Vol. 44 (2013) *ACTA PHYSICA POLONICA B* No 5

EINSTEIN RELATION IN SYSTEMS WITH ANOMALOUS DIFFUSION*

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(Received April 11, 2013)

We discuss the role of non-equilibrium conditions in the context of anomalous dynamics. We study in detail the response properties in different models, featuring subdiffusion and superdiffusion: in such models, the presence of currents induces a violation of the Einstein relation. We show how in some of them it is possible to find the correlation function proportional to the linear response, in other words, we have a generalized fluctuation-response relation.

DOI:10.5506/APhysPolB.44.899
PACS numbers: 05.40.–a, 05.60.–k, 05.70.Ln

1. Introduction

One of the most important and useful results of statistical mechanics is the fluctuation-dissipation relation (FDR). Its relevance, both practical and conceptual, is in the link between the statistical properties of an unperturbed system and the behavior of the relaxation of a perturbation. The first example of FDR has been found by Einstein in his seminal work on the Brownian Motion: it is possible to understand the response to an external field in terms of the statistical features of the unperturbed system. Namely denoting by x the position of the Brownian particle, we have that in the unperturbed system $\langle x(t)\rangle = 0$ and $\langle x^2(t)\rangle \simeq 2Dt$; once a small constant external force F is applied, one has $\overline{\delta x}(t) = \langle x(t)\rangle_F - \langle x(t)\rangle \simeq \mu F t$,

* Presented at the XXV Marian Smoluchowski Symposium on Statistical Physics, "Fluctuation Relations in Nonequilibrium Regime", Kraków, Poland, September 10–13, 2012.

where $\langle \ldots \rangle_F$ indicates the average on the perturbed system, and one has the proportionality between $\overline{\delta x}(t)$ and $\langle x^2(t) \rangle$

$$\frac{\langle x^2(t) \rangle}{\overline{\delta x}(t)} = \frac{2}{\beta F}\,. \tag{1}$$

Therefore, the mobility μ is determined by D: $\mu = D/k_{\rm B}T$.

In the last decades, many researchers have studied anomalous diffusion, *i.e.*

$$\langle x^2(t) \rangle \sim t^{2\nu} \quad \text{with} \quad \nu \neq 1/2\,. \tag{2}$$

When the diffusion is sublinear, $\nu < 1/2$, the dynamics is called sub-diffusive, a phenomenon found, for instance, in disordered systems where the leading mechanism for the relaxation is barriers jumping or in chaotic systems with long-time correlations. Differently, when $\nu > 1/2$, the systems features superdiffusion. The latter phenomenon characterizes particles diffusing in media where the distribution of free jumps is scale free (Lévy flights), that can be either porous media or systems with turbulent flows. It is quite natural to wonder if the Einstein relation (1) holds even in the presence of anomalous diffusion.

In this contribution, we discuss the FDR for systems with anomalous diffusive dynamics. Some results in this context have been already obtained for specific systems [1, 2] and for non-standard noise [3]. The main aim of this paper is to discuss the role of out-of-equilibrium conditions.

In Secs. 2 and 3, we study transport properties of the comb lattice. In such a system, in the absence of an external drift, we have a subdiffusive behavior $\langle x^2(t) \rangle \sim t^{1/2}$, but, in spite of this, the Einstein relation (1) holds. This relation fails when one perturbs a state where a current is already present. Quite remarkably in this case, the Einstein relation (1) can be substituted with a generalized FDR which is always valid, even in the presence of non-equilibrium currents.

Section 4 is devoted to the study of a system, the continuous time random walk (CTRW), showing superdiffusion. Also, in this case, in the absence of an external drift, the relation (1) holds, while it is violated when a drift is present. At variance with the diffusion on the comb lattice, in the superdiffusive case, we are not able to find in an explicit way how the response formula Eq. (1) can be modified by adding correlations with physical observables in the presence of non-equilibrium currents.

In Sec. 5, we will illustrate the behavior of the single-file model, which is a one-dimensional gas on a ring, coupled with a thermostat, where the collisions between particles can be taken either elastic or inelastic. In such a model, which also shows subdiffusive dynamics, even in the absence of directional current, an out-of-equilibrium regime is present due to a homogeneous flux of energy passing through the system. In this case, we will show how in the out-of-equilibrium regimes a generalized FDR can be recovered with a *minimal* assumption on the velocities probability density function (PDF) and by exploiting the response formula introduced in [4]. Finally, some conclusions are drawn.

