

Ultrametricity and energy landscape in photonic spin glasses

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Hierarchy is the architecture of complex systems. While Euclidean metric describes homogeneous and static systems, ultrametric geometry is for heterogeneous ensembles, hierarchic relations, and dynamic contexts. Applications spread across all fields: communication technologies, information retrieval, clusterwise regression, analysis, and synthesis of narratives and emotions. The paradigmatic model describing complex systems is the spin glass theory, according to which the topology of the many metastable energy states is intrinsically ultrametric, as a consequence of the breaking of their symmetry. Despite the importance, a real experimental observation of ultrametricity in spin glasses with a large number of spins has not yet been reported, due to the difficulty of finding a physical system in which the order parameter can be easily obtained from direct measurements. Random lasers have recently been proposed as excellent photonic counterparts of spin glasses with robust replica symmetry breaking features. Here, we report the ultrametric structure of the replica space with clustered domains in random lasers, as predicted in the Parisi ansatz. We show that the number of states forming isosceles triangles in the metric space increases through the laser threshold giving rise to the construction of a genealogical tree. Moreover, from the hierarchical topology of the states, we obtain a direct observation of the complex energy landscape with evident breaking of ergodicity in the glassy regime. These results mark a key advance in complexity science and invigorate current efforts to harness physical systems for the realization of optical computers and quantum optimization processors.

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I. INTRODUCTION

Ultrametricity is a mathematical concept that dates back to the late 19th century with the introduction of the topology of the so-called p -adic numbers [1,2]. The basic concept is as follows: Given three points A , B , and C , the distance d between them obeys the ultrametric triangular inequality $d(A, C) \leq \max\{d(A, B), d(B, C)\}$. This inequality is stronger than the triangular one describing the usual metric of Archimedean space $d(A, C) \leq \{d(A, B) + d(B, C)\}$. In the ultrametric space, all triangles are either equilateral or isosceles with the third side shorter than the two equal ones [3].

The ultrametric topology is typical of hierarchical trees. For this reason, ultrametricity first appeared in taxonomy [4]. The evolutionary relationships of biological species relying on their genetic features are a phylogenetic tree, which shows the history of connections of species, or taxa [5].

Besides biological classification, ultrametricity came into dataset analysis in the 1960s. It has a central and powerful role in the evaluation of big data clusterability, a topic extremely relevant in deep and complex neural networks within machine learning [6].

In physics, ultrametricity made its entry in the mean-field theory of spin glasses (SGs), which is the fundamental theory of order-to-disorder transitions [7], and recently, significant progress in the mathematical study of this concept has been presented [8]. In spin glasses, the coexistence of disorder and frustration is responsible for a complex energy landscape, with many metastable minima that are pure states of the out-of-equilibrium system. The mathematical transcription of the existence of many metastable states is the breaking of the symmetry of replica, identical copies of the same disordered system. Physical states and replica have the same order parameter and in the low-temperature SG phase replica hierarchy appears [9].

The overlap matrix $q_{\alpha\beta}$ quantifies how similar two replicas α and β are. In the glassy phase, the elements $q_{\alpha\beta}$ fluctuate widely from sample to sample, producing a probability distribution with a rich, nontrivial structure. This behavior is a hallmark of ergodicity breaking in the thermodynamic limit [9]. In simple terms, the system can no longer explore all of its configuration space, and instead becomes confined within many long-lived metastable states. To explain this complex

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landscape, Giorgio Parisi—later awarded the Nobel Prize—introduced a remarkably powerful ansatz for the structure of the overlap matrix [10]. His construction breaks the global permutation symmetry among replicas and encodes the existence of an entire hierarchy of pure states. This hierarchical organization is what allows the replica approach to correctly describe the rugged free-energy landscape of spin glasses. A major breakthrough came when Mézard *et al.* [11] showed that the space of pure states has an intrinsically ultrametric structure. Ultrametricity arises naturally from the combined effects of quenched randomness and the law of large numbers. In this picture, pure states are arranged as leaves of a genealogical tree: Each leaf corresponds to a state, and the branching structure reflects how states group into clusters. The “distance” between two states is then defined by how far up the tree one must go to reach their most recent common ancestor—i.e., by the number of generations separating them. This treelike geometry captures the deep hierarchical organization characteristic of mean-field spin glasses.

Ultrametricity in spin glasses comes directly from the calculation of the three-state overlap distribution $P(q_{\alpha\beta}, q_{\beta\gamma}, q_{\alpha\gamma})$. For pure states α, β , and γ , the three-replica overlap distribution of spin glass phase has been calculated [11] as

$$\begin{aligned} P(q_{\alpha\beta}, q_{\beta\gamma}, q_{\alpha\gamma}) &= 1/2 P(q_{\alpha\beta}) x(q_{\alpha\beta}) \delta(q_{\alpha\beta} - q_{\beta\gamma}) \delta(q_{\beta\gamma} - q_{\alpha\gamma}) \\ &+ 1/2 [P(q_{\alpha\beta}) P(q_{\beta\gamma}) \theta(q_{\alpha\beta} - q_{\beta\gamma}) \delta(q_{\beta\gamma} - q_{\alpha\gamma}) \\ &+ \text{permutations}], \end{aligned} \quad (1)$$

where $\theta(x)$ is the Heaviside theta function and $x(q) = \int_{-1}^q P(q') dq'$. Assuming without loss of generality that $q_{\alpha\beta} \geq q_{\beta\gamma} \geq q_{\alpha\gamma}$, Eq. (1) implies that $P(q_{\alpha\beta}, q_{\beta\gamma}, q_{\alpha\gamma})$ is nonzero only for

$$q_{\alpha\beta} \geq q_{\beta\gamma} = q_{\alpha\gamma}, \quad (2)$$

which is the ultrametric condition for overlaps.

The ultrametric structure of the energy states in SGs has been widely investigated in numerical simulations of Sherrington-Kirkpatrick and Edwards-Anderson models. It has been shown that the finite size of the system strongly compromises the occurrence of ultrametricity. Deviations from the Ghirlanda-Guerra predictions of ultrametricity are shown to get smaller as system size increases [12–18].

In addition, the ultrametric structure of the energy landscape has been reported in numerical simulations of jamming systems [19,20], providing direct evidence for the existence of the marginal Gardner phase. This is the glassy phase with infinitely broken replica symmetry and each subbasin of the energy landscape is itself broken up into many subbasins with a hierarchical structure. Contrarily, a stable glass phase is characterized by the single replica symmetry breaking (RSB) solution reflecting distinct, infinitely long-lived energy basins. Refs. [19] and [20], the minima of the complex landscape are calculated, and the structure of the distance metric shows that the energy landscape of overjammed three-dimensional configurations becomes hierarchical and ultrametric in the thermodynamic limit. Experimental evidence of the existence of the marginal Gardner phase is reported in colloidal and

granular materials [21,22]. The authors measured the mean square displacement at short timescales and length scales of colloidal particles by increasing density and of granulars under vibration. They obtained the phase diagram and showed clearly the transition from stable glass to marginal glass during jamming.

Moreover, experiments on the study of aging phenomena have been reported showing the hierarchical organization of the metastable states as a function of temperature [23]. This picture is reminiscent of the hierarchical organization of the mean-field pure states as a function of their overlap and the results were extrapolated over very extended range of times compared to the measured timescales [24]. The hierarchy of metastable states has also been described in terms of the hierarchy of measured correlation lengths even though with many limitations and assumptions [25]. In addition, an ultrametric arrangement for spin glass states in magnetic alloys under magnetic field has been used to represent aging. It was shown that energy barrier heights increase linearly with the Hamming distance between the states of the system during the aging time [26,27]. In all these reports, the off-equilibrium dynamics of disordered magnetic alloys gave only indirect evidence of ultrametricity in experimental spin glasses. Very recently, RSB and ultrametricity have been shown in a vector of eight quantum-optical spins [28,29].

So far, the experimental proof of ultrametricity and the direct demonstration of the hierarchical organization of replica with broken symmetry in a system with many interacting spins are lacking.

