a dotted square has neither two nor three neighbouring blank squares.

See in Fig.6 some steps of a LIFEG sequence executed and displayed by computer; a long sequence when filmed (our display is slow) can be suggestive, provided that the grid is very dense (not our case, unfortunately).

Infact this abstract process becomes interesting with high order grids and very long sequences, because then the lack of an analytical model gives rise to several deep problems such as the specificity, the stability, the exact- or the quasi-periodicity, and the ergodicity of patterns: actually the only known (to us) model of this process is the process itself, though perfectly deterministic.

Here one can see an interesting, though rather subtle, didactic value of LIFEG: the official culture may be divided on the concept of Causality but common sense is still predominantly Aristotelian about; so, it can be an illuminating surprise that of looking at a very simple graphic process like LIFEG and realizing that it is unpredictable though strictly causal.

At a more concrete didactic level, instead, the teacher may present LIFEG to the student in order to suggest analogies from bio- and socio-economic contexts, where the 'neighbourhood'rules lend themselves to various interpretations, more or less approximate but often significant; for instance, to ecological comparisons.

E x p e r i e n c e a n d C o s t s : The CAI systems presented here have been tried singly: EXER with 1-st year University students of the Algebra course, Info Sci.Dep't; ELGE and MAGA occasionally with high school students.

About EXER we noticed the lack of 'motivational' informal frames to introduce formal developments, besides other prospective developments of which later. About ELGE we were glad to observe the students' interest aroused by computer animated ELGE films, as shown by us on TVCC in schools with no computer nor terminal (as they are bound to remain for long in this country).

Thus ELGE proves effective at a perceptual level even when the student cannot intervene actively with the light pen, possibly because of the variety of unusual, often complex, patterns and of the transforms realizable, though only isometrical.

About costs, we do not quote ours as they would not be indicative due to a special convention, but we can suggest a fairly general formula to compute costs in the case of a terminal consôle in time sharing with a large computer.

Assume that you want to evaluate the total cost C_t for a given CAI course taken by N students during m months, and that each student is allotted the same total time T_c , in hours, of connection with the computer, and the same total T_{CPU} time of Central Processing Unit at his disposal, usually shorter than the T_c time when other computer resources are utilized.

Let us now call C_c the hourly connection cost and C_{CPU} the hourly CPU cost, L_t the (fixed) monthly amount for the terminal lease and C_{VM} the monthly amount for the Virtual Machine lease (in the computer storage). Then the total cost required shall be

$$C_t = (T_c C_c + T_{CPU} C_{CPU}) N + (C_{VM} + L_t) m$$

P r o s p e c t i v e   d e v e l o p m e n t s : Summing up, with little or no help from the teacher, ELGE would help the student to define properties or, more technically, to conceptualize: EXER 1-a) from a few formal contexts, EXER 1 -b) from one empirical context; ELGE would help the student to recognize properties through 'association by perception' while conceptualization is 'abstraction by induction'; finally MAGA would stir the student to act, easy when he acts for playing and only then. To make one think while doing, is often hard; to make one think before doing or without doing is harder.

Thus we have here several but isolated instances of didactic action at the perceptual and the conceptual level, and in various modalities: tutorial, drill-and-practice, 'Socratic' or dialogue (there is some in EXER 1-a)), gaming, and simulation (the weakest one but not completely absent). How can we develop these initial CAI realizations into a unique 'comprehensive' system? i.e. a system presenting the above levels and modalities well balanced, inter-dependent, and integrated?

EXER: more syntactical structuring and recognition to promote this system fully from the tutorial to the 'Socratic' modality, together with motivational presentation from various empirical contexts to help forming (abstract) concepts. Adding to the list of contents dealt with by EXER is not the main problem, once realized some improvements in linguistic analysis to ensure uniformity of presentation and handling.

