APPLICATION OF THE ENDOCHRONIC THEORY OF PLASTICITY BY THE FINITE ELEMENT METHOD.

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2. Preliminary notations and hypotheses

Let σ_{ij} $(i,j=1,2,3)$ be the stress tensor and ε_{ij} the strain tensor. We indicate as deviatoric stress the tensor $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_0$ where δ_{ij} is Kronecker's symbol and $\sigma_0 = \sigma_{ii}/3$.

In the same way we define the deviatoric strain tensor as $e_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon_0$ where $\varepsilon_0 = \varepsilon_{ii}/3$.

The quantities σ_0 and ε_0 are called the hydrostatic parts of the respective tensors (*)

The basic hypotheses about the elasto-plastic behaviour of a material are as follows:

- the material behaves isotropically in the sense there are no privileged directions for the load or the strain;
- the deformation process takes place in an isothermal way;
- the deformations are small enough so that we can consider them as infinitesimal;
- the strain state does not depend on the rate at which the deformation develops itself;
- during the entire load process, the hydrostatic part of the stress is a linear function of the hydrostatic strain, i.e.:

$$\sigma_0 = 3\varkappa\varepsilon_0 \qquad (1.1)$$

where $\varkappa$ is the bulk modulus of the material.

(*) In what follows, σ, ε and all related quantities will generally be considered as vectors in a nine dimension vector space.

Following the development of a plastic strain, the elastic range P generally modifies the value of its radius and the position of its centre according to the hardening rules for the material.

b) Hardening rules.

In the present theory, the material hardening is governed by the following assumptions:

1) during the entire load process the position of the centre and the value of the radius of the elastic range P are functions of a unique parameter ζ defined by:

$$\zeta = \int \sqrt{d\underline{e}^p \cdot d\underline{e}^p} \quad ; \quad (*) \qquad (3.3)$$

2) there are two material characteristic functions $\varphi(\zeta)$ and $\psi(\zeta)$ which are fully deducible from a uniaxial test and:

- $\varphi(\zeta)$ is not decreasing with $\varphi(0) = \sqrt{2}$, (3.4)

- $\psi(\zeta) = \sum_{0}^{n} {}_i\, \eta_i \exp(-\alpha_i \zeta)$, $\eta_i \geq 0$, $\alpha_i \geq 0$ for every i, (3.5)

if $\underline{c}$ is the centre and ϱ is the radius of P, then:

$$- \underline{c} = 2\mu_0 \int_0^{\zeta} \psi(\zeta - \zeta') \frac{d\underline{e}^p}{d\zeta'} d\zeta' \quad ; \qquad (3.6)$$

$$- \varrho(\zeta) = \tau_0 \varphi(\zeta) . \quad (**) \qquad (3.7)$$

If $\underline{s}$ lies on the boundary of P, from (3.6) and (3.7) we obtain:

$$||\underline{s} - \underline{c}|| = \tau_0 \varphi(\zeta) , \qquad (3.8)$$

and from (3.3) and from Prandtl-Reuss's hypothesis:

(*) if $\underline{a}$ and $\underline{b}$ are vectors we denote their scalar product as $\underline{a} \cdot \underline{b}$

(**) the relations (3.6) and (3.7) determine the material hardening status as function of ζ and its history. ζ is independent from the clock time and constitutes a measure of the permanent deformation level; therefore ζ can be interpreted as an "intrinsic time" from which the name "endochronic" is taken for this theory of the elasto-plastic behaviour. It is to be observed that this definition of intrinsic time is the same definition used by Valanis in reference [2].

$$d\underline{e}^p = \frac{\underline{s} - \underline{c}}{\tau_0 \varphi(\zeta)} d\zeta \quad . \tag{3.9}$$

We shall examine now some particular cases showing how the φ and ψ functions govern the hardening mechanism.

A) Let us suppose $\psi(\zeta)\equiv 0$. Hence $\underline{c}(\zeta)\equiv 0$ and (3.2) gives:

$$||\underline{s}|| = \tau_0 \varphi(\zeta) \quad . \tag{3.10}$$

In this case the centre of P remains fixed at the origin of the S space during the entire load process and the radius expands itself following the value of $\varphi(\zeta)$. We have the case of a pure isotropic hardening. If, in particular, $\varphi(\zeta) = \sqrt{2}$ = constant, we will have the case of the classical ideal plasticity.

B) Let us suppose $\varphi(\zeta) = \sqrt{2}$ = constant and $\psi(\zeta)\neq 0$. In this case, during the load process, the elastic range P carries out a translation in the S space, and keeps constant its radius.

It is convenient that the two cases be distinguished as follows:

B1) $\psi(\zeta) = \psi(0)$ = constant (i.e. $a_i = 0$ for every i). We deduce from (3.6):

$$d\underline{c}(\zeta) = 2\mu_0 \psi(0)\, d\underline{e}^p \quad . \tag{3.11}$$

Therefore P carries out a translation in the direction of its outer normal vector $\underline{n}$. We have the kinematic hardening of Prager.

B2) $\psi(\zeta)$ not constant. In this case we deduce from (3.6):

$$\frac{d\underline{c}}{d\zeta} = 2\mu_0 \int_0^\zeta \psi'(\zeta-\zeta')\frac{d\underline{e}^p}{d\zeta'} d\zeta' + 2\mu_0 \psi(0)\frac{d\underline{e}^p}{d\zeta} \quad , \tag{3.12}$$

where $\psi'(\zeta) = d\psi/d\zeta$

Indicating:

$$\underline{h}(\zeta) = \int_0^\zeta \psi'(\zeta-\zeta')\frac{d\underline{e}^p}{d\zeta'} d\zeta' \quad , \tag{3.13}$$

we obtain:

$$d\underline{c} = 2\mu_0 (\psi(0)\, d\underline{e}^p + \underline{h}\, d\zeta) . \tag{3.14}$$

We have now a more general kinematic hardening because the translation direction of the centre of P is generally different from $\underline{n}$ and depends on the whole history of the deformation.

4. Elasto-plastic constitutive relationships

In this paragraph we deduce an incremental method which permits the stress increment $d\underline{\sigma}$ to be calculated for every given strain increment $d\underline{\varepsilon}$.

Because the linear relation (1.1) connects the hydrostatic parts, the problem is reduced to the determination of $d\underline{s}$ for a given $d\underline{e}$.

Let us suppose that during the load process the conditions shown in fig. (4.1) have been reached.