Recently, several concepts related to SG theory have been successfully applied in the field of random photonics [30], specifically for random lasers (RLs) [31–41], and nonlinear waves [42]. The experiments provide direct access to the complex configuration of the replicas and their fluctuations. For the first time and after 40 years, the distribution of the Parisi overlap $q_{\alpha\beta}$ has been obtained from random laser measurements and RSB has been experimentally proven [31].

Here, we demonstrate the ultrametric structure of the RL overlap matrix that emerges above the laser threshold when the system enters the glassy regime. Our approach is completely different from the proposed ones, because it is based on the direct measurement of the overlap matrix and the distance between replicas. We report the progressive clustering through the order-disorder transition reproducing experimentally the Parisi ansatz formulated for spin glasses [10,11]. The related dendrogram evidences the hierarchical topology of the observed replicas proving the predicted universal behavior in the SG theory. The representation of data by dendrograms has been shown theoretically to be a powerful tool for the demonstration of hierarchic clustering in nonergodic systems [43].

Indeed, the topology of the energy landscape of a complex system can be obtained from the hierarchical clustering of the states. Different representations of the energy landscape have been reported in the literature. They include funnels, clusters of minima, tendrils, and disconnective graphs and are widely used in the fields of chemistry and biology [44,45]. They describe the general shape and overall connectivity that characterize rugged energy surfaces. An open experimental challenge is achieving a direct measurement of the topology

of the replica landscape. Rugged funnel shapes of phase space energy landscape in spin glass systems have been numerically obtained, where a hierarchical structure of energy scales is found at the different levels of the barrier tree, with a single long branch (the global minimum) and many shorter branches separating from it [19,46,47]. Hierarchical clustering leads to ultrametric energy landscape. The local minima in the nonhierarchical clustering system result from the original zero energy mode of the classical energy Hamiltonian and the energy minima are clustered into different branches, following the topology of the replica distances.

From the complex ultrametric structure of the overlap matrix, we achieve the experimental observation of the energy landscape with evident breaking of ergodicity in the RL glassy phase.

A section is devoted to the demonstration of disorder-averaged overlap distribution $P(q)$. Because $P(q)$ is non-self-averaging and exhibits persistent sample-to-sample fluctuations even in the thermodynamic limit, the ultrametric structure can only be accessed through disorder averages or statistical analyses across realizations. This is the first time that such an average has been computed experimentally in the context of random lasers.

In addition, we perform a graph representation of RSB and we calculate the connectivity degree of replica with broken symmetry, meaning with high overlap. We find that about 50% of the replica are linked (overlapped) at the transition point, reaching a saturated value of 60% in the glassy regime.

II. RESULTS AND DISCUSSION

A. RSB from overlap graphs

RLs [48–52] are laser sources where the optical feedback for stimulated emission is the multiple scattering of light. RLs differ from conventional lasers with defined cavities. Our RL is a compact rhodamine 6G-doped silica powder fabricated by a conventional sol-gel synthesis at room temperature. Details on fabrication and standard characterization are reported in Appendix A.

The RL complex material has a broad fluorescence emission centered at 590 nm at low pumping and incoherent laser emission when increasing pumping above threshold. The transition to the laser regime at 0.5 mJ is evidenced by intense fluence and spectral narrowing, as shown in Appendix B.

When the stimulated emission occurs, we observe significant spectral fluctuations [Figs. 13(b) and 13(c)] from one laser shot to another under identical conditions. In the framework of the SG theory, each spectrum is a different configuration of the same system with quenched disorder, i.e., a replica. In each replica, a large number of electromagnetic modes coexist and their interaction grows by increasing the injected energy. At high pumping, modes compete in energy similarly to magnetic spins at low temperature.

The mode energy at wavelength λ is given by the spectral intensity $I(\lambda)$ [31,53,54]. The Parisi overlap $q_{\alpha\beta}$ between two replicas α and β is

$$q_{\alpha\beta} = \frac{\sum_{\lambda=1}^N \Delta_{\alpha}(\lambda) \Delta_{\beta}(\lambda)}{\sqrt{\sum_{\lambda=1}^N \Delta_{\alpha}^2(\lambda) \sum_{\lambda=1}^N \Delta_{\beta}^2(\lambda)}}. \quad (3)$$

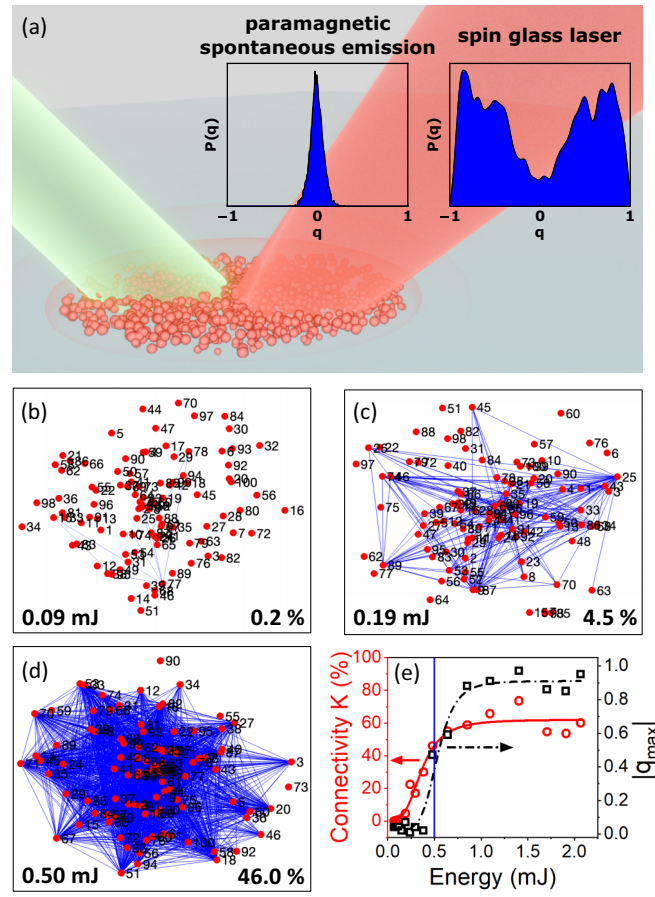


FIG. 1. Graph representation of the replica symmetry breaking. (a) Random laser at low pumping and in the paramagnetic regime and at high pumping with replica symmetry breaking in spin glass regime. Our RL is made of dyed powder pumped by an external green laser and with red emission. (b)–(d) Graphs with connection between 100 replicas when varying the pumping energy. Edges connect states with overlap $|q| \geq 0.5$. (e) Connectivity number K (black squares) and $|q_{\max}|$ (red circles) vs pumping energy. The RSB threshold 0.5 mJ is defined for $|q_{\max}| = 0.5$ and the saturation of K . The value $|q_{\max}|$ is defined as the most probable $|q|$ at various pumping energy [31] and is obtained from the overlap distribution $P(q)$ as demonstrated in Appendix B. The laser threshold corresponds to the replica symmetry breaking point of $|q_{\max}| = 0.5$ [38]. Above threshold, the connectivity K reaches saturation.

In Eq. (3), the variation

$$\Delta_{\alpha}(\lambda) = I_{\alpha}(\lambda) - \bar{I}(\lambda) \quad (4)$$

is, for a replica α , the difference between each spin $I(\lambda)$ and the average value $\bar{I}(\lambda) = 1/N_s \sum_{\alpha=1}^{N_s} I_{\alpha}(\lambda)$, calculated over N_s replicas (spectra).

The overlap matrix $q_{\alpha\beta}$ is the correlation between the fluctuations of replica pairs. When the replica fluctuations become uncorrelated and lose their similarity, the system enters a glassy disordered regime. Such replica symmetry breaking is evidenced by a nontrivial shaped probability distribution of the overlap [Figs. 1(a), 13(e), and 13(f)].

Here, we give a graph representation of RSB in replica network obtained from the RL spectra.