2. Comb: diffusion and response function

The comb lattice is a discrete structure consisting of an infinite linear chain (backbone), the sites of which are connected with other linear chains (teeth) of length L [5]. We denote by $x \in (-\infty, \infty)$ the position of the particle performing the random walk along the backbone and with $y \in [-L, L]$, that along a tooth. The transition probabilities from (x, y) to (x', y') are: $W^d[(x, 0) \to (x \pm 1, 0)] = 1/4 \pm d$, $W^d[(x, 0) \to (x, \pm 1)] = 1/4$ and $W^d[(x, y) \to (x, y \pm 1)] = 1/2$ for $y \neq 0, \pm L$. On the boundaries of each tooth, $y = \pm L$, the particle is reflected with probability 1. Here, we consider a discrete time process and, of course, the normalization $\sum_{(x',y')} W^d[(x, y) \to (x', y')] = 1$ holds. The parameter $d \in [0, 1/4]$ allows us to consider also the case where a constant external field is applied along the x axis, producing a non-zero drift of the particle. A state with a non-zero drift can be considered as a perturbed state (in that case, we denote the perturbing field by ε), or it can be itself the starting state where a further perturbation can be added changing $d \to d + \varepsilon$. Let us start by considering the case $d = 0$.

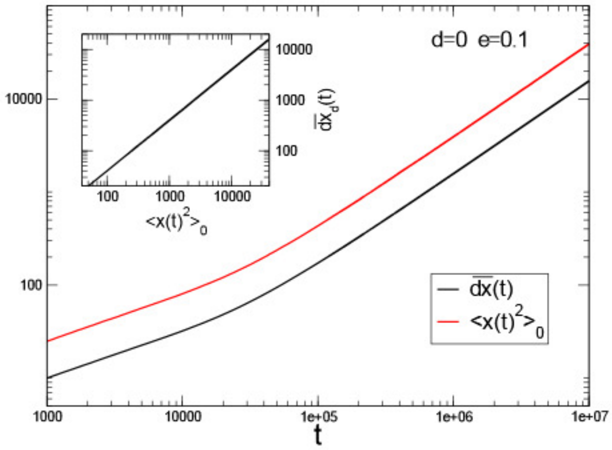

Fig. 1. $\langle x^2(t) \rangle_0$ and the response function $\overline{\delta x}(t)$ for $L = 512$. In the inset, the parametric plot $\overline{\delta x}(t)$ *versus* $\langle x^2(t) \rangle_0$ is shown. The scaling behavior of the crossover is commented in [6].

In the limit of infinite teeth, $L \to \infty$, one has a subdiffusive behavior [7]: $\langle x^2(t)\rangle_0 \sim t^{1/2}$. The FDR in its standard form is fulfilled at any time, namely, if we apply a constant perturbation ε pulling the particles along the backbone, one has numerical evidence that $\langle x^2(t)\rangle_0 \simeq C\overline{\delta x}(t) \sim t^{1/2}$. In the following section, we derive this result from a generalized FDR. Moreover, the proportionality between $\langle x^2(t)\rangle_0$ and $\overline{\delta x}(t)$ is fulfilled also with $L < \infty$, where both the mean square displacement and the drift with an applied force exhibit the same crossover from subdiffusive, $\sim t^{1/2}$, to diffusive, $\sim t$ (see Fig. 1). Therefore, the FDR is somehow "blind" to the dynamical crossover experienced by the system. When the perturbation is applied to a state without any current, the proportionality between response and correlation holds despite anomalous transport phenomena.

On the contrary, in the presence of a non-zero drift [8], the emergence of a dynamical crossover is connected to the breaking of the FDR. Indeed, what we found in the infinite L comb model is

$$\left\langle x^2(t)\right\rangle_d \sim a\, t^{1/2} + b\, t\,, \qquad \overline{\delta x}_d(t) \sim \varepsilon t^{1/2}\,, \tag{3}$$

with a and b two constants and $\overline{\delta x}_d(t) = \langle x(t)\rangle_{d+\varepsilon} - \langle x(t)\rangle_d$: at large times the Einstein relation breaks down (see Fig. 2). The proportionality between response and fluctuations cannot be recovered by simply replacing $\langle x^2(t)\rangle_d$ with $\langle x^2(t)\rangle_d - \langle x(t)\rangle_d^2$, as it happens for Gaussian processes, namely we find

