ADAPTIVE FINITE ELEMENT METHOD FOR THE MAXWELL EIGENVALUE PROBLEM

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Abstract. In this paper we prove the optimal convergence of a standard adaptive scheme based on edge finite elements for the approximation of the solutions of the eigenvalue problem associated with Maxwell's equations. The proof uses the known equivalence of the problem of interest with a mixed eigenvalue problem.

1. Introduction

In this paper we present an adaptive scheme, based on standard three dimensional edge elements, for the approximation of the Maxwell eigenvalue problem and analyze its convergence.

A posteriori error estimates for Maxwell's equations have been studied by several authors for the source problem (see, in particular [32, 3, 37, 19, 36, 20, 38, 22, 14, 23, 42, 15, 21, 18] and the references therein). The eigenvalue problem has been studied only recently in [13, 12] where residual type error indicators are considered and proved to be equivalent to the actual error in the standard framework of efficiency and reliability. The analysis relies on the classical equivalence with a mixed variational formulation [10]. The numerical results presented in [12] confirm that the adaptive scheme driven by our error indicator converges in three dimensions with optimal rate with respect to the number of degrees of freedom.

In [11] it was presented the first convergence analysis for an adaptive scheme applied to the Laplace eigenvalue problem in mixed form. The main tools for the analysis originate from various papers related to adaptive finite elements, in particular from [17, 29]. Thanks to the well-known isomorphism between the spaces $\mathbf{H}(\mathrm{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$ in two space dimensions, the result for the Laplacian implies the convergence of the 2D adaptive scheme for Maxwell's eigenproblem: actually, the isomorphism carries over to the corresponding mixed formulation as well as to the error indicators. In this paper we extend the results of [11] to the mixed formulation associated with Maxwell's eigenproblem in three dimensions; as we will notice, such extension is not trivial: several technical details have to be checked and suitably designed interpolation operators are used to complete the analysis. Useful results in this direction are reported in [38, 43].

It is well understood that the convergence analysis of the adaptive scheme for eigenvalue problems has to consider multiple eigenvalues and clusters of eigenvalues in order to prevent subtoptimal convergence. In particular, degeneracy of the convergence may be observed when the error indicator is computed by taking into

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account only a subset of the discrete eigenmodes approximating the eigensolutions we are interested in (multiple or belonging to a cluster) [40, 28, 9].

Starting from this remark, the analysis performed in [11] has been carried on for generic clusters of eigenvalues. This approach has the inconvenience of adding a heavy notation dealing with deep technicalities. For this reason, we decided in this paper to develop our theory in the case of simple eigenvalues. We believe that the presentation in the case of a simple eigenvalue highlights better the novelties with respect to the previous results for the mixed Laplacian, that would rather be hidden by the technical machinery related to clusters of eigenvalues. Nevertheless, the general case can be dealt with by using similar arguments as in [11].

In Section 2 we recall Maxwell's eigenvalue problem, its standard variational formulation, and the equivalent mixed formulation, together with their finite element discretizations. Section 3 defines our error indicator and describes the adaptive scheme. Reliability and efficiency from [12] are recalled and the theory concerning the convergence of the adaptive method is described. The auxiliary results needed for the convergence proof are collected in Section 4. These include in particular discrete reliability, quasi-orthogonality, and contraction property.

2. Maxwell's eigenvalue problem and its finite element discretization

In this paper we deal with the well known eigenvalue problem associated with the Maxwell equations (see, for instance, [30, 33, 7]).

Given a polyhedral domain Ω, after eliminating the magnetic field, the problem reads: find $\omega \in \mathbb{R}$ and $\boldsymbol{u} \neq 0$ such that

$$\begin{aligned} &\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}) = \omega^2\varepsilon\boldsymbol{u} && \text{in } \Omega \\ &\operatorname{div}(\varepsilon\boldsymbol{u}) = 0 && \text{in } \Omega \\ &\boldsymbol{u}\times\boldsymbol{n} = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ represents the electric field, μ and ε the magnetic permittivity and electric permeability, respectively, and $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$, the boundary of Ω. For general inhomogeneous, anisotropic materials μ and ε are 3-by-3 positive definite and bounded matrix functions. We are considering for simplicity the case when Ω is simply connected: more general situations will be described in Remark 3.

A standard variational formulation of our eigenvalue problem is obtained by considering the functional space $\mathbf{H}_0(\mathbf{curl};\Omega)$ of vector fields in $L^2(\Omega)^3$ with $\mathbf{curl}$ in $L^2(\Omega)^3$ and vanishing tangential component along $\partial\Omega$. The formulation reads: find $\omega \in \mathbb{R}$ with $\omega > 0$ and $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ with $\boldsymbol{u} \neq 0$ such that

$$(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) = \omega^2(\varepsilon\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega). \tag{2}$$

It is well known, in particular, that the condition $\omega^2 \neq 0$ is equivalent to the divergence condition $\operatorname{div}(\varepsilon\boldsymbol{u}) = 0$ due to the Helmholtz decomposition (see also Remark 1). We assume that the domain Ω and the coefficients ε, μ are such that the problem is associated with a compact solution operator. The eigenvalues can then be numbered in an increasing order as follows:

$$0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_j \leq \ldots,$$

where the same eigenvalue is repeated as many times as its algebraic multiplicity. The associated eigenfunctions are denoted by $\boldsymbol{u}_j$ and normalized according to the L^2 norm, that is $\|\varepsilon^{1/2}\boldsymbol{u}_j\|_0 = 1$.

A powerful tool for the analysis of this problem is a mixed formulation introduced in [10]. With the notation $\boldsymbol{\sigma} = \omega\boldsymbol{u}$, $\boldsymbol{p} = -\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{u}/\omega$, and $\lambda = \omega^2$, the variational formulation (2) is equivalent to the following mixed eigenproblem: find $\lambda \in \mathbb{R}$ and $(\boldsymbol{\sigma}, \boldsymbol{p}) \in \mathbf{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{Q}$ with $\boldsymbol{p} \neq 0$ such that

$$\begin{aligned}&(\varepsilon\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{p}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl};\Omega)\\&(\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\sigma}, \boldsymbol{q}) = -\lambda(\boldsymbol{p}, \boldsymbol{q}) \qquad \forall \boldsymbol{q} \in \boldsymbol{Q},\end{aligned} \tag{3}$$

where $\boldsymbol{Q} = \mu^{-1/2}\,\mathbf{curl}(\mathbf{H}_0(\mathbf{curl};\Omega))$.

The eigenvalues of (3) are denoted by

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \dots.$$

Given j, we use the notation $\boldsymbol{p}_j = -\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{u}_j/\omega_j$ and $\boldsymbol{\sigma}_j = \omega_j\boldsymbol{u}_j$ with $\lambda_j = \omega_j^2$, so that $(\lambda_j, \boldsymbol{\sigma}_j, \boldsymbol{p}_j)$ solves (3) and the following normalization holds true for the eigenfunction: $\|\boldsymbol{p}_j\|_0 = 1$.

The finite element approximation of (2) is usually performed with edge elements. Given a tetrahedral decomposition of Ω, we consider Nédélec edge elements introduced in [34, 35]. More general families of finite elements could be considered in the spirit of [2]. More precisely, the general situation can be described by adopting the following standard notation related to de Rham complex:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{H}^1_0(\Omega) & \xrightarrow{\nabla} & \mathbf{H}_0(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \mathbf{H}_0(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \mathbb{R}\\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_h & \xrightarrow{\nabla} & \boldsymbol{\Sigma}_h & \xrightarrow{\mathbf{curl}} & \boldsymbol{F}_h & \xrightarrow{\mathrm{div}} & DG_h & \longrightarrow & \mathbb{R}. \end{array} \tag{4}$$

In the case when $\boldsymbol{\Sigma}_h$ is a sequence of tetrahedral edge finite elements the remaining finite element spaces will be composed by nodal Lagrange elements N_h, face elements $\boldsymbol{F}_h$, and discontinuous elements DG_h, respectively. The corresponding diagrams in the case of Nédélec elements of the first and second family can be found in (2.5.58) and (2.5.59) of [8], respectively.

The discretization of (2) reads: find $\omega_h \in \mathbb{R}$ with $\omega_h > 0$ and $\boldsymbol{u}_h \in \boldsymbol{\Sigma}_h$ with $\boldsymbol{u}_h \neq 0$ such that

$$(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{v}) = \omega_h^2(\varepsilon\boldsymbol{u}_h, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{\Sigma}_h. \tag{5}$$

The corresponding mixed formulation is: find $\lambda_h \in \mathbb{R}$ and $(\boldsymbol{\sigma}_h, \boldsymbol{p}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{Q}_h$ with $\boldsymbol{p}_h \neq 0$ such that

$$\begin{aligned}&(\varepsilon\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\tau}, \boldsymbol{p}_h) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\\&(\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\sigma}_h, \boldsymbol{q}) = -\lambda_h(\boldsymbol{p}_h, \boldsymbol{q}) \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}_h,\end{aligned} \tag{6}$$

where $\boldsymbol{Q}_h = \mu^{-1/2}\,\mathbf{curl}(\boldsymbol{\Sigma}_h)$. In particular, we have that $\boldsymbol{Q}_h$ is a subspace of $\mu^{-1/2}\boldsymbol{F}_h$ and it can be easily seen that $\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\sigma}_h = -\lambda_h\boldsymbol{p}_h$.