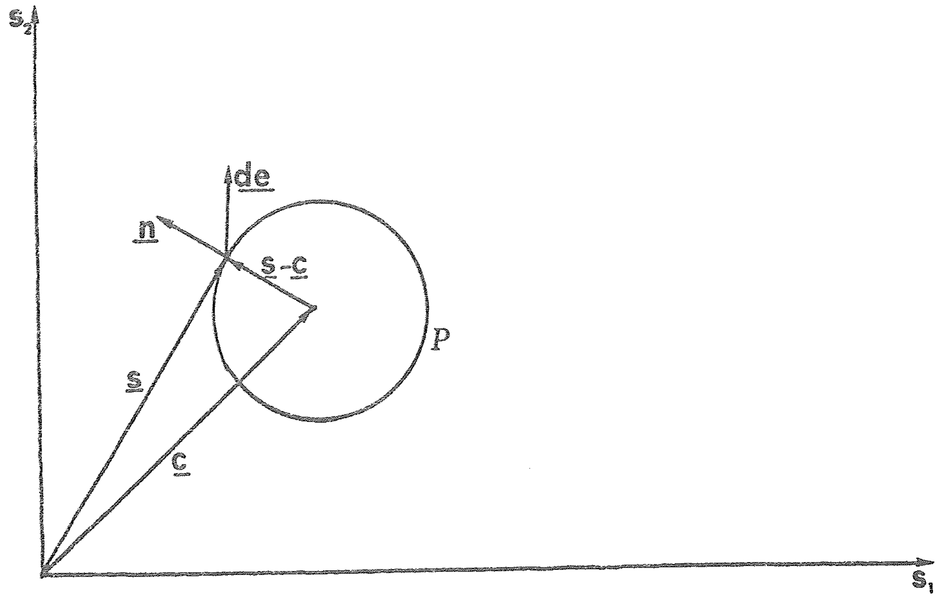


Fig. 4.1

The values of ζ, $\underline{s}$, $\underline{c}$, $\underline{e}^p$, $\underline{h}$ are known and $\underline{s}$ lies on the boundary of P. For a given $d\underline{e}$ we want to calculate the corresponding $d\zeta$, $d\underline{s}$, $d\underline{c}$, $d\underline{e}^p$ and $d\underline{h}$.

Two cases can be given:

A) $d\underline{e}.\underline{n} \leq 0$,

B) $d\underline{e}.\underline{n} > 0$.

A) We have the case of elastic unload, hence it holds:

$$d\zeta = ||d\underline{e}^p|| = 0, \qquad (4.1)$$

from (3.2):

$$d\underline{s} = 2\mu_o d\underline{e}, \qquad (4.2)$$

and from (3.13) and (3.14):

$$d\underline{h} = 0, \qquad (4.3)$$

$$d\underline{c} = 0. \tag{4.4}$$

B) In this case, from (3.8) the increments $d\underline{s}$, $d\underline{c}$ and $d\zeta$ have to verify the following:

$$||\underline{s}+d\underline{s}-\underline{c}-d\underline{c}||^2 = \tau_0^2\varphi^2(\xi+d\zeta). \tag{4.5}$$

Expanding $\varphi^2(\zeta+d\zeta)$ to the first order as $\varphi^2(\zeta)+2\varphi(\zeta)\varphi'(\zeta)d\zeta$ and remembering that if $\underline{a}$ and $\underline{b}$ are two vectors, it holds:

$$||\underline{a}-\underline{b}||^2 = ||a||^2 + ||b||^2 - 2\,\underline{a}.\underline{b},$$

we have, from (4.5):

$$||\underline{s}-\underline{c}||^2+||d\underline{s}-d\underline{c}||^2-2(\underline{s}-\underline{c}).(d\underline{s}-d\underline{c}) = \tau_0^2\varphi^2(\zeta)+2\tau_0^2\varphi(\zeta)\varphi'(\zeta)d\zeta.$$

Neglecting the higher order terms and knowing that

$||\underline{s}-\underline{c}||^2 = \tau_0^2\varphi^2(\zeta)$, we have:

$$(\underline{s}-\underline{c}).(d\underline{s}-d\underline{c}) = \tau_0^2\varphi(\zeta)\varphi'(\zeta)d\zeta. \tag{4.6}$$

Moreover, from (3.2) and (3.14) we have:

$$d\underline{s}-d\underline{c} = 2\mu_0(d\underline{e}-d\underline{e}^p-\psi(0)d\underline{e}^p-\underline{h}d\zeta). \tag{4.7}$$

Therefore, from (4.6), (4.7):

$$(\underline{s}-\underline{c}).d\underline{e}-(1+\psi(0))(\underline{s}-\underline{c}).d\underline{e}^p-(\underline{s}-\underline{c}).\underline{h}d\zeta = \frac{\tau_0^2\varphi(\zeta)\varphi'(\zeta)}{2\mu_0}d\zeta.$$

Taking into account the relation (3.9) and dividing by $\tau_0\varphi(\zeta)$, we have:

$$\frac{(\underline{s}-\underline{c}).d\underline{e}}{\tau_0\varphi(\zeta)}-(1+\psi(0))d\zeta-\frac{(\underline{s}-\underline{c}).\underline{h}}{\tau_0\varphi(\zeta)}d\zeta = \frac{\tau_0\varphi'(\zeta)}{2\mu_0}d\zeta,$$

and solving for $d\zeta$, we have:

$$d\zeta = \frac{(\underline{s}-\underline{c}).d\underline{e}}{\tau_0\varphi(\zeta)\left(1+\psi(0)+\frac{(\underline{s}-\underline{c}).\underline{h}}{\tau_0\varphi(\zeta)}+\frac{\tau_0\varphi'(\zeta)}{2\mu_0}\right)}\ (*). \tag{4.8}$$

(*) keeping in mind the definition of $\underline{h}$ and the expression of $\psi(\zeta)$, we can easily demonstrate that the quantity:

$$\vartheta = \left(1+\psi(0)+\frac{(\underline{s}-\underline{c}).\underline{h}}{\tau_0\varphi(\zeta)}+\frac{\tau_0\varphi'(\zeta)}{2\mu_0}\right)$$

is ≥ 1 for every value of ζ, so that the relation (4.8) is always well defined.

From (3.8), (3.2) and (3.14) we have:

$$d\underline{e}^{p} = \frac{\underline{s}-\underline{c}}{\tau_0 \varphi(\zeta)} d\zeta \tag{4.9}$$

$$d\underline{s} = 2\mu_0 (d\underline{e}-d\underline{e}^{p}), \tag{4.10}$$

$$d\underline{c} = 2\mu_0 (\psi(0)\, d\underline{e}^{p} + \underline{h}\, d\zeta). \tag{4.11}$$

There remains $d\underline{h}$ to be calculated. From (3.5) and (3.13) we have:

$$\underline{h} = \sum_{0}^{n} {}_i -\alpha_i \eta_i \int_0^{\zeta} \exp(-\alpha_i(\zeta-\zeta')) \frac{d\underline{e}^{p}}{d\zeta'} d\zeta',$$

hence:

$$\frac{d\underline{h}}{d\zeta} = \psi'(0) \frac{d\underline{e}^{p}}{d\zeta} + \int_0^{\zeta} \psi''(\zeta-\zeta') \frac{d\underline{e}^{p}}{d\zeta'} d\zeta' =$$

$$= \psi'(0) \frac{d\underline{e}^{p}}{d\zeta} + \sum_{0}^{n} {}_i \left(\alpha_i^2 \eta_i \int_0^{\zeta} \exp(-\alpha_i(\zeta-\zeta')) \frac{d\underline{e}^{p}}{d\zeta'} d\zeta'\right) =$$

$$= \psi'(0) \frac{d\underline{e}^{p}}{d\zeta} - \sum_{0}^{n} {}_i\, \alpha_i \underline{h}_i(\zeta),$$

where $\underline{h}_i(\zeta) = -\alpha_i \eta_i \int_0^{\zeta} \exp(-\alpha_i(\zeta-\zeta')) \frac{d\underline{e}^{p}}{d\zeta'} d\zeta'$.