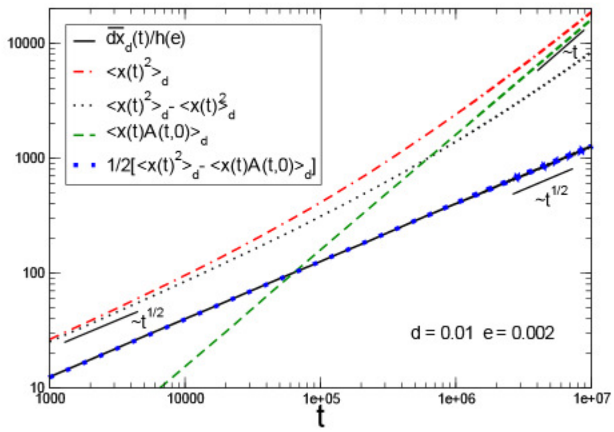

Fig. 2. Response function (black line), mean squared displacement (dash-dotted/red line) and second cumulant (dotted/black line) measured in the comb model with $L = \infty$, field $d = 0.01$ and perturbation $\varepsilon = 0.002$. The correlation with the quantity A defined in the text (dashed/green line) yields the right correction to recover the full response function (squares/blue), in agreement with the FDR (6).

numerically

$$\left\langle [x(t) - \langle x(t)\rangle_d]^2 \right\rangle_d \sim a' t^{1/2} + b' t \,, \tag{4}$$

where a' and b' are two constants, as reported in Fig. 2.

3. Comb: application of a generalized FDR

In order to find out a relation between $\langle x^2(t)\rangle_d$ and $\langle x(t)\rangle_d$, we need a generalized fluctuation-dissipation relation. In the comb model discussed here, the relation between transition rates in presence of small perturbation d, W^d, and with a slightly enhanced one $W^{d+\varepsilon}$ can be written as

$$W^{d+\varepsilon}\left[(x,y) \to (x',y')\right] = W^d\left[(x,y) \to (x',y')\right] e^{\frac{h(\varepsilon)}{2}(x'-x)} \tag{5}$$

with $h(\varepsilon) = 8\varepsilon$. Such a relation between transition probabilities in presence of a different perturbation is usually called in the literature *local detail balance condition*. For general models, where the perturbation enters the transition probabilities according to such a local detailed balance condition, the following formula for the integrated linear response function has been derived [9–13]

$$\begin{aligned}\frac{\overline{\delta \mathcal{O}}_d}{h(\varepsilon)} &= \frac{\langle \mathcal{O}(t)\rangle_{d+\varepsilon} - \langle \mathcal{O}(t)\rangle_d}{h(\varepsilon)} \\ &= \tfrac{1}{2}\left[\langle \mathcal{O}(t)x(t)\rangle_d - \langle \mathcal{O}(t)x(0)\rangle_d - \langle \mathcal{O}(t)A(t,0)\rangle_d\right] ,\end{aligned} \tag{6}$$

where $\mathcal{O}$ is a generic observable, and $A(t,0) = \sum_{t'=0}^{t} B(t')$, with

$$B[(x,y)] = \sum_{(x',y')} (x' - x)\, W^d\left[(x,y) \to (x',y')\right] . \tag{7}$$

The above observable yields an effective measure of the propensity of the system to leave a certain state (x,y). Recalling the definitions for the transition probabilities from the above equation, we have $B[(x,y)] = 2d\delta_{y,0}$ and, therefore, the sum on B has an intuitive meaning: it counts the time spent by the particle on the x axis. The results described in the previous section can be then read in the light of the fluctuation-dissipation relation (6):

1. Putting $\mathcal{O}(t) = x(t)$ in the case without drift, *i.e.* $d = 0$, one has $B = 0$ and, recalling the choice of the initial condition $x(0) = 0$,

$$\frac{\overline{\delta x}}{h(\varepsilon)} = \frac{\langle x(t)\rangle_\varepsilon - \langle x(t)\rangle_0}{h(\varepsilon)} = \tfrac{1}{2}\left\langle x^2(t)\right\rangle_0 . \tag{8}$$

   This explains the behavior $\langle x^2(t)\rangle_0 \simeq C\overline{\delta x}(t) \sim t^{1/2}$ even in the anomalous regime and predicts the correct proportionality factor, $\overline{\delta x}(t) = \varepsilon\langle x^2(t)\rangle_0 / W^0$.

2. Putting $\mathcal{O}(t) = x(t)$ in the case with $d \neq 0$, one has

$$\frac{\overline{\delta x_d}}{h(\varepsilon)} = \frac{1}{2}\left[\langle x^2(t)\rangle_d - \langle x(t)A(t,0)\rangle_d\right] . \tag{9}$$

This explains the observed behaviors (3): the leading behavior at large times of $\langle x^2(t)\rangle_d \sim t$, turns out to be exactly canceled by the term $\langle x(t)A(t,0)\rangle_d$, so that the relation between response and unperturbed correlation functions is recovered (see Fig. 2).