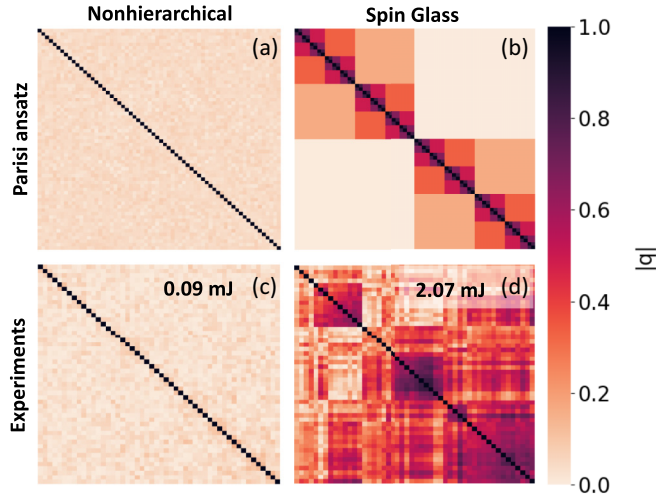


FIG. 2. Hierarchical clustering in spin glass RL reproducing Parisi overlap ansatz. Theoretical Parisi ansatz of overlap matrix in paramagnetic, nonhierarchical regime [panel (a)] and in SG hierarchical regime [panel (b)]. Overlap matrix $|q_{\alpha\beta}|$: at low input energy 0.09 mJ photonic analogous of paramagnetic phase [panel (c)]; and at high energy 2.07 mJ spin glass RL phase [panel (d)]. In the matrices, black pixels on the diagonal correspond to the self-overlap $|q_{\alpha\alpha}| = 1$ ($\alpha = \beta$), and the color of the off-diagonal pixel changes from dark purple to white as $|q|$ decreases. The order of the states in the experiments is given by applying the average linkage agglomerative clustering algorithm.

Figures 1(b)–1(d) show a set of 100 points randomly distributed in a spherical geometry. They represent 100 replicas, i.e., single-shot measurements, obtained at the identical pumping energy. Each displayed edge is the connection between two replicas with $|q| \geq |q_{\max}| = 0.5$. The value $|q_{\max}|$ is defined as the most probable $|q|$ at various pumping energy [31] and is obtained from the overlap distribution $P(q)$ as demonstrated in Appendix B. The laser threshold corresponds to the replica symmetry breaking point of $|q_{\max}| = 0.5$ [38].

At low pumping energy 0.09 mJ [Fig. 1(b)], only ten connections are obtained, i.e., 0.2% of the total of the $[N_s(N_s - 1)/2]$ possible $|q|$ values. As the energy increases, the number of connections increases: In Fig. 1(c), at 0.19 mJ, we have 224 connections (4.5%), at 0.50 mJ, 2279 connections (46.0%) [Fig. 1(d)].

In Fig. 1(e), we plot the connectivity K (black squares), that is, the percentage of graph edges, and $|q_{\max}|$ (red circles) versus pumping energy.

The trends in Fig. 1(e) show that the percentage of connected replica at RSB point, i.e., the laser threshold of 0.5 mJ, almost reaches its maximum saturation value. There is not a sharp transition of K ; the amount of connected replica with broken symmetry grows through the glass transition before saturation.

With these results, we propose the graph method as a different way to show RSB and RL activity.

B. Ultrametricity

The hierarchical clustering of the states can be visualized by building the matrix of the overlap q . In Figs. 2(a) and 2(b),

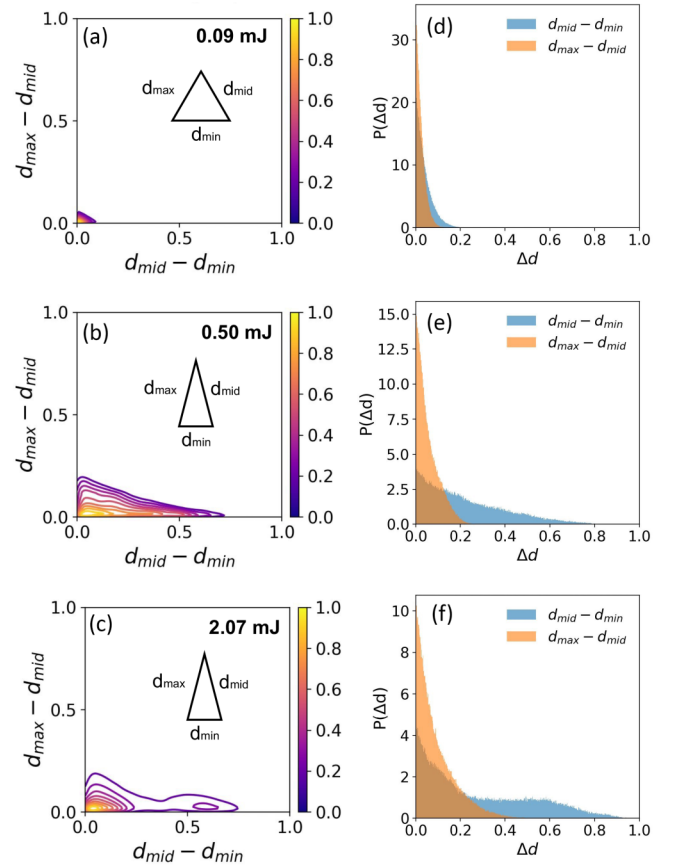


FIG. 3. SG ultrametricity from experimental data. Probability density of triangle side lengths, in the plane $(d_{\text{mid}} - d_{\text{min}}, d_{\text{max}} - d_{\text{mid}})$ at (a) 0.09 mJ, (b) 0.50 mJ, and (c) 2.07 mJ. Correspondent histograms of $[d_{\text{mid}} - d_{\text{min}}]$ and $[d_{\text{max}} - d_{\text{mid}}]$ at (d) 0.09 mJ, (e) 0.50 mJ, and (f) 2.07 mJ.

we show the well-known theoretical Parisi ansatz [10]. In the paramagnetic regime [panel (a)], all $|q_{\alpha\beta}|$ are close to 0, while in the SG [panel (b)], the formation of clusters with $|q_{\alpha\beta}|$ close to 1 values is an intrinsic characteristic of the system.

To obtain the photonic experimental counterpart, we organize the 100×100 matrix of overlap $q_{\alpha\beta}$, for 100 single-shot measurements, by using the average linkage agglomerative clustering algorithm; details are provided in Appendix C. Three representative matrices below, at and above laser threshold, in Fig. 15, clearly show the progressive cluster formation through the glass transition. For better visualization, a close-up of the results is reported in Figs. 2(c) and 2(d). In Fig. 2(d), we give evidence of the cluster structure at high pumping energy where the system is in the glassy regime. Contrarily, in Fig. 2(c) at low energy the structure is flat with all overlap values close to 0.

This is the first experimental spin glass replica ansatz obtained directly from the analysis of the emission spectra.

The ultrametric topology of the spin states is given by the demonstration of the ultrametric triangular inequality between their distances.

To this aim, the Hamming distance d [55,56] is calculated from the experiments, as $d_{\alpha\beta} = (1 - |q_{\alpha\beta}|)$ [55], for each pair of states α and β . By substituting this relation in Eq. (2), we

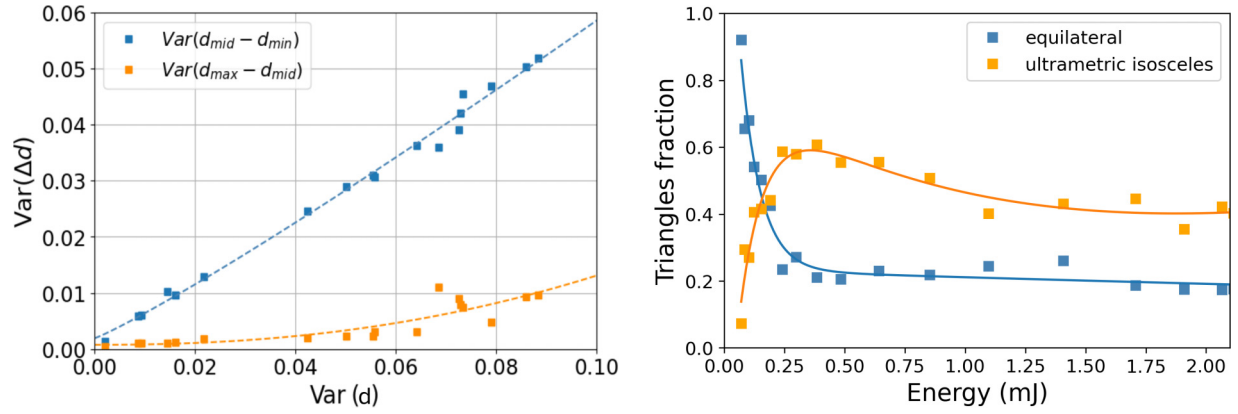


FIG. 4. SG ultrametricity from experimental data. (a) Relative variance of the two lengths $[d_{\text{mid}} - d_{\text{min}}]$ and $[d_{\text{max}} - d_{\text{mid}}]$ vs the variance of all calculated Hamming distances $\text{Var}(d)$. (b) Fraction of the triangles population calculated from the histograms at different input energies.

obtain $d_{\alpha\gamma} \leq d_{\alpha\beta} = d_{\beta\gamma}$ for the distances [55,56]. This is the ultrametric triangular inequality, which results to be intrinsic to the SG theory.

