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OPTIMAL CONTROL OF A RATE-INDEPENDENT EVOLUTION EQUATION VIA VISCOUS REGULARIZATION

Ulisse Stefanelli

University of Vienna, Faculty of Mathematics
Oskar-Morgenstern-Platz 1, 1090 Wien, Austria and
Istituto di Matematica Applicata e Tecnologie Informatiche *E. Magenes* - CNR
via Ferrata 1, 27100 Pavia, Italy

Daniel Wachsmuth*

Institute of Mathematics, University of Würzburg
Emil-Fischer-Str. 40, 97074 Würzburg, Germany

Gerd Wachsmuth

Technische Universität Chemnitz, Faculty of Mathematics
09107 Chemnitz, Germany

Abstract. We study the optimal control of a rate-independent system that is driven by a convex quadratic energy. Since the associated solution mapping is non-smooth, the analysis of such control problems is challenging. In order to derive optimality conditions, we study the regularization of the problem via a smoothing of the dissipation potential and via the addition of some viscosity. The resulting regularized optimal control problem is analyzed. By driving the regularization parameter to zero, we obtain a necessary optimality condition for the original, non-smooth problem.

1. Introduction.

Let a Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $T > 0$ be given and set $I := (0, T)$. We study the optimal control of a non-smooth evolution problem given by the non-smooth dissipation

$$\mathcal{D} : H_0^1(\Omega) \to \mathbb{R}, \quad \mathcal{D}(\dot{z}) := \int_\Omega |\dot{z}| \, \mathrm{d}x \tag{1}$$

and the quadratic energy

$$\mathcal{E} : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}, \quad \mathcal{E}(z, g) := \int_\Omega \frac{1}{2} |\nabla z|^2 - z\, g \, \mathrm{d}x, \tag{2}$$

which give rise to the differential inclusion

$$0 \in \partial|\dot{z}| - \Delta z - g \quad \text{in } H^{-1}(\Omega), \text{ a.e. in } I, \tag{3}$$

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* Corresponding author: D. Wachsmuth.

to be complemented by the initial condition $z(0) = 0$. Here, $z \in H^1(I; H_0^1(\Omega))$ has the role of the state variable, whereas $g \in H^1(I; L^2(\Omega))$ is the control. The optimal control problem under consideration reads:

$$\text{Minimize } J(z, g) \text{ subject to (3) and } g(0) = 0, \tag{P}$$

where J denotes a suitable objective functional, see (6) below. The requirement $g(0) = 0$ arises as compatibility condition implying the *stability* of the initial state $z(0) = 0$.

The aim of this article is to derive necessary optimality conditions. This turns out to be a quite demanding task, even in the basic setting of (3), for the dependence of the state on the control is non-smooth. This reflects the non-smoothness of the dissipation, which on the other hand is the trademark of rate-independent evolution. In this connection, we refer the reader to the recent monograph by [36], where a thorough discussion of the current state of the art on rate-independent systems is recorded.

Let us sketch the strategy of our method. Under rather mild assumptions, the optimal control problem (P) admits global solutions. By letting $(\bar{z}, \bar{g})$ be locally optimal for the original optimal control problem, we find $\delta > 0$ such that $J(\bar{z}, \bar{g}) \leq J(z, g)$ for all (z, g) with $\|g - \bar{g}\|_{H^1(I;L^2(\Omega))} \leq \delta$ and satisfying the constraints in (P). In order to prove necessary optimality conditions to be satisfied by $(\bar{z}, \bar{g})$ we consider the regularized problem

$$\min J(z, g) + \frac{1}{2}\|g - \bar{g}\|^2_{H^1(I;L^2(\Omega))} \tag{4}$$

subject to $\|g - \bar{g}\|_{H^1(I;L^2(\Omega))} \leq \delta$, $g(0) = 0$, and the regularized problem

$$0 = \partial|\dot{z}|_\rho - \rho\Delta\dot{z} - \Delta z - g \quad \text{ in } H^{-1}(\Omega), \text{ a.e. in } (0, T), \quad z(0) = 0. \tag{5}$$

Here, $|\cdot|_\rho$ is a smooth approximation of the modulus $|\cdot|$. The regularized state equation (5) is smooth. Hence, necessary optimality conditions for (P_ρ) can be derived by standard techniques. The main challenge is then to pass to the limit as $\rho \searrow 0$ in the optimality system.

As already mentioned above, the structure of the state equation (3) is inspired by the theory of rate-independent systems. These arise ubiquitously in applications, ranging from mechanics and electromagnetism to economics and life sciences, see [36] besides the classical monographs [46, 6, 30]. In particular, the presence of the elliptic operator (3) can be put in relation with the occurrence of exchange energy term in micromagnetics [14] or with gradient plasticity theories [38].

Our method is based on regularizing the equation by adding some viscosity. This relates with the classical *vanishing-viscosity* approach to rate-independent systems. Pioneered by [16], evolutions of this technology in the abstract setting are in a series of papers by [34, 35, 37]. See also [31] for an existence theory for discontinuous loadings based on Kurzweil integration.

Vanishing viscosity has been applied in a number of mechanical contexts ranging from plasticity with softening [12], generalized materials driven by nonconvex energies [19], crack propagation [7, 25, 28, 32, 33, 39, 45], nonassociative plasticity of Cam-clay [13], Armstrong-Frederick [20], cap type [2], and heterogeneous materials [43]. An application to adhesive contact is in [42], and damage problems via vanishing viscosity are studied in [26, 27]. In all of these settings, the vanishing-viscosity approach has served as a tool to circumvent non-convexity of the energy toward existence of solutions. Our aim here is clearly different for the energy $\mathcal{E}$ is convex. In particular, we exploit vanishing viscosity in order to regularize the control-to-state mapping and deriving optimality conditions.

Optimal control of finite-dimensional rate-independent processes has been considered in [3, 4, 5] and we witness an increasing interest for the optimal control of sweeping processes, see [8, 9, 10, 11]. In the infinite-dimensional setting, the available results are scant. The existence of optimal controls, also in combination with approximations, was first studied by [40, 41] and subsequently applied in the context of shape memory materials by [17, 18, 44]. In these works, no optimality conditions were given.

To our knowledge, optimality conditions in the time-continuous, rate-independent, infinite-dimensional setting were firstly derived in [47, 48, 49] in the context of quasi-static plasticity, see also [24]. Let us however mention other works addressing optimality conditions for control problem for rate-independent systems in combination with time-discretizations, namely [29, 22, 23, 1].

The plan of the paper is as follows. We firstly derive an optimality system for (P) by means of formal calculations in Section 2. The argument is then made rigorous along the paper and brings to the proof of our main result, namely Theorem 5.2. The existence of a solution of (P) is at the core of Section 3, see Lemma 3.5. In Section 4, we address the regularization of (P) instead. We study the regularized state equation, and derive an optimality system for the regularized control problem by means of the regularized adjoint equations. Eventually, in Section 5 we pass to the limit in the regularized control problem and rigorously obtain optimality conditions for (P) in Theorem 5.2.

Notation. Recall that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $T > 0$, and let $I := (0, T)$ and $Q := I \times \Omega$. We work with standard function spaces like $H_0^1(\Omega)$ and $L^2(\Omega)$. The space $H_0^1(\Omega)$ is equipped with the norm and scalar product

$$(z, w)_{H_0^1(\Omega)} := \int_\Omega \nabla z \cdot \nabla w \, \mathrm{d}x \qquad \text{and} \qquad \|z\|^2_{H_0^1(\Omega)} := (z, z)_{H_0^1(\Omega)},$$

respectively. Throughout the text, $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ denotes the distributional Laplacian and $\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ denotes its inverse.

Moreover, we use the Bochner spaces $L^p(I; H)$, $W^{1,p}(I; H)$, and $H^1(I; H)$, where H is a Hilbert space. By $W^{-1,p}(I; H')$ we denote the dual space of $W^{1,p'}(I; H)$, where $1/p + 1/p' = 1$, $p, p' \in (1, \infty)$. Since our state equation is equipped with homogeneous initial conditions, we also use

$$H^1_\star(I; H) := \{v \in H^1(I; H) : v(0) = 0\}.$$

We will consider optimal control problems with an objective functional of the type

$$J(z, g) := j_1(z) + j_2(z(T)) + \frac{1}{2}\|g\|^2_{H^1(I;L^2(\Omega))}, \tag{6}$$

where the functions $j_1 : L^2(I; H_0^1(\Omega)) \to \mathbb{R}$ and $j_2 : H_0^1(\Omega) \to \mathbb{R}$ are assumed to be continuously Fréchet differentiable and bounded from below.