ELGE: point generation by direct pointing of the light pen, instead of the current Cartesian assignments, is an immediate objective, to be followed (now that a more powerful display is at last available to us in Pisa) by similitude transforms i.e. dilations. Then ELGE would be of a great help in Linear Algebra and also in Complex Variable (with Lord Kelvin transform).

EXER-ELGE: Analytical-Synthetical integration, i.e. EXER-ELGE interaction between Linear Algebra and Similitude Geometry is the basic objective. We want a 'black' box - speaking for a moment from the student viewpoint - whose alphanumerical input {output} is an equation $Y = A X$, where Y,X are vectors and A is a matrix, whose graphic output {input} is the corresponding transform.

Furthermore, we want this 'box''grey'and then 'white' for helping the student to work by himself at such an output once given such an input. And we want this interaction both for the Gaussian Optics of EXER-b) and for an extension of EXER-a).

EXER-MAGA: the objectives have been outlined at points 2), 3) of page 7. Here also the main problem is not adding some more games (trying our hand at linguistic rather than numerical games) but developing EXER Socratically and integrating it with MAGA in order to offer the student two parallel 'boxes':
one black, against which he plays and the other one grey and/or white, such that he becomes aware of - and understands - the gaming procedures of his opponent.

M o t i v a t i o n : obviously we are not expounding our basic motivation for didactic research work to dedicated people at this Summer School, but there are two side aspects non negligible. They can be subsumed under two distinct meanings of 'comprehensive' when applying this attribute to a large CAI system as an assemblage of several programmes at various levels (perceptual, conceptual) and modalities (tutorial, Socratic, etc.).

A) The first meaning is less important: we think it worthwhile to realize an assorted 'package' (not only for solving more flexibly CAI problems but also) for presenting it to school authorities and to people with decisional power over school matters.
Infact we do not trust much any more in spontaneous reverberation of CAI work among the teachers as the sole mean of action for a badly needed school reform in Italy. Therefore we also want to convince and/or persuade people higher up, without trying too hard at guessing beforehand which CAI feature might turn out less useful in school practice.

B) In the second meaning 'comprehensive' implies the concept of 'learner's control'. Indeed, the widest possible choice of levels and modalities plus the obvious widest possible choice of *examples* and *pertinent* student requests, both seem to us as the right interpretation of 'learner's control', rather than choice of *the content* itself. 'Spontaneism' was fine when rebelling against the authoritarian teaching of the past but, if meant as free choice of didactic content on the student side, would seem to us as closely linked to the concept of Biological evolution. While we are dealing here with problems of Cultural evolution.

('learner's control' has a special meaning when the learner is learning to teach. We are thinking of a 'Teacher Oriented CAI' but this is another story).

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PROPOSIZIONI - INSIEMI - CALCOLO COMBINATORIO - PERMUTAZIONI

HAI GIA' SEGUITO QUESTO CORSO ?

NO

(4)-VUOI SEGUIRE INTEGRALMENTE TUTTO IL CORSO ?
(5)-VUOI FARE SOLO GLI ESERCIZI ?

4

PER SEGUIRE QUESTO CORSO E' SUFFICIENTE CONOSCERE LE REGOLE DEL CALCOLO PROPOSIZIONALE, LE OPERAZIONI ELEMENTARI SUGLI INSIEMI, ELEMENTI DI CALCOLO COMBINATORIO E LE PERMUTAZIONI.

GLI ESERCIZI SONO 9 :

1- TABELLE DI VERITA'
2- FORME CANONICHE
3- SISTEMI UNIVERSALI
4- PREDICATI
5-.6- OPERAZIONI FRA INSIEMI
7- COMBINAZIONI
8-9- PERMUTAZIONI

LE ISTRUZIONI NECESSARIE PER ANDARE AVANTI NEL CORSO TI SARANNO DATE MAN MANO CHE SERVONO.

RIASSUMIAMO QUI LE PIU' IMPORTANTI; IN PARTICOLARE:

-CONT- PASSA AD UN ARGOMENTO SUCCESSIVO

-SALT- TORNA SU ARGOMENTI GIA' TRATTATI.

