

Finite-time singularity formation for scalar stretching equations

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Abstract

We consider equations of the type:

$$\partial_t \omega = \omega R(\omega),$$

for general linear operators R in any spatial dimension. We prove that such equations almost always exhibit finite-time singularities for smooth and localised solutions. Singularities can even form in settings where solutions dissipate an energy. Such equations arise naturally as models in various physical settings such as inviscid and complex fluids.

Keywords: singularity formation, Vortex stretching, complex fluids, Riesz transform

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1. Introduction

Singularity formation in non-linear PDEs is the source of a number of interesting phenomena. In general, we would like to know what the main mechanisms are that lead to singularity formation. The purpose of this work is to show that singularities are inherent to a certain

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type of equation and that they appear whenever possible. In particular, imposing energetic constraints (as long as they do not *a-priori* exclude singularity formation) is not sufficient to prevent singularity formation. Specifically, we consider solutions to scalar evolution equations of the form:

$$\partial_t \omega = \omega R(\omega),$$

where R is a general linear operator. Note that this includes a number of well-known equations like the Burgers equation, some thin film equations, crystal growth models [7], models of vorticity stretching in incompressible fluids [1, 8], and models of complex fluids [2].

1.1. Organisation of the paper

Section 2 gives a general criterion, theorem 2.1, that implies finite-time singularity formation for any equation for the form (2.1) with R a general linear operator. We then mention a few applications of this criterion. Section 3 deals with a particular setting that is not easily recovered from the general theorem 2.1 where solutions dissipate an energy and the main result there is theorem 3.1. A slightly counter-intuitive theorem 3.2 of possibly independent interest, relating the positivity of a function and its Riesz transform, is established in the course of the proof of theorem 3.1.

2. General case

This section concerns solutions to the general equation:

$$\begin{cases} \partial_t \omega = \omega R(\omega), \\ \omega(0, x) = \omega_0(x), \end{cases} \quad (2.1)$$

on \mathbb{R}^d for some $d \geq 1$, where R is a (possibly unbounded) linear operator³ on L^2 . Let R^* be the formal adjoint of R . Let us make the following hypothesis on R and ω_0 :

Hypothesis 1. Assume that there exist:

- a non-negative $W_2 \in L^1(\mathbb{R}^d)$;
- an ω_0 for which $R^*(W_2)\omega_0 \geq 0$

satisfying

$$-\infty < \int \log \left(\frac{\omega_0 R^*(W_2)}{W_2} \right) W_2, \quad 0 < \int \omega_0 R^*(W_2) < \infty.$$

Note that we are implicitly assuming in the Hypothesis that $R^*(W_2)$ is well-defined. It is easy to see that most linear operators that arise in applications satisfy this hypothesis (see also section 2.1). We now have the following general theorem.

Theorem 2.1. *Assume that ω_0, R satisfy the Hypothesis. Then, any smooth solution to (2.1) must develop a singularity in finite time.*

³ We could think of R as a Fourier multiplier.

Proof. Set $W_1 = R^*(W_2)$ and assume without loss of generality that $\int_{\mathbb{R}^d} W_2 = 1$. We have that

$$\omega = \omega_0 \exp\left(\int_0^t R(\omega)\right).$$

Let us integrate against W_1 . Then we get

$$\begin{aligned} \int \omega W_1 &= \int \omega_0 \exp\left(\int_0^t R(\omega)\right) W_1 = \int \exp\left(\log\left(\frac{\omega_0 W_1}{W_2}\right)\right. \\ &\quad \left. + \int_0^t R(\omega)\right) W_2. \end{aligned}$$

Now we apply Jensen's inequality with the measure $W_2 dx$ and get:

$$\begin{aligned} \int \omega W_1 &\geq \exp\left(\int \log\left(\frac{\omega_0 W_1}{W_2}\right) W_2\right) \exp\left(\int_0^t \int R(\omega) W_2\right) \\ &= c_* \exp\left(\int_0^t \int \omega W_1\right), \end{aligned}$$

where we used that $R^*(W_2) = W_1$ in the final equality and denoted

$$c_* = \exp\left(\int \log\left(\frac{\omega_0 W_1}{W_2}\right) W_2\right) = \exp\left(\int \log\left(\frac{\omega_0 R^*(W_2)}{W_2}\right) W_2\right),$$

with $c_* > 0$ by the Hypothesis.