2. **Formal derivation of an optimality system.**

In this section, we *formally* derive an optimality system. It is clear that the resulting system may not be a necessary optimality condition. However, this derivation sheds some light on the situation and we get an idea what relations can be expected as necessary conditions.

We start by (formally) restating the optimal control problem by

$$\begin{aligned} &\text{Minimize} && J(z,g) \\ &\text{such that} && (\dot z(t,x),\, g(t,x)+\Delta z(t,x)) \in M \;\forall (t,x) \in (0,T)\times\Omega. \end{aligned}$$

Here,

$$\begin{aligned} M &:= \operatorname{gph} \partial|\cdot| = \{(u,v)\in\mathbb{R}^2 : v \in \partial|u|\} \\ &= \big((-\infty,0]\times\{-1\}\big) \cup \big(\{0\}\times[-1,1]\big) \cup \big([0,\infty)\times\{+1\}\big). \end{aligned}$$

The Lagrangian for this optimization problem is given by

$$\mathcal{L}(z,g,q,\xi) = J(z,g) - (q,\dot z)_{L^2(Q)} + (\xi, g+\Delta z)_{L^2(Q)}.$$

As (formal) optimality conditions, we would expect

$$0 = \frac{\partial}{\partial z}\mathcal{L}(z,g,q,\xi), \tag{7a}$$

$$0 = \frac{\partial}{\partial g}\mathcal{L}(z,g,q,\xi), \tag{7b}$$

$$(-q(t,x),\xi(t,x)) \in N_M(\dot z(t,x),\, g(t,x)+\Delta z(t,x)). \tag{7c}$$

Here, N_M is a normal-cone mapping associated with the closed set $M \subset \mathbb{R}^2$. Since M is not convex, the different normal cones of variational analysis, namely Fréchet, Clarke, Mordukhovich, do not coincide. In particular, by using the Fréchet normal cone, which is the smallest among these, we would expect the relations

$$\dot z(t,x) > 0,\ g(t,x)+\Delta z(t,x) = 1 \quad\Longrightarrow\quad q(t,x) = 0, \tag{8a}$$

$$\dot z(t,x) = 0,\ g(t,x)+\Delta z(t,x) = 1 \quad\Longrightarrow\quad q(t,x) \ge 0,\ \xi(t,x) \ge 0, \tag{8b}$$

$$\dot z(t,x) = 0,\ |g(t,x)+\Delta z(t,x)| < 1 \quad\Longrightarrow\quad \xi(t,x) = 0, \tag{8c}$$

$$\dot z(t,x) = 0,\ g(t,x)+\Delta z(t,x) = -1 \quad\Longrightarrow\quad q(t,x) \le 0,\ \xi(t,x) \le 0, \tag{8d}$$

$$\dot z(t,x) < 0,\ g(t,x)+\Delta z(t,x) = -1 \quad\Longrightarrow\quad q(t,x) = 0. \tag{8e}$$

The above equations (7a)–(7b) for q,ξ could be written as

$$q(T) = j_2'(z(T)) \qquad \text{a.e. on } \Omega, \tag{9a}$$

$$-\dot q = j_1'(z) + \Delta\xi \qquad \text{a.e. on } Q, \tag{9b}$$

$$-\ddot g + g + \xi = 0 \qquad \text{a.e. on } Q. \tag{9c}$$

Here, (9c) is equipped with the boundary conditions $g(0) = \dot g(T) = 0$. Hence, this formal derivation suggests that for each local solution (z,g) of (P), there exist functions q,ξ such that (8) and (9) are satisfied.

3. **Unregularized optimal control problem.** In this section, we give some first results concerning the optimal control problem (P). We recall some known results for the state equation and prove the existence of solutions to (P).

A concept tailored to rate-independent systems is the notion of *energetic solutions*, see [36, Section 1.6]. Since the energy (2) is convex, our situation is much more comfortable and we can use the formulation (3), which is strong in time. Indeed, for every $g \in H^1(I;H^{-1}(\Omega))$ with $\|g(0)\|_{L^\infty(\Omega)} \le 1$, there is a unique energetic solution $\mathcal{S}(g) := z \in H^1_\star(I;H^1_0(\Omega))$ and this is the unique solution to (3), see [36, Section 1.6.4, Theorem 3.5.2].

The requirement $\|g(0)\|_{L^\infty(\Omega)} \le 1$ is needed as a compatibility condition. Indeed, it ensures that $\Delta z(0) + g(0) = g(0)$ is in the range of $\partial_{\dot z}\mathcal{D} = \partial\|\cdot\|_{L^1(\Omega)}$. Hence, we define

$$\mathcal{G}_0 := \{g \in C(\bar I; H^{-1}(\Omega)) : \|g(0)\|_{L^\infty(\Omega)} \le 1\}.$$

Due to the quadratic nature of the energy, it is possible to recast the state equation as an evolution variational inequality in the sense of [30].

Lemma 3.1. *Let $z \in H^1_\star(I; H^1_0(\Omega))$ and $g \in H^1(I; H^{-1}(\Omega)) \cap \mathcal{G}_0$ be given. Then, the state equation* (3) *in $H^{-1}(\Omega)$ is equivalent to*

$$\dot z \in N^{\text{Hilbert}}_{\tilde K}(-z - \Delta^{-1} g) \quad \text{in } H^1_0(\Omega), \text{ a.e. on } (0,T) \tag{10}$$

and to

$$\dot z \in N_K(\Delta z + g) \quad \text{in } H^1_0(\Omega), \text{ a.e. on } (0,T). \tag{11}$$

Here,

$$N^{\text{Hilbert}}_{\tilde K}(v) := \{w \in H^1_0(\Omega) : (w, \tilde v - v)_{H^1_0(\Omega)} \le 0 \quad \forall \tilde v \in \tilde K\}$$

is the (Hilbert space) normal cone of the set

$$\tilde K := \{w \in H^1_0(\Omega) :\ \Delta w \in L^2(\Omega),\ -1 \le \Delta w \le 1 \text{ a.e. in } \Omega\} = \Delta^{-1}(K)$$

at $v \in \tilde K$ and

$$N_K(v) := \{w \in H^1_0(\Omega) : \langle \tilde v - v, w\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \le 0 \quad \forall \tilde v \in K\}$$

is the normal cone of the set

$$K := \{v \in H^{-1}(\Omega) : v \in L^2(\Omega) \text{ and } -1 \le v \le 1 \text{ a.e. in } \Omega\} = \Delta(\tilde K)$$

at $v \in K$.

Proof. The assertion follows directly from standard results in convex analysis by using the definition of the dissipation (1) and of the energy (2). □

The mapping $(-\Delta^{-1} g) \mapsto z$ is also known as the *play operator*, see [30, Section I.3]. From [30, Remark I.3.10, Theorem I.3.12] we find the following regularity results for equation (10).

Lemma 3.2. *The control-to-state map $\mathcal{S}$ is continuous from $H^1(I; H^{-1}(\Omega)) \cap \mathcal{G}_0$ to $H^1_\star(I; H^1_0(\Omega))$ and Lipschitz continuous from the space $W^{1,1}(I; H^{-1}(\Omega)) \cap \mathcal{G}_0$ to $L^\infty(I; H^1_0(\Omega))$.*

The next lemma provides the energy equality (12), which will be crucial to prove the consistency of the regularization in $H^1_\star(I; H^1_0(\Omega))$, cf. Theorem 4.9.

Lemma 3.3. *Let $g \in H^1(I; H^{-1}(\Omega)) \cap \mathcal{G}_0$ be given and set $z = \mathcal{S}(g)$. Then, we have*

$$\langle \dot z(t), \Delta \dot z(t) + \dot g(t)\rangle_{H^1_0(\Omega), H^{-1}(\Omega)} = 0 \qquad \text{for a.a. } t \in I. \tag{12}$$

Proof. Using (11) and $\Delta z(s) + g(s) \in K$ for all $s \in I$, we find

$$\langle \dot z(t), \Delta z(t \pm h) + g(t \pm h) - (\Delta z(t) + g(t))\rangle_{H^1_0(\Omega), H^{-1}(\Omega)} \le 0$$

for almost all $t \in I$ and all $h > 0$ such that $t \pm h \in I$. Using Lebesgue's differentiation theorem, see [15, Theorem II.2.9] for the version with Bochner integrals, we can pass to the limit $h \searrow 0$. This yields the claim, see also [30, (I.3.22)(ii)]. □

We note that the a-priori energy estimate

$$\|\dot z(t)\|_{H_0^1(\Omega)} = \|\Delta \dot z(t)\|_{H^{-1}(\Omega)} \le \|\dot g(t)\|_{H^{-1}(\Omega)} \qquad \text{for a.a. } t \in I \tag{13}$$

follows immediately from (12).