4. A model for superdiffusion

We consider an ensemble of probe particles of mass m endowed with scalar velocity v and position x, interacting with particles of mass M and velocity V extracted from an equilibrium bath at temperature T. The scattering probability does not depend on the relative velocity between the probe particle and the colliders as, for instance, in the case of Maxwell-molecule models [14]. Velocity of the probe particle changes from v to v' at each collision, according to the rule

$$v' = \gamma v + (1-\gamma)V , \tag{10}$$

where $\gamma = (\zeta - \alpha)/(1+\zeta)$, with $\zeta = m/M$, and α is the coefficient of restitution ($\alpha = 1$ for an elastic collision). The velocity V of the bath particles is a random variable generated from a Gaussian distribution with zero mean and variance T/M

$$P_S(V) = \sqrt{\frac{M}{2\pi T}} \exp\left\{-\frac{M}{2T}V^2\right\} . \tag{11}$$

The elementary steps of the dynamics are: *(i)* a flight, $x(t+\tau) = x(t) + v'\tau + 1/2\mathcal{E}\tau^2$, where $x(t)$ is the position of the probe particle at time t, with τ taken from a distribution $P_\tau(\tau)$ and $\mathcal{E}$ a constant acceleration, followed by *(ii)* a collision $v' = \gamma v + (1-\gamma)V$, with V taken from the Gaussian distribution (11). In the specific case $\alpha = 1$ and $M = m$, one has $\gamma = 0$ and the collision rule (10) results in a random update of the velocity according to the distribution (11). The duration of each flight, τ, is an independent identically distributed random variable with probability

$$P_\tau(\tau) \sim \tau^{-g} \tag{12}$$

with $g > 1$. This kind of process is called Lévy walk collision process [15], and *may* be interpreted as due to scattering centers randomly distributed on a fractal spatial structure as, for instance, in the case of molecular diffusion in porous media [16]. If $\alpha \neq 1$ or $m \neq M$, with $\alpha \neq \zeta$, a dependence on the last velocity before the collision remains.

According to the dynamic rules of the process described above, the displacement of the probe particle is always finite in a finite time. The anomalous dynamics of such a model has been studied in [17], showing that the process is an example of "strong" anomalous diffusion, namely that it is not possible to find a scaling for the PDF of displacements. Such a collision process becomes a standard diffusive system when P_τ decays fast enough: in this regime, the dynamics is qualitatively equivalent to that of a system with exponential P_τ studied, for instance, in [18–20]. Extended investigations on the dynamics of Levý walks, see for instance [21], showed how analytical expressions for the mean square displacement are available.

We recall here a simple argument [17] to study the asymptotic behavior of higher order moments of the displacements distribution; such an argument can be easily applied also to the case with an external perturbing field discussed here in Sec. 4.2. In order to obtain in a simple way the dominant asymptotic behavior of $\langle x^2(t)\rangle$, we introduce a cut-off t_c for $P_\tau(\tau)$

$$P_\tau(\tau) \sim \begin{cases} \tau^{-g} & \text{if } \tau < t_\mathrm{c}\,; \\ 0 & \text{if } \tau > t_\mathrm{c}\,. \end{cases} \tag{13}$$

Assuming $x = 0$ as initial condition for each trajectory, the mean square displacement after the time t, where $N(t)$ collisions occurred, can be written in full generality as

$$\left\langle x^2(t)\right\rangle = \left\langle \left[\sum_{i=1}^{N(t)} v_i\tau_i\right]^2\right\rangle \simeq \sum_{i=1}^{\overline{N}(t)} \left\langle v_i^2\tau_i^2\right\rangle + 2\overline{N}(t)\sum_{i=1}^{\overline{N}(t)} \langle v_i v_0 \tau_i \tau_0\rangle\,. \tag{14}$$

Here, v_i denotes the velocity of the probe particle after the ith collision, τ_i is the time elapsed between the collisions i and $i+1$ and $\overline{N}(t)$ is the average number of collisions occurred up to time t. The average $\langle\cdots\rangle$ is taken over the distributions (11) and (12). From Eq. (13), we have for $n+1-g>0$

$$\langle\tau^n\rangle_\mathrm{c} \sim t_\mathrm{c}^{n+1-g}, \tag{15}$$

where $\langle\cdots\rangle_\mathrm{c}$ denotes an average over the distribution (13) with the cut-off t_c.