It follows that ω must develop a singularity in finite time, since the quantity $f(t) := \int_0^t \int \omega W_1$ satisfies:

$$f'(t) \geq c_* \exp(f(t)),$$

so long as the solution exists. □

Let us mention a first interesting corollary:

Corollary 2.2. Consider (2.1) posed on \mathbb{T}^d . Assume that R is a (possibly unbounded) linear operator on $L^2(\mathbb{T}^d)$ for which R, R^* map analytic functions on \mathbb{T}^d to analytic functions on \mathbb{T}^d . Either $R \equiv 0$ or there exists an analytic ω_0 so that any solution to (2.1) must develop a singularity in finite time.

Remark 2.3. For general R , this equation might not even be locally solvable in the space of analytic functions. The case of solutions posed on \mathbb{R}^d is similar.

Proof. Consider $R^*(1)$. If it is not identically zero, then we may take $\omega_0 = R^*(1)$, and $\log(R^*(1)^2)$ is integrable [5]. Thus, the condition is satisfied and we have a singularity. Otherwise $R^*(1) = 0$. If $R^* \not\equiv 0$, then we fix an analytic \tilde{W}_2 with $R^*(\tilde{W}_2) \not\equiv 0$ and define $W_2 = M + \tilde{W}_2$ for M large so that $W_2 > 0$. Then we take $\omega_0 = W_2 R^*(W_2)$ and note that $\log(R^*(W_2)^2)$ is integrable. Thus the hypothesis is satisfied and we have a singularity. □

2.1. Some examples

Let us give a few examples.

2.1.1. The Burgers equation. This is the case where $R = \partial_x$ on \mathbb{R} or \mathbb{T} . In this case, we may take $W_2(x) = \frac{1}{(1+x^2)^2}$ and then $W_1(x) = -W_2'(x) = \frac{4x}{(1+x^2)^3}$. Then we see the condition implies that ω_0 should satisfy:

$$\left| \int_{-\infty}^{\infty} \log \left(\frac{4x\omega_0(x)}{1+x^2} \right) \frac{1}{(1+x^2)^2} dx \right| < \infty.$$

Taking $\omega_0 = W_1(x) \exp(-x^2)$ does the job. Note, as a sanity check, that the integrability condition implies that ω_0 must be non-positive on $(-\infty, 0]$ and non-negative on $[0, \infty)$. This implies that $\omega_0'(x) > 0$ for some x . This is consistent with the classical fact that solutions to the Burgers equation (with a minus sign!) become singular if and only if there exists a point where the derivative of the data is positive.

2.1.2. The Constantin–Lax–Majda equation. This is the case where $R = H$ is the Hilbert transform. The problem can be posed on \mathbb{R} or on \mathbb{T} . The classical proofs of singularity formation for the Constantin–Lax–Majda equation all rely on *non-linear* identities related to the Hilbert transform. By taking $W_2(x) = \frac{1}{1+x^2}$, we see that $H(W_2) = c \frac{x}{1+x^2}$. Consequently, we deduce singularity formation for *any* ω_0 for which

$$\left| \int \log(cx\omega_0(x)) \frac{1}{1+x^2} dx \right| < \infty.$$

Note, as another sanity check, that the condition implies that $cx\omega_0(x) \geq 0$, which in particular implies that ω_0 vanishes at $x=0$ and that its Hilbert transform is positive at $x=0$ (this is consistent with the result of [1]). On \mathbb{T}^2 , we similarly note that we may take $W_2(x) = 1 + \cos(x)$ and $W_1 = -H(W_2) = -\sin(x)$. We thus see that the condition on ω_0 becomes

$$\int \log(-\sin(x)\omega_0(x)) (1 + \cos(x)) dx > -\infty,$$

which is manifestly true, for example, when $\omega_0(x) = -\sin(x)$.