In order to prove the existence of solutions of the optimal control problem (P), we need to show a weak continuity result for $\mathcal{S}$. Recall, that $H^1(I; L^2(\Omega))$ is *not* compactly embedded in $H^1(I; H^{-1}(\Omega))$, hence, the following result is not a simple consequence of Lemma 3.2. Similarly, it does not directly follow from Helly's selection theorem, which would only give pointwise weak convergence of the state variable. We note that a similar argument was used in [47, Theorem 2.3, Section 2.3].

Lemma 3.4. *Let $\{g_n\}_{n\in\mathbb{N}} \subset H^1(I; L^2(\Omega)) \cap \mathcal{G}_0$ be given such that $g_n \rightharpoonup g$ in $H^1(I; L^2(\Omega))$. Then, $z_n := \mathcal{S}(g_n) \rightharpoonup \mathcal{S}(g) =: z$ in $H^1_\star(I; H^1_0(\Omega))$ and $z_n \to z$ in $C(\bar I; H_0^1(\Omega))$.*

Proof. The assumptions imply that $g_n(0) \rightharpoonup g(0)$ in $H^{-1}(\Omega)$. Hence, $g(0)$ belongs to $\mathcal{G}_0$, which makes $z = \mathcal{S}(g)$ well-defined. Due to (13), the sequence $\{z_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_\star(I; H^1_0(\Omega))$.

From (11) we find for arbitrary $t \in \bar I$

$$\int_0^t \langle \dot z_n, (\Delta z + g) - (\Delta z_n + g_n)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \,\mathrm{d}s \le 0,$$
$$\int_0^t \langle \dot z, (\Delta z_n + g_n) - (\Delta z + g)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \,\mathrm{d}s \le 0.$$

Adding these inequalities yields

$$\int_0^t (\nabla(\dot z_n - \dot z), \nabla(z_n - z))_{L^2(\Omega)} \,\mathrm{d}s \le \int_0^t \langle \dot z_n - \dot z, g_n - g\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \,\mathrm{d}s,$$

which gives

$$\begin{aligned}\frac12 \|z_n(t) - z(t)\|^2_{H_0^1(\Omega)} &\le \int_0^t \langle \dot z_n - \dot z, g_n - g\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \,\mathrm{d}s \\ &\le \|\dot z_n - \dot z\|_{L^2(I;H_0^1(\Omega))} \|g_n - g\|_{L^2(I;H^{-1}(\Omega))}.\end{aligned}$$

Owing to (13), we have

$$\frac12 \|z_n(t) - z(t)\|^2_{H_0^1(\Omega)} \le \big(\|g_n\|_{H^1(I;H^{-1}(\Omega))} + \|g\|_{H^1(I;H^{-1}(\Omega))}\big) \|g_n - g\|_{L^2(I;H^{-1}(\Omega))}.$$

Due to the compact embedding $H^1(I; L^2(\Omega)) \hookrightarrow L^2(I; H^{-1}(\Omega))$, we can pass to the limit to obtain the convergence $z_n \to z$ in $C(\bar I; H_0^1(\Omega))$. Since $\{z_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_\star(I; H^1_0(\Omega))$, the weak convergence $z_n \rightharpoonup z$ in $H^1_\star(I; H^1_0(\Omega))$ follows. □

Now, we are in the position to prove the existence of solutions of (P).

Lemma 3.5. *There exists a (global) optimal control of* (P).

The proof is standard, but included for the reader's convenience.

Proof. We denote by j the infimal value of the optimal control problem and by $\{(z_n, g_n)\}_{n\in\mathbb{N}}$ a minimizing sequence. By the boundedness of the controls $\{g_n\}_{n\in\mathbb{N}}$ in $H^1(I; L^2(\Omega))$ we obtain the weak convergence of a subsequence (without relabeling) in $H^1(I; L^2(\Omega))$ towards $\bar g$.

Now, we have $z_n = \mathcal{S}(g_n) \to \mathcal{S}(\bar g)$ in $C(\bar I; H_0^1(\Omega))$ and $z_n(T) \to z(T)$ in $H_0^1(\Omega)$ due to Lemma 3.4. This implies

$$J(\bar z, \bar g) \le \liminf_{n\to\infty} J(z_n, g_n) = j.$$

Hence, $(\bar z, \bar g)$ is globally optimal for (P). □

4. **Regularized optimal control problem.** In this section, we study the regularized optimal control problem.

4.1. **Regularized dissipation.** For given parameter $\rho > 0$, let us define the regularized dissipation by

$$\mathcal{D}_\rho : H_0^1(\Omega) \to \mathbb{R}, \quad \mathcal{D}_\rho(\dot z) := \int_\Omega |\dot z|_\rho + \frac{\rho}{2} |\nabla \dot z|^2 \,\mathrm{d}x. \tag{14}$$

Note that the additional quadratic term in $\mathcal{D}_\rho$ will add some viscosity to our state equation. In the regularization (14), $|\cdot|_\rho$ is a regularized version of the modulus function $|\cdot| : \mathbb{R} \to \mathbb{R}$ satisfying the following assumption:

Assumption 4.1. *The family* $\{|\cdot|_\rho\}_{\rho>0}$ *satisfies*

1. $|\cdot|_\rho$ *is* $C^{2,1}(\mathbb{R},\mathbb{R})$ *and convex,*
2. $|v|_\rho = |-v|_\rho$ *for all* $v \in \mathbb{R}$*,*
3. $|v|_\rho = |v|$ *for all* $v \in \mathbb{R}$ *with* $|v| \ge \rho$*, and*
4. $|v|''_\rho \le \frac{2}{\rho}$ *for all* $v \in \mathbb{R}$*.*

Note that this assumption implies

$$|v|'_\rho \in [-1, 1], \quad \text{and} \quad |v|''_\rho \ge 0 \quad \forall v \in \mathbb{R}$$

by convexity of $|\cdot|_\rho$.

Lemma 4.2. *Let* $|\cdot|_\rho$ *satisfy Assumption 4.1. Then it holds*

$$|v| \le |v|_\rho \le |v| + \rho \quad \text{and} \quad |v|'_\rho\, v \ge |v| - \rho \quad \forall v \in \mathbb{R}. \tag{15}$$

Proof. The first inequality follows from convexity and Property 3. The second inequality obviously holds for $|v| \ge \rho$ due to Property 3. Now let $v \in [-\rho, \rho]$ be given. Using the monotonicity of $|\cdot|'_\rho$ due to Property 1, we have

$$|v|'_\rho\, v = |v|'_\rho\,(v - 0) \ge |0|'_\rho\,(v - 0) = 0 \ge |v| - \rho,$$

since $|0|'_\rho = 0$ follows from Property 2. □

Let us remark that Assumption 4.1 is satisfied, e.g., by

$$|v|''_\rho := 2\,\rho^{-2} \max\big(\rho - |v|, 0\big), \quad |v|'_\rho := \int_0^v |s|''_\rho \,\mathrm{d}s, \quad |v|_\rho := \rho + \int_{-\rho}^v |s|'_\rho \,\mathrm{d}s.$$

4.2. **Regularized state equation.** Let us now discuss the regularized state equation. In particular, we will prove the differentiability of the solution map $\mathcal{S}_\rho$ and show a-priori stability results.