Following [10, Th. 2.1], the equivalence between (5) and (6) can be proved using the definition of $\boldsymbol{Q}_h$ and the identities $\boldsymbol{\sigma}_h = \omega_h\boldsymbol{u}_h$, $\boldsymbol{p}_h = -\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{u}_h/\omega_h$, and $\lambda_h = \omega_h^2$.

With natural notation, we denote by $0 < \omega_{h,1} \le \omega_{h,2} \le \cdots \le \omega_{h,N(h)}$ the eigenvalues of (5) and by $0 < \lambda_{h,1} \le \lambda_{h,2} \le \cdots \le \lambda_{h,N(h)}$ those of (6). Analogously, the corresponding eigenfunctions are denoted by $\boldsymbol{u}_{h,j}$ and $(\boldsymbol{\sigma}_{h,j}, \boldsymbol{p}_{h,j})$, respectively $(j = 1, \dots, N(h))$ with $\|\varepsilon^{1/2}\boldsymbol{u}_{h,j}\|_0 = \|\boldsymbol{p}_{h,j}\|_0 = 1$. The number of discrete frequencies, repeated according to their multiplicity, is given by $N(h) = \dim \boldsymbol{Q}_h$. We discuss this fact in the next remark.

Remark 1. It is straightforward to check that the number of real eigenvalues of problem (6) is equal to $N(h) = \dim \boldsymbol{Q}_h$. Indeed, the matrix form of (6) is, with obvious notation,

$$\begin{pmatrix} \mathsf{A} & \mathsf{B}^\top \\ \mathsf{B} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mathsf{p} \end{pmatrix} = \lambda_h \begin{pmatrix} 0 & 0 \\ 0 & -\mathsf{M} \end{pmatrix} \begin{pmatrix} \sigma \\ \mathsf{p} \end{pmatrix}.$$

The number of real eigenvalues of this problem is equal to the size $N(h)$ of the matrix M, as it is evident by looking at its equivalent formulation written in terms of the Schur complement

$$\mathsf{B}\mathsf{A}^{-1}\mathsf{B}^\top \mathsf{p} = \lambda_h \mathsf{M}\mathsf{p}$$
$$\sigma = -\mathsf{A}^{-1}\mathsf{B}^\top \mathsf{p}.$$

The size of the matrix problem corresponding to (5) is equal to the dimension of the space $\boldsymbol{\Sigma}_h$. The Helmholtz decomposition in the case of simply connected domains implies that $\dim(\boldsymbol{\Sigma}_h) = \dim(\boldsymbol{\nabla}(N_h)) + N(h)$. Since the space $\boldsymbol{\nabla}(N_h)$ is the kernel of the **curl** operator, it follows that the number of eigenvalues corresponding to $\omega_h > 0$ is equal to $N(h)$. For an additional discussion about this count when the domain is multiply connected, the reader is referred to Remark 3.

Remark 2. It can be useful to recall that the mixed formulations (3) and (6) are not used for the definition of the method (nor for its implementation), but are crucial ingredients for its analysis.

Remark 3. It is well known that if the domain is not topologically trivial, then the first row of the diagram presented in Equation (4) is not an exact sequence. More precisely, the following space of *harmonic forms* plays an important role

$$\mathcal{H} = \{\mathfrak{h} \in \mathbf{H}_0(\mathbf{curl}; \Omega) : \mathbf{curl}\, \mathfrak{h} = 0,\ \mathrm{div}(\varepsilon \mathfrak{h}) = 0 \text{ in } \Omega\}$$

and corresponds to the one form cohomology of the de Rham complex. The Helmholtz decomposition in this case has the following form:

$$L^2(\Omega)^3 = \boldsymbol{\nabla}(\mathrm{H}^1_0(\Omega)) \oplus \mathcal{H} \oplus \varepsilon^{-1}\, \mathbf{curl}(\mathbf{H}(\mathbf{curl}; \Omega)),$$

where the three components of the decomposition are ε-orthogonal, that is they are orthogonal with respect to the scalar product $(\varepsilon\, \cdot, \cdot)$.

It turns out that in the general case the formulation (2) is not the variational formulation of (1) any more. Indeed, functions in $\mathcal{H}$ are eigenfunctions of (1) with vanishing frequency. In this case, if we are not interested in the approximation of the space of harmonic functions $\mathcal{H}$, we can disregard the zero frequency and use formulations (2) and (3) for the analysis of the rest of the spectrum. The approximation of harmonic functions is out of the aims of this paper. We point the reader to possible approaches for the approximation of $\mathcal{H}$: a direct discretization of the space has been proposed in [1]; an adaptive algorithm has been presented in [25]; another indirect approach may be the use of the following alternative mixed formulation known as *Kikuchi formulation* (see [31, 6]): find $\lambda \in \mathbb{R}$ such that for $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ and $p \in \mathrm{H}^1_0(\Omega)$, with $\boldsymbol{u} \neq 0$, it holds

$$\begin{aligned} &(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) + (\boldsymbol{\nabla}\, p, \varepsilon \boldsymbol{v}) = \lambda(\varepsilon \boldsymbol{u}, \boldsymbol{v}) && \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega)\\ &(\boldsymbol{\nabla}\, q, \varepsilon \boldsymbol{u}) = 0 && \forall q \in \mathrm{H}^1_0(\Omega). \end{aligned}$$

It is not difficult to see that any solution of the Kikuchi formulation satisfies $p = 0$ (take $\boldsymbol{v} = \boldsymbol{\nabla}\, p$ in the first equation). Hence, it is immediate to check that the Kikuchi formulation is equivalent to the standard variational formulation (2) with the additional solution $\lambda = 0$ corresponding to $\boldsymbol{u} \in \mathcal{H}$.

3. Error indicator and adaptive method

We are going to study and analyze an adaptive finite element scheme in the framework of [26, 24, 17, 29, 11]. The scheme is based on the following local error indicator (see [12])

$$\begin{aligned} \tilde\eta_K^2 &= h_K^2 \|\varepsilon \boldsymbol{u}_h - \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h)/\omega_h^2\|^2_{0,K} + h_K^2 \|\operatorname{div}(\varepsilon \boldsymbol{u}_h)\|^2_{0,K} \\ &\quad + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left[h_F \left\| [\![(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h/\omega_h^2) \times \boldsymbol{n}]\!] \right\|^2_{0,F} + h_F \left\| [\![\varepsilon \boldsymbol{u}_h \cdot \boldsymbol{n}]\!] \right\|^2_{0,F} \right], \end{aligned} \tag{7}$$

where K is an element of our triangulation $\mathcal{T}_h$, $\mathcal{F}_{\mathrm{I}}(K)$ is the set of inner faces of K, h_K and h_F the diameters of K, and F, respectively, and $[\![\cdot]\!]$ the jump across an inner face F.

Given a set of elements $\mathcal{M}$, we use the notation

$$\tilde\eta(\mathcal{M})^2 = \sum_{K \in \mathcal{M}} \tilde\eta_K^2$$

and we write $\tilde\eta = \tilde\eta(\mathcal{T}_h)$ for the global error indicator when no confusion arises. Moreover, a subscript κ is used when $\tilde\eta_\kappa$ refers to the mesh $\mathcal{T}_\kappa$.

Given an initial mesh $\mathcal{T}_0$ and a bulk parameter $\theta \in \mathbb{R}$, with $0 < \theta \le 1$, we compute a sequence of meshes $\{\mathcal{T}_\ell\}$, solutions $\{(\omega_\ell^2, \boldsymbol{u}_\ell)\}$, and indicators $\{\tilde\eta(\mathcal{T}_\ell)\}$ according to the standard solve/estimate/mark/refine strategy (see [26]). In particular, at a given level ℓ, the marking step consists in choosing a minimal subset $\mathcal{M}_\ell$ of $\mathcal{T}_\ell$ such that

$$\theta \tilde\eta_\ell^2(\mathcal{T}_\ell) \le \tilde\eta_\ell^2(\mathcal{M}_\ell).$$

The new mesh $\mathcal{T}_{\ell+1}$ is given by the smallest admissible refinement of $\mathcal{T}_\ell$ satisfying $\mathcal{M}_\ell \cap \mathcal{T}_{\ell+1} = \emptyset$ according to the rules defined in [4, 41].

Considering the equivalence between the standard formulation (5) and the mixed formulation (6), the local error indicator for the mixed problem takes the following form:

$$\begin{aligned} \eta_K^2 &= h_K^2 \|\varepsilon \boldsymbol{\sigma}_h + \mathbf{curl}(\mu^{-1/2}\boldsymbol{p}_h)\|^2_{0,K} + h_K^2 \|\operatorname{div}(\varepsilon \boldsymbol{\sigma}_h)\|^2_{0,K} \\ &\quad + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left(h_F \left\| [\![(\mu^{-1/2}\boldsymbol{p}_h) \times \boldsymbol{n}]\!] \right\|^2_{0,F} + h_F \left\| [\![\varepsilon \boldsymbol{\sigma}_h \cdot \boldsymbol{n}]\!] \right\|^2_{0,F} \right) \end{aligned} \tag{8}$$

It is easy to check that the following relation between the two indicators holds true

$$\tilde\eta_K^2 = \frac{1}{\lambda_h} \eta_K^2 \quad \forall K \in \mathcal{T}_h.$$

In particular, the comments stated in Remark 2 can be extended to the error indicators: our analysis will be performed by using the mixed formulation (6) and the indicator (8) even if the scheme is originally defined in terms of the standard formulation (5) and the indicator (7).