2.1.3. A 2d equation. The next example we give resolves a question raised by Kiselev in [6]. Here we take $R = R_{12}$, the composition of the Riesz transforms R_1 and R_2 . Whether a finite-time singularity for this particular equation occurs was raised also in [4], where it was remarked that the C^α theory developed there implies that C^α solutions can develop a singularity in finite time. Using the above theorem, it is not difficult to exhibit a *smooth* and rapidly decaying initial datum for which the unique local solution develops a singularity. Indeed, we may take $W_2(x) = \frac{1}{(1+|x|^2)^3}$ and it is not difficult to show that the data $\omega_0(x_1, x_2) = R_{12}(W_2) \exp(-|x|^2)$ satisfies the Hypothesis. Similarly, on \mathbb{T}^2 , we may take $W_2 = 1 + \cos(x) \cos(y)$. Then, $R_{12}(W_2) = \sin(x) \sin(y)$, so that we may take $\omega_0(x, y) = \sin(x) \sin(y)$ and get a singularity.

2.1.4. Vortex stretching in swirl-free solutions to the 3d Euler equation. When modelling just the effect of vortex stretching in the swirl-free axi-symmetric Euler equation, R is taken to be the linear map that takes $\omega^\theta \rightarrow \frac{u_r}{r}$, where

$$u_r(r, z) = \int_{-\infty}^{\infty} \int_0^{\infty} K(r, r', z, z') \omega^\theta(r', z') dr' dz'.$$

Taking $W_2(r, z) = \frac{1}{(1+r^2+z^2)^3}$ and setting $\omega_0^\theta = R^*(W_2) \exp(-(r^2 + z^2))$ gives us a singularity. This recovers the result of [8]. Note that the hypotheses are easily seen to hold arguing as in the proof of corollary 2.2 and in the examples above.

2.15. *A non-example when we put further conditions on ω_0 .* In the case where $R = -Id$ and we consider *non-negative* initial data, it is obviously *impossible* to find ω_0 that is non-negative for which $\omega_0 R^*(W_2) \geq 0$. This is consistent with the fact that non-negative solutions in this case always satisfy $0 \leq \omega \leq \omega_0$. This simple example is meant to motivate the coming section.

3. An equation with an energy

Let us now consider a slightly more complicated setting, which is related to the non-example above. Consider non-negative solutions to the following model on \mathbb{R}^2 :

$$\begin{cases} \partial_t \omega = \omega R_1^2 \omega, \\ \omega(0, x) = \omega_0(x). \end{cases} \tag{3.1}$$

Here, R_1 is the first component of the Riesz transform. Since R_1^2 is a negative operator on L^2 , we get the following identities:

$$\begin{aligned} \frac{d}{dt} |\omega|_{L^1} &= -|R_1 \omega|_{L^2}^2, \\ \frac{d}{dt} |R_1 \omega|_{L^2}^2 &= -2 \int \omega (R_1^2 \omega)^2 \leq 0. \end{aligned}$$

Using these dissipative properties, Constantin and Sun [2] deduced global regularity for a wide class of non-negative solutions (in fact, for a wider class of equations when R_1 is replaced by more general anti-symmetric operators). Equation (3.1) can be seen as a toy model of the Oldroyd-B system [2]. We now show that general smooth non-negative solutions to (3.1) can develop singularities in finite time.

Theorem 3.1. *There exists a non-negative, smooth, and compactly supported ω_0 so that the unique solution to (3.1) develops a singularity in finite time.*

The proof given in section 2 does not seem to carry over here because we are searching for non-negative solutions and R_1^2 is a negative operator. While similar in spirit, the proof we will give here is a bit more involved and it will rely on a number of observations specific to the operator R_1^2 . A particularly important observation is that for a specific type of data ω_0 , which is highly concentrated in a certain way around the origin and supported in a particular conical region, $R_1^2 \omega_0$ restricted to a small ball around the origin is large and non-negative. A version of the leading order expansions given in [4] is used to establish this. In fact, it can be seen from the proof that we actually establish the following.

Theorem 3.2. *Consider R_1^2 , the composition of the first component of the Riesz transform with itself. There exists a non-trivial and non-negative $\tilde{W} \in L^1(\mathbb{R}^2)$ for which $R_1^2(\tilde{W})$ is non-negative on the support of \tilde{W} .*

This lemma may seem to contradict the fact that R_1^2 is a negative operator on L^2 , since we then should have that $(R_1^2(f), f)_{L^2} \leq 0$; the point is that $\tilde{W} \notin L^2$. Since \tilde{W} is algebraically unbounded and in L^1 , it actually belongs to L^p for some $1 < p < 2$. We are not aware of such a phenomenon being studied in the harmonic analysis literature, and it may be interesting to

study further questions related to the signs of W and $R(W)$ under various assumptions on W and R .