We start by considering the case with independent velocities v_i, this corresponds to $\gamma = 0$. Then, estimating $\langle x^2(t)\rangle$ at a time $t \gg t_\mathrm{c}$, so that the average number of collisions along the trajectory can be approximated with $\overline{N}(t) \simeq t/\langle\tau\rangle_\mathrm{c}$, and considering that the cross terms in Eq. (14) are zero, we can write

$$\left\langle x^2(t)\right\rangle \simeq \frac{t}{\langle\tau\rangle_\mathrm{c}}\left\langle v^2\right\rangle\left\langle\tau^2\right\rangle_\mathrm{c}\,. \tag{16}$$

In the case of $g > 3$, both $\langle\tau\rangle_\mathrm{c}$ and $\langle\tau^2\rangle_\mathrm{c}$ are finite, even in the limit $t_\mathrm{c}\to\infty$, so that we find the simple diffusive behavior $\langle x^2(t)\rangle \sim t$. For t □ t_c and

$1 < g < 3$ instead of (16), we expect

$$\left\langle x^2(t) \right\rangle \sim t^{2\nu}\,. \tag{17}$$

One can easily find the exponent ν with a matching argument. Comparing (16) and (17) at $t \sim t_{\rm c}$ and using (15), we obtain $\nu = 1/2$ for $g > 3$, $\nu = 2 - g/2$ for $2 < g < 3$, and $\nu = 1$ for $1 < g < 2$ (logarithmic corrections appear at the values $g = 2$ and $g = 3$ [21]).

4.1. Einstein relation

We discuss here the validity of the Einstein relation for superdiffusive anomalous dynamics. In particular, such a relation can be checked in two different situations:

(A) The drift due to the external force is compared with the mean square displacement (MSD) of the probe particle in the absence of any pulling force. This case corresponds to a fluctuation-dissipation experiment realized by switching on from zero the external perturbation.

(B) The drift is compared to the MSD in a state where a current is already present. This procedure corresponds to increase the intensity of the perturbation in a state already perturbed and compare the average current in this state with the fluctuations in the initial reference state.

In the following, we will refer to situation (A) as a test of the fluctuation-dissipation relation at *equilibrium* while to case (B) as a test *out of equilibrium*. We will show that these two cases are very different.

4.2. Perturbation of a state without current

The argument used in Sec. 4 to study the MSD can be applied to the drift, yielding

$$\langle x(t) \rangle_{\mathsf{E}} = \left\langle \sum_{i=1}^{N(t)} \left(v_i \tau_i + \frac{\mathcal{E}}{2}\tau_i^2 \right) \right\rangle = \frac{t}{\langle \tau \rangle_{\rm c}} \left[\langle \tau \rangle_{\rm c} \langle v \rangle + \frac{\mathcal{E}}{2} \langle \tau^2 \rangle_{\rm c} \right] = \frac{\mathcal{E}}{2} \frac{t}{\langle \tau \rangle_{\rm c}} \left\langle \tau^2 \right\rangle_{\rm c}\,, \tag{18}$$

which perfectly matches the result for the MSD found in Eq. (16). Therefore, when we perturb an equilibrium state, namely a state without currents, we have for any value of $g > 1$

$$\frac{\left\langle x^2(t) \right\rangle}{\langle x(t) \rangle_{\mathsf{E}}} = \text{const.} \tag{19}$$

Let us note that the Einstein relation holds quite generally, namely it persists even if we make our process non-time-independent by allowing some memory across collisions, that is, we put $\gamma > 0$ in the collision rule Eq. (10), preventing a complete reshuffling of velocities [22].

4.3. Perturbation of a state with a current

In order to study the Einstein relation out of equilibrium, let us first consider a simple Gaussian process, namely the Brownian motion of a colloidal particle when we add a constant force pulling the particle. In this case, it is sufficient to replace the MSD around the average position $\langle|\delta x(t)]^2\rangle_{\mathsf{E}} = \langle x^2(t)\rangle_{\mathsf{E}} - \langle x(t)\rangle_{\mathsf{E}}^2$ in place of the simple MSD, in order to recover the Einstein relation with the drift $\langle x(t)\rangle_{\mathsf{E}+\delta\mathsf{E}} - \langle x(t)\rangle_{\mathsf{E}}$ (here $\langle\cdots\rangle_{\mathsf{E}+\delta\mathsf{E}}$ denotes the average over a state where the further perturbation $\delta\mathcal{E}$ is applied). In what follows, we consider that a non-trivial violation of the Einstein relation happens when also the MSD around the drift lacks the proportionality with the drift itself. This is indeed the case of superdiffusive dynamics.