From the previous relation, we obtain:

$$d\underline{h} = \psi'(0)\, d\underline{e}^{p} - \sum_{0}^{n} {}_i\, \alpha_i \underline{h}_i(\zeta)\, d\zeta. \tag{4.12}$$

The relations (4.8), (4.9), (4.10), (4.11), (4.12) are a complete system of incremental constitutive relations. In the following paragraph we will show how these relations will be used to find the expression of the elasto-plastic constitutive matrix D^{ep}, i.e. the matrix defined by:

$$D^{ep} = d\underline{\sigma}/d\underline{\varepsilon}.$$

5. The D^{ep} matrix

In a finite element program which uses the tangent stiffness matrix K_t, it is necessary to calculate the elasto-plastic constitutive matrix D^{ep} defined by:

$$D^{ep} = d\underline{\sigma}/d\underline{\varepsilon} \,, \qquad (5.1)$$

relating stress and strain increments [3].

The basic relations used to calculate D^{ep} in a full tridimensional case are:

$$d\underline{s} = 2\mu_0 (d\underline{e} - d\underline{e}^p) \,, \qquad (5.2)$$

$$d\underline{e}^p = \frac{\underline{s}-\underline{c}}{\tau_0 \varphi(\zeta)} d\zeta \,, \qquad (5.3)$$

$$d\zeta = \frac{(\underline{s}-\underline{c})^t \, d\underline{e}}{\vartheta \tau_0 \varphi(\zeta)} \,, \quad (*) \qquad (5.4)$$

where:

$$\vartheta = 1 + \psi(0) + \frac{(\underline{s}-\underline{c})^t \, \underline{h}(\zeta)}{\tau_0 \varphi(\zeta)} + \frac{\tau_0 \varphi'(\zeta)}{2\mu_0} \,. \qquad (5.5)$$

Taking into account the linear relation between the hydrostatic parts of $\underline{\sigma}$ and $\underline{\varepsilon}$, we have:

$$d\underline{s} = A d\underline{\sigma} - 3\varkappa P d\underline{\varepsilon} \qquad (5.6)$$

$$d\underline{e} = M d\underline{\varepsilon} \qquad (5.7)$$

where A, P and M are the following matrices:

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \,,$$

(*) In this paragraph, we denote with $\underline{a}^t \underline{b}$ the scalar product of the two column vectors $\underline{a}$ and $\underline{b}$, where $\underline{a}^t$ is the transpose of $\underline{a}$.

$$P = \frac{1}{3}\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$M = A - P$.

From (5.6), we have:

$$A\frac{d\underline{\sigma}}{d\underline{\varepsilon}} = AD^{ep} = \frac{d\underline{s}}{d\underline{\varepsilon}} + 3\kappa P = \frac{d\underline{s}}{d\underline{e}}\,\frac{d\underline{e}}{d\underline{\varepsilon}} + 3\kappa P = \frac{d\underline{s}}{d\underline{e}} M + 3\kappa P. \qquad (5.8)$$

From (5.2), (5.3) and (5.4) we can write:

$$\frac{d\underline{s}}{d\underline{e}} = 2\mu_0\left(I - \frac{d\underline{e}^p}{d\underline{e}}\right) = 2\mu_0\left(I - \frac{d\underline{e}^p}{d\zeta}\,\frac{d\zeta}{d\underline{e}}\right) =$$

$$= 2\mu_0\left(I - \frac{(\underline{s}-\underline{c})\,(\underline{s}-\underline{c})}{\vartheta\,\tau_0^2\,\varphi^2(\zeta)}\right) = 2\mu_0\,(I-H), \qquad (5.9)$$

where H is the following matrix:

$$H = \frac{(\underline{s}-\underline{c})\,(\underline{s}-\underline{c})}{\vartheta\,\tau_0^2\,\varphi^2(\zeta)} . \qquad (5.10)$$

Therefore from (5.8) and (5.9) we have:

$$AD^{ep} = 2\mu_0\,(I-H)\,M + 3\kappa P . \qquad (5.11)$$

D^{ep} can be easily found by eliminating the last three rows and columns from the matrix on the right hand side of (5.11).

In bidimensional problems as the plane-strain or the axisymmetric cases, the D^{ep} matrix can be easily obtained from the tridimensional one.

The plane-stress case requires some calculation because the plane-stress definition concerns the stresses.

In this case, it is convenient to start with the relation obtained inverting (5.1), i.e.,

$$d\underline{\varepsilon} = [D^{ep}]^{-1}\, d\underline{\sigma} . \qquad (5.12)$$

Once the calculus of $[D^{ep}]^{-1}$ has been done, we can apply the conditions obtained from the definition of plane-stress, eliminating some rows and columns. This reduced matrix can now be inverted to obtain the D^{ep} matrix useful for plane-stress.

6. Determination of the φ and ψ functions from a uniaxial test

Let us suppose that the results of a uniaxial traction test are known, as shown in fig. (6.1).

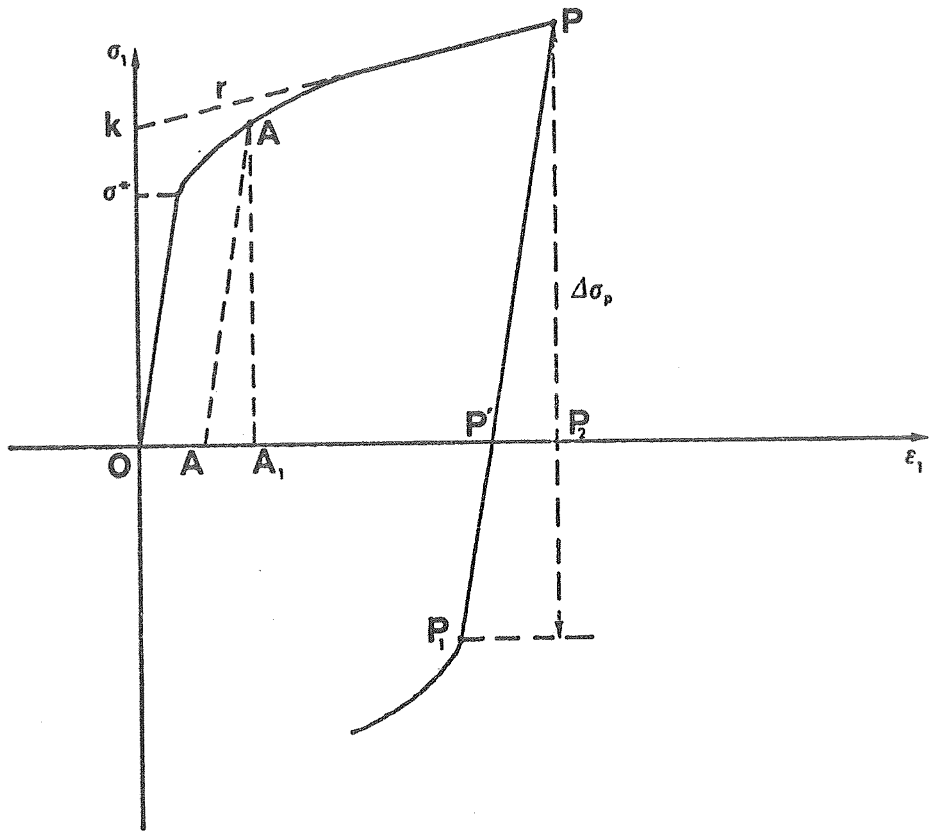


Fig. 6.1

For simplicity, let us suppose the functions $\varphi(\zeta)$ and $\psi(\zeta)$ have the following expressions:

$$\varphi(\zeta) = \sqrt{2}(1+\beta\zeta) , \tag{6.1}$$

$$\psi(\zeta) = \eta_1 + \eta_2 \exp(-\alpha\zeta) . \tag{6.2}$$

Therefore we have to determine the β, η_1, η_2, α parameters.