To demonstrate ultrametricity in our photonic SG, we calculate the distribution of triangles with vertices given by three random states α, β, γ and sides $d_{\alpha\beta}, d_{\alpha\gamma}, d_{\beta\gamma}$ [55].

From the experiments, we obtain $[N_s(N_s - 1)(N_s - 2)]/3!$ triplets of $d_{\alpha\gamma}$, with $N_s = 100$ single-shot spectra. For any triplet, the Hamming distances between the states, forming three sides of a triangle, are sorted as $0 \leq d_{\text{min}} \leq d_{\text{mid}} \leq d_{\text{max}} \leq 1$. The objective is to distinguish the so-called trivial ultrametricity, in which all the triangles are equilateral, and nontrivial ultrametricity, in which a fraction of triangles is isosceles with $d_{\text{min}} \leq d_{\text{mid}} = d_{\text{max}}$.

In Figs. 3(a)–3(c), we show the contour plots of the probability density of the triangles in the plane $(d_{\text{mid}} - d_{\text{min}}, d_{\text{max}} - d_{\text{mid}})$ for three different energies.

At low energy, in Fig. 3(a), all points are grouped near the origin, which corresponds to equilateral triangles with $d_{\text{mid}} \simeq d_{\text{min}} \simeq d_{\text{max}}$. As the energy increases, the points significantly spread toward the horizontal axis and slightly toward the vertical one, as in Figs. 3(b) and 3(c).

To quantify the variations, we report the histograms $P(\Delta d)$ of the differences between side lengths $\Delta d = d_{\text{mid}} - d_{\text{min}}$ (blue) and $\Delta d = d_{\text{max}} - d_{\text{mid}}$ (orange) in Figs. 3(d)–3(f). It is evident that the distribution of $\Delta d = d_{\text{max}} - d_{\text{mid}}$ is much more concentrated close to 0, while the distribution of $\Delta d = d_{\text{mid}} - d_{\text{min}}$ is spread on a larger scale in the glassy regime, in accordance with the numerical results reported by Parisi and co-workers in Ref. [57]. The nonfully ultrametric behavior given by the marginal spread of $[d_{\text{max}} - d_{\text{mid}}]$ was shown to be related to the finite volume of the system [16,57–59].

As further proof, we report in Fig. 4(a) the normalized variance of the two distances $\text{Var}(\Delta d)$ versus the variance of all calculated Hamming distances $\text{Var}(d)$ [57]. The growth of the spread of $\Delta d = d_{\text{mid}} - d_{\text{min}}$ (blue) is much faster than the slight growth of $\Delta d = d_{\text{max}} - d_{\text{mid}}$ (orange). At low $\text{Var}(d)$, the difference between the equal sides of isosceles triangles is negligible and grows slightly for higher $\text{Var}(d)$ due to the finite size of the system [57].

Moreover, in Fig. 4(b), the fraction of equilateral and isosceles triangles versus the input energy is reported, showing the increase of replica ultrametricity in the glassy regime.

These results demonstrate that more isosceles triangles are formed with two sides $d_{\text{mid}} \simeq d_{\text{max}}$ and larger than the third one d_{min} and give the experimental proof of replica ultrametricity in spin glasses.

It is interesting to note that in Fig. 4(b) all the variations occur before the threshold of 0.5 mJ and that the variables after the transition, in the glassy regime, go to saturation. This agrees with the results obtained from the graphs and shown in Fig. 1(e).

C. Topological energy landscape

In the previous paragraph, the hierarchical cluster structure and the ultrametric architecture of spin glass replicas have been evidenced. Here, we obtain the topology of the SG energy landscape directly from the clustering of the photonic spin states.

The average linkage agglomerative clustering algorithm, used in the previous section and described in Appendix C, furnishes a treelike structure of all the distances $d_{\alpha\beta}$ between spin states called dendrogram. We depict in Figs. 5(a)–5(c) three examples for three different input energies showing the complex structure below, at, and above the laser threshold. The ordering of the branches of the tree provides a natural ordering of the states. Longer branches correspond to larger distances between the connecting points. The reported results clearly illustrate the gradual occurrence of hierarchical relationship between the states from the paramagnetic regime [Fig. 5(a)] toward the random laser SG phase [Fig. 5(c)]. At threshold [Fig. 5(b)], replicas start to form the first large clusters.

To quantify the clustering formation toward glass transition from the dendrograms, we calculate the clustering entropy as in Ref. [60]. For a given dendrogram height h , the hierarchical tree is cut horizontally, resulting in $N_c(h)$ clusters. Each cluster k contains $N_r^k(h)$ replicas. The fraction of replicas belonging to cluster k is therefore $p_k(h) = N_r^k(h)/N$, where N is the total number of replicas. The clustering entropy at height h is then defined as

$$S(h) = - \sum_{k=1}^{N_c(h)} p_k(h) \log_{10} p_k(h). \quad (5)$$

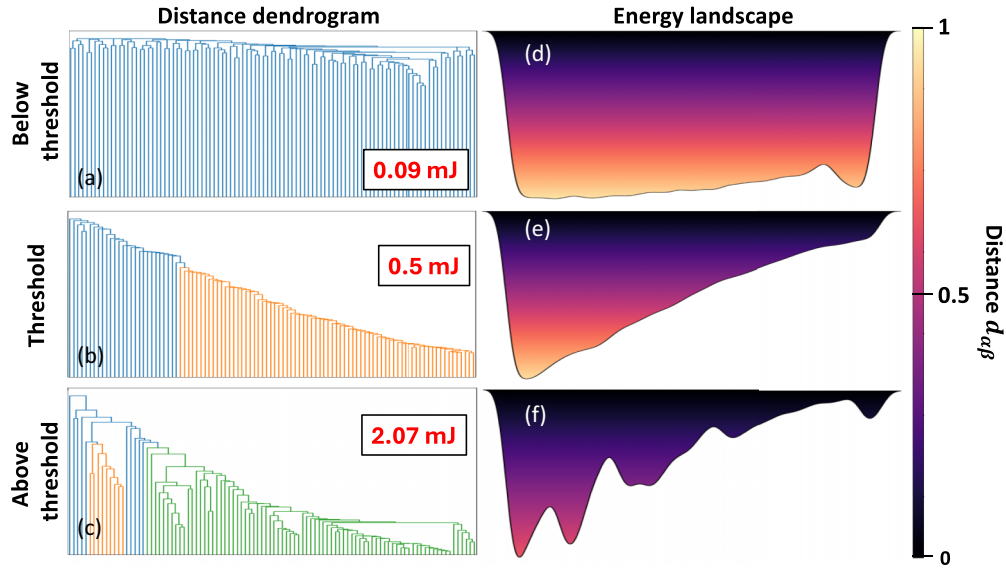


FIG. 5. Experimental energy landscape representation across the phase transition. (a)–(c) Distance $d_{\alpha\gamma}$ dendrograms below, at, and above the RL threshold. Dendrograms are calculated by sorting the Hamming distance of the states from the experiments with the average linkage agglomerative clustering algorithm. (d)–(f) Energy landscape profiles obtained directly from the clustering dendrograms of the left panel, respectively. See Appendix D for details. Input energy is 0.09 mJ [panels (a) and (d)], 0.5 mJ [panels (b) and (e)], and 2.07 mJ [panels (c) and (f)].