In the rest of this section we present our main result in the case of an eigenvalue of multiplicity one, since we believe that in this case it is easier to describe the main arguments leading to the optimal convergence of the adaptive scheme. Moreover, for ease of notation, we assume from now on that ε and μ are scalar and $\varepsilon = \mu = 1$. This assumption does not reduce the relevance of our result: more general situations can be dealt with by adopting similar arguments as in [10] or [16].

Let $\omega = \omega_j$ be a simple eigenvalue of (2) and $\widetilde{W} = \mathrm{span}\{\boldsymbol{u}_j\}$ the associated one-dimensional eigenspace. Let $\omega_\ell = \omega_{\ell,j}$ be the j-th discrete eigenvalue of (5) computed with the adaptive scheme on the mesh $\mathcal{T}_\ell$ and $\widetilde{W}_\ell = \mathrm{span}\{\boldsymbol{u}_{\ell,j}\}$ the corresponding eigenspace. The gap between $\widetilde{W}$ and $\widetilde{W}_\ell$ is measured by

$$\delta(\widetilde{W}, \widetilde{W}_\ell) = \sup_{\substack{\boldsymbol{u}\in\widetilde{W}\\ \|\boldsymbol{u}\|_{\mathbf{curl}}=1}} \inf_{\boldsymbol{u}_\ell\in\widetilde{W}_\ell} \|\boldsymbol{u} - \boldsymbol{u}_\ell\|_{\mathbf{curl}}.$$

For the reader's convenience, we recall the reliability and efficiency properties proved in [12]. As it is common for eigenvalue problems, the efficiency property is not local in the sense that it relies on the difference between ω and ω_h which is a global quantity.

Proposition 1. *Let $(\omega, \boldsymbol{u})$ and $(\omega_h, \boldsymbol{u}_h)$ be solutions of Problems 2 and 5, respectively, such that the latter approximates the former as h goes to zero. Then, there exists C such that for h small enough*

Reliability:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{curl}} \le C\tilde{\eta} \qquad |\omega^2 - \omega_h^2| \le C\tilde{\eta}^2.$$

Efficiency:

$$\tilde{\eta}_K \le C\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,K'} + \|\mathbf{curl}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,K'} + h_K \left\|\omega^2\boldsymbol{u} - \omega_h^2\boldsymbol{u}_h\right\|_{0,K'}\right),$$

where K' denotes the union of the elements sharing a face with K.

Proof. See Propositions 5 and 6 of [12]. □

The convergence of the adaptive scheme is usually described by making use of the nonlinear approximation classes discussed in [4]. Denoting by $\mathbb{T}(m)$ the set of admissible refinements of $\mathcal{T}_0$ whose cardinality differs from that of the initial triangulation by less than m, the best algebraic convergence rate $s \in (0, +\infty)$ for the approximation of functions belonging to a space $\boldsymbol{\mathcal{W}}$ is characterized in terms of the following seminorm

$$|\boldsymbol{\mathcal{W}}|_{\mathcal{A}_s} = \sup_{m\in\mathbb{N}} m^s \inf_{\mathcal{T}\in\mathbb{T}(m)} \delta(\boldsymbol{\mathcal{W}}, \boldsymbol{\Sigma}_{\mathcal{T}}),$$

where $\boldsymbol{\Sigma}_{\mathcal{T}}$ is the edge finite element space on the mesh $\mathcal{T}$.

The main result of our paper, stated in the next theorem, shows that if $\widetilde{W}$ has bounded $\mathcal{A}_s$-seminorm for some s, then the optimal convergence order s is obtained by the sequence of solutions constructed by the adaptive procedure described above.

Theorem 2. *Provided the meshsize of the initial mesh $\mathcal{T}_0$ and the bulk parameter θ are small enough, if the eigenspace satisfies $|\widetilde{W}|_{\mathcal{A}_s} < \infty$, then the sequence of* *discrete eigenspaces $\widetilde{W}_\ell$ computed on the mesh $\mathcal{T}_\ell$ fulfills the optimal estimate*

$$\delta(\widetilde{W},\widetilde{W}_\ell) \le C(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^{-s}|\widetilde{W}|_{\mathcal{A}_s}.$$

Moreover, the eigenvalue satisfies the optimal double order rate of convergence

$$|\omega - \omega_\ell| \le C\delta(\widetilde{W},\widetilde{W}_\ell)^2.$$

The proof of Theorem 2 is based on the corresponding result written in terms of the mixed formulations (3) and (6).

Let $\lambda = \lambda_j$ be a simple eigenvalue of (3) and $W = \mathrm{span}\{(\boldsymbol{\sigma}_j, \boldsymbol{p}_j)\}$ the associated one-dimensional eigenspace. Let $\lambda_\ell = \lambda_{\ell,j}$ be the j-th discrete eigenvalue corresponding to the ℓ-th level of refinement in the adaptive scheme and $W_\ell = \mathrm{span}\{(\boldsymbol{\sigma}_{\ell,j}, \boldsymbol{p}_{\ell,j})\}$ the associated eigenspace. The gap between W and W_ℓ is measured by

$$\delta(W, W_\ell) = \sup_{\substack{(\boldsymbol{\sigma},\boldsymbol{p})\in W\\ \|\boldsymbol{p}\|_0=1}} \inf_{(\boldsymbol{\sigma}_\ell,\boldsymbol{p}_\ell)\in W_\ell} \left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_\ell\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_\ell\|_0^2\right)^{1/2}.$$

We recall the reliability and efficiency properties proved in [12]. It turns out that in the case of the mixed formulation it is possible to obtain a local efficiency estimate.

Proposition 3. *Let $(\lambda, \boldsymbol{\sigma}, \boldsymbol{p})$ and $(\lambda_h, \boldsymbol{\sigma}_h, \boldsymbol{p}_h)$ be solutions of Problems (3) and (6), respectively, such that the latter approximates the former as h goes to zero.*

Reliability: *there exist $\rho_{\mathrm{rel1}}(h)$ and $\rho_{\mathrm{rel2}}(h)$ tending to zero as $h \to 0$ and positive constants C independent of the mesh size such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 \le C\eta + \rho_{\mathrm{rel1}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0)$$
$$|\lambda - \lambda_h| \le C\eta^2 + \rho_{\mathrm{rel2}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0)^2.$$

Efficiency: *for each $K \in \mathcal{T}_h$,*

$$\eta_K \le C(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K'} + \|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,K'}),$$

where K' is the union of the tetrahedra sharing a face with K.

Proof. See Theorems 3 and 4 of [12]. The estimate for $|\lambda - \lambda_h|$ is an immediate consequence of (10) (see next section).

□

The counterpart of Theorem 2 in the framework of the mixed formulation is stated as follows.

Theorem 4. *Provided the meshsize of the initial mesh $\mathcal{T}_0$ and the bulk parameter θ are small enough, if the eigenspace satisfies $|W|_{\mathcal{A}_s} < \infty$, then the sequence of discrete eigenspaces W_ℓ corresponding to the solution computed on the mesh $\mathcal{T}_\ell$ fulfills the optimal estimate*

$$\delta(W, W_\ell) \le C(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^{-s}|W|_{\mathcal{A}_s}.$$

Moreover, the eigenvalue satisfies the optimal double order rate of convergence

$$|\omega - \omega_\ell| \le C\delta(W, W_\ell)^2.$$

The proof of our main result has the same structure as the one presented in [11], based on [28] and [17]. For this reason, we do not repeat it here, but we conclude this section by listing some keystone properties that are essential for the proof of our main result. We refer the interested reader to [11] and to the references therein for a rigorous proof of how to combine them in order to get the result of Theorem 4.

The following properties involve quantities related to meshes that will be denoted by $\mathcal{T}_H$, $\mathcal{T}_h$, or $\mathcal{T}_\ell$. In general, $\mathcal{T}_h$ denotes an arbitrary refinement of a fixed mesh $\mathcal{T}_H$, while $\mathcal{T}_\ell$ refers to the sequence of meshes designed by the adaptive procedure. The eigenmode approximating $\{\lambda, (\boldsymbol{\sigma}, \boldsymbol{p})\}$ will be indicated by $\{\lambda_\kappa, (\boldsymbol{\sigma}_\kappa, \boldsymbol{p}_\kappa)\}$ where κ may be H, h, or ℓ, respectively. We assume that the sign of $(\boldsymbol{\sigma}_\kappa, \boldsymbol{p}_\kappa)$ is chosen in such a way that the scalar product between $\boldsymbol{p}$ and $\boldsymbol{p}_\kappa$ is positive (so that the same is true for the scalar product between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_\kappa$).

Property 1 (Discrete reliability). There exists a constant C_{drel} and a function $\rho_{\mathrm{drel}}(H)$ tending to zero as H goes to zero, such that, for a sufficiently fine mesh $\mathcal{T}_H$ and for all refinements $\mathcal{T}_h$ of $\mathcal{T}_H$, it holds

$$\begin{aligned}\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0 &\le C_{\mathrm{drel}}\eta_H(\mathcal{T}_H \setminus \mathcal{T}_h)\\ &+ \rho_{\mathrm{drel}}(H)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_H\|_0).\end{aligned}$$

Property 2 (Quasi-orthogonality). There exists a function $\rho_{\mathrm{qo}}(h)$ tending to zero as h goes to zero, such that

$$\begin{aligned}\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 &\le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_H\|_0^2 - \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 - \|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2\\ &+ \rho_{\mathrm{qo}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_H\|_0^2).\end{aligned}$$

Property 3 (Contraction). If the initial mesh $\mathcal{T}_0$ is sufficiently fine, there exist constants $\beta \in (0, +\infty)$ and $\gamma \in (0, 1)$ such that the term

$$\xi_\ell^2 = \eta(\mathcal{T}_\ell)^2 + \beta(\|\boldsymbol{\sigma}_\ell - \boldsymbol{\sigma}_{\ell+1}\|_0^2 + \|\boldsymbol{p}_\ell - \boldsymbol{p}_{\ell+1}\|_0^2)$$

satisfies for all integers ℓ

$$\xi_{\ell+1}^2 \le \gamma\xi_\ell^2.$$

In the next section we will show how to prove the above properties. While in some cases these are natural extensions of the analogous results for the Laplace eigenproblem in mixed form (see [11]), we will see that in particular the *discrete reliability* property requires a more careful analysis.