In the present case we have $\sigma_1 \neq 0$, and:

$$s_1 = 2\,\sigma_1/3,$$

$$s_2 = s_3 = -\,\sigma_1/3 = -s_1/2, \tag{6.3}$$

$$e_2^p = e_3^p = -e_1^p\ /2,$$

$$\frac{de_1^p}{d\zeta} = \sqrt{2/3} \quad \text{(during the load phase).}$$

Because it holds:

$$\underline{\underline{s}}-\underline{\underline{c}} = ||\underline{\underline{s}}-\underline{\underline{c}}||\ \underline{\underline{n}} = ||\underline{\underline{s}}-\underline{\underline{c}}||\ d\underline{\underline{e}}^{P}\ /d\zeta \qquad (6.4)$$

from (3.6), (3,8) and (6.3) we have:

$$s_1 = \tau_0\varphi(\zeta)\sqrt{2/3} + 2\mu_0\int_0^{\zeta}\psi(\zeta-\zeta')\sqrt{2/3}\ d\zeta', \qquad (6.5)$$

and from (6.1) and (6.2):

$$s_1 = 2\ \tau_0(1+\beta\zeta)/\sqrt{3} + 2\mu_0\sqrt{2/3}\int_0^{\zeta}(\eta_1+\eta_2\exp(-\alpha(\zeta-\zeta')))\,d\zeta',$$

or from the first of (6.3):

$$\sigma_1 = \sqrt{3}\ \tau_0(1+\beta\zeta)+\sqrt{3/2}\ 2\mu_0\int_0^{\zeta}(\eta_1+\eta_2\exp(-\alpha(\zeta-\zeta')))\,d\zeta'. \qquad (6.6)$$

From (6.6) τ_0 can be calculated, because we have:

$$\tau_0 = \sigma^*/\sqrt{3}, \qquad (6.7)$$

and σ^* can be read on the diagram of fig. (6.1).

1. Determination of β parameter

In the case of uniaxial traction, we have:

$$||\underline{\underline{s}}||^2 = s_1^2+s_2^2+s_3^2 = 3s_1^2/2, \text{ or}$$

$$||\underline{\underline{s}}|| = \sqrt{2/3}\ \sigma_1. \qquad (6.8)$$

Because it is $||\underline{\underline{s}}-\underline{\underline{c}}|| = \tau_0\varphi(\zeta)$, we have:

$$\Delta\sigma_p/2 = \sqrt{3/2}\ \tau_0\varphi(\zeta_p),$$

hence,

$$\Delta\sigma_p/2 = \sigma^*(1+\beta\zeta_p). \qquad (6.9)$$

From (6.9) we can calculate β, because $\zeta_p = \sqrt{3/2}\ e^P(P) = \sqrt{3/2}\ OP'$, and OP' and $\Delta\sigma_p$ can be read from the diagram in fig. (6.1).

2. Determination of η_1, η_2 and α.

To calculate η_1, η_2 and α, we suppose for the moment to have the diagram (σ_1 - e_1^p) for the same test as shown in fig. (6.2).

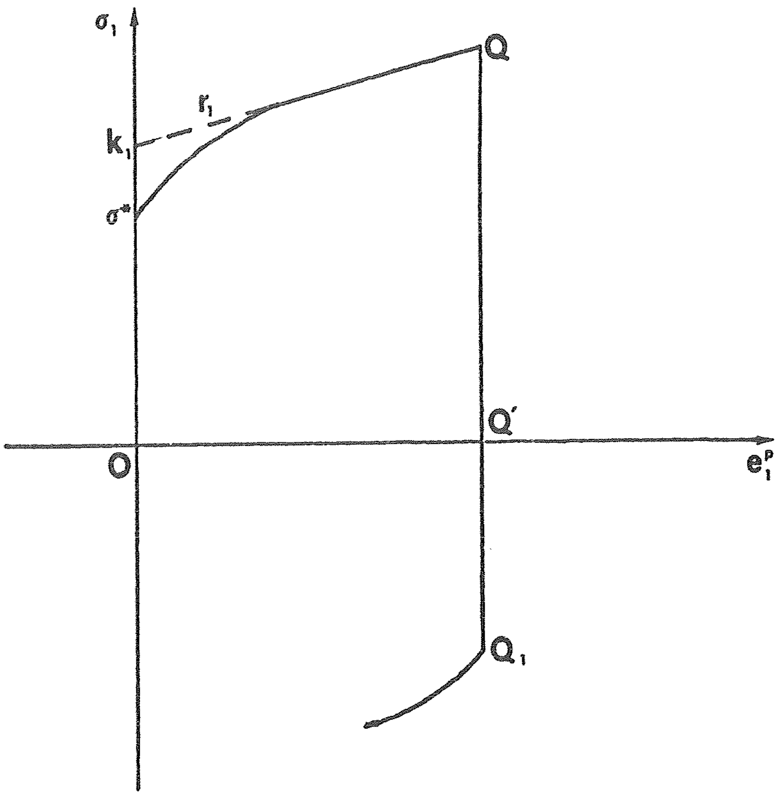


Fig. 6.2

Because it is $\varphi(\zeta) = \sqrt{2}(1+\beta\zeta)$, we have from (6.6), during the load phase:

$$\sigma_1(\zeta) = \sigma^*(1+\beta\zeta) + \sqrt{6}\mu_0 \int_0^{\zeta} (\eta_1 + \eta_2 \exp(-\alpha(\zeta-\zeta'))) d\zeta',$$

or after some calculations:

$$\sigma_1(e_1^p) = \sigma^*(1+\beta\sqrt{3/2}\; e_1^p) + 3\mu_0\eta_1 e_1^p +$$
$$+ \sqrt{6}\;\mu_0\;\eta_2/\alpha\;(1-\exp(-\sqrt{3/2}\;\alpha\;e_1^p)). \qquad (6.10)$$

If e_1^p is high enough the exponential term can be disregarded, we have:

$$\sigma_1(e_1^p) = \sigma^*(1+\beta\sqrt{3/2}\;e_1^p) + 3\mu_0\;\eta_1 e_1^p + \sqrt{6}\;\mu_0\;\eta_2/\alpha, \qquad (6.11)$$

which represents the equation of r_1 straight line in fig. (6.2). If k_1 and m_1 are respectively the intercept and the slope of r_1, we have:

$$\eta_1 = \frac{m_1 - \sqrt{3/2}\,\sigma^*\beta}{3\mu_0}, \tag{6.12}$$

$$\eta_2/\alpha = \frac{(k_1 - \sigma^*)}{\sqrt{6}\,\mu_0}. \tag{6.13}$$

After the determination of η_1 and η_2/α, we choose a point A on the curved part of the diagram in fig. (6.1). We measure the ordinate AA_1 and the corresponding e_1^p (A) (given by the abscissa A' of the crossing point between the abscissa axis ε_1 and the parallel to OY passing through A).