To construct such a weight \tilde{W} , we have to carefully design its angular and radial dependence. The two key points are that \tilde{W} is sufficiently unbounded as $r \rightarrow 0$ and it is supported in a cone about the angle $\theta = \frac{\pi}{2}$ (the reason for this latter choice has to do with the nature of R_1^2). Here, (r, θ) are the usual polar radius and angle on \mathbb{R}^2 .

3.1. Angular decomposition and localisation

We now turn to give the proof of theorem 3.1. For $k \in \mathbb{N}$ and $f: [0, \infty) \times \mathbb{S}^1 \rightarrow \mathbb{R}$, we write:

$$\begin{aligned} f_0(r) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r, \theta) \, d\theta, \\ f_k(r) &:= \frac{1}{\pi} \int_{-\pi}^{\pi} f(r, \theta) \cos(k\theta) \, d\theta, \quad k \geq 1. \end{aligned} \tag{3.2}$$

We will consider even Fourier modes in θ .

Lemma 3.3. *Solve $\Delta\psi = \omega$ on \mathbb{R}^2 . Then, we have that*

$$\begin{aligned} \psi_0(r) &= \int_0^r s^{-1} \int_0^s \tau \omega_0(\tau) \, d\tau; \\ \psi_{2k}(r) &= r^{2k} \int_r^\infty s^{-4k-1} \int_0^s \tau^{2k+1} \omega_{2k}(\tau) \, d\tau, \quad k \geq 1. \end{aligned} \tag{3.3}$$

Let us now focus our attention on *non-negative* solutions that are even in θ and supported in the region

$$\mathcal{C} := \left\{ (r, \theta) : \frac{3\pi}{8} < |\theta| < \frac{5\pi}{8} \right\}.$$

A key property is that for $(r, \theta) \in \mathcal{C}$, we have that

$$|\cos(2k\theta)| \leq \sqrt{2} |\cos(2\theta)|$$

for any k . It follows that

$$|\omega_{2k}(r)| \leq \sqrt{2} |\omega_2(r)|,$$

for all $r \in [0, \infty)$ whenever ω is non-negative on \mathbb{R}^2 and supported in \mathcal{C} . Now we have an important result.

Lemma 3.4. *Fix a non-negative $W \in C^4([\frac{3\pi}{8}, \frac{5\pi}{8}])$, vanishing to fourth order on the boundary. Then, there exist universal constants $c, C > 0$ so that*

$$\begin{aligned} \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} R_1^2 \omega W(\theta) \, d\theta &\geq c \int_r^\infty \frac{|\omega_2(s)|}{s} \, ds - C (|\omega_2(r)| \\ &\quad + \frac{1}{r} \int_0^r |\omega_2(s)| \, ds + r \int_r^\infty \frac{|\omega_2(s)|}{s^2} \, ds), \end{aligned}$$

for all ω non-negative and supported in \mathcal{C} .

Proof. To compute $R_1^2\omega$, we write:

$$R_1^2\omega = -\partial_{xx} \sum_k \psi_{2k}(r) \cos(2k\theta),$$

where ψ_{2k} is in (3.3). Now we simply use that

$$\left| \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \cos(2k\theta) W(\theta) d\theta \right| \leq \frac{C}{1+k^4}$$

and that $|\omega_{2k}| \leq \sqrt{2}|\omega_2|$ to estimate all terms with $k \neq 1$. For $k = 1$, we get the most singular term only when both x derivatives hit $r^2 \cos(2\theta)$ and we must investigate

$$\mathcal{S} := \int_r^\infty s^{-5} \int_0^s \tau^3 \omega_2(\tau) d\tau.$$

Upon integrating by parts, we see that

$$\mathcal{S} = \frac{1}{4} \int_r^\infty \frac{\omega_2(s)}{s} ds + \frac{1}{4} r^{-4} \int_0^r s^3 \omega_2(s) ds,$$

the latter term can be estimated by:

$$r^{-4} \int_0^r s^3 |\omega_2(s)| ds \leq \frac{1}{r} \int_0^r |\omega_2(s)| ds.$$