4. Proof of the main results

We start this section recalling some known results for the approximation of problem (3). The first one is a superconvergence estimate which has been proved in [12, Lemma 9].

Lemma 5. *Let* $(\lambda, \boldsymbol{\sigma}, \boldsymbol{p})$ *and* $(\lambda_h, \boldsymbol{\sigma}_h, \boldsymbol{p}_h)$ *be solutions of equations* (3) *and* (6), *respectively, with* $\|\boldsymbol{p}\|_0 = \|\boldsymbol{p}_h\|_0 = 1$ *and such that the latter approximates the former as* h *goes to zero. Then, there exists a function* $\rho_{\mathrm{sc}}(h)$ *tending to zero as* $h \to 0$ *such that*

$$\|\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h\|_0 \le \rho_{\mathrm{sc}}(h)(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0), \tag{9}$$

where $\boldsymbol{P}_h$ *denotes the* L^2*-projection onto* $\boldsymbol{Q}_h$.

If $(\lambda, \boldsymbol{\sigma}, \boldsymbol{p})$ and $(\lambda_h, \boldsymbol{\sigma}_h, \boldsymbol{p}_h)$ are as in Lemma 5, thanks to the definition of $\boldsymbol{Q}_h$, it is not difficult to verify that the following equations hold true (see [27, Lemma 4])

$$\begin{aligned} \lambda - \lambda_h &= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 - \lambda_h \|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2 \\ \lambda_h - \lambda_H &= \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 - \lambda_H \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2. \end{aligned} \tag{10}$$

It is useful to recall the source problem associated with (3): given $\mathbf{g} \in L^2(\Omega)^3$, find $(\boldsymbol{\sigma}_g, \boldsymbol{p}_g) \in \mathbf{H}_0(\mathbf{curl}; \Omega) \times \boldsymbol{Q}$ such that

$$\begin{aligned} &(\boldsymbol{\sigma}_g, \boldsymbol{\tau}) + (\mathbf{curl}\, \boldsymbol{\tau}, \boldsymbol{p}_g) = 0 && \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \\ &(\mathbf{curl}\, \boldsymbol{\sigma}_g, \boldsymbol{q}) = -(\mathbf{g}, \boldsymbol{q}) && \forall \boldsymbol{q} \in \boldsymbol{Q}, \end{aligned} \tag{11}$$

Since we have taken $\mu = 1$, it turns out that $\boldsymbol{Q} = \mathbf{curl}(\mathbf{H}_0(\mathbf{curl}; \Omega)) = \mathbf{H}_0(\mathrm{div}^0; \Omega)$, that is the space of vectorfields in $L^2(\Omega)^3$ with zero divergence and vanishing normal component along the boundary.

Standard regularity results for (11) imply that, if Ω is a Lipschitz polyhedron, then both components of the solution of (11) are in $\mathbf{H}^s(\Omega)$ for some $s > 1/2$ (see, for instance, the discussion related to [13, Theorem 2.1]).

The discretization of (11) reads: find $(\boldsymbol{\sigma}_{g,h}, \boldsymbol{p}_{g,h}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{Q}_h$ such that

$$\begin{aligned} &(\boldsymbol{\sigma}_{g,h}, \boldsymbol{\tau}) + (\mathbf{curl}\, \boldsymbol{\tau}, \boldsymbol{p}_{g,h}) = 0 && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \\ &(\mathbf{curl}\, \boldsymbol{\sigma}_{g,h}, \boldsymbol{q}) = -(\mathbf{g}, \boldsymbol{q}) && \forall \boldsymbol{q} \in \boldsymbol{Q}_h, \end{aligned} \tag{12}$$

The following error estimate is well known (see [5])

$$\|\boldsymbol{\sigma}_g - \boldsymbol{\sigma}_{g,h}\|_0 + \|\boldsymbol{p}_g - \boldsymbol{p}_{g,h}\|_0 \le C h^s \|\mathbf{g}\|_0, \quad s > 1/2. \tag{13}$$

4.1. **Proof of Property 1.** The proof of Property 1 (*Discrete reliability*) constitutes the main novelty with respect to the results present in the literature. The structure of the proof is a combination of the analogous proof in [11] and of some of the results in [12]. However, some new estimates are needed that will be detailed in this section. Since the proof is composed of several steps, we summarize in Table 1 the structure of the proof.

Let us start with the estimate of $\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0$. We split $\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H$ using a discrete Helmholtz decomposition as

$$\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H = \boldsymbol{\nabla}\, \alpha_h + \boldsymbol{\zeta}_h, \tag{14}$$

where $\alpha_h \in \mathrm{H}_0^1(\Omega)$ is a Lagrange finite element in N_h and $\boldsymbol{\zeta}_h$ is an edge element in $\boldsymbol{\Sigma}_h$ satisfying

$$(\boldsymbol{\nabla}\, \alpha_h, \boldsymbol{\nabla}\, \psi_h) = (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H, \boldsymbol{\nabla}\, \psi_h) \quad \forall \psi_h \in N_h \tag{15}$$

and, for some $\boldsymbol{r}_h \in \boldsymbol{Q}_h$,

$$\begin{aligned} &(\boldsymbol{\zeta}_h, \boldsymbol{\tau}) + (\mathbf{curl}\, \boldsymbol{\tau}, \boldsymbol{r}_h) = 0 && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \\ &(\mathbf{curl}\, \boldsymbol{\zeta}_h, \boldsymbol{q}) = (\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H), \boldsymbol{q}) && \forall \boldsymbol{q} \in \boldsymbol{Q}_h. \end{aligned} \tag{16}$$

In particular, $(\boldsymbol{\zeta}_h, \boldsymbol{r}_h)$ approximates the solution of the mixed problem (11) with source term $\mathbf{g} = -\,\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H)$.

Clearly, we have

$$\|\boldsymbol{\nabla}\, \alpha_h\|_0 \le C \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0, \qquad \|\boldsymbol{\zeta}_h\|_{\mathbf{curl}} + \|\boldsymbol{r}_h\|_0 \le C \|\,\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H)\|_0.$$

Let us estimate the first term of (14). By standard procedure, defining α_H as the Scott–Zhang interpolant of α_h on $\mathcal{T}_H$ (see [39]), we have

$$\|\boldsymbol{\nabla}\, \alpha_h\|_0^2 = (\boldsymbol{\nabla}\, \alpha_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H) = -(\boldsymbol{\nabla}\, \alpha_h, \boldsymbol{\sigma}_H) = -(\boldsymbol{\nabla}(\alpha_h - \alpha_H), \boldsymbol{\sigma}_H),$$

$$\begin{aligned}\mathscr{E} :=& \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0\\ \leq& \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_H\|_0\end{aligned}$$

Property 1: $\mathscr{E} \leq C\eta_H(\mathcal{T}_H \setminus \mathcal{T}_h) + \rho(H)\mathscr{E}$

- $\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H = \boldsymbol{\nabla}\,\alpha_h + \boldsymbol{\zeta}_h$ (14)
  - ◇ $\|\boldsymbol{\nabla}\,\alpha_h\|_0 \leq C\eta_H(\mathcal{T}_H \setminus \mathcal{T}_h)$ (17)
  - ◇ $\|\boldsymbol{\zeta}_h\|_0^2 = \mathrm{I} + \mathrm{II} + \mathrm{III}$ (18)
    - ∗ $\mathrm{I} \leq \rho(H)\mathscr{E}\|\boldsymbol{\zeta}_h\|_0$ (19)
    - ∗ $\mathrm{II} \leq CH^{1/2}\mathscr{E}\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0$ (20)
    - ∗ $\mathrm{III} \leq C\|\boldsymbol{p}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0\|\boldsymbol{\zeta}_h\|_0$ (21)
  - $\|\boldsymbol{\zeta}_h\|_0^2 \leq \rho_1(H)\mathscr{E}^2 + C\lambda_H^2\|\boldsymbol{p}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2$ (22)
    - ∗ $\|\boldsymbol{p}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0 \leq \|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0 + \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0 = \mathrm{IV} + \mathrm{V}$
    - ∗ $\mathrm{IV} \leq \rho(H)\mathscr{E} + C\mathrm{V}$ (24)
    - ∗ $\mathrm{V}^2 \leq \mathrm{A}_1\mathrm{A}_2 + \mathrm{B}_1\mathrm{B}_2$ (26)
      - · $\mathrm{A}_1 \leq \rho(H)\mathscr{E} + C\mathrm{V}$ (27)
      - · $\mathrm{A}_2 \leq C(h^s + H^s)\mathrm{V}$ (28)
      - · $\mathrm{B}_1 \leq C(\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0 + \mathscr{E}^2 + \mathrm{V})$ (29)
      - · $\mathrm{B}_2 \leq C(h^s + H^s)\mathrm{V}$ (30)
    - $\mathrm{V} \leq CH^s\mathscr{E}$ (32)
  - $\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 \leq C\eta_H(\mathcal{T}_H \setminus \mathcal{T}_h) + \rho(H)\mathscr{E} + C\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0$
- $\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 = \mathrm{I} + \mathrm{II} + \mathrm{III}$ (36)
  - ◇ $\mathrm{I} \leq \rho(H)\mathscr{E}\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0$
  - ◇ $|\mathrm{II} + \mathrm{III}| \leq C\eta_H(\mathcal{T}_H \setminus \mathcal{T}_h)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0$
  - $\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0 \leq C\eta_H(\mathcal{T}_H \setminus \mathcal{T}_h) + \rho(H)\mathscr{E}$ (37)