From (6.10) we have the following relation:

$$\exp(-\sqrt{3/2}\,\alpha\, e_1^p\ (A)) = \frac{k_1 + m_1 e_1^p\ (A) - AA_1}{k_1 - \sigma^*}. \tag{6.14}$$

Using the relation (6.14), α can be easily calculated.

Until now we have used k_1 and m_1 which are seldom available because we have only the $(\sigma_1 - \varepsilon_1)$ diagram as shown in fig. (6.1). Now we have to calculate k_1 and m_1 from some measurements done on the diagram in fig. (6.1).

Let r be the asymptote of the curve in fig. (6.1), and m and k be respectively the slope and its intercept. We have:

$$d\sigma_1 = m(d\varepsilon_1^e + de_1^p), \tag{6.15}$$

or

$$d\sigma_1 = m(d\sigma_1/E + d\sigma_1/m), \tag{6.16}$$

where $d\varepsilon_1^e$ is the elastic part of the strain and E is Young's modulus.

From (6.16), we have:

$$m_1 = mE/(E-m), \tag{6.17}$$

so that m_1 can be calculated, having measured m on the fig. (6.1).

For k_1, we have from fig. (6.2):

$$\sigma_1(Q) - k_1 = m_1\ e_1^p\ (Q'). \tag{6.18}$$

Because m_1 is known, and from fig. (6.1), $\sigma_1(Q) = PP_2$, $e_1^p\ (Q') = OP'$, we have:

$$k_1 = PP_2 - OP'(mE)/(E-m). \tag{6.19}$$

This completes the calculation of the characteristic parameters of the material.

7. Some examples

Example 1

Plane stress. Infinite plate with a circular hole and internal pressure load.

Geometrical data : radius of the hole = 10. cm,
thickness = 1. cm.

Material properties : $E = 2.1 * 10^6$ kg/cm^2,
= 0.3,
= 1000. kg/cm^2,
ideal plasticity.

The applied load is a uniform pressure p = 1591. kg/cm^2. With the value of the applied pressure we can calculate [4], the radius of the plastic zone and we found:

r = 13.536 cm.

Figs. (7.1) and (7.2) show respectively the behaviour of the radial and tangential stress versus the distance from the centre of the hole.

The theoretical results, as obtained from reference [4], are indicated by the continuous line.

The results otained using the program founded on the theory presented in this paper are indicated by o points.

The results obtained using the MARC program are indicated by + points.

We have a good agreement between the results of the two programs and the theoretical forecasts, and the practical identity between the results of the two programs. This perfect correspondence between the results of the two programs will be maintained in all ideal plasticity examples proposed here.

Example 2

Axi-symmetric. Thick tube with internal pressure.

Geometrical data : length = 1. in,
inner radius = 1. in,
outer radius = 2. in.

Material properties : $E = 30. * 10^6$ psi,
= 0.3,
= 25981. psi,
ideal plasticity.

Internal pressure value: p = 26839. psi.

With this load value we can calculate [4] the radius of the plastic zone and we found:

r = 1.227 in

Figs. (7.3) and (7.4) show respectively the behaviour of the radial and tangential stress versus the distance from the symmetry axis. The notations are the same as in the previous example.

Example 3

Plane-strain. Thick tube with internal pressure.

The structure is the same as in the previous example, but it is seen as a plane-strain case. The behaviour of radial and tangential stress versus the radius is shown in figs. (7.5) and (7.6) respectively.

Example 4

Tridimensional. Thick tube with torsion load

Geometrical data : length = 2. in,
inner diameter = 2. in,
outer diameter = 2.2 in.

Material properties : E = 5798. tons/sq.in,
= 0.3,
= 5. tons/sq.in.

Pure kinematic hardening: $\beta = 0$, $\alpha = 500.$, $\eta_1 = 0.0484$, $\eta_2 = 0.3363$.

The middle cross section of the tube is held fixed, and a uniformly distributed twisting moment is applied on the two end faces.

The behaviour of stress τ_{12} versus the plastic strain γ_{12}^{p} is shown in fig. (7.7).

The upper curve shows the behaviour of stress in an annealed specimen without any initial strain.

The lower curve shows the behaviour of stress in the case the twisting load is applied on a specimen which has a permanent initial deformation due to the application and successive removal of a uniform traction load up to 1.5 times the yield value.