We recall the regularized problem (5)

$$0 = \partial_{\dot z} \mathcal{D}_\rho(\dot z) + \partial_z \mathcal{E}(z, g) \quad \text{in } H^{-1}(\Omega), \text{ a.e. on } I, \quad z(0) = 0.$$

By using the differentiability of $|\cdot|_\rho$, we obtain the equivalent formulation

$$|\dot z|'_\rho - \rho\,\Delta \dot z - \Delta z = g \quad \text{in } H^{-1}(\Omega), \text{ a.e. on } I. \tag{16}$$

This equation can be written as the system

$$\dot z = w \qquad \text{in } H_0^1(\Omega), \quad \text{a.e. on } I, \tag{17a}$$

$$-\rho\,\Delta w + |w|'_\rho = \Delta z + g \quad \text{in } H^{-1}(\Omega), \quad \text{a.e. on } I, \tag{17b}$$

equipped with the initial condition $z(0) = 0$. In order to discuss the solvability of (17), we first analyze the semilinear equation

$$-\,\rho\,\Delta w + |w|'_\rho = v \quad \text{in } H^{-1}(\Omega). \tag{18}$$

Due to the monotonicity of $|\cdot|'_\rho$, this equation has a unique weak solution $w \in H_0^1(\Omega)$ for all $v \in H^{-1}(\Omega)$. Moreover, the solution depends Lipschitz continuously on the right-hand side. Let us denote by $T_\rho := (-\rho\,\Delta + |\cdot|'_\rho)^{-1}$ the associated solution mapping, which is globally Lipschitz continuous from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$ for fixed, positive ρ.

Using this mapping, equation (17) can be written as

$$\dot z = T_\rho(g + \Delta z) \quad \text{in } H_0^1(\Omega), \text{ a.e. on } I, \tag{19}$$

which is an ODE in $H_0^1(\Omega)$. Due to the global Lipschitz continuity of T_ρ, we have the following classical result.

Theorem 4.3. *Let $\rho > 0$ be given. For each $g \in L^2(I; H^{-1}(\Omega))$, there exists a unique solution $z \in H^1_\star(I; H_0^1(\Omega))$ of the regularized state equation (5). The mapping $\mathcal{S}_\rho$, which maps g to z, is continuous with respect to these spaces.*

Proof. The result follows directly from [21, Satz 1.3, p. 166]. □

In the next step, we will investigate the differentiability of $\mathcal{S}_\rho$. Due to the properties of $|\cdot|'_\rho$, the operator T_ρ is Fréchet differentiable from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$. Let $v, h \in H^{-1}(\Omega)$ be given with $w = T_\rho(v)$. By standard arguments it can be proven that $y = T'_\rho(v)\,h$ is given as the unique weak solution of the equation

$$-\,\rho\,\Delta y + |w|''_\rho\, y = h. \tag{20}$$

Moreover due to $|w|''_\rho \ge 0$, we can bound the norm of $T'_\rho(v)$ uniformly with respect to v by

$$\|y\|_{H_0^1(\Omega)} = \|T'_\rho(v)\,h\|_{H_0^1(\Omega)} \le \rho^{-1}\,\|h\|_{H^{-1}(\Omega)}.$$

Hence, the linearized ODE

$$\dot\zeta = T'_\rho(g + \Delta z)(h + \Delta\zeta) \quad \text{in } H_0^1(\Omega), \text{ a.e. on } I$$

with the initial condition $\zeta(0) = 0$ is uniquely solvable provided $g \in L^2(I; H^{-1}(\Omega))$, $z = \mathcal{S}_\rho(g)$, and $h \in L^2(I; H^{-1}(\Omega))$, see again [21]. Summarizing these arguments leads to the following differentiability result.

Theorem 4.4. *Let $\rho > 0$ be given. The regularized control-to-state map $\mathcal{S}_\rho$ is Fréchet differentiable from $L^2(I; H^{-1}(\Omega))$ to $W^{1,p}(I; H_0^1(\Omega))$ for all $p \in (1, 2)$. The directional derivative $\zeta = \mathcal{S}'_\rho(g)\,h$ satisfies the system*

$$\dot\zeta = \omega \qquad \textit{in } H_0^1(\Omega), \ \textit{a.e. on } I, \tag{21a}$$

$$-\rho\,\Delta\omega + |\dot z|''_\rho\,\omega = \Delta\zeta + h \quad \textit{in } H^{-1}(\Omega), \ \textit{a.e. on } I, \tag{21b}$$

$$\zeta(0) = 0, \tag{21c}$$

where z is given by $z = \mathcal{S}_\rho(g)$.

Proof. For $h \in L^2(I; H^{-1}(\Omega))$, we set $z_h = \mathcal{S}_\rho(g+h)$ and $\zeta_h = \mathcal{S}_\rho'(g)\,h$, which is defined via the solution of (21). Then, the error $e_h = z_h - z - \zeta_h$ satisfies the ODE

$$\dot e_h = T_\rho'(g+\Delta z)\Big(\Delta e_h - \big(|\dot z_h|_\rho' - |\dot z|_\rho' - |\dot z|_\rho''\,(\dot z_h - \dot z)\big)\Big).$$

Using that $T_\rho'(v)$ is bounded from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$ uniformly w.r.t. v, we find

$$\|e_h\|_{W^{1,p}(I;H_0^1(\Omega))} \le C\,\||\dot z_h|_\rho' - |\dot z|_\rho' - |\dot z|_\rho''\,(\dot z_h - \dot z)\|_{L^p(I;H^{-1}(\Omega))}$$

for $p \in (1,2)$. By using the differentiability of the Nemytskii operator associated with $|\cdot|_\rho'$ from $L^2(I;H_0^1(\Omega))$ to $L^p(I;H^{-1}(\Omega))$ we find

$$\begin{aligned}\|e_h\|_{W^{1,p}(I;H_0^1(\Omega))} &\le C\,\||\dot z_h|_\rho' - |\dot z|_\rho' - |\dot z|_\rho''\,(\dot z_h - \dot z)\|_{L^p(I;H^{-1}(\Omega))}\\ &= o(\|\dot z_h - \dot z\|_{L^2(I;H_0^1(\Omega))}).\end{aligned}$$

Finally, the Lipschitz continuity of $\mathcal{S}_\rho$ implies

$$\|e_h\|_{W^{1,p}(I;H_0^1(\Omega))} = o(\|\dot z_h - \dot z\|_{L^2(I;H_0^1(\Omega))}) = o(\|h\|_{L^2(I;H^{-1}(\Omega))}). \qquad \square$$

Now, we show a regularized counterpart to the Lipschitz continuity of $\mathcal{S}$, cf. Lemma 3.2.

Lemma 4.5. *Let $g_1, g_2 \in W^{1,1}(I;H^{-1}(\Omega))$ and $\rho > 0$ be given. Then it holds*

$$\|\mathcal{S}_\rho(g_2) - \mathcal{S}_\rho(g_1)\|_{C(\bar I;H_0^1(\Omega))} \le C\,\|g_2 - g_1\|_{W^{1,1}(I;H^{-1}(\Omega))}$$

with $C > 0$ solely depending on T.

Proof. By testing the state equations (16) for $z_1 := \mathcal{S}_\rho(g_1)$ and $z_2 := \mathcal{S}_\rho(g_2)$ by $\dot z_2 - \dot z_1$, integrating over $(0,t)$, and taking the difference, we get

$$\begin{aligned}&\int_0^t \int_\Omega \big(|\dot z_2|_\rho' - |\dot z_1|_\rho'\big)\big(\dot z_2 - \dot z_1\big)\,\mathrm{d}x\,\mathrm{d}s + \rho\,\|\dot z_2 - \dot z_1\|^2_{L^2(0,t;H_0^1(\Omega))}\\ &\qquad + \int_0^t (\nabla(z_2 - z_1), \nabla(\dot z_2 - \dot z_1))\,\mathrm{d}s = \int_0^t \langle g_2 - g_1, \dot z_2 - \dot z_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}s.\end{aligned}$$

Using the monotonicity of $|\cdot|_\rho'$ and $z_1(0) = z_2(0) = 0$, we get for all $t \in \bar I$

$$\begin{aligned}&\frac{1}{2}\,\|z_2(t) - z_1(t)\|^2_{H_0^1(\Omega)} \le \int_0^t \langle g_2 - g_1, \dot z_2 - \dot z_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}s\\ &= -\int_0^t \langle \dot g_2 - \dot g_1, z_2 - z_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}s + \langle g_2(t) - g_1(t), z_2(t) - z_1(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\ &\le \|z_2 - z_1\|_{L^\infty(I;H_0^1(\Omega))}\left(\|\dot g_2 - \dot g_1\|_{L^1(I;H^{-1}(\Omega))} + \|g_2 - g_1\|_{L^\infty(I;H^{-1}(\Omega))}\right).\end{aligned}$$

Taking the supremum on the left-hand side, we obtain

$$\|z_2 - z_1\|_{L^\infty(I;H_0^1(\Omega))} \le 2\left(\|\dot g_2 - \dot g_1\|_{L^1(I;H^{-1}(\Omega))} + \|g_2 - g_1\|_{L^\infty(I;H^{-1}(\Omega))}\right),$$

which shows the assertion. $\square$

As last result in this section, we provide some a-priori estimates and, in particular, provide the boundedness of $z = \mathcal{S}_\rho(g)$ in $H^1(I;H_0^1(\Omega))$ independent of ρ.