TABLE 1. Structure of the proof of Property 1

since $(\boldsymbol{\nabla}\,\alpha_h, \boldsymbol{\sigma}_h) = (\boldsymbol{\nabla}\,\alpha_H, \boldsymbol{\sigma}_H) = 0$ from the first equation of (6). Integrating by parts element by element, we get

$$\begin{aligned}\|\boldsymbol{\nabla}\,\alpha_h\|_0^2 &= \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}\Big((\alpha_h - \alpha_H, \operatorname{div}\boldsymbol{\sigma}_H) - \frac{1}{2}\sum_{F\in K}\int_F(\alpha_h - \alpha_H)[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\Big)\\ &\leq C\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}\Big(\|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K}H_K\|\boldsymbol{\nabla}\,\alpha\|_{0,K}\\ &\qquad + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)}\|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F}H_F^{1/2}\|\alpha_h\|_{1,K}\Big)\\ &\leq C\|\boldsymbol{\nabla}\,\alpha_h\|_0\Bigg(\Big(\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}H_K^2\|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K}^2\Big)^{1/2}\\ &\qquad + \Big(\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}\sum_{F\in\mathcal{F}_I(K)}H_F\|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F}^2\Big)^{1/2}\Bigg)\\ &\leq C\|\boldsymbol{\nabla}\,\alpha_h\|_0\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h).\end{aligned} \tag{17}$$

The estimate of the second term in (14) requires a more careful analysis.

$$\begin{aligned}\|\boldsymbol{\zeta}_h\|_0^2 &= (\boldsymbol{\zeta}_h,\boldsymbol{\zeta}_h) = -(\mathbf{curl}\,\boldsymbol{\zeta}_h,\boldsymbol{r}_h) = -(\mathbf{curl}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H),\boldsymbol{r}_h)\\ &= (\lambda_h\boldsymbol{p}_h-\lambda_H\boldsymbol{p}_H,\boldsymbol{r}_h)\\ &= (\lambda_h-\lambda_H)(\boldsymbol{p}_h,\boldsymbol{r}_h)+\lambda_H(\boldsymbol{p}_h-\boldsymbol{P}_H\boldsymbol{p}_h,\boldsymbol{r}_h)+\lambda_H(\boldsymbol{P}_H\boldsymbol{p}_h-\boldsymbol{p}_H,\boldsymbol{r}_h).\end{aligned}\tag{18}$$

We bound the three terms in the last line separately.

From classical inf-sup condition involving edge and face elements (see, for instance, [10]), we have

$$\begin{aligned}(\boldsymbol{p}_h,\boldsymbol{r}_h) &\le \|\boldsymbol{p}_h\|_0\|\boldsymbol{r}_h\|_0 = \|\boldsymbol{r}_h\|_0\\ &\le C\sup_{\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h}\frac{(\mathbf{curl}\,\boldsymbol{\tau}_h,\boldsymbol{r}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{curl}}} = C\sup_{\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h}\frac{(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{curl}}}\le C\|\boldsymbol{\zeta}_h\|_0.\end{aligned}$$

Hence, using the second of (10) we conclude the estimate of the first term in (18) as follows

$$(\lambda_h-\lambda_H)(\boldsymbol{p}_h,\boldsymbol{r}_h)\le C\big(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0^2+\lambda_H\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2\big)\|\boldsymbol{\zeta}_h\|_0.\tag{19}$$

We now estimate the second term in (18). Recalling the mixed problem (16) defining $\boldsymbol{\zeta}_h$ and $\boldsymbol{r}_h$, we denote by $\boldsymbol{\zeta}_H\in\boldsymbol{\Sigma}_H$ and $\boldsymbol{r}_H\in\boldsymbol{Q}_H$ the corresponding solution on the mesh $\mathcal{T}_H$ with $\mathbf{g}=-(\mathbf{curl}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H)$. Moreover, we denote by $\boldsymbol{\zeta}\in\mathbf{H}_0(\mathbf{curl};\Omega)$ and $\boldsymbol{r}\in\boldsymbol{Q}$ the solution of the continuous problem satisfying

$$\begin{aligned}(\boldsymbol{\zeta},\boldsymbol{\tau})+(\mathbf{curl}\,\boldsymbol{\tau},\boldsymbol{r}) &= 0 &&\forall\boldsymbol{\tau}\in\mathbf{H}_0(\mathbf{curl};\Omega)\\ (\mathbf{curl}\,\boldsymbol{\zeta},\boldsymbol{q}) &= (\mathbf{curl}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H),\boldsymbol{q}) &&\forall\boldsymbol{q}\in\boldsymbol{Q}.\end{aligned}$$

It is clear that we have

$$\begin{aligned}\lambda_H(\boldsymbol{p}_h-\boldsymbol{P}_H\boldsymbol{p}_h,\boldsymbol{r}_h) &= \lambda_H(\boldsymbol{p}_h-\boldsymbol{P}_H\boldsymbol{p}_h,\boldsymbol{r}_h-\boldsymbol{P}_H\boldsymbol{r}_h)\\ &\le \lambda_H\|\boldsymbol{p}_h-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\|\boldsymbol{r}_h-\boldsymbol{P}_H\boldsymbol{r}_h\|_0\\ &\le \lambda_H\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\|\boldsymbol{r}_h-\boldsymbol{r}_H\|_0.\end{aligned}$$

The term $\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0$ will be estimated later. The other term can be bounded as follows using known results for mixed problems, together with regularity results (see the discussion after (11))

$$\begin{aligned}\|\boldsymbol{r}_h-\boldsymbol{r}_H\|_0 &\le \|\boldsymbol{r}_h-\boldsymbol{r}\|_0+\|\boldsymbol{r}-\boldsymbol{r}_H\|_0\\ &\le CH^{1/2}\|\boldsymbol{r}\|_{\mathbf{H}^{1/2}(\Omega)}\le CH^{1/2}\|\,\mathbf{curl}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H)\|_0\\ &= CH^{1/2}\|\lambda_h\boldsymbol{p}_h-\lambda_H\boldsymbol{p}_H\|_0\\ &\le CH^{1/2}\big(|\lambda_h-\lambda_H|+\lambda_H\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\big)\\ &\le CH^{1/2}\big(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0^2+\lambda_H\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2+\lambda_H\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\big)\\ &\le CH^{1/2}\big((\lambda_h+\lambda_H)\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+3\lambda_H\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\big).\end{aligned}\tag{20}$$

The last term in (18) can be estimated using again the inf-sup condition related to edge and face elements

$$\lambda_H(\boldsymbol{P}_H\boldsymbol{p}_h-\boldsymbol{p}_H,\boldsymbol{r}_h)\le\lambda_H\|\boldsymbol{p}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\|\boldsymbol{r}_h\|_0\le C\lambda_H\|\boldsymbol{p}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\|\boldsymbol{\zeta}_h\|_0.\tag{21}$$

Putting together the estimates of the three terms in (18), and using the a priori error estimate for the eigenvalue problem (see, for instance [10] or [7]), we obtain by a suitable definition of $\rho_1(H)$

$$\|\boldsymbol{\zeta}_h\|_0^2\le\rho_1(H)\big(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0^2+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2\big)+C\lambda_H^2\|\boldsymbol{p}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0^2+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2.\tag{22}$$

The estimate of $\|\boldsymbol{p}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0$ can be obtained in several steps. We consider the auxiliary problem: find $(\widehat{\boldsymbol{\sigma}}_H, \widehat{\boldsymbol{p}}_H) \in \boldsymbol{\Sigma}_H \times \boldsymbol{Q}_H$ such that

$$
\begin{aligned}
&(\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \widehat{\boldsymbol{p}}_H) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_H\\
&(\mathbf{curl}\,\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{q}) = -\lambda_h(\boldsymbol{p}_h, \boldsymbol{q}) \quad \forall \boldsymbol{q} \in \boldsymbol{Q}_H.
\end{aligned}
\tag{23}
$$

By triangular inequality we have

$$
\|\boldsymbol{p}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0 \le \|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0 + \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0.
$$

We first show that $\|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0$ is bounded by $\|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0$ plus a term which asymptotically behaves like the above estimate of $\|\boldsymbol{r}_h - \boldsymbol{r}_H\|_0$. Let $\{\lambda_{H,i}, (\boldsymbol{\sigma}_{H,i}, \boldsymbol{p}_{H,i})\}$ $(i = 1, \dots, N(H))$ be the family of eigensolutions of problem (6) related to the mesh $\mathcal{T}_H$ (recall that $\lambda_H = \lambda_{H,j}$). We have

$$
\|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0^2 = \sum_{i=1}^{N(H)} a_i^2, \qquad a_i = (\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i}).
$$