For simplicity, we will refer to the case $\gamma = 0$ but the physical picture remains the same also when the case with memory is considered. In our model, by applying the constant field $\mathcal{E} > 0$, we have

$$\begin{aligned}\left\langle x^2(t)\right\rangle_{\mathsf{E}} &= \left\langle \left| \sum_{i=1}^{N(t)} \left(v_i \tau_i + \frac{\mathcal{E}}{2}\tau_i^2 \right) \right]^2 \right\rangle_{t_{\mathsf{c}}} \\ &\simeq \frac{\mathcal{E}^2}{4} t^2 \frac{\left\langle \tau^2 \right\rangle_{\mathsf{c}}^2}{\langle \tau \rangle_{\mathsf{c}}^2} + t \left(\frac{\mathcal{E}^2}{4} \frac{\left\langle \tau^4 \right\rangle_{\mathsf{c}} - \left\langle \tau^2 \right\rangle_{\mathsf{c}}^2}{\langle \tau \rangle_{\mathsf{c}}} + \frac{\left\langle v^2 \right\rangle \left\langle \tau^2 \right\rangle_{\mathsf{c}}}{\langle \tau \rangle_{\mathsf{c}}} \right), \end{aligned} \tag{20}$$

$$\left\langle|\delta x(t)]^2\right\rangle_{\mathsf{E}} = \left\langle x^2(t)\right\rangle_{\mathsf{E}} - \langle x(t)\rangle_{\mathsf{E}}^2 \simeq t \left(\frac{\mathcal{E}^2}{4} \frac{\left\langle \tau^4 \right\rangle_{\mathsf{c}} - \left\langle \tau^2 \right\rangle_{\mathsf{c}}^2}{\langle \tau \rangle_{\mathsf{c}}} + \frac{\left\langle v^2 \right\rangle \left\langle \tau^2 \right\rangle_{\mathsf{c}}}{\langle \tau \rangle_{\mathsf{c}}} \right). \tag{21}$$

In the case of $2 < g < 3$, namely when the distribution $P_\tau(\tau)$ has finite mean and infinite variance, imposing the cut-off, the diffusion around the average position behaves asymptotically as

$$\left\langle|\delta x(t)]^2\right\rangle_{\mathsf{E}} \simeq t \left(\frac{\mathcal{E}^2}{4} \frac{t_{\mathsf{c}}^{5-g} - t_{\mathsf{c}}^{6-2g}}{\langle \tau \rangle_{\mathsf{c}}} + \frac{\left\langle v^2 \right\rangle t_{\mathsf{c}}^{3-g}}{\langle \tau \rangle_{\mathsf{c}}} \right). \tag{22}$$

Considering, for instance, the case $g = 5/2$, by applying the matching argument to Eq. (22), we find that the leading behaviors are

$$\left\langle x^2(t)\right\rangle_{\mathsf{E}} \sim t^3, \qquad \left\langle|\delta x(t)]^2\right\rangle_{\mathsf{E}} \sim t^{7/2}, \tag{23}$$

whereas, from Eq. (18), we have that

$$\langle x(t)\rangle_{\mathsf{E}+\delta\mathsf{E}} - \langle x(t)\rangle_{\mathsf{E}} \propto \langle x(t)\rangle_{\mathsf{E}} \sim t^{3/2}\,, \tag{24}$$

as shown in Fig. 3. The Einstein relation is, therefore, violated in the out-of-equilibrium regime for both the MSD and MSD around the average current for all the values of the flight time distribution exponent $g < 5$.

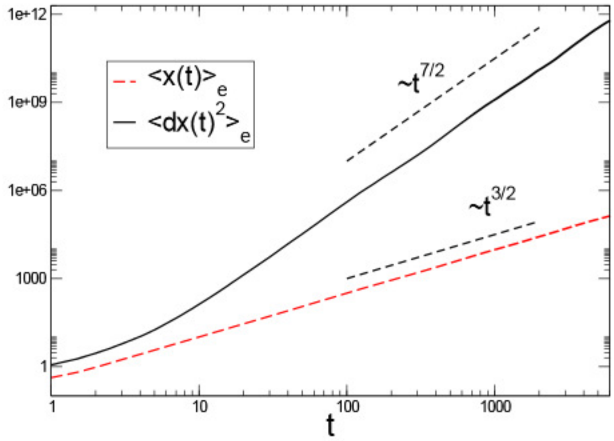

Fig. 3. Log–log plot of the MSD (black line) in the presence of a constant external field $\mathcal{E} = 1.5$ and drift (dashed/red line) with the same value of the field, in the case of superdiffusive dynamics with $g = 5/2$. We observe a breaking of the Einstein relation with leading asymptotic behaviors, $\langle[\delta x(t)]^2\rangle_{\mathsf{E}} \sim t^{7/2}$, see Eq. (22), and $\langle x(t)\rangle_{\mathsf{E}} \sim t^{3/2}$, see Eq. (18).