-RIPE- RIPETE UNO STESSO ARGOMENTO.

-STOP- INTERROMPE LA SEDUTA.

FIG.1

ESERCIZIO N 4

DIRE PER QUALI VALORI DI X IL SEGUENTE PREDICATO DIVENTA UNA PROPOSIZIONE VERA:

(X×X-1×X-20<0) ∧ (X>¯3 ∧ X<5)

[¯3,5]

ERRORE SULLA NATURA DEGLI ESTREMI

(¯3,5]

ERRORE SULLA NATURA DEGLI ESTREMI

AIUTO

RISOLVI INTANTO

(X×X-1×X-20<0)

(¯5,4)

L'INTERVALLO NON E' QUESTO
IL RISULTATO E' :(¯4,5)
ALLORA QUALE E' IL RISULTATO DELL'ESERCIZIO?

[¯3,5)

ERRORE SULLA NATURA DEGLI ESTREMI

(¯3,5)

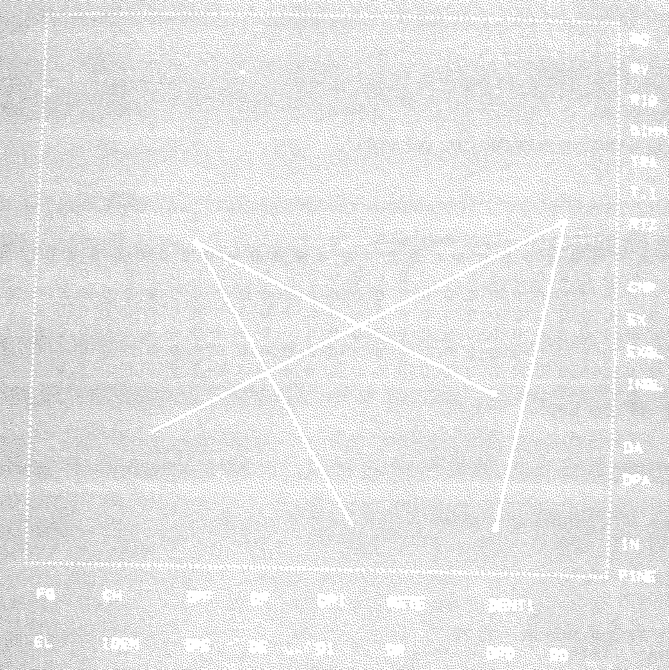
ESATTO

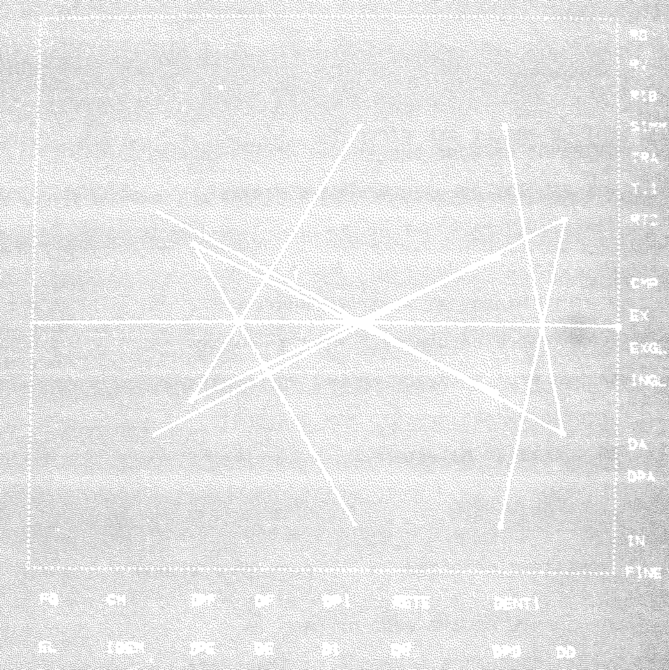
*** PER CONTINUARE CONT; PER RIPETERE L'ESERCIZIO RIPE; PER SALTARE SALT; PER FERMARTI STOP.