For $i = j$

$$
\begin{aligned}
a_j &= (\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H, \boldsymbol{p}_H) = 1 - (\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_H) = 1 + \frac{1}{\lambda_H}(\widehat{\boldsymbol{p}}_H, \mathbf{curl}\,\boldsymbol{\sigma}_H) = 1 - \frac{1}{\lambda_H}(\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{\sigma}_H)\\
&= 1 + \frac{1}{\lambda_H}(\boldsymbol{p}_H, \mathbf{curl}\,\widehat{\boldsymbol{\sigma}}_H) = 1 - \frac{\lambda_h}{\lambda_H}(\boldsymbol{p}_h, \boldsymbol{p}_H) = 1 - \frac{\lambda_h}{\lambda_H} + \frac{\lambda_h}{\lambda_H}\big(1 - (\boldsymbol{p}_h, \boldsymbol{p}_H)\big)\\
&= \frac{\lambda_H - \lambda_h}{\lambda_H} + \frac{\lambda_h}{2\lambda_H}\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 = \left(1 + \frac{\lambda_h}{2\lambda_H}\right)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 - \frac{1}{\lambda_H}\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2.
\end{aligned}
$$

For $i \ne j$, since $a_i = -(\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i})$, we can proceed with the following estimate

$$
\begin{aligned}
\lambda_{H,i}(\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i}) &= -(\mathbf{curl}\,\boldsymbol{\sigma}_{H,i}, \widehat{\boldsymbol{p}}_H) = (\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{\sigma}_{H,i}) = -(\mathbf{curl}\,\widehat{\boldsymbol{\sigma}}_H, \boldsymbol{p}_{H,i})\\
&= \lambda_h(\boldsymbol{p}_h, \boldsymbol{p}_{H,i}) = \lambda_h(\boldsymbol{P}_H\boldsymbol{p}_h, \boldsymbol{p}_{H,i}),
\end{aligned}
$$

which gives

$$
(\lambda_{H,i} - \lambda_h)(\widehat{\boldsymbol{p}}_H, \boldsymbol{p}_{H,i}) = -\lambda_h(\boldsymbol{p}_{H,i}, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h).
$$

Hence,

$$
\begin{aligned}
\sum_{i\ne j} a_i^2 &= \sum_{i \ne j} a_i \frac{\lambda_h}{\lambda_{H,i} - \lambda_h}(\boldsymbol{p}_{H,i}, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h)\\
&\le \max_{i\ne j}\left|\frac{\lambda_h}{\lambda_{H,i} - \lambda_h}\right| \left(\sum_{i\ne j} a_i^2\right)^{1/2} \left(\sum_{i \ne j}(\boldsymbol{p}_{H,i}, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h)^2\right)^{1/2}\\
&\le \max_{i\ne j}\left|\frac{\lambda_h}{\lambda_{H,i} - \lambda_h}\right| \left(\sum_{i\ne j} a_i^2\right)^{1/2} \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0.
\end{aligned}
$$

Putting things together, we get

$$
\begin{aligned}
\|\boldsymbol{p}_H - \widehat{\boldsymbol{p}}_H\|_0^2 &= \sum_{i=1}^{N(H)} a_i^2\\
&\le C(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2)^2 + \max_{i \ne j}\left|\frac{\lambda_h}{\lambda_{H,i} - \lambda_h}\right|^2 \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2.
\end{aligned}
\tag{24}
$$

If H is small enough (remember that we have assumed in Theorem 2 that the initial mesh is fine enough), then the denominator $\lambda_{H,i} - \lambda_h$ is bounded away from zero for all $i \neq j$ and for all h.

From the a priori error estimate for the eigenvalue problem and a suitable definition of $\rho_2(H)$ we get the estimate

$$(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2)^2 \leq \rho_2(H)(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2).$$

Now we use a duality argument in order to get a bound for $\|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0$. Let $\boldsymbol{\xi} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ and $\boldsymbol{w} \in \boldsymbol{Q}$ be the solution of

$$\begin{aligned} &(\boldsymbol{\xi},\boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau},\boldsymbol{w}) = 0 && \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl};\Omega) \\ &(\mathbf{curl}\,\boldsymbol{\xi},\boldsymbol{q}) = (\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h,\boldsymbol{q}) && \forall \boldsymbol{q} \in \boldsymbol{Q} \end{aligned}$$

and $\boldsymbol{\xi}_h \in \boldsymbol{\Sigma}_h$ and $\boldsymbol{w}_h \in \boldsymbol{Q}_h$ the corresponding discrete solution on the mesh $\mathcal{T}_h$. We observe that $(\boldsymbol{\xi},\boldsymbol{w})$ is the solution of (11) when $\mathbf{g} = -(\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h)$, hence both the components of the solution belong to $\mathbf{H}^s(\Omega)$ for some $s > 1/2$. Let $\boldsymbol{\Pi}_H^F$ be the Fortin operator introduced in [5] associated with problem (11), with the following properties:

$$\begin{aligned} &\boldsymbol{\Pi}_H^F : \mathbf{H}^s(\Omega) \to \boldsymbol{\Sigma}_H \\ &(\mathbf{curl}(\boldsymbol{\xi} - \boldsymbol{\Pi}_H^F\boldsymbol{\xi}),\boldsymbol{q}) = 0 \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}_H,\ \forall \boldsymbol{\xi} \in \mathbf{H}^s(\Omega) \\ &\|\boldsymbol{\Pi}_H^F\boldsymbol{\xi}\|_{\mathbf{curl}} \leq C\|\boldsymbol{\xi}\|_{\mathbf{H}}^s(\Omega) \\ &\|\boldsymbol{\xi} - \boldsymbol{\Pi}_H^F\boldsymbol{\xi}\|_0 \leq C\|\boldsymbol{\xi}\|_{\mathbf{H}}^s(\Omega). \end{aligned} \tag{25}$$

We have

$$\begin{aligned} \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2 &= (\mathbf{curl}\,\boldsymbol{\xi}_h, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h) \\ &= (\mathbf{curl}\,\boldsymbol{\Pi}_H^F\boldsymbol{\xi}_h, \widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h) && \text{def. of } \boldsymbol{\Pi}_H^F \\ &= (\mathbf{curl}\,\boldsymbol{\Pi}_H^F\boldsymbol{\xi}_h, \widehat{\boldsymbol{p}}_H - \boldsymbol{p}_h) \\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\Pi}_H^F\boldsymbol{\xi}_h) && \text{err. eq. (23)-(6)} \\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\Pi}_H^F\boldsymbol{\xi}_h - \boldsymbol{\xi}_h) - (\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\xi}_h) \\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\Pi}_H^F\boldsymbol{\xi}_h - \boldsymbol{\xi}_h) + (\mathbf{curl}(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h), \boldsymbol{w}_h) && \text{duality arg.} \\ &= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h, \boldsymbol{\Pi}_H^F\boldsymbol{\xi}_h - \boldsymbol{\xi}_h) + (\mathbf{curl}(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h), \boldsymbol{w}_h - \boldsymbol{P}_H\boldsymbol{w}_h). && \text{err. eq. (23)-(6)} \end{aligned}$$

By Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \|\widehat{\boldsymbol{p}}_H - \boldsymbol{P}_H\boldsymbol{p}_h\|_0^2 \leq \|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h\|_0 &\|\boldsymbol{\Pi}_H^F\boldsymbol{\xi}_h - \boldsymbol{\xi}_h\|_0 \\ &+ \|\,\mathbf{curl}(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h)\|\|\boldsymbol{w}_h - \boldsymbol{P}_H\boldsymbol{w}_h\|_0, \end{aligned} \tag{26}$$

and we estimate separately the four norms on the right hand side. The triangular inequality gives

$$\|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_h\|_0 \leq \|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{\sigma}_H - \boldsymbol{\sigma}_h\|_0.$$

We only need to estimate the first term, for which we proceed as before by expanding it in terms of the eigensolutions on the mesh $\mathcal{T}_H$. Since $\|\boldsymbol{\sigma}_{H,i}\|_0^2 = \lambda_{H,i}$, we set $\widetilde{\boldsymbol{\sigma}}_{H,i} = \boldsymbol{\sigma}_{H,i}/\sqrt{\lambda_{H,i}}$. We have

$$\|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0^2 = \sum_{i=1}^{N(H)} b_i^2, \qquad b_i = (\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H, \widetilde{\boldsymbol{\sigma}}_{H,i}).$$

For $i = j$

$$\begin{aligned}
b_j &= (\widehat{\boldsymbol\sigma}_H - \boldsymbol\sigma_H, \widetilde{\boldsymbol\sigma}_H) = (\widehat{\boldsymbol\sigma}_H, \widetilde{\boldsymbol\sigma}_H) - \sqrt{\lambda_H} = -(\mathbf{curl}\,\widetilde{\boldsymbol\sigma}_H, \widehat{\boldsymbol p}_H) - \sqrt{\lambda_H}\\
&= \lambda_H\left(\frac{\boldsymbol p_H}{\sqrt{\lambda_H}}, \widehat{\boldsymbol p}_H\right) - \sqrt{\lambda_H} = \sqrt{\lambda_H}\big((\boldsymbol p_H, \widehat{\boldsymbol p}_H) - 1\big)\\
&= -\sqrt{\lambda_H}\left(\frac{\lambda_H - \lambda_h}{\lambda_H} + \frac{\lambda_h}{2\lambda_H}\|\boldsymbol p_h - \boldsymbol p_H\|_0^2\right).
\end{aligned}$$

Using (10) we get

$$b_j \le C\big(\|\boldsymbol\sigma_h - \boldsymbol\sigma_H\|_0^2 + \|\boldsymbol p_h - \boldsymbol p_H\|_0^2\big).$$