The study of the Einstein relation in a state with non-zero current induced by a constant field $\mathcal{E}$ allows us to show that there is some "anomaly" in the dynamics also when the exponent of the power law distribution of times is $g > 3$. More precisely, when $3 < g < 5$, at equilibrium, *i.e.* in the absence of current, a fluctuation-dissipation experiment would not show any anomaly in the dynamics, because $\langle[\delta x(t)]^2\rangle \sim \langle x(t)\rangle_{\mathsf{E}}$. On the other hand, the same experiment done out of equilibrium, *i.e.* comparing the MSD around the drift with the drift itself, shows an evident violation of the Einstein relation

$$\frac{\langle[\delta x(t)]^2\rangle_{\mathsf{E}}}{\langle x(t)\rangle_{\mathsf{E}}} \sim t^{5-g}\,. \tag{25}$$

5. Single file model: the role of correlations

Let us now discuss the single-file model, consisting of N Brownian rods on a ring of length L interacting with elastic or inelastic collisions and coupled with a thermal bath. The equation of motion for the single particle velocity between collisions is

$$m\dot{v}(t) = -\gamma v(t) + \eta(t)\,, \tag{26}$$

where m is the mass, $\gamma \equiv 1/\tau_{\rm b}$ is the friction coefficient (with $\tau_{\rm b}$ the typical interaction time with the bath), and η is a white noise with variance $\langle \eta(t)\eta(t')\rangle = 2T\gamma\delta(t-t')$. The combined effect of collisions, noise and geometry (since the system is one-dimensional the particles cannot overcome each other) produces a non-trivial behavior. In the thermodynamic limit, *i.e.* L, $N \to \infty$ with $N/L \to \rho$, a subdiffusive behavior occurs [7].

Differently from the comb model, where we studied the effect of a constant field, we apply here an impulsive small force, which produces an instantaneous variation of the velocity of a certain particle, $\delta v_i(0)$, and we compare the response $R(t) = \overline{\delta v_i}(t)/\delta v_i(0)$ to the correlation $C(t) = \langle v_i(t)v_i(0)\rangle/\langle v_i(0)^2\rangle$. When collision are elastic, the stationary PDF of velocities is factorized in the contribution of each particle, which provides a Gaussian term, so that, by applying a perturbation of the kind just described, one finds that the Einstein relation is always fulfilled [23–26].

In Fig. 4 (left), the parametric plot of response *versus* correlation is displayed for cases where the Einstein relation is no more verified. The departure from the equality $R(t) = C(t)$ can be quite strong: for fixed inelasticity it increases with the packing fraction ϕ.

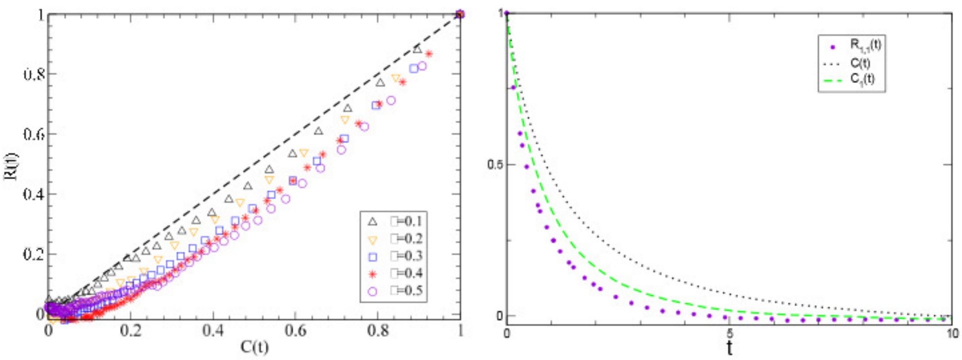

Fig. 4. Left: parametric plot of response $R(t)$ *versus* normalized autocorrelation $C(t)$. The dashed line is the Einstein relation $R \equiv C$. All data are obtained with restitution coefficient $r = 0.6$. $\tau_{\rm b} = 1$ is kept constant, while ϕ is changed. Right: comparison between $R(t)$ and correlations $C(t)$ and $C_1(t) \equiv \Sigma_{1,1}\langle v_1(t)v_1(0)\rangle + 2\Sigma_{1,2}\langle v_1(t)v_2(0)\rangle$.