□

3.2. Jensen in θ and reduction to an equation on ω_2

Let us take the initial data

$$\omega_0(r, \theta) = F_0(r) \Gamma(\theta),$$

with $\Gamma \in C^\infty(\mathbb{S}^1)$ and supported in \mathcal{C} , even in θ and π -periodic. This simply means that the Fourier expansion in θ of Γ (and thus $\omega(r, \cdot)$, by inspection) only contains terms of the form $\cos(2k\theta)$ for $k \in \mathbb{N} \cup \{0\}$. Note that, in order that ω_0 be C^∞ , we will need F_0 to vanish to infinite order at $r=0$. This can be easily arranged. Now, formally solving the equation (3.1), we get that

$$\omega = \omega_0 \exp\left(\int_0^t R_1^2\omega\right).$$

Note that ω_0 being supported in \mathcal{C} implies that ω is supported in \mathcal{C} for all time. Let us now assume that

$$\int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \log(\Gamma(\theta)) d\theta = -M > -\infty.$$

Now, let us multiply by a weight W as in lemma 3.4 only stipulating further that

$$\int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} W(\theta) d\theta = 1,$$

and integrate in θ only:

$$\begin{aligned} \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \omega W d\theta &= F_0 \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \Gamma W \exp\left(\int_0^t R_1^2 \omega\right) d\theta \\ &= F_0 \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} W \exp\left(\log(\Gamma) + \int_0^t R_1^2 \omega\right) d\theta. \end{aligned}$$

Now we apply Jensen's inequality and deduce that

$$\int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \omega W d\theta \geq c(M, W) F_0 \exp\left(\int_0^t \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} R_1^2 \omega W d\theta ds\right).$$

Next, we invoke lemma 3.4 and we deduce:

$$\begin{aligned} \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \omega W d\theta &\geq c(M, W) F_0 \exp\left(\int_0^t \left[c \int_r^\infty \frac{|\omega_2(s)|}{s} ds - C(|\omega_2(r)| \right. \right. \\ &\quad \left. \left. + \frac{1}{r} \int_0^r |\omega_2(s)| ds + r \int_r^\infty \frac{|\omega_2(s)|}{s^2} ds \right] d\tau\right). \end{aligned}$$

Observing that

$$\int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \omega W d\theta \leq C|\omega_2(r)|,$$

we deduce (upon calling $|\omega_2(t, r)| = f(t, r)$),

$$\begin{aligned} f(r, t) &\geq cF_0(r) \exp\left(\int_0^t \left[c \int_r^\infty \frac{f(s, \tau)}{s} ds - C(f(r, \tau) \right. \right. \\ &\quad \left. \left. + \frac{1}{r} \int_0^r f(s, \tau) ds + r \int_r^\infty \frac{f(s, \tau)}{s^2} ds \right] d\tau\right). \end{aligned} \tag{3.4}$$

To simplify the notation, let us define the linear operator L :

$$L(g) := c \int_r^\infty \frac{g(s)}{s} ds - C\left(g(r) + \frac{1}{r} \int_0^r g(s) ds + r \int_r^\infty \frac{g(s)}{s^2} ds\right).$$

3.3. C^α argument for C^∞ solutions

Now we make a choice of a weight in (r, θ) that will allow us to deduce singularity formation as before. Interestingly, the radial part of the weight that we choose is inspired by the C^α singularities constructed in [4] even though we construct here smooth solutions that develop a singularity.

Lemma 3.5. *For any $c, C > 0$, there exists a weight W_2 that is positive and integrable on $[0, \infty)$ for which*

$$W_1 := L^*(W_2) > 0,$$

where L^* is the L^2 adjoint of L .