We show in Fig. 6 the clustering entropy $S(h)$, computed from the dendrograms, as a function of the normalized merge height $h/h_{\max}$, for different pumping regimes. Here, the height h represents the intercluster distance at which clusters merge, thus acting as a resolution parameter for the hierarchical organization of the replicas. In the average linkage scheme, the value of h is computed by explicitly averaging all pairwise replica-level distances $d_{\alpha\beta}$ between replicas belonging to the two clusters [60]. Above threshold (0.5 mJ), the curves exhibit a slower and more structured decay, characterized by the evident presence of plateaus in the curves. In contrast, below threshold the curves are smoother, and the presence of clusters is not distinguishable. Such behavior is consistent with the development of a complex ultrametric organization, as expected in a replica-symmetry-broken phase.

In Figs. 5(d)–5(f), we report the measured energy landscape from the distance dendrograms of Figs. 5(a)–5(c), considering the energy barrier values proportional to the distances. Details are described in Appendix D. For low pumping energy [Fig. 5(d)], the clustering dendrogram gives a flat nonhierarchical energy landscape and, as the input pumping increases, the landscape shows a rugged funnel-type topology as expected for spin glass systems [Figs. 5(e) and 5(f)]. Barriers and local minima are evident in the high-energy glassy regime [Fig. 5(f)].

To the best of our knowledge, this is the first experimental energy landscape representation of a complex SG system. Figure 5 shows the transition from ergodic to nonergodic by changing a measurable physical system, emblematic of theoretical spin ensemble.

We may notice that the progressive cluster formation and ultrametric replica topology toward the glassy regime are robust with respect to the choice of the distance d ; the main features are retained even for $d = 1 - q$ (see Sec. I of the Supplemental Material [61]).

D. Disorder average

In mean-field spin glasses, the complex organization of pure states is reflected in the structure of the overlap distribution function $P(q)$, which provides statistical information about the mutual overlap between equilibrium configurations. In particular, the hierarchical and ultrametric arrangement of states predicted by replica symmetry breaking manifests itself through a nontrivial form of $P(q)$. An important distinction must be made between thermodynamic observables that are self-averaging and those that are not. The free energy is a self-averaging quantity: In the thermodynamic limit, its value for a single disorder realization converges to the average over disorder. In contrast, the overlap distribution function $P(q)$ does not self-average. Even in the thermodynamic limit, its detailed shape depends on the specific realization of the quenched disorder and exhibits persistent sample-to-sample

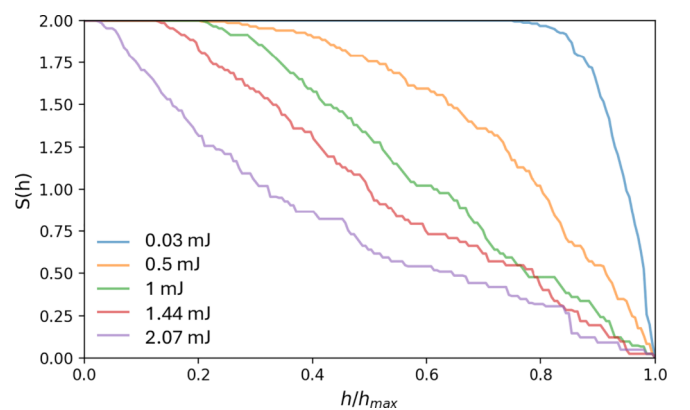


FIG. 6. Clustering entropy. The clustering entropy $S(h)$ vs the normalized height $h/h_{\max}$ of the branches in the dendrograms for different input energies.

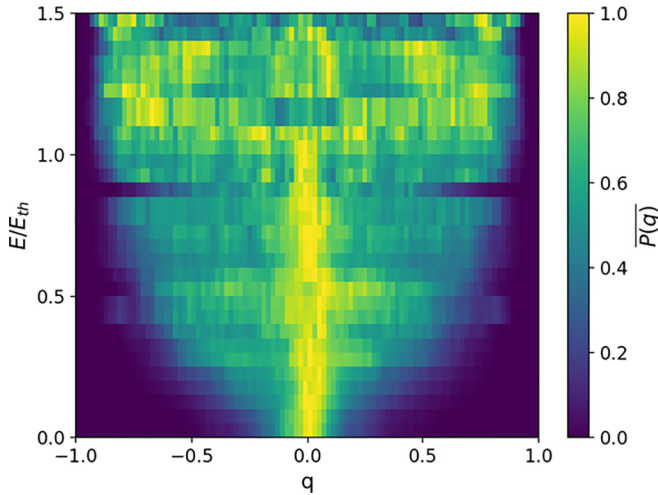


FIG. 7. Disorder-averaged overlap distribution. The average overlap distribution $\overline{P(q)}$ calculated over six different disorder realizations (samples) at different input energies on the vertical axis. E_{th} ranges from 0.3 to 2 mJ depending on the disordered sample, and is estimated as described in Appendix B.

fluctuations. Meaningful information about the underlying ultrametric structure can only be obtained by considering appropriate disorder averages or by studying quantities specifically designed to capture the statistical properties of $P(q)$ across different samples [7].

To this aim, we perform same type of measurements and analysis on six different samples, which are different realizations of the disorder. All of them show RSB above the energy threshold in the glassy regime.

We calculate the average overlap distribution $\overline{P(q)} = \sum_{j=1}^6 P_j(q)$, where $P_j(q)$ denotes the overlap distribution for a given realization J of the quenched disorder and the overline indicates the average over disorder realizations. The average $\overline{P(q)}$ is obtained by doing the histogram of all q values from different datasets with an appropriate binning. In Fig. 7, we report $\overline{P(q)}$ at growing normalized input energy E/E_{th} , with E_{th} the energy threshold measured for each dataset. The energy threshold of each sample is estimated as described in Appendix B. It is sample-specific and depends on the microscopic structure and the scattering strength.

The occurrence of replica symmetry breaking across the threshold energy is clearly evident. In Fig. 8, we present representative overlap profiles at three different energy

values. Continuous lines and symbols correspond to disorder-averaged quantities, whereas the shaded regions quantify the sample-to-sample fluctuations associated with different realizations of the quenched disorder, highlighting the non-self-averaging nature of the overlap distribution.

In a similar way, we calculate the disorder-averaged distribution $P(\Delta d) = \sum_{j=1}^6 (\Delta d)_j$ of the replica triangle edges $\Delta d = [d_{max} - d_{mid}]; [d_{mid} - d_{min}]$ for the demonstration of average ultrametricity. The behavior observed for a single sample in Fig. 3 is confirmed by the average over six datasets as shown in Fig. 9. As the energy E/E_{th} increases, two sides (d_{max} and d_{mid}) are nearly equal while the difference with the third (d_{min}) grows, indicating an increasing prevalence of isosceles triangles.

In Sec. II of the Supplemental Material [62], we show that the clustering entropy also fluctuates across different disorder realizations, while still preserving the common overall trend, that is, the emergence of the characteristic steplike shape, signature of cluster formation. In Fig. 10, we report the area of average entropy calculated over six samples, computed as $\int \overline{S(h)} dh$, as a function of the input energy. The mean entropy decreases as the hierarchical structure of the replicas develops and the states begin to form larger clusters.

III. CONCLUSIONS

In this work, we used random laser emission to demonstrate the ultrametric nature of the spin glass states. We present an approach for the investigation of the replica symmetry breaking based on the graph representation of the overlap parameter and we measure the profile of the energy landscape from the experimental SG overlaps.

Spin glass theories are the foundation of complex systems and Parisi replica trick is a fundamental paradigm: The many metastable states of a glass are treated as replicas and the overlap between replicas is identified as the order parameter. The distribution of the overlap defines the thermodynamics of the complex system. In out-of-equilibrium glassy regime, such distribution has a hierarchical ultrametric structure and it is signature of the breaking of the replica symmetry.