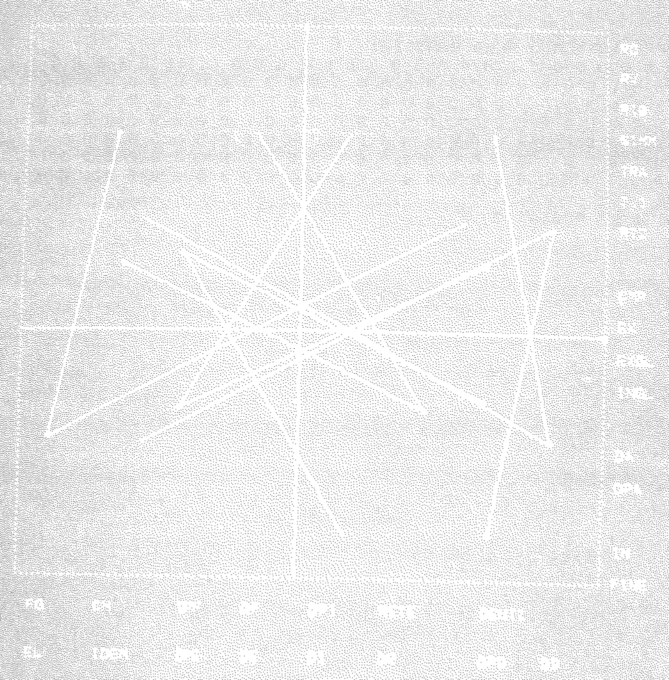
FIG. 2

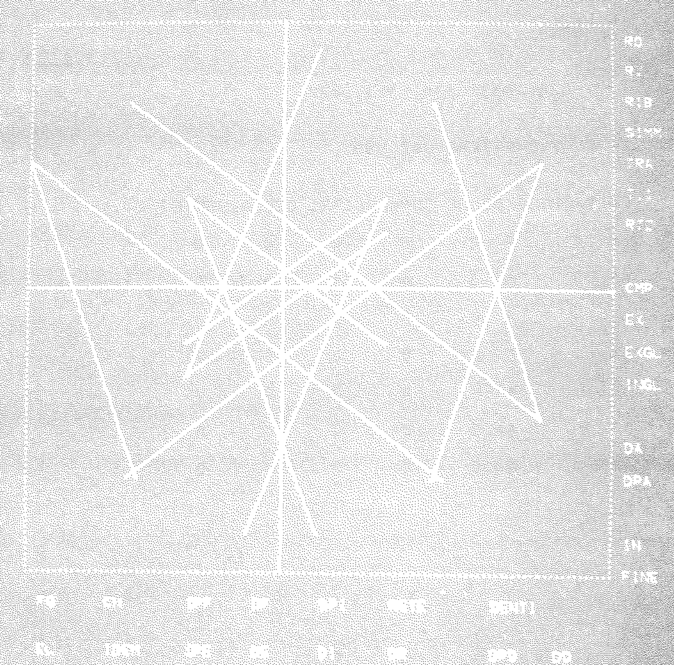
```
****LEGGI ATTENTAMENTE E POI PREMI IL RITORNO CARRELLO


 POSSIAMO DIRE CHE IL GENERICO RAGGIO E' INDIVIDUATO DA DUE PARAMETRI:
1-IL PUNTO D'INTERSEZIONE SUL DIOTTRO, Y
2-L'INCLINAZIONE  U  CON CUI INCIDE SUL DIOTTRO
 SI PUO' DUNQUE PORRE UNA CORRISPONDENZA BIUNIVOCA TRA RAGGI E VETTORI
 DI UNO SPAZIO BIDIMENSIONALE:

                 GENERICO RAGGIO  <========> (Y,U)

 LA RIFRAZIONE INTRODUCE UNA TRASFORMAZIONE IN QUESTO SPAZIO:

                 (   Y'=Y                        LE LETTERE PRIMATE SI RIFERISCONO
                 )                               AL SECONDO MEZZO
                 (   U'=U+F×Y

 CHE IN FORMA MATRICIALE SI SCRIVE:

                 ( Y' )    ( 1 0 )    ( Y )
                 )    ( = )      ( × )    (
                 ( U' )    ( F 1 )    ( U )

  SI HA COSI' LA SEPARAZIONE TRA LE PROPRIETA' DEL DIOTTRO, CHE SONO NELLA
 MATRICE, E QUELLE DEL RAGGIO NEL VETTORE. ( IN EFFETTI QUESTO CONTIENE ANCHE N
 TRAMITE  U, MA SE NE POTREBBE FARE A MENO PERDENDO PERO' DI SEMPLICITA.'
  (NOTA CHE LA MATRICE HA DET=1)
```