For $i \ne j$

$$\begin{aligned}
b_i &= (\widehat{\boldsymbol\sigma}_H, \widetilde{\boldsymbol\sigma}_{H,i}) = -(\mathbf{curl}\,\widetilde{\boldsymbol\sigma}_{H,i}, \widehat{\boldsymbol p}_H) = \sqrt{\lambda_{H,i}}(\boldsymbol p_{H,i}, \widehat{\boldsymbol p}_H)\\
&= \sqrt{\lambda_{H,i}}\frac{\lambda_h}{\lambda_h - \lambda_{H,i}}(\boldsymbol p_{H,i}, \widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h),
\end{aligned}$$

where in the last two estimates we took advantage of the already computed bounds for $\big((\boldsymbol p_H, \widehat{\boldsymbol p}_H) - 1\big)$ and $(\boldsymbol p_{H,i}, \widehat{\boldsymbol p}_H)$. Hence

$$\begin{aligned}
\sum_{i\ne j} b_i^2 &= \sum_{i\ne j} b_i \frac{\sqrt{\lambda_{H,i}}\lambda_h}{\lambda_h - \lambda_{H,i}}(\boldsymbol p_{H,i}, \widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h)\\
&= \max_{i\ne j} \frac{\sqrt{\lambda_{H,i}}\lambda_h}{|\lambda_h - \lambda_{H,i}|}\Big(\sum_{i\ne j} b_i^2\Big)^{1/2}\|\widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h\|_0.
\end{aligned}$$

As we have already observed, the denominator of the last expression is always bounded away from zero for H small enough. It follows that

$$\begin{aligned}
\Big(\sum_{i\ne j} b_i^2\Big)^{1/2} &\le \max_{i\ne j} \frac{\sqrt{\lambda_{H,i}}\lambda_h}{|\lambda_h - \lambda_{H,i}|}\|\widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h\|_0\\
&\le \max_{i\ne j}\left(\lambda_h/\sqrt{\lambda_{H,i}}\right)\frac{\lambda_{H,i}/\lambda_h}{|1 - \lambda_{H,i}/\lambda_h|}\|\widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h\|_0\\
&\le C\|\widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h\|_0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\widehat{\boldsymbol\sigma}_H - \boldsymbol\sigma_H\|_0^2 &\le C\big(\|\boldsymbol\sigma_h - \boldsymbol\sigma_H\|_0^4 + \|\boldsymbol p_h - \boldsymbol p_H\|_0^4 + \|\widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h\|_0^2\big)\\
&\le \rho_3(H)\big(\|\boldsymbol\sigma_h - \boldsymbol\sigma_H\|_0^2 + \|\boldsymbol p_h - \boldsymbol p_H\|_0^2\big) + C\|\widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h\|_0^2.
\end{aligned} \tag{27}$$

To bound the second term in (26), we use the triangle inequality, the error estimates for the mixed source problem (13), the properties of the Fortin operator (25)

$$\begin{aligned}
\|\boldsymbol\Pi_H^F \boldsymbol\xi_h - \boldsymbol\xi_h\|_0 &\le \|\boldsymbol\Pi_H^F(\boldsymbol\xi_h - \boldsymbol\xi)\|_0 + \|\boldsymbol\Pi_H^F\boldsymbol\xi - \boldsymbol\xi\|_0 + \|\boldsymbol\xi - \boldsymbol\xi_h\|_0\\
&\le C\|\boldsymbol\xi - \boldsymbol\xi_h\|_0 + \|\boldsymbol\Pi_H^F\boldsymbol\xi - \boldsymbol\xi\|_0\\
&\le C(h^s + H^s)\|\widehat{\boldsymbol p}_H - \boldsymbol P_H \boldsymbol p_h\|_0.
\end{aligned} \tag{28}$$

From the definition of the discrete spaces, since $\boldsymbol Q_h = \mathbf{curl}(\boldsymbol\Sigma_h)$ for any choice of the mesh, it is clear that

$$\mathbf{curl}(\widehat{\boldsymbol\sigma}_H) = -\lambda_h \boldsymbol P_H \boldsymbol p_h, \qquad \mathbf{curl}(\boldsymbol\sigma_h) = -\lambda_h \boldsymbol p_h.$$

Therefore, from (24) we obtain

(29)
$$\begin{aligned}\|\operatorname{\mathbf{curl}}(\boldsymbol{\sigma}_h-\widehat{\boldsymbol{\sigma}}_H)\|_0 &\le \lambda_h\|\boldsymbol{p}_h-\boldsymbol{P}_H\boldsymbol{p}_h\|_0 \le \lambda_h(\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0+\|\boldsymbol{p}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0)\\ &\le \lambda_h\big(\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0+C(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0^2+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2)\\ &\quad +C\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\big).\end{aligned}$$

Considering again the definition of the solution of the dual problem, the last term in (26) can be bounded by using (13) and the properties of the projection operator $\boldsymbol{P}_H$:

$$\begin{aligned}\|\boldsymbol{w}_h-\boldsymbol{P}_H\boldsymbol{w}_h\|_0 &\le \|\boldsymbol{w}_h-\boldsymbol{w}\|_0+\|\boldsymbol{w}-\boldsymbol{P}_H\boldsymbol{w}\|_0+\|\boldsymbol{P}_H(\boldsymbol{w}-\boldsymbol{w}_h)\|_0\\ &\le C\|\boldsymbol{w}-\boldsymbol{w}_h\|_0+\|\boldsymbol{w}-\boldsymbol{P}_H\boldsymbol{w}\|_0\\ &\le C(h^s+H^s)\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0.\end{aligned} \tag{30}$$

Putting together the estimates of the four norms in (26), we arrive at

$$\begin{aligned}\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0^2 &\le CH^s\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0\big(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\big)\\ &\quad +C(h^s+H^s)\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0^2,\end{aligned}$$

which implies that, for H sufficiently small, we have

$$\|\widehat{\boldsymbol{p}}_H-\boldsymbol{P}_H\boldsymbol{p}_h\|_0 \le CH^s\big(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0\big). \tag{31}$$

It turns out that the final estimate for (18) is obtained from (19), (20), and (31) as follows:

$$\|\boldsymbol{\zeta}_h\|_0 \le \rho_4(H)(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0)+C\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0 \tag{32}$$

with an appropriate definition of $\rho_4(H)$.

Finally, from (14), (17), and (32), we have

$$\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0 \le C\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h)+\rho_4(H)(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H\|_0+\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0)+C\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0. \tag{33}$$

We now move to the term $\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0$. We consider the following auxiliary problem: find $\boldsymbol{\chi}_h\in\boldsymbol{\Sigma}_h$ and $\boldsymbol{z}_h\in\boldsymbol{Q}_h$ such that

$$\begin{aligned}&(\boldsymbol{\chi}_h,\boldsymbol{\tau})+(\operatorname{\mathbf{curl}}\boldsymbol{\tau},\boldsymbol{z}_h)=0 &&\forall\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h\\ &(\operatorname{\mathbf{curl}}\boldsymbol{\chi}_h,\boldsymbol{q})=(\boldsymbol{p}_h-\boldsymbol{p}_H,\boldsymbol{q}) &&\forall\boldsymbol{q}\in\boldsymbol{Q}_h.\end{aligned} \tag{34}$$

Therefore, we have $\operatorname{\mathbf{curl}}\boldsymbol{\chi}_h=\boldsymbol{p}_h-\boldsymbol{p}_H$ and $\|\boldsymbol{\chi}_h\|_{\mathbf{curl}}\le C\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0$.

We are going to use a technical tool introduced in [43, Theorem 4.1]. More precisely, if $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, there exists an operator $\boldsymbol{\mathcal{P}}_H:\boldsymbol{\Sigma}_h\to\boldsymbol{\Sigma}_H$ such that for all $\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h$ it holds $\boldsymbol{\mathcal{P}}_H\boldsymbol{\tau}=\boldsymbol{\tau}$ on the elements of $\mathcal{T}_H$ that have not been refined (more precisely, on the elements of $\mathcal{T}_H$ whose closures have no intersection with the closures of any refined elements). Such operator is stable in the H(**curl**)-norm, i.e., $\|\boldsymbol{\mathcal{P}}_H\boldsymbol{\tau}\|_{\mathbf{curl}}\le C\|\boldsymbol{\tau}\|_{\mathbf{curl}}$ for all $\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h$.

We get

(35)
$$\begin{aligned}\|\boldsymbol{p}_h-\boldsymbol{p}_H\|_0^2 &= (\boldsymbol{p}_h-\boldsymbol{p}_H,\operatorname{\mathbf{curl}}\boldsymbol{\chi}_h)=-(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\operatorname{\mathbf{curl}}\boldsymbol{\chi}_h)\\ &=-(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\operatorname{\mathbf{curl}}(\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\operatorname{\mathbf{curl}}\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)\\ &=-(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\operatorname{\mathbf{curl}}(\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)+(\boldsymbol{\sigma}_H,\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)\\ &=-(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h)-(\boldsymbol{p}_H,\operatorname{\mathbf{curl}}(\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)-(\boldsymbol{\sigma}_H,\boldsymbol{\chi}_h-\boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h)\end{aligned}$$

Let us set $\boldsymbol{\vartheta}_h = \boldsymbol{\chi}_h - \boldsymbol{\mathcal{P}}_H\boldsymbol{\chi}_h$ and denote by $\boldsymbol{\mathcal{S}}_H$ the operator introduced in [38, Theorem 1] mapping $\mathbf{H}_0(\mathbf{curl};\Omega)$ into the space of lowest order Nédélec elements so that there exist $\varphi \in \mathrm{H}^1_0(\Omega)$ and $\boldsymbol{s} \in \mathbf{H}^1_0(\Omega)$ satisfying

$$
\begin{aligned}
&\boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h = \boldsymbol{\nabla}\varphi + \boldsymbol{s}\\
&h_K^{-1}\|\varphi\|_{0,K} + \|\boldsymbol{\nabla}\varphi\|_{0,K} \le C\|\boldsymbol{\vartheta}_h\|_{0,K'}\\
&h_K^{-1}\|\boldsymbol{s}\|_{0,K} + \|\boldsymbol{\nabla}\boldsymbol{s}\|_{0,K} \le C\|\mathbf{curl}\,\boldsymbol{\vartheta}_h\|_{0,K'}
\end{aligned}
$$

for all $K \in \mathcal{T}_h$ and with K' denoting the union of elements in $\mathcal{T}_h$ sharing at least a vertex with K.