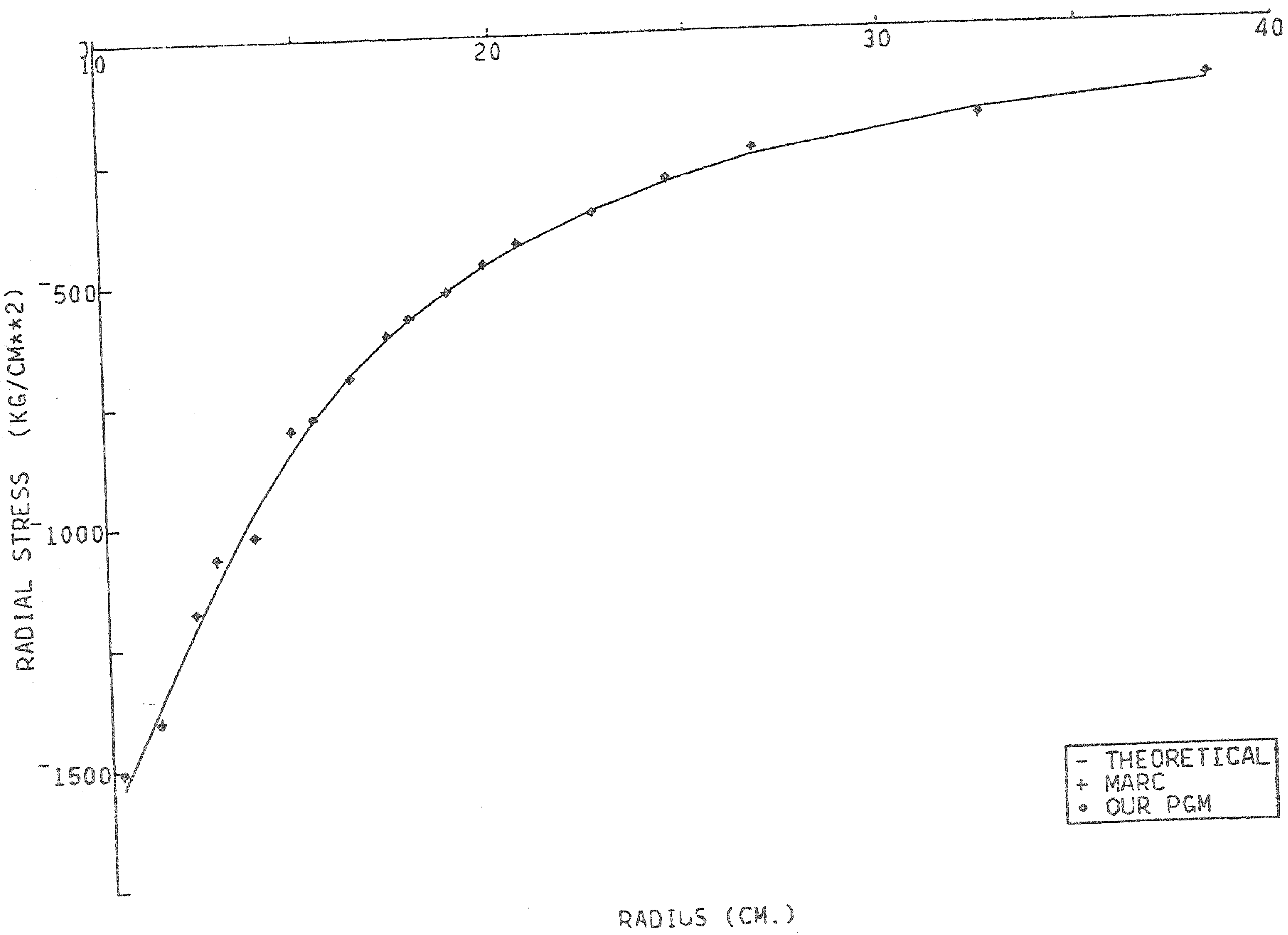

10
20
30
40
-500
-1000
-1500
RADIAL STRESS (KG/CM**2)
RADIUS (CM.)
- THEORETICAL
+ MARC
◦ OUR PGM

Fig. 7.1

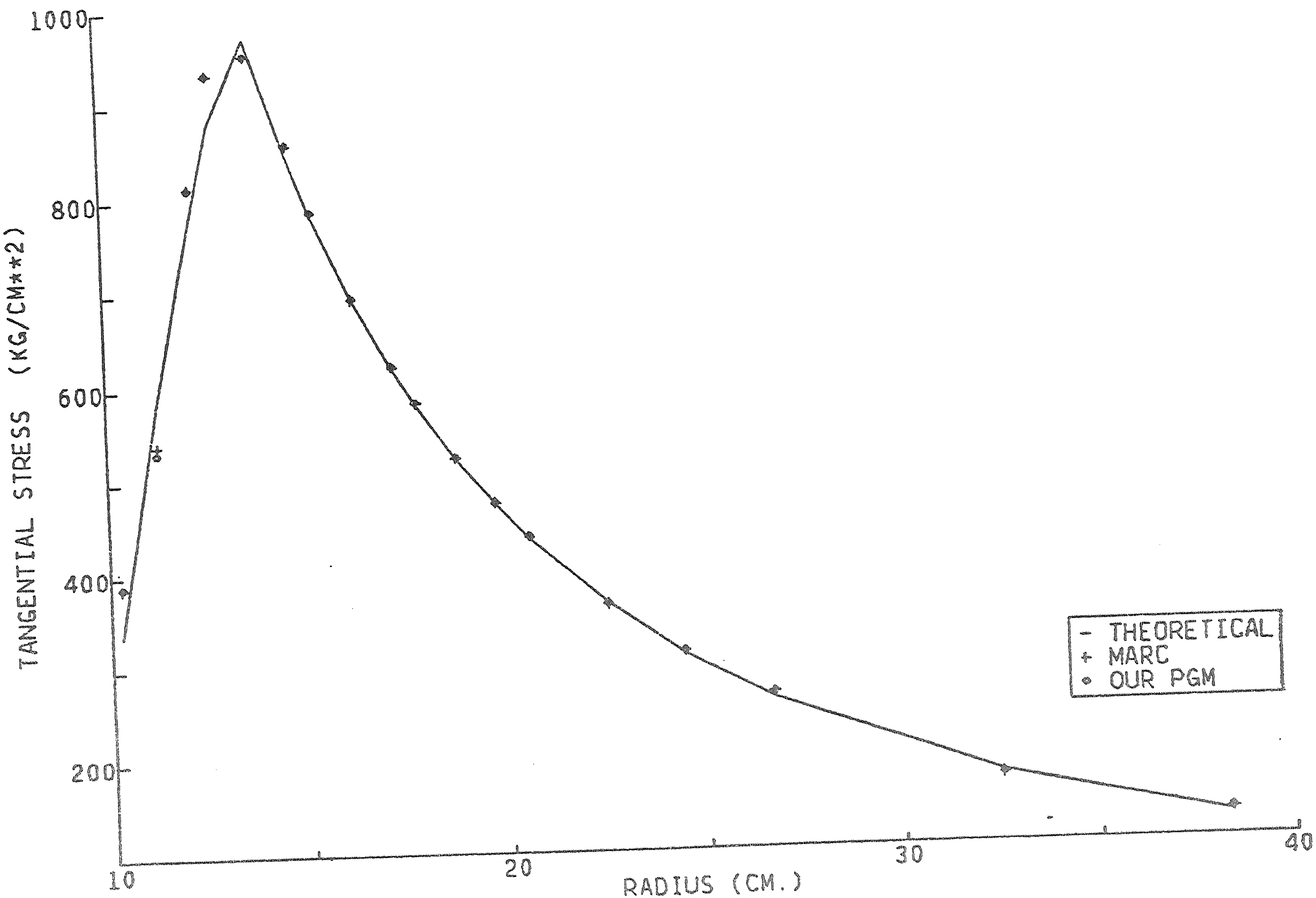

1000
800
600
400
200
TANGENTIAL STRESS (KG/CM**2)
10
20
30
40
RADIUS (CM.)
- THEORETICAL
+ MARC
◇ OUR PGM

Fig. 7.2