FIG.3


RIB
T.1
CMP
EX
EXGL
INGL
DA
DPA
IN
FINE
FQ
CH
DENTI
EL
IDEM
DD


RIB
TRA
T.1
CMP
EX
EXGL
INGL
DA
DPA
IN
FINE
FQ
CH
DENTI
EL
IDEM
DPD
DD


EX
EXGL
INGL
DA
DPA
IN
FINE
FQ
CH
DENTI
EL
IDEM
DD


RIB
TRA
CMP
EX
EXGL
INGL
DA
DPA
IN
FINE
FQ
CH
DENTI
EL
IDEM
DPD
DD

```
RUN
APILES

     SAI LE REGOLE DEL GIOCO ?
     RISPONDI SI O NO ?NO

     GLI OGGETTI DEL GIOCO SONO N PEDINE

     (1<=N<=9); DA PORSI SU 3 FILE

     INIZIALMENTE SONO POSTE SULLA FILA

     PIU' IN BASSO. SI POSSONO SPOSTARE

     PEDINE SOLO DA UNA FILA PIU' IN BASSO

     AD UNA PIU' IN ALTO PURCHE' NELLE FILE

     RIMANGA UN'ORDINAMENTO NON CRESCENTE

     IN QUANTITA' DI PEDINE.

          0                            00
          00    123                    0     213
          000                          000

      POSIZIONE CORRETTA          POSIZIONE ERRATA

     CON QUANTE PEDINE VUOI GIOCARE ?6

 *** QUESTO PROGRAMMA APPRENDE GIOCANDO ***

     IL GIOCO HA  7   POSIZIONI POSSIBILI

     CHE SONO :
 1    2     3     4     5     6     7

006  015   024   033   114   123   222

     SE VUOI SMETTERE DAI 999

     LA MATRICE DELLE CONNESSIONI DEL GIOCO E' :

     DA  1     A  2     3     4     0     0

     DA  2     A  3     4     5     0     0

     DA  3     A  4     6     7     5     0

     DA  4     A  6     0     0     0     0

     DA  5     A  6     0     0     0     0

     DA  6     A  7     0     0     0     0

     DA  7     A  0     0     0     0     0
```

```
     IL CAMMINO MINIMO HA VERTICI

      1     3     7

     APPRENDIMENTO ALLA PARTITA   0
2 3 4 2 3 4 2 3 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3 4 5 3 4 5 3 4 5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
4 6 7 5 4 6 7 5 4 6 7 5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
6 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
6 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
7 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

     A TE LA MOSSA  ?006


     000000

     LA MIA MOSSA E'  024


     00
     0000

     A TE LA MOSSA  ?222
     HAI VINTO

     00
     00
     00

     APPRENDIMENTO ALLA PARTITA   1
2 4 2 3 4 2 3 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3 4 5 3 4 5 3 4 5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
4 6 7 5 4 6 7 5 4 6 7 5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
6 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
6 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
7 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

     A TE LA MOSSA  ?
```

FIG. 5