In agreement with the observation in [26] for higher dimensions, the main source of breakdown of the FDR for the single file model is a stationary PDF that couples the velocities of different particles. We know [4] that a generalized response formula for systems with a stationary state is

$$R_{i,j}(t) = \frac{\overline{\delta v_i(t)}}{\delta v_j(0)} = -\left\langle v_i(t) \frac{\partial \ln \rho_v(\{v\})}{\partial v_j} \right\rangle_{t=0}, \tag{27}$$

where $\rho_v(\{v\})$ is the joint distribution of velocities. Clearly, if the PDF of velocities is not factorized, we cannot simply have $\langle v_i(t) v_j(0) \rangle$ on the right of Eq. (27), and hence the FDR relation is broken.

The most natural way to proceed in the response analysis is to make some *Ansatz* on the coupled probability distribution of velocities. For instance, one can consider the Gaussian approximation

$$\ln \rho_v(\{v\}) \simeq -\tfrac{1}{2} \sum_{i,j} v_i \Sigma_{i,j} v_j + \text{const.}\,, \tag{28}$$

and, as a first correction, one can take $\Sigma_{i,j}$ different from zero only for neighbor particles. By inserting the assumption of Eq. (28) in the generalized response formula Eq. (27), one has

$$\frac{\overline{\delta v_1(t)}}{\delta v_1(0)} \simeq \Sigma_{1,1} \langle v_1(t) v_1(0) \rangle + 2\Sigma_{1,2} \langle v_1(t) v_2(0) \rangle\,. \tag{29}$$

Using such an approximation, it is possible to obtain a good prediction of the response, as shown in Fig. 4 (right). Thus we have that, even if the interest is for the autoresponse function, it is necessary to take into account the correlations with the other variables.

6. Conclusions

In the last decades, many efforts have been devoted to the study of anomalous diffusion. A natural issue is whether the presence of anomalous diffusion can significantly change the response scenario, in both equilibrium and non-equilibrium setups. With this problem in mind, we have studied three models exhibiting anomalous diffusion with different dynamical features. In the comb model, the distribution of the position shows strong deviations from the Gaussian behavior [27]. On the contrary, the single file model is Gaussian, but a power-law tail is present in the autocorrelation of velocities [23]. The superdiffusive model is maybe more pathological, since it belongs to the class of strong anomalous diffusion [28].

Despite these differences, the emerging scenario is essentially unified: the fluctuation-dissipation properties are somehow blind to the presence of the anomalous diffusion in the underlying dynamics if the perturbation is added to an equilibrium or to a zero current state. In these cases, the proportionality between response and correlation is not altered by the presence of anomalous dynamics. This result is quite robust and it is valid also in presence of finite size effects and dynamical transitions from anomalous to standard diffusion. A different scenario holds when one perturbs a state out-of-equilibrium or already provided with a finite current. In the latter case, we have shown the validity of generalized response relations, which go beyond the Einstein response formula, and which amount to the fact that response can be written in terms of correlators computed on the unperturbed dynamics: this happens for the comb model and the single file model, both showing subdiffusion. They are somehow representative of the two classes of models where generalized response formulas are easily writable: comb is a paradigmatic example of stochastic models where the perturbations modify in a simple way the transition rates of the process (local detail balance is fulfilled); single file is representative of models where stationary non-equilibrium is characterized by a probability distribution which can be easily guessed although only approximately for the single-file. In these two models, there are the two most important mechanisms by which an ordinary response formula like the Einstein relation breaks down: either the presence of directed current in the reference state (comb) or the presence of nontrivial out-of-equilibrium correlations among the degrees of freedom (single file). For the Lévy-walk model, where the violation of the Einstein relation is also due to the presence of a current, we did not find a general response formula, for a state already with a current, in terms of correlators. Probably this pathological behavior is due to the "strong" anomalous nature of this model, which amounts to the absence of a single scaling *ansatz* valid for the PDF of displacements. This last results suggest how the out-of-equilibrium response of systems with anomalous dynamics has a sensitive dependence on the kind of anomaly that affects dynamics. A more systematic study on the situations where one finds a strong anomalous behavior remains therefore as an interesting field to be developed.

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