Let us start by observing that

$$L^*(g) = \frac{1}{2r} \int_0^r g(s) \, ds - C \left(g(r) + \int_r^\infty \frac{g(s)}{s} \, ds + \frac{1}{r^2} \int_0^r sg(s) \, ds \right).$$

Proof. For α small, take

$$\tilde{W}_2(r) = \frac{r^{-1+\alpha}}{1+r^{2\alpha}}$$

and let $W_2(r)$ be the normalization of $\tilde{W}_2(r)$. In this case, the first term in $L^*(g)$ can be computed directly:

$$\frac{c}{r} \int_0^r \tilde{W}_2(s) \, ds = \frac{c}{r} \int_0^r \frac{s^{-1+\alpha}}{1+s^{2\alpha}} \, ds = \frac{c}{\alpha r} \int_0^{r^\alpha} \frac{ds}{1+s^2} = \frac{c \arctan(r^\alpha)}{\alpha r}.$$

It is not difficult to show that for α sufficiently small, this term dominates all the other terms. In particular,

$$L^*(\tilde{W}_2)(r) > \frac{c \arctan(r^\alpha)}{2\alpha r}$$

for all $r \geq 0$. □

Now let us establish singularity formation for (3.1) by studying (3.4). First, let us choose $F_0 \in C^\infty([0, \infty))$ vanishing to infinite order at $r = 0$ with the property that

$$\int \log\left(\frac{F_0 W_1}{W_2}\right) W_2 = -K > -\infty.$$

The existence of such an F_0 is easy to see since W_1 and W_2 are positive everywhere and algebraic as $r \rightarrow 0$ and $r \rightarrow \infty$. Now, testing (3.4) with W_1 , we see that

$$\int f W_1 \geq c \int F_0 W_1 \exp\left(\int_0^t L(f)\right) = c \int \exp\left(\log\left(\frac{F_0 W_1}{W_2}\right) + \int_0^t L(f)\right) W_2.$$

Applying Jensen's inequality and using the definition of W_1 , we see that

$$\int f W_1 \geq c \exp\left(\int_0^t \int f W_1\right).$$

Singularity formation now follows.

3.4. The case of strictly positive solutions?

The proof of theorem 3.1 uses that the data is compactly supported (specifically that it is supported in the cone \mathcal{C}). It may be asked whether it is possible to get a singularity for *positive* solutions, say, on \mathbb{T}^2 . Observe that, setting $\omega = \exp(f)$, positive solutions to (3.1) are equivalent to solutions of:

$$\partial_t f = R_1^2(\exp(f)).$$

Let us multiply this equation by Δf . Then we see that

$$\frac{1}{2} \frac{d}{dt} \|f\|_{H^1}^2 = - \int R_1^2 (\exp(f)) \Delta f = \int \partial_{xx} \exp(f) f = - \int \exp(f) (\partial_x f)^2.$$

It follows that $\|f\|_{H^1}$ is non-increasing. It is not difficult to show that this implies that, for positive solutions, we have *a-priori* control of all the L^p norms of ω for $p < \infty$ (this follows from the Moser–Trudinger inequality applied to $f = \log \omega$). While this does not necessarily imply global regularity, as far as we can tell, it does indicate that there may be a serious difference between considering *positive* solutions and non-negative solutions (see also Question 5 of [3]). Certainly, such bounds rule out the existence of positive self-similar blow-up solutions.

4. Some concluding remarks

We gave a new technique to establish singularity formation in *scalar* stretching problems. The technique was flexible enough to handle even equations with a dissipative structure and answer a few open problems. Two important directions for future consideration include the matrix version:

$$\partial_t A = R(A)A, \tag{4.1}$$

with R some operator acting on matrices as well as the advective problem both in the scalar case and the matrix/vector case:

$$\partial_t f + u \cdot \nabla f = fR(f). \tag{4.2}$$

For the matrix case, (4.1), it is possible to consider the evolution of the determinant:

$$\frac{d}{dt} \det(A) = \operatorname{tr}(R(A)) \det(A).$$

Using this idea, one can derive similar singularity results for certain classes of systems. Unfortunately, for most of the systems we are concerned with (such as the evolution of the gradient of the flow-map in incompressible flows), the determinant is conserved and so no singularity can be deduced so simply. Still, it is possible that there are other extensions of the ideas presented above that do not require us to reduce to a scalar problem. Investigating this question is of great interest. For the case with transport given in (4.2), there are obvious extensions once some assumptions are made on the relationship between R and u . It remains to be seen whether such ideas can be used to establish singularity formation in relevant physical systems like the incompressible Euler equation.

Data availability statement

No new data were created or analysed in this study.

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