Lemma 4.6. *Let $\rho > 0$ and $g \in H^1(I;H^{-1}(\Omega)) \cap \mathcal{G}_0$ be given, and let $z = \mathcal{S}_\rho(g)$. Then it holds $z \in H^2(I;H_0^1(\Omega))$. In addition, there is a constant $C > 0$ independent of ρ (and g) such that*

$$\rho\,\|\dot z(T)\|^2_{H_0^1(\Omega)} + \|z\|^2_{H^1(I;H_0^1(\Omega))} \le C\left(\rho + \|g\|^2_{H^1(I;H^{-1}(\Omega))}\right).$$

and

$$\|\dot z(0)\|_{H_0^1(\Omega)} \le C.$$

Proof. We start by showing $\dot z \in H^1(I; H_0^1(\Omega))$. Since T_ρ is globally Lipschitz continuous, we have

$$\begin{aligned}\|\dot z(t_2) - \dot z(t_1)\|_{H_0^1(\Omega)} &= \|T_\rho(g(t_2) + \Delta z(t_2)) - T_\rho(g(t_1) + \Delta z(t_1))\|_{H_0^1(\Omega)} \\ &\le L_\rho \, \|g(t_2) + \Delta z(t_2) - g(t_1) - \Delta z(t_1)\|_{H^{-1}(\Omega)}.\end{aligned}$$

with a ρ-dependent constant L_ρ. Since both $\dot g$ and $\Delta \dot z$ are in $L^2(I; H^{-1}(\Omega))$, one can prove with the help of finite differences that it holds $\dot z \in H^1(I; H_0^1(\Omega))$.

Moreover, we obtain $\dot z(0) = T_\rho(\Delta z(0) + g(0)) = T_\rho(g(0))$ by continuity. Testing the associated semilinear elliptic equation by $\dot z(0)$ and using $z(0) = 0$ yields

$$\int_\Omega |\dot z(0)|'_\rho \, \dot z(0) \, \mathrm{d}x + \rho \, \|\dot z(0)\|^2_{H_0^1(\Omega)} = \int_\Omega g(0) \, \dot z(0) \, \mathrm{d}x.$$

By using the second inequality in (15) for $|\cdot|_\rho$ as well as the compatibility assumption $\|g(0)\|_{L^\infty(\Omega)} \le 1$ we obtain

$$\|\dot z(0)\|_{L^1(\Omega)} + \rho \, \|\dot z(0)\|^2_{H_0^1(\Omega)} \le \|\dot z(0)\|_{L^1(\Omega)} + \rho \, \mathrm{meas}(\Omega),$$

which implies

$$\|\dot z(0)\|^2_{H_0^1(\Omega)} \le \mathrm{meas}(\Omega). \tag{22}$$

Now, let us differentiate (16) w.r.t. t to obtain

$$|\dot z|''_\rho \, \ddot z - \rho \, \Delta \ddot z - \Delta \dot z = \dot g.$$

Testing with $\dot z$ and integrating, we find

$$\int_Q |\dot z|''_\rho \, \dot z \, \ddot z + \rho \, \nabla \ddot z \cdot \nabla \dot z + |\nabla \dot z|^2 \, \mathrm{d}x \, \mathrm{d}t = \int_I \langle \dot g, \dot z \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}t.$$

Let us introduce the function

$$f_\rho(r) = \int_0^r |s|''_\rho \, s \, \mathrm{d}s.$$

This construction implies

$$\frac{\mathrm{d}}{\mathrm{d}t} f_\rho(\dot z) = f'_\rho(\dot z) \, \ddot z = |\dot z|''_\rho \, \dot z \, \ddot z.$$

Consequently, we find

$$\begin{aligned}&\int_\Omega f_\rho(\dot z(T)) \, \mathrm{d}x + \frac{\rho}{2} \, \|\dot z(T)\|^2_{H_0^1(\Omega)} + \|\dot z\|^2_{L^2(I; H_0^1(\Omega))} \\ &\qquad \le \int_I \langle \dot g, \dot z \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}t + \int_\Omega f_\rho(\dot z(0)) \, \mathrm{d}x + \frac{\rho}{2} \, \mathrm{meas}(\Omega),\end{aligned}$$

where we used in addition the estimate (22) of $\dot z(0)$. Due to the assumptions on $|\cdot|_\rho$, the auxiliary function f_ρ is bounded, and it holds $0 \le f_\rho(s) \le \rho$. Hence, we obtain

$$\frac{\rho}{2} \, \|\dot z(T)\|^2_{H_0^1(\Omega)} + \|\dot z\|^2_{L^2(I; H_0^1(\Omega))} \le \int_I \langle \dot g, \dot z \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}t + \frac{3}{2} \, \rho \, \mathrm{meas}(\Omega). \tag{23}$$

Using Young's inequality, we finally obtain

$$\rho \, \|\dot z(T)\|^2_{H_0^1(\Omega)} + \|\dot z\|^2_{L^2(I; H_0^1(\Omega))} \le \|\dot g\|^2_{L^2(I; H^{-1}(\Omega))} + 3 \, \rho \, \mathrm{meas}(\Omega).$$

This shows the claim. □

We emphasize that the compatibility condition $g \in \mathcal{G}_0$, i.e., $\|g\|_{L^\infty(\Omega)} \le 1$, is crucial for the validity of Lemma 4.6.

4.3. **Convergence of the regularization of the state equation.** In this section, we show that the sequence $\{\mathcal{S}_\rho(g)\}_{\rho>0}$ converges towards the solution $\mathcal{S}(g)$ of the unregularized system.

Lemma 4.7. *Let $\{g_n\}_{n\in\mathbb{N}} \in H^1(I;L^2(\Omega)) \cap \mathcal{G}_0$ be given, such that $g_n \rightharpoonup g$ in $H^1(I;L^2(\Omega))$. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a positive sequence with $\rho_n \searrow 0$. For $n \in \mathbb{N}$ we set $z_n := \mathcal{S}_{\rho_n}(g_n)$ and $z := \mathcal{S}(g)$. Then, $z_n \rightharpoonup z$ in $H^1_\star(I;H^1_0(\Omega))$ and $z_n \to z$ in $C(\bar{I};H^1_0(\Omega))$.*

Moreover, in case $g_n \equiv g$, we have the estimate

$$\|z - z_n\|_{C(\bar{I};H^1_0(\Omega))} \le C\,(1 + \|z\|_{H^1(I;H^1_0(\Omega))})\,\rho_n^{1/2}, \tag{24}$$

with $C > 0$ independent of g, ρ_n.

Proof. By Lemma 4.6 we find that the sequence of states $\{z_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(I;H^1_0(\Omega))$. From the state equation (3), we find

$$\langle \Delta z + g, \dot{z}_n - \dot{z}\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \|\dot{z}\|_{L^1(\Omega)} \le \|\dot{z}_n\|_{L^1(\Omega)}.$$

Similarly, by testing the regularized equation (16) by $\dot{z}_n - \dot{z}$ we obtain

$$\langle \Delta z_n + g_n + \rho_n\,\Delta\dot{z}_n, \dot{z} - \dot{z}_n\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_\Omega |\dot{z}_n|_{\rho_n}\,\mathrm{d}x \le \int_\Omega |\dot{z}|_{\rho_n}\,\mathrm{d}x.$$

Adding both inequalities and integrating on $(0,t)$ for $t \in \bar{I}$ yields

$$\begin{aligned}
&\rho_n\,\|\dot{z} - \dot{z}_n\|^2_{L^2(0,t;H^1_0(\Omega))} + \frac{1}{2}\,\|z(t) - z_n(t)\|^2_{H^1_0(\Omega)}\\
&\le \int_0^t \int_\Omega |\dot{z}_n| - |\dot{z}_n|_{\rho_n} + |\dot{z}|_{\rho_n} - |\dot{z}|\,\mathrm{d}x\,\mathrm{d}s + \rho_n\,\|\dot{z}\|_{L^2(0,t;H^1_0(\Omega))}\,\|\dot{z} - \dot{z}_n\|_{L^2(0,t;H^1_0(\Omega))}\\
&\quad + \int_0^t \langle g - g_n, \dot{z} - \dot{z}_n\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}s.
\end{aligned}$$