This fascinating and universal theory lacks direct experimental proof, as it is difficult to access the replica space of a large number of spins through experiments.

Random lasers offer the great advantage of measuring the overlap parameter directly from emission spectra and thus are known to be excellent examples of SGs.

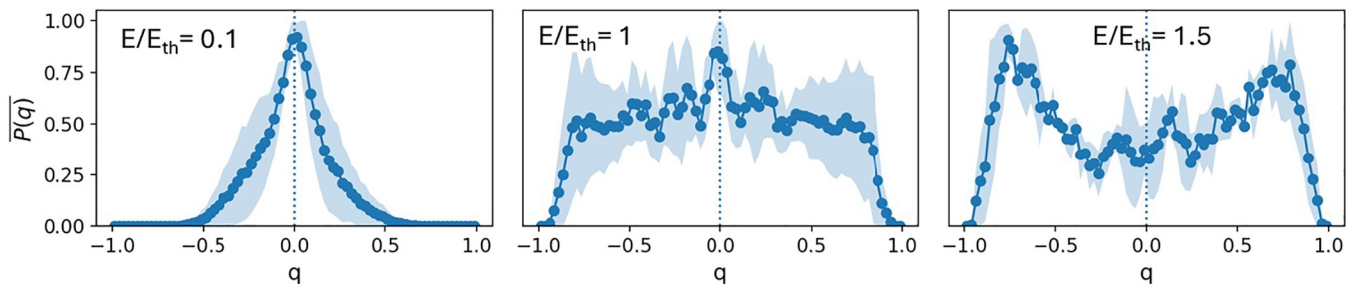


FIG. 8. Disorder-averaged overlap distribution. The average overlap distribution $\overline{P(q)}$ calculated over six different disorder realizations (samples) at three different input energies taken from Fig. 7.

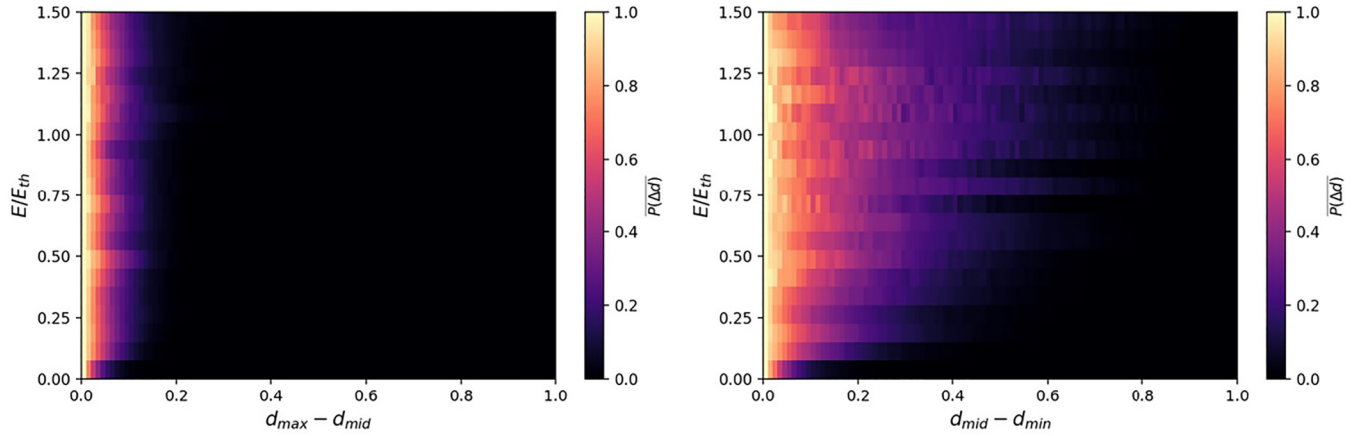


FIG. 9. Disorder-averaged overlap distribution. The average overlap distribution $\overline{P(q)}$ calculated over six different disorder realizations (samples) at three different input energies taken from Fig. 7. E_{th} ranges from 0.3 to 2 mJ depending on the disordered sample, and is estimated as described in Appendix B.

In our experiments, we measure the distance between the replicas from the overlap, we obtain the equivalent of the Parisi ansatz showing the clustering of the states, and we attain the topology of the complex energy landscape through the phase transition. In the ordered paramagnetic regime, one large basin is obtained, and above threshold in the disordered phase, many metastable minima emerge.

The most relevant result is the experimental evidence of replica ultrametricity in spin glasses. We demonstrate it by measuring the distribution of the replica triplets forming isosceles and equilateral triangles. This is a powerful way to assess the topology of the replica space and it has been reported only in numerical simulations.

Here, we show the increase of the number of isosceles triangles and the decrease of the equilateral ones at high energy as a signature of the ultrametric landscape of the states in the glassy regime.

Our results, in addition to advancing complexity science, contribute to the ongoing research into physical systems that has, in recent years, led to the development of optical computers and quantum annealers [63–67].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: SAMPLE PREPARATION AND CHARACTERIZATION

Silica xerogel powders were obtained by sol-gel synthesis performed at room temperature. Silica sol-gel synthesis was performed by the controlled hydrolysis of the alkoxide tetraethyl orthosilicate (TEOS), followed by hydrolysis and condensation reactions in an aqueous solution. In this work, the synthesis was acid catalyzed by the addition of nitric acid. At low pH levels, silica tends to form linear molecules that are occasionally cross-linked. These molecular chains

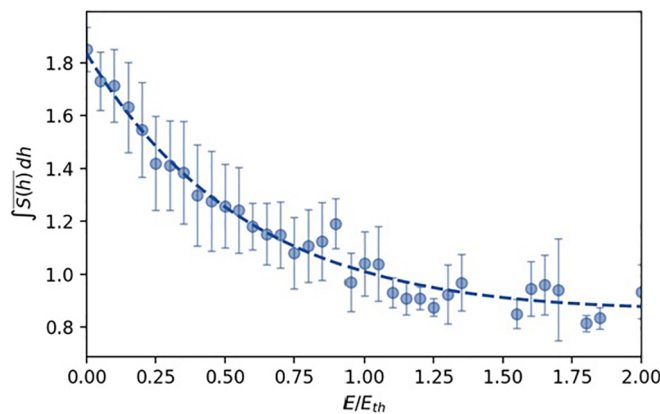


FIG. 10. Disorder-averaged clustering entropy area. The area of average $S(h)$ calculated over six different disorder realizations (samples) at different input energies.

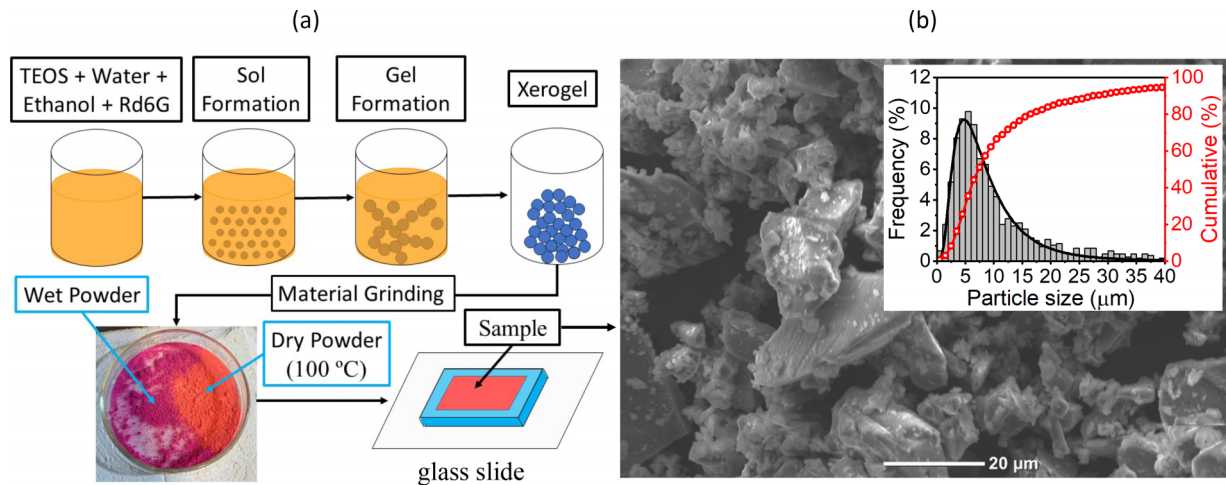


FIG. 11. Sample preparation and characterization. (a) Sol-gel process and sample preparation. (b) Powder morphology by SEM. Inset: size distribution of the powder grains.

become entangled and form additional branches, resulting in gelation [68]. Initially, 5.5 ml of TEOS and 8.5 ml of ethanol-rhodamine 6G solution at 10^{-3} mol/l are mixed. This solution stays under constant agitation for approximately 15 min. Then, 5.5 ml of deionized water, 0.15 ml of nitric acid, and 1.9 ml of dimethylformamide are added into the initial solution. The system is kept under stirring at room temperature for about 10 min for homogenization and formation of the initial sol. The sol is placed in drying oven (50°C) for two days, resulting in a dry silica xerogel.