From the first equation in (6) is follows that $(\boldsymbol{\sigma}_H, \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h) + (\mathbf{curl}\,\boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h, \boldsymbol{p}_H) = 0$. This implies that (35) gives

$$
\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 = -(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H, \boldsymbol{\chi}_h) - (\boldsymbol{p}_H, \mathbf{curl}(\boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)) - (\boldsymbol{\sigma}_H, \boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h). \tag{36}
$$

The first term can be estimated as follows.

$$
\begin{aligned}
-(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H, \boldsymbol{\chi}_h) &= -(\boldsymbol{\sigma}_h - \widehat{\boldsymbol{\sigma}}_H, \boldsymbol{\chi}_h) - (\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H, \boldsymbol{\chi}_h)\\
&= (\mathbf{curl}(\boldsymbol{\sigma}_h - \widehat{\boldsymbol{\sigma}}_H), \boldsymbol{z}_h) - (\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H, \boldsymbol{\chi}_h) \quad \text{Eq. (34)}\\
&= -(\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H, \boldsymbol{\chi}_h) \quad \text{def. of } \widehat{\boldsymbol{\sigma}}_H\\
&\le \|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0\|\boldsymbol{\chi}_h\|_0\\
&\le \|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0.
\end{aligned}
$$

The estimate for $\|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0$ follows from (27) and (31) and is given by

$$
\|\widehat{\boldsymbol{\sigma}}_H - \boldsymbol{\sigma}_H\|_0 \le \rho_5(H)(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0).
$$

The remaining two terms in (36) can be bounded together.

$$
\begin{aligned}
&(\boldsymbol{p}_H, \mathbf{curl}(\boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)) + (\boldsymbol{\sigma}_H, \boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)\\
&\quad= \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}\Big(\int_K \mathbf{curl}\,\boldsymbol{p}_H\cdot\boldsymbol{s} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)}\int_F [\![\boldsymbol{p}_{\mathrm{H}}\times\boldsymbol{n}]\!]\cdot\boldsymbol{s}\Big) + (\boldsymbol{\sigma}_H, \boldsymbol{s}) + (\boldsymbol{\sigma}_H, \boldsymbol{\nabla}\varphi)\\
&\quad= \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}\Big(\int_K (\boldsymbol{\sigma}_H + \mathbf{curl}\,\boldsymbol{p}_H)\cdot\boldsymbol{s} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)}\int_F [\![\boldsymbol{p}_{\mathrm{H}}\times\boldsymbol{n}]\!]\cdot\boldsymbol{s}\Big)\\
&\qquad+ \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h}\Big(-\int_K \operatorname{div}\boldsymbol{\sigma}_H\varphi + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)}\int_F [\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\varphi\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|(\boldsymbol{p}_H, \mathbf{curl}(\boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)) + (\boldsymbol{\sigma}_H, \boldsymbol{\vartheta}_h - \boldsymbol{\mathcal{S}}_H\boldsymbol{\vartheta}_h)\right| \\
&\quad \le \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(\|\boldsymbol{\sigma}_H + \mathbf{curl}\,\boldsymbol{p}_H\|_{0,K}\|\boldsymbol{s}\|_{0,K} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \|[\![\boldsymbol{p}_{\mathrm{H}}\times\boldsymbol{n}]\!]\|_{0,F}\|\boldsymbol{s}\|_{0,F}\Big) \\
&\qquad + \sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(\|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K}\|\varphi\|_{0,K} + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} \|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F}\|\varphi\|_{0,F}\Big) \\
&\quad \le C\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(H_K\|\boldsymbol{\sigma}_H + \mathbf{curl}\,\boldsymbol{p}_H\|_{0,K}\|\,\mathbf{curl}\,\boldsymbol{\vartheta}_h\|_{0,K'} \\
&\qquad + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} H_F^{1/2}\|[\![\boldsymbol{p}_{\mathrm{H}}\times\boldsymbol{n}]\!]\|_{0,F}\|\,\mathbf{curl}\,\boldsymbol{\vartheta}_h\|_{0,K'}\Big) \\
&\qquad + C\sum_{K\in\mathcal{T}_H\setminus\mathcal{T}_h} \Big(H_K\|\operatorname{div}\boldsymbol{\sigma}_H\|_{0,K}\|\boldsymbol{\vartheta}_h\|_{0,K'} \\
&\qquad + \frac{1}{2}\sum_{F\in\mathcal{F}_I(K)} H_F^{1/2}\|[\![\boldsymbol{\sigma}_H\cdot\boldsymbol{n}]\!]\|_{0,F}\|\boldsymbol{\vartheta}_h\|_{0,K'}\Big) \\
&\qquad \le C\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h)\|\boldsymbol{\vartheta}_h\|_{\mathbf{curl}} \le C\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0.
\end{aligned}
$$

Finally, Equation (36) becomes

$$
\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0 \le C\eta_H(\mathcal{T}_H\setminus\mathcal{T}_h) + \rho_5(H)(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0 + \|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0). \tag{37}
$$

Putting things together, estimates (33) and (37) give the final result.

4.2. **Proof of Property 2.** The proof of Property 2 (*Quasi-orthogonality*) can be obtained after appropriate modification of the analogous result in [11].

By direct computation we have

$$
\begin{aligned}
\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 &= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_0^2 - \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 - 2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H) \\
\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 &= \|\boldsymbol{p} - \boldsymbol{p}_H\|_0^2 - \|\boldsymbol{p} - \boldsymbol{p}_h\|_0^2 - 2(\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p}_h - \boldsymbol{p}_H).
\end{aligned}
$$

Since $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, we have that $\boldsymbol{\sigma}_H \in \boldsymbol{\Sigma}_h$, hence the error equations relative to (3) and (6) give

$$
\begin{aligned}
(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H) &= -(\mathbf{curl}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H), \boldsymbol{p} - \boldsymbol{p}_h) \\
&= (\lambda_h\boldsymbol{p}_h - \lambda_H\boldsymbol{p}_H, \boldsymbol{p} - \boldsymbol{p}_h) \\
&= (\lambda_h\boldsymbol{p}_h - \lambda_H\boldsymbol{p}_H, \boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h).
\end{aligned}
$$

Using Lemma 5 and the equalities in (10), we obtain

$$
\begin{aligned}
&(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H) + (\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p}_h - \boldsymbol{p}_H) \\
&\quad = (\lambda_h\boldsymbol{p}_h - \lambda_H\boldsymbol{p}_H, \boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h) + (\boldsymbol{p}_h - \boldsymbol{p}_H, \boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h) \\
&\quad \le \big(|\lambda_h - \lambda_H| + (1+\lambda_H)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0\big)\|\boldsymbol{P}_h\boldsymbol{p} - \boldsymbol{p}_h\|_0 \\
&\quad \le \big(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2 + \lambda_H\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0^2 + (1+\lambda_H)\|\boldsymbol{p}_h - \boldsymbol{p}_H\|_0\big) \\
&\qquad\quad \rho_{\mathrm{sc}}(h)\big(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{p} - \boldsymbol{p}_h\|_0\big)
\end{aligned}
$$

which, using Young inequality, gives the desired result.

4.3. **Proof of Property 3.** The contraction property is quite standard in the framework of adaptive schemes, see [17]. It is a consequence of the following error estimator reduction property: there exist constants $\beta_1 \in (0,+\infty)$ and $\gamma_1 \in (0,1)$ such that, if $\mathcal{T}_{\ell+1}$ is the refinement of $\mathcal{T}_\ell$ generated by the adaptive scheme, it holds

$$\eta(\mathcal{T}_{\ell+1})^2 \le \gamma_1 \eta(\mathcal{T}_\ell)^2 + \beta_1 \big(\|\boldsymbol{\sigma}_\ell - \boldsymbol{\sigma}_{\ell+1}\|_0^2 + \|\boldsymbol{p}_\ell - \boldsymbol{p}_{\ell+1}\|_0^2 \big).$$

In our case, the proof can be obtained with natural modifications from the one outlined in [11] and using the following notation:

$$e_\ell^2 = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_\ell\|_0^2 + \|\boldsymbol{p} - \boldsymbol{p}_\ell\|_0^2, \qquad \mu_\ell^2 = \eta(\mathcal{T}_\ell)^2.$$

5. Conclusions

In this paper we have proved the optimal convergence of an adaptive finite element scheme for the approximation of the eigensolutions of the Maxwell system. The scheme makes use of Nédélec edge finite element in three space dimensions and a standard residual-based error indicator. The proof is based on an equivalent mixed formulation. The most challenging part of the proof consists in showing a suitable discrete reliability property.

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