With Lemma 4.2, we can estimate the first integral. Applying Young's inequality we obtain

$$\begin{aligned}
&\frac{1}{2}\,\|z(t) - z_n(t)\|^2_{H^1_0(\Omega)}\\
&\qquad \le \rho_n\ \mathrm{meas}(Q) + \frac{\rho_n}{4}\,\|z\|^2_{H^1(I;H^1_0(\Omega))} + \int_0^t \langle g - g_n, \dot{z} - \dot{z}_n\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}.
\end{aligned}$$

which proves the convergence claim due to $g_n \to g$ in $L^2(I;H^{-1}(\Omega))$, see the end of the proof of Lemma 3.4, as well as estimate (24). ☐

Corollary 4.8. *Let $\rho > 0$ and let $g, g_\rho \in H^1(I;L^2(\Omega)) \cap \mathcal{G}_0$ be given. We set $z_\rho := \mathcal{S}_\rho(g_\rho)$ and $z := \mathcal{S}(g)$. Then it holds*

$$\|z - z_\rho\|_{C(\bar{I};H^1_0(\Omega))} \le C\,\big((1 + \|z\|_{H^1(I;H^1_0(\Omega))})\,\rho^{1/2} + \|g - g_\rho\|_{W^{1,1}(I;H^{-1}(\Omega))}\big).$$

with $C > 0$ independent of ρ, g, g_ρ.

Proof. Combine Lemmas 4.5 and 4.7. ☐

Theorem 4.9. *Let $\{g_n\}_{n\in\mathbb{N}} \subset H^1(I;H^{-1}(\Omega)) \cap \mathcal{G}_0$ be given such that $g_n \to g$ in $H^1(I;H^{-1}(\Omega))$. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a positive sequence with $\rho_n \searrow 0$. Then, we have $z_n := \mathcal{S}_{\rho_n}(g_n) \to \mathcal{S}(g) =: z$ in $H^1_\star(I;H^1_0(\Omega))$.*

Proof. By Corollary 4.8, we obtain the convergence $z_n \to z$ in $C(\bar{I}; H_0^1(\Omega))$. Due to Lemma 4.6, the sequence $\{z_n\}_{n\in\mathbb{N}}$ is bounded in $H_\star^1(I; H_0^1(\Omega))$. Thus, it converges weakly towards z in $H_\star^1(I; H_0^1(\Omega))$.

As in the proof of Lemma 4.6, see, e.g., (23), we obtain

$$\|\dot{z}_{\rho_n}\|^2_{L^2(I;H_0^1(\Omega))} \leq \int_I \langle \dot{g}_{\rho_n}, \dot{z}_{\rho_n}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + \frac{3}{2}\,\rho_n\,\mathrm{meas}(\Omega).$$

As $g_{\rho_n} \to g$ in $H^1(I; H^{-1}(\Omega))$ and $\dot{z}_{\rho_n} \rightharpoonup \dot{z}$ in $L^2(I; H^{-1}(\Omega))$, we can pass to the limit to find

$$\limsup_{n\to\infty} \|\dot{z}_{\rho_n}\|^2_{L^2(I;H_0^1(\Omega))} \leq \int_I \langle \dot{g}, \dot{z}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\mathrm{d}t.$$

Together with (12) this implies

$$\limsup_{n\to\infty} \|\dot{z}_{\rho_n}\|^2_{L^2(I;H_0^1(\Omega))} \leq \int_I \langle \dot{g}, \dot{z}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\mathrm{d}t = \int_I \|\dot{z}\|^2_{H_0^1(\Omega)}\mathrm{d}t = \|\dot{z}\|^2_{L^2(I;H_0^1(\Omega))},$$

which shows $\|\dot{z}_n\|_{L^2(I;H_0^1(\Omega))} \to \|\dot{z}\|_{L^2(I;H_0^1(\Omega))}$. The assertion follows from the weak convergence of z_n to z in $H_\star^1(I; H_0^1(\Omega))$. □

4.4. **Regularized optimal control problem.** In this section, let the pair $(\bar{z}, \bar{g}) \in H_\star^1(I; H_0^1(\Omega)) \times H_\star^1(I; L^2(\Omega))$ be a fixed local solution of (P). Then there is $\delta > 0$ such that $J(\bar{z}, \bar{g}) \leq J(z, g)$ for all (z, g) with $\|g - \bar{g}\|_{H^1(I;L^2(\Omega))} \leq \delta$ and satisfying (3). Let us consider the relaxed optimal control problem with the regularized state equation (17) as constraint

$$\begin{aligned} &\text{Minimize} && J(z,g) + \frac{1}{2}\|g - \bar{g}\|^2_{H^1(I;L^2(\Omega))} \\ &\text{subject to} && \|g - \bar{g}\|_{H^1(I;L^2(\Omega))} \leq \delta,\ g(0) = 0, \text{ and (17).} \end{aligned} \tag{P$_\rho$}$$

Note that $\bar{g}$ is a feasible control for this problem. With similar arguments as in the proof of Lemma 3.5 we can show the existence of global solutions of (P_ρ).

Lemma 4.10. *There exists a (global) optimal control of* (P_ρ).

Due to special construction of (P_ρ), we can prove convergence of global minimizers to the local solution $(\bar{z}, \bar{g})$.

Theorem 4.11. *Let* $\{(z_\rho, g_\rho)\}_{\rho>0}$ *denote a family of global solutions of* (P_ρ). *Then it holds* $g_\rho \to \bar{g}$ *and* $z_\rho \to \bar{z}$ *for* $\rho \searrow 0$ *in* $H_\star^1(I; L^2(\Omega))$ *and* $C(\bar{I}; H_0^1(\Omega))$, *respectively.*

Proof. Due to the constraints of (P_ρ), the controls $\{g_\rho\}_{\rho>0}$ are uniformly bounded in the space $H_\star^1(I; L^2(\Omega))$. Let now $\{\rho_k\}_{k\in\mathbb{N}}$ with $\rho_k > 0$ and $\rho_k \searrow 0$ such that g_{ρ_k} converges weakly in $H_\star^1(I; L^2(\Omega))$ to $\hat{g}$. By Lemma 4.7 the associated sequence $\{z_{\rho_k}\}_{k\in\mathbb{N}}$ converges weakly in $H_\star^1(I; H_0^1(\Omega))$ to $\hat{z}$, with $\hat{z} = \mathcal{S}(\hat{g})$, thus $(\hat{z}, \hat{g})$ satisfies the state equation (3). Moreover, $z_{\rho_k} \to \hat{z}$ in $C(\bar{I}; H_0^1(\Omega))$.

For $\rho > 0$ let $\bar{z}_\rho$ denote the solution of the regularized equation (17) to the fixed control $\bar{g}$. Then by Lemma 4.7, it holds $\bar{z}_\rho \to \bar{z}$ in $C(\bar{I}; H_0^1(\Omega))$. This implies the convergence $J(\bar{z}_\rho, \bar{g}) \to J(\bar{z}, \bar{g})$.

The optimality of g_{ρ_k} yields

$$J(z_{\rho_k}, g_{\rho_k}) + \frac{1}{2}\|g_{\rho_k} - \bar{g}\|^2_{H^1(I;L^2(\Omega))} \leq J(\bar{z}_{\rho_k}, \bar{g}).$$

Passing to the limit $k \to \infty$ it follows by lower-semicontinuity that

$$J(\hat{z}, \hat{g}) + \frac{1}{2}\|\hat{g} - \bar{g}\|^2_{H^1(I;L^2(\Omega))} \leq J(\bar{z}, \bar{g}).$$

The optimality of $(\bar z, \bar g)$ implies $J(\bar z, \bar g) \le J(\hat z, \hat g)$, which yields $\hat g = \bar g$ and $\hat z = \bar z$. Moreover, the strong convergence $g_\rho \to \bar g$ in $H^1_\star(I; L^2(\Omega))$ follows from

$$\limsup_{k\to\infty} \frac{1}{2} \|g_{\rho_k} - \bar g\|^2_{H^1(I;L^2(\Omega))} \le \limsup_{k\to\infty} \big(J(\bar z_{\rho_k}, \bar g) - J(z_{\rho_k}, g_{\rho_k})\big) = 0. \qquad \square$$

This result shows that for all $\rho > 0$ which are sufficiently small the constraint $\|g - \bar g\|_{H^1(I;L^2(\Omega))} \le \delta$ of (P_ρ) is immaterial.