The xerogel is then heat treated at 100°C for 1 h to remove the remaining solvents. Then, the solid material is minced using a mortar and pestle to obtain a homogeneous powder, which is carefully deposited in a sample chamber made of two microscope slides sealed by double-sided tape with a thickness of $250\text{ }\mu\text{m}$. Figure 11(a) summarizes the sol-gel process and sample preparation and Fig. 11(b) shows the powder morphology by scanning electron microscope (SEM). The powder presents a polydisperse particle size distribution, with an average size of about $5\text{ }\mu\text{m}$; it is possible to observe a fraction of larger particles, with dimensions up to $20\text{ }\mu\text{m}$ (top inset).

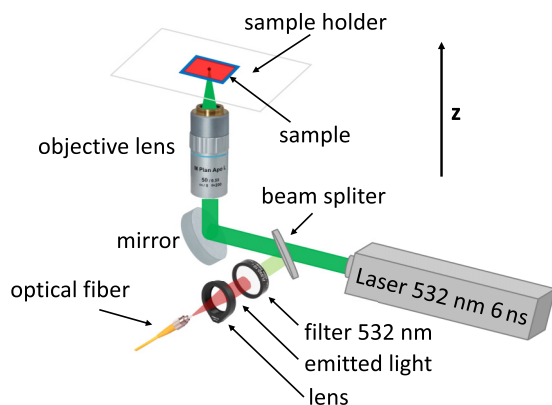


FIG. 12. Scheme of the random laser setup.

APPENDIX B: RL EXPERIMENTS

RL setup. A sketch of the optical setup is reported in Fig. 12.

The sample is pumped by a Continuum-Surelite frequency-doubled Q -switched Nd:YAG pulsed laser emitting at $\lambda = 532\text{ nm}$, with 10 Hz repetition rate, and 6 ns pulse duration. The pumping beam is focused on the sample by a microscope objective with numerical aperture of 0.55 and we use an illuminated area of about $500\text{ }\mu\text{m}$ of diameter. The scattered light emitted from the sample is collected from its front surface and focused into an optic fiber connected to an Andor Shamrock SR-303i-A spectrometer with a Czerny-Turner arrangement, 303 mm focal length and diffraction grating density of 1200 grooves/mm, equipped with electrically cooled iDus camera detector.

The acquisition is triggered by the pump laser flash lamp at 10 Hz repetition rate. The sample is pumped perpendicularly along the z direction and all reported spectra are taken from single pumping shots. A notch filter is positioned at the entrance of the fiber to remove the residual incident laser beam.

RL measurements. We illustrate in Figs. 13(a)–13(c) emissions of subsequent 100 shots taken at 0.09, 0.50, and 2.07 mJ, respectively. The spectra are characterized by a broad line emission, typical for rhodamine 6G molecules at low energy [Fig. 13(a)].

As the pumping energy increases, random laser line width reduction is observed, as shown in Figs. 13(b) and 13(c). The absence of spikes in the emission spectra is characteristic of nonresonant feedback in which light propagates in a diffusive regime. Moreover, the growth of spectral fluctuations with increasing pumping is evident.

The replica symmetry breaking behavior results are shown in Figs. 13(d)–13(f). We present three examples of the overlap distribution $P(q)$. At low pumping energy [Fig. 13(d)], most of the overlaps q are centered around the zero value, meaning that the emitted modes weakly interact. As the energy increases, the modes are coupled by the nonlinearity and q assumes all possible values in the range $[-1, 1]$. Such behavior characterizes the breaking of the replica symmetry.

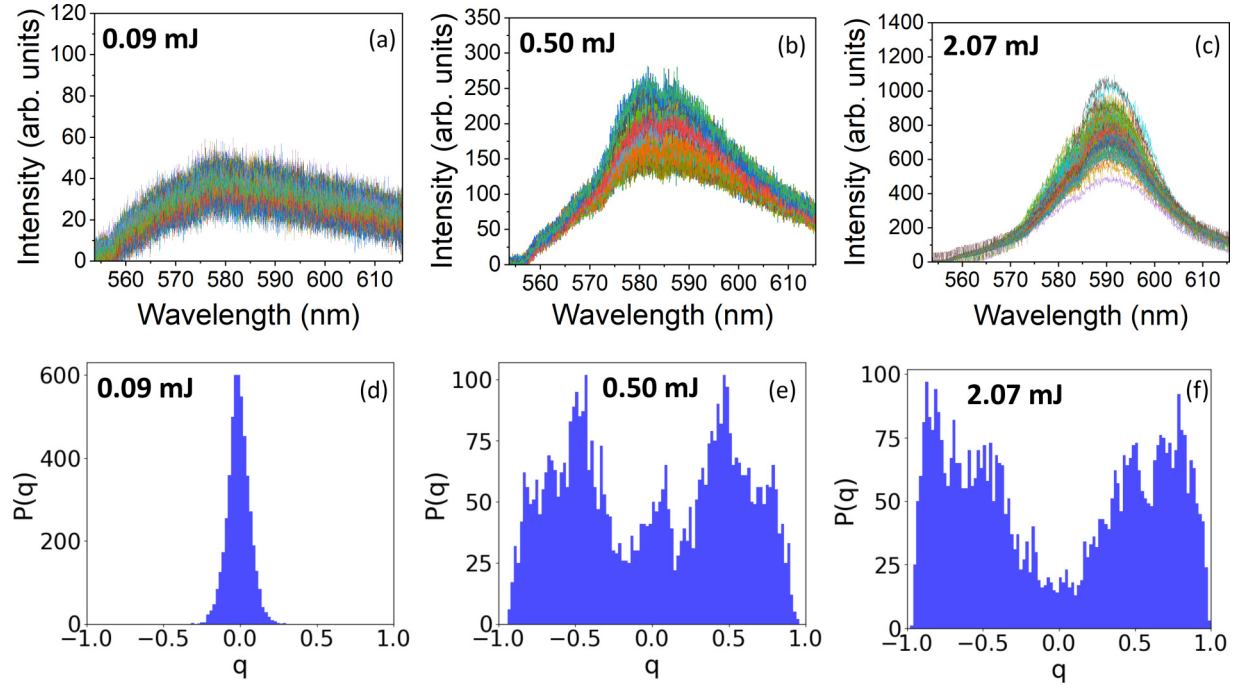


FIG. 13. Replica symmetry breaking from $P(q)$. (a)–(c) Three representative series of emission spectra (100 shots) at various input energies. (d)–(f) Overlap distribution $P(q)$ at 0.09, 0.50, and 2.07 mJ, respectively.

Figure 14(a) shows the growth of the mean intensity $\sum_{\lambda=1}^N \bar{I}(\lambda)/N$ over replicas and wavelength (black curve) and the respective variance (red curve) for increasing input energy. In both curves, a threshold from the spontaneous regime to stimulated emission at 0.5 mJ input energy is observed.

In Fig. 14(b), the full width at half maximum (FWHM) of the emission band (red curve) and the $|q_{\max}|$ (black curve) as a function of the input energy are depicted. $|q_{\max}|$ is the absolute value of the most probable overlap q [31]. Bandwidth narrowing from about 50 nm to about 25 nm (reduction of 50%) and a clear transition from $|q_{\max}| \simeq 0$ to $|q_{\max}| \simeq 1$ at 0.50 mJ are evidenced.