Remark 4.12. In view of the assumptions of Theorem 4.9, we could relax the constraint in (P) and (P_ρ) on $g(0)$ to $\|g(0)\|_{L^\infty(\Omega)} \le 1$.

4.5. **Regularized optimality system.** Let us now turn to the first-order optimality system of (P_ρ). At first, we study the regularity of solutions of the adjoint system to (17). For given $\rho > 0$ and $z_\rho \in H^1(I; H^1_0(\Omega))$ it reads

$$-\dot q_\rho = \Delta \xi_\rho + j_1'(z_\rho) \qquad \text{in } H^{-1}(\Omega), \text{ a.e. on } I, \tag{25a}$$

$$q_\rho(T) = j_2'(z_\rho(T)) \qquad \text{in } H^{-1}(\Omega), \tag{25b}$$

$$-\rho\, \Delta \xi_\rho + |\dot z_\rho|''_\rho\, \xi_\rho = q_\rho \qquad \text{in } H^{-1}(\Omega), \text{ a.e. on } I. \tag{25c}$$

With the help of the adjoint variables we will express derivatives of the reduced objective functional. Let us first prove existence and uniqueness of solutions.

Lemma 4.13. *Let $(z_\rho, g_\rho) \in H^1(I; H^1_0(\Omega)) \times H^1(I; H^{-1}(\Omega))$ be given. Then there exists a unique solution $(q_\rho, \xi_\rho) \in H^1(I; H^{-1}(\Omega)) \times L^2(I; H^1_0(\Omega))$ of the adjoint system* (25).

Proof. Using the operator $T_\rho'(g_\rho(t) + \Delta z_\rho(t))$, we can eliminate ξ_ρ with (25c) to rewrite (25a) as a differential equation in $H^{-1}(\Omega)$:

$$-\dot q_\rho(t) = \Delta T_\rho'(g_\rho(t) + \Delta z_\rho(t))\, q_\rho(t) + j_1'(z_\rho(t)) \quad \text{in } H^{-1}(\Omega), \text{ f.a.a. } t \in I.$$

By [21, Satz 1.3, p. 166], it follows that there exists a uniquely defined solution $q_\rho \in H^1(I; H^{-1}(\Omega))$. This implies $\xi_\rho := T_\rho'(\cdot)\, q_\rho \in L^2(I; H^1_0(\Omega))$. $\square$

As a consequence, we can derive first-order optimality conditions for (P_ρ).

Theorem 4.14. *For $\rho > 0$, let (z_ρ, g_ρ) be a local optimal solution for the regularized optimal control problem (P_ρ) with $\|g_\rho - \bar g\|_{H^1(I;L^2(\Omega))} < \delta$. Then there exist uniquely determined functions $(q_\rho, \xi_\rho) \in H^1(I; H^{-1}(\Omega)) \times L^2(I; H^1_0(\Omega))$ satisfying* (25) *as well as*

$$-2\,\ddot g_\rho + \ddot{\bar g} + 2\, g_\rho - \bar g + \xi_\rho = 0 \text{ in } H^{-1}(I; L^2(\Omega)), \quad g_\rho(0) = 0, \quad 2\, \dot g_\rho(T) - \dot{\bar g}(T) = 0. \tag{26}$$

Proof. Let $h \in H^1_\star(I; L^2(\Omega))$ be arbitrary. Since (z_ρ, g_ρ) is locally optimal for (P_ρ) and $\|g_\rho - \bar g\|_{H^1(I;L^2(\Omega))} < \delta$, we have

$$J'(z_\rho, g_\rho)\, (\mathcal{S}_\rho'(g_\rho)\, h, h) + (g_\rho - \bar g, h)_{H^1(I;L^2(\Omega))} = 0.$$

Here, we used the Fréchet differentiability of $\mathcal{S}_\rho$ from Theorem 4.4 and the Fréchet differentiability of J. We set $\zeta_\rho := \mathcal{S}_\rho'(g_\rho)\, h$. Using the structure of J, we get

$$j_1'(z_\rho)\, \zeta_\rho + j_2'(z_\rho(T))\, \zeta_\rho(T) + (2\, g_\rho - \bar g, h)_{H^1(I;L^2(\Omega))} = 0.$$

Now, we use the definition of the adjoint variables and (25a)–(25b) implies

$$\int_I \langle -\dot q_\rho - \Delta \xi_\rho, \zeta_\rho \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}\, \mathrm{d}t + \langle q_\rho(T), \zeta_\rho(T) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + (2\, g_\rho - \bar g, h)_{H^1(I;L^2(\Omega))} = 0.$$

Via integration by parts, we obtain

$$\int_I \langle q_\rho, \dot\zeta_\rho\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} - \langle \xi_\rho, \Delta\zeta_\rho\rangle_{H^1_0(\Omega),H^{-1}(\Omega)} \,\mathrm{d}t + (2\,g_\rho - \bar g, h)_{H^1(I;L^2(\Omega))} = 0.$$

Using (25c), we get

$$\int_I \langle -\rho\,\Delta\xi_\rho + |\dot z_\rho|''_\rho\,\xi_\rho, \dot\zeta_\rho\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} - \langle \xi_\rho, \Delta\zeta_\rho\rangle_{H^1_0(\Omega),H^{-1}(\Omega)} \,\mathrm{d}t \\ + (2\,g_\rho - \bar g, h)_{H^1(I;L^2(\Omega))} = 0.$$

Hence, the linearized state equation (21a)–(21b) yields

$$(\xi_\rho, h)_{L^2(I;L^2(\Omega))} + (2\,g_\rho - \bar g, h)_{H^1(I;L^2(\Omega))} = 0$$

for arbitrary $h \in H^1_\star(I;L^2(\Omega))$ and this is the weak formulation of (26). □

Let us now derive bounds on ξ_ρ and q_ρ that are explicit with respect to ρ.

Lemma 4.15. *Let $z_\rho \in C(\bar I; H^1_0(\Omega))$ be given. Let (q_ρ, ξ_ρ) be the associated solution of* (25)*. Then there is a constant $C > 0$ independent of ρ and z_ρ such that*

$$\|q_\rho\|_{L^\infty(I;H^{-1}(\Omega))} + \rho^{1/2}\,\|\xi_\rho\|_{L^2(I;H^1_0(\Omega))} + \big\||\dot z_\rho|''_\rho\,\xi_\rho^2\big\|^{1/2}_{L^1(Q)} \\ \le C\left(\|j_2'(z_\rho(T))\|_{H^{-1}(\Omega)} + \|j_1'(z_\rho)\|_{L^2(I;H^{-1}(\Omega))}\right). \quad (27)$$

Proof. Testing (25a) and (25c) by $(-\Delta)^{-1}q_\rho$ and ξ_ρ, respectively, adding both equations and integrating on (t,T) yields

$$\|q_\rho(t)\|^2_{H^{-1}(\Omega)} + \int_t^T \int_\Omega \rho\,|\nabla\xi_\rho|^2 + |\dot z_\rho|''_\rho\,\xi_\rho^2 \,\mathrm{d}x\,\mathrm{d}s \\ = \|j_2'(z_\rho(T))\|^2_{H^{-1}(\Omega)} + \int_t^T \langle j_1'(z_\rho), (-\Delta)^{-1}q_\rho\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}s \\ \le \|j_2'(z_\rho(T))\|^2_{H^{-1}(\Omega)} + \|j_1'(z_\rho)\|^2_{L^2(I;H^{-1}(\Omega))} + \int_t^T \|q_\rho(s)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}s.$$

The claim follows now from Gronwall's inequality. □

It remains to get an estimate for ξ_ρ.

Corollary 4.16. *Let $z_\rho \in C(\bar I; H^1_0(\Omega))$ be given. Let (q_ρ, ξ_ρ) be the associated solution of* (25)*. Then there is a constant $C > 0$ independent of ρ and z_ρ such that it holds*

$$\|\xi_\rho\|_{W^{-1,p}(I;H^1_0(\Omega))} \le C\left(\|j_2'(z_\rho(T))\|_{H^{-1}(\Omega)} + \|j_1'(z_\rho)\|_{L^p(I;H^{-1}(\Omega))}\right).$$

for all $p \in [2,\infty)$.

Proof. For all $v \in W^{1,p'}(I; H^{-1}(\Omega))$, where $1 = 1/p + 1/p'$, by (25a) and (25b) we have

$$\begin{aligned}
\langle \xi_\rho, v\rangle_{W^{-1,p}(I;H_0^1(\Omega)),W^{1,p'}(I;H^{-1}(\Omega))} &= \int_I \langle \xi_\rho, v\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}\,\mathrm{d}t \\
&= \int_I \big\langle \Delta^{-1}\left(-\dot q_\rho + j_1'(z_\rho)\right), v\big\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}\,\mathrm{d}t \\
&= \int_I \big\langle \Delta^{-1} q_\rho, \dot v\big\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}\,\mathrm{d}t - \big\langle \Delta^{-1} q_\rho(T), v(T)\big\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \\
&\quad + \big\langle \Delta^{-1} q_\rho(0), v(0)\big\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + \int_I \big\langle \Delta^{-1} j_1'(z_\rho), v\big\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}\,\mathrm{d}t \\
&\le C\left(\|q_\rho\|_{L^\infty(I;H^{-1}(\Omega))} + \|j_2'(z_\rho(T))\|_{H^{-1}(\Omega)}\right)\|v\|_{W^{1,1}(I;H^{-1}(\Omega))} \\
&\quad + \|j_1'(z_\rho)\|_{L^p(I;H^{-1}(\Omega))}\,\|v\|_{W^{1,p'}(I;H^{-1}(\Omega))}.
\end{aligned}$$

Hence, the claim follows from (27). □

5. Passing to the limit.

In this final section we investigate the limit $\rho \searrow 0$ and prove our main result, namely Theorem 5.2.

Lemma 5.1. *Let $\{z_\rho\}_{\rho>0}$ be given such that $z_\rho \to z$ in $H^1_\star(I; H_0^1(\Omega))$. Further, let $\{(q_\rho, \xi_\rho)\}_{\rho>0}$ be the family of solutions of the adjoint system* (25)*. If $q_\rho \rightharpoonup q$ in $L^2(I; H^{-1}(\Omega))$ then it holds*

$$\langle q, \phi\,|\dot z|\rangle = 0 \quad \forall \phi \in C_0^\infty(Q).$$

Proof. Let $\phi \in C_0^\infty(Q)$. Then it holds $\phi\,|\dot z_\rho| \in L^2(I; H_0^1(\Omega))$, and $\nabla(\phi\,|\dot z_\rho|)$ is bounded in $L^2(Q)$ uniformly with respect to ρ. Testing (25c) with $\phi\,|\dot z_\rho|$ yields

$$\langle q_\rho, \phi\,|\dot z_\rho|\rangle = \int_Q \rho \nabla\xi_\rho \nabla(\phi\,|\dot z_\rho|) + |\dot z_\rho|''_\rho\,\xi_\rho\,\phi\,|\dot z_\rho|\,\mathrm{d}x\,\mathrm{d}t.$$

The first addend on the right-hand side tends to zero for $\rho \searrow 0$ as $\rho^{1/2}\,\nabla\xi_\rho$ and $\nabla(\phi\,|\dot z_\rho|)$ are bounded in $L^2(Q)$ uniformly in ρ, see Lemma 4.15. To bound the second addend, observe that $\sqrt{|\dot z_\rho|''_\rho}\,|\dot z_\rho|$ is pointwise bounded by $\rho^{1/2}$ due to Assumption 4.1. As $\sqrt{|\dot z_\rho|''_\rho}\,\xi_\rho$ is uniformly bounded in $L^2(Q)$ by Lemma 4.15, the second addend vanishes for $\rho \searrow 0$ as well. □

In particular, Lemma 5.1 shows that (8a) and (8e) hold in a distributional sense. We are now in the position to formulate and prove the main result of this article.

Theorem 5.2. *Let $(\bar z, \bar g)$ be a local solution of* (P)*. Then there are adjoint variables $q \in L^\infty(I; H^{-1}(\Omega))$ and $\xi \in W^{-1,p}(I; H_0^1(\Omega))$ (for all $p \in [2, \infty)$) such that*

$$-\dot q = \Delta\xi + j_1'(\bar z), \tag{28a}$$

$$q(T) = j_2'(\bar z(T)), \tag{28b}$$

$$-\ddot{\bar g} + \bar g + \xi = 0, \quad \bar g(0) = 0, \quad \dot{\bar g}(T) = 0, \tag{28c}$$

$$\langle q, \phi\,|\dot{\bar z}|\rangle = 0 \qquad \forall \phi \in C_0^\infty(Q) \tag{28d}$$

is satisfied. Here, (28a)–(28b) *have to be understood in the following very weak sense: For all* $\phi \in H^1_\star(\bar{I}; H^1_0(\Omega))$ *it holds*

$$\int_I \langle q, \dot{\phi}\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t - \langle \phi, \Delta\xi\rangle_{H^1(I;H^1_0(\Omega)),H^{-1}(I;H^{-1}(\Omega))}$$
$$= \langle j_2'(\bar{z}(T)), \phi(T)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_I \langle j_1'(\bar{z}), \phi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t. \quad (28e)$$

Proof. Let (z_ρ, g_ρ) be a family of global solutions of (P_ρ) such that $g_\rho \to \bar{g}$ and $z_\rho \to \bar{z}$ for $\rho \searrow 0$ in $H^1(I; L^2(\Omega))$ and $C(\bar{I}; H^1_0(\Omega))$, respectively. This is possible due to Theorem 4.11. Let now (q_ρ, ξ_ρ) be the associated adjoint states provided by Theorem 4.14.

Due to Lemma 4.15 and Corollary 4.16, we find that the dual quantities (q_ρ, ξ_ρ) are uniformly bounded in $L^\infty(I; H^{-1}(\Omega)) \times W^{-1,p}(I; H^1_0(\Omega))$, respectively. Thus we can pass to the limit in (26) and (25a) to obtain (28c) and (28e). Lemma 5.1 proves (28d). □

Remark 5.3. 1. In Theorem 5.2, we have rigorously checked relations (8a) and (8e) from (8). As a next step, it would be desirable to prove also (8c) in variational terms. However, due to the low regularity of the involved quantities, $g \in H^1(I; L^2(\Omega))$, $\Delta z \in H^1(I; H^{-1}(\Omega))$ and $\xi \in W^{-1,p}(I; H^1_0(\Omega))$ it is not clear how (8c) could be formulated.

Let us highlight two possible approaches which might be tractable. First, one could take $\phi \in H^1(I; L^2(\Omega))$ with $\xi = 0$ on the set where $|g + \Delta z| = 1$ and show $\langle \phi, \xi\rangle = 0$. However, since $g_\rho + \Delta z_\rho$ does only converge in $H^1(I; H^{-1}(\Omega))$, it is not clear how this complementarity could be derived.

Alternatively, one could choose $\varphi \in C^\infty(\mathbb{R}; \mathbb{R})$ with $\mathrm{supp}(\varphi) \subset [-1, 1]$ and show $\langle \varphi(g + \Delta z), \xi\rangle = 0$. However, this formulation requires $\varphi(g + \Delta z) \in H^1(I; L^2(\Omega))$, thus $z \in H^1(I; H^2(\Omega))$. This regularity of z, however, seems to be not available.

2. The low regularity of the adjoint variables q, ξ is not surprising for it has already been observed in connection with other optimality systems for rate-independent evolutions, see, e.g., [3, Satz 8.12], [5, Theorem 5.2], [49, Theorem 3.1].

6. Conclusions and outlook.

In this work, we have derived optimality conditions for the optimal control of a rate-independent process. The full set of conditions has been formally derived and we have succeeded in presenting rigorous arguments for the validity of a specific subset of those.

The verification of the remaining optimality conditions as well as their possible validity in a stronger regularity setting will be the object of further research. A time-discretization or a decoupling of the smoothing of $|\cdot|$ and the additional viscosity might offer the chance of deriving the complementarity (8c) as well. Note however that this task is challenging by the low regularity of the adjoint variables.

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E-mail address: `ulisse.stefanelli@univie.ac.at`
E-mail address: `daniel.wachsmuth@mathematik.uni-wuerzburg.de`
E-mail address: `gerd.wachsmuth@mathematik.tu-chemnitz.de`