APPENDIX C: LINKAGE AGGLOMERATIVE CLUSTERING ALGORITHM

The linkage agglomerative clustering algorithm begins with a forest of clusters and each state belongs to its own

cluster. When two closest clusters s and t are combined into a single cluster u , they are removed from the forest, and u is added to the forest. The distance between two clusters is the average distance between all pairs of members of the clusters. This process continues, with the definition of intercluster distances given by the average of all the distances between elements of each cluster. When only one cluster remains in the forest, the algorithm stops, and this cluster becomes the root. At each interaction, the algorithm must update the distance matrix to reflect the distance of the newly formed cluster u with the remaining clusters in the forest [15,16].

We apply this algorithm to obtain the overlap matrices reported in Fig. 15, where the progressive cluster formation through the RL threshold (0.5 mJ) is demonstrated, as the experimental example of the Parisi ansatz.

To demonstrate the stability of the results with respect to the choice of clustering algorithm, we evaluate four different methods: average, complete, weighted, and single linkage.

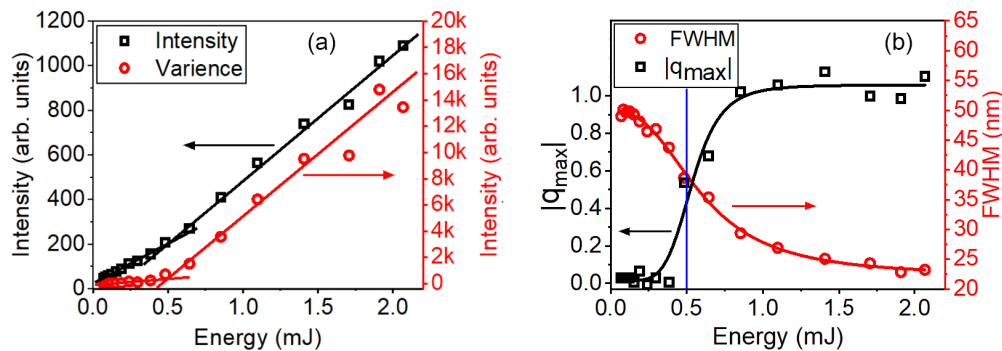


FIG. 14. RL threshold characterization. (a) Mean emission intensity (black curve) and variance (red curve) vs. pumping energy. (b) $|q_{\max}|$ (black curve) and FWHM (red curve) vs. pumping energy.

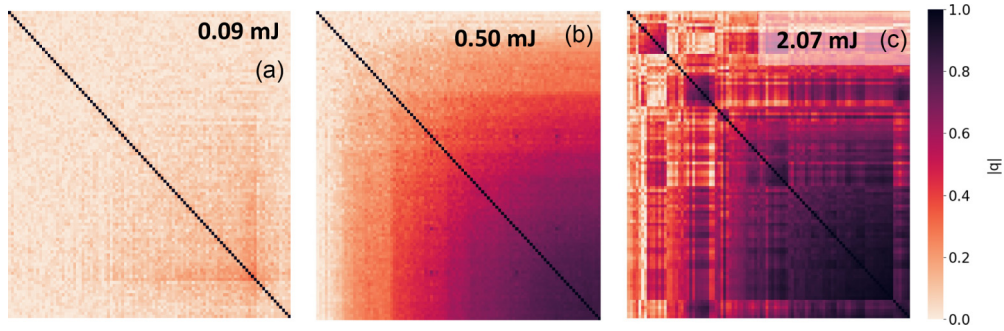


FIG. 15. Hierarchical clustering. 100×100 matrix of overlap q , for 100 single-shot measurements, obtained by using the average linkage agglomerative clustering algorithm, at different input energy of 0.09 mJ: below threshold [panel (a)], 0.50 mJ: at threshold [panel (b)], and 2.07 mJ: above threshold [panel (c)].

The consistency between the conceptually different agglomerative procedures is shown in Sec. III of the Supplemental Material [69] and demonstrates that the observed hierarchical organization and ultrametric features are stable with respect to the choice of clustering algorithm. We also assessed the cophenetic correlation coefficient [70], which reveals the highest values for the average clustering algorithm, independently of the pumping energy.

APPENDIX D: ENERGY LANDSCAPE CONSTRUCTION

We construct the schematic energy landscape based on the distance dendrogram, considering that the energy barrier

values are proportional to the distance $d_{\alpha\beta}$ between replicas and d_{st} between the generated clusters.

We show the procedure in Fig. 16 for the dendrogram obtained at 2.07 mJ injected energy, as an example.

In the dendrogram, the length of each vertical line is a distance between replica (or clusters of replica) and we regard them at the energy barrier between the considered replica (or clusters). To obtain a smooth landscape of the barriers and valleys, we replace the vertical lines with Gaussian curves and then we connect them with a smooth curve.

We first position on the horizontal axis of the energy landscape representation all the end points in the dendrogram and keeping the same order, as shown in Fig. 16(a). Their distance is arbitrary. Then, Gaussian curves are generated centered on each horizontal coordinate and with amplitude equal to the correspondent branch length (vertical coordinate). The sum of the Gaussian funnels gives the energy landscape profile.

The Gaussian width w is variable and we indicate w^* as the smallest significant width.

When w is small, the individual Gaussians are narrower and have less overlap. This results in sharper and more separate peaks in the sum, as depicted in Fig. 16(a), where $w = w^*$. As the width increases [Fig. 16(b)], the Gaussian funnels begin to overlap more significantly, creating a smoother and more continuous sum, where the cluster structure emerges. The generated image is filled with a color map showing the variation in amplitude (distance in the dendrogram). In this work, we choose $w = 20 w^*$ [Figs. 4(d)–4(f) and 16(b)], which provides a sufficiently high degree of clustering to clearly reveal a topology characterized by valleys and barriers in the glassy state. In Sec. IV of the Supplemental Material [71], other profiles with different values of w are reported. For $w < 20 w^*$, finer and less relevant minima appear within the main basins, resulting in noisier landscape profiles. In our construction, the parameter w controls the smoothness of the landscape profile and is therefore chosen primarily to improve visualization.

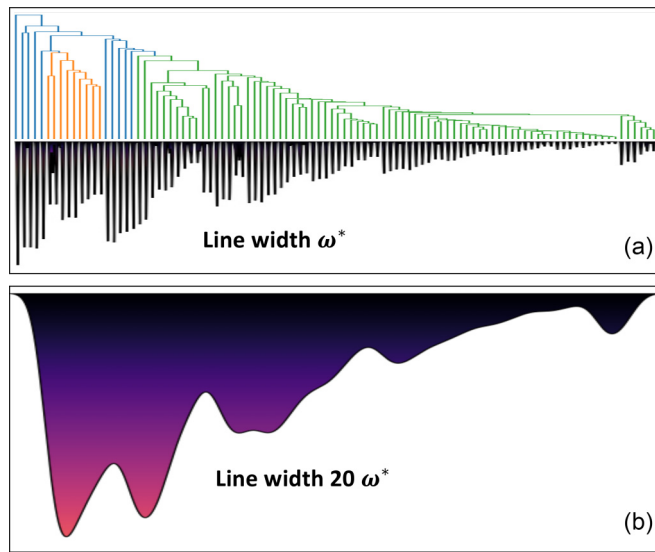


FIG. 16. Energy landscape from dendrogram. (a) Dendrogram at input energy 2.07 mJ, the same of Fig. 4, and correspondent array of lines with position at each end point of the dendrogram and height equal to the respective height in the dendrogram. Here, each line is a Gaussian curve with width w^* . (b) Same as panel (a) with Gaussian width $w = 20 w^*$. This is the one we consider as the topological energy landscape of the glassy RL.

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- [62] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/b3py-tyv8> for plots of clustering entropy $S(h)$ calculated for six disorder realizations (six samples), highlighting that the clustering entropy retains a strong dependence on the specific disorder realization, even though the transition to a glassy behavior with a hierarchical structure of replicas is common to all samples.
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