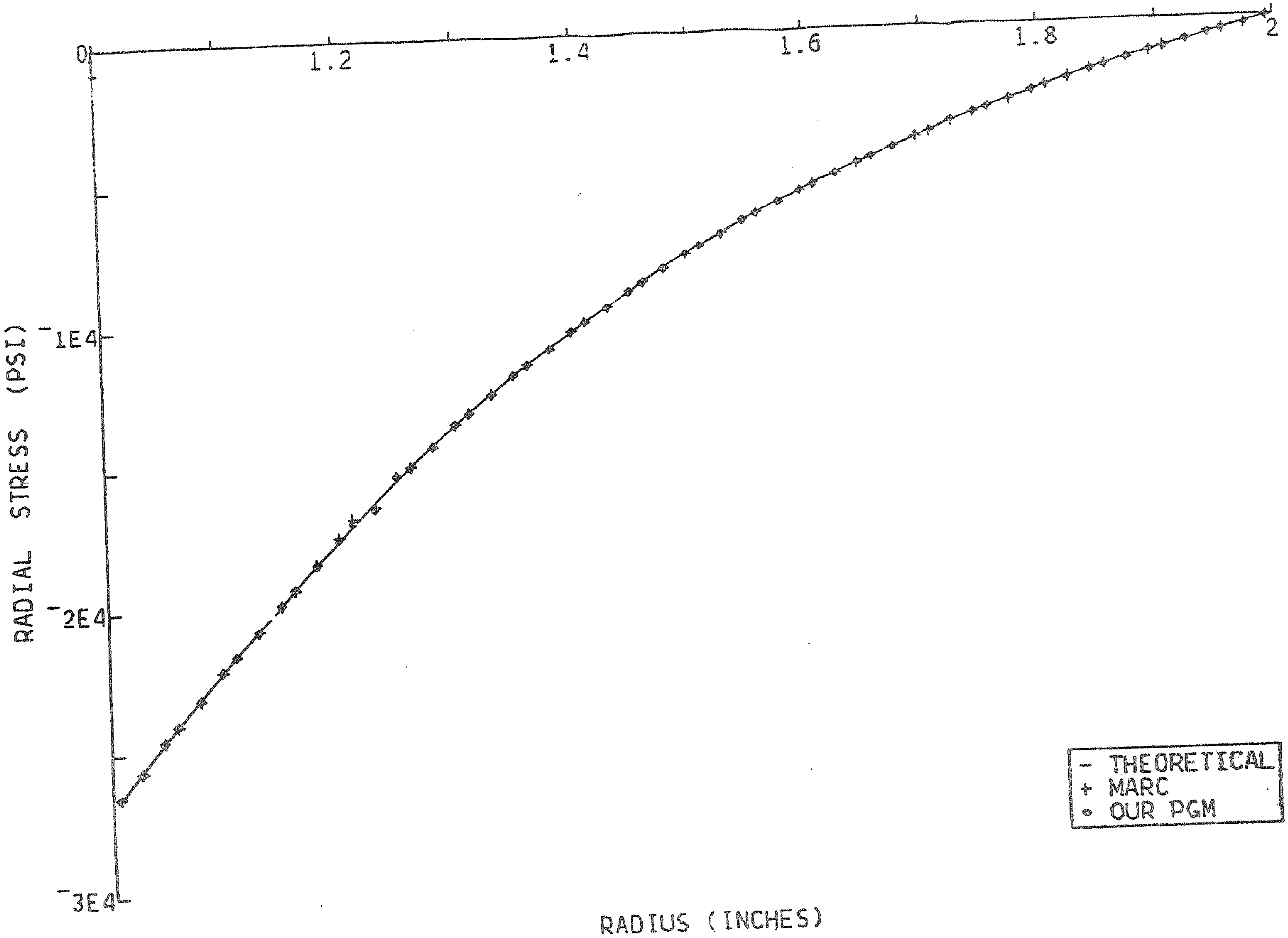

0
1.2
1.4
1.6
1.8
2
-1E4
-2E4
-3E4
RADIAL STRESS (PSI)
RADIUS (INCHES)
- THEORETICAL
+ MARC
• OUR PGM

Fig. 7.3

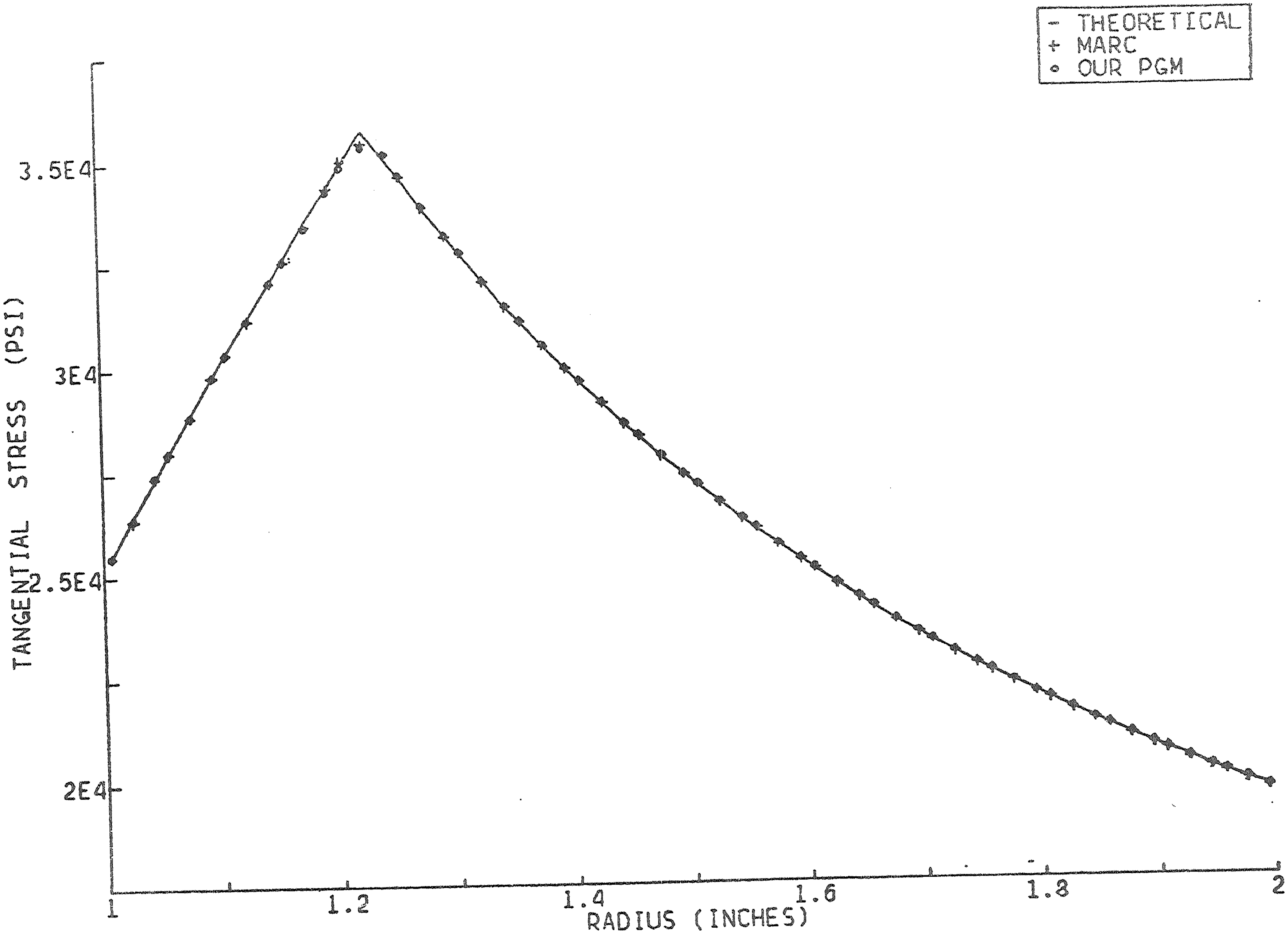
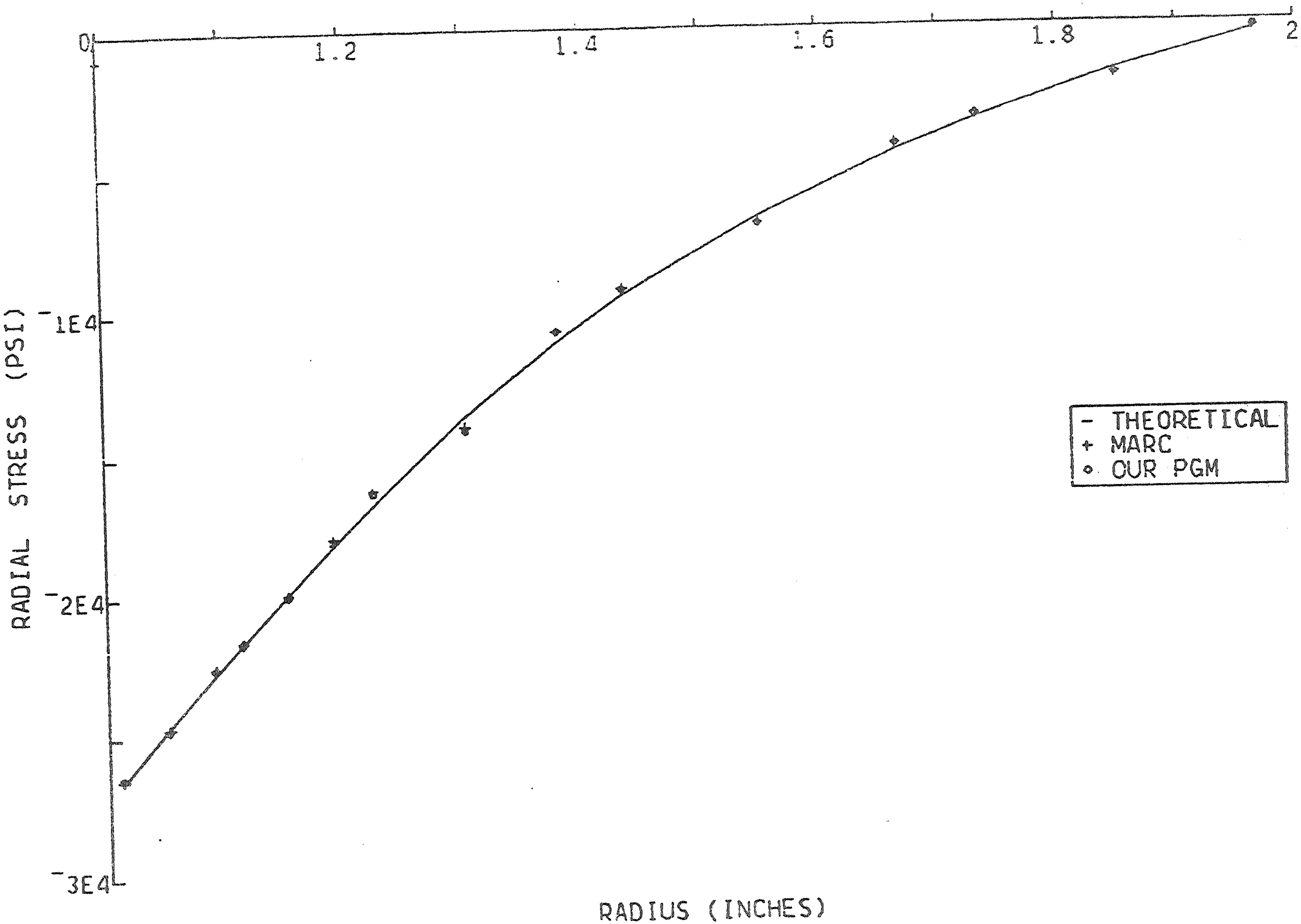
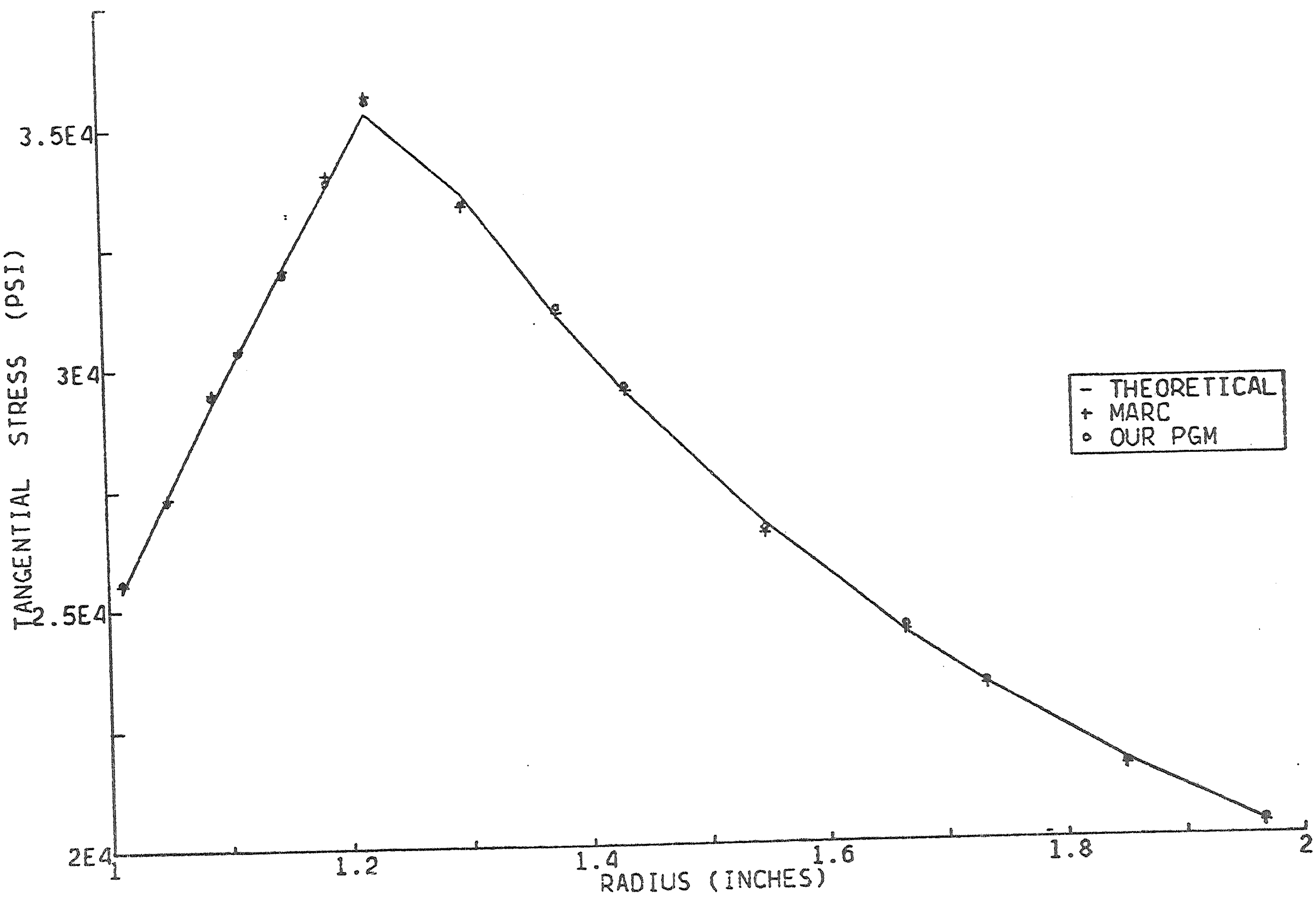

- THEORETICAL
+ MARC
∘ OUR PGM
3.5E4
3E4
2.5E4
2E4
TANGENTIAL STRESS (PSI)
1
1.2
1.4
1.6
1.8
2
RADIUS (INCHES)

Fig. 7.4



0
1.2
1.4
1.6
1.8
2
-1E4
-2E4
-3E4
RADIAL STRESS (PSI)
RADIUS (INCHES)
- THEORETICAL
+ MARC
◇ OUR PGM

Fig. 7.5



TANGENTIAL STRESS (PSI)
3.5E4
3E4
2.5E4
2E4
1
1.2
1.4
1.6
1.8
2
RADIUS (INCHES)
- THEORETICAL
+ MARC
∘ OUR PGM

Fig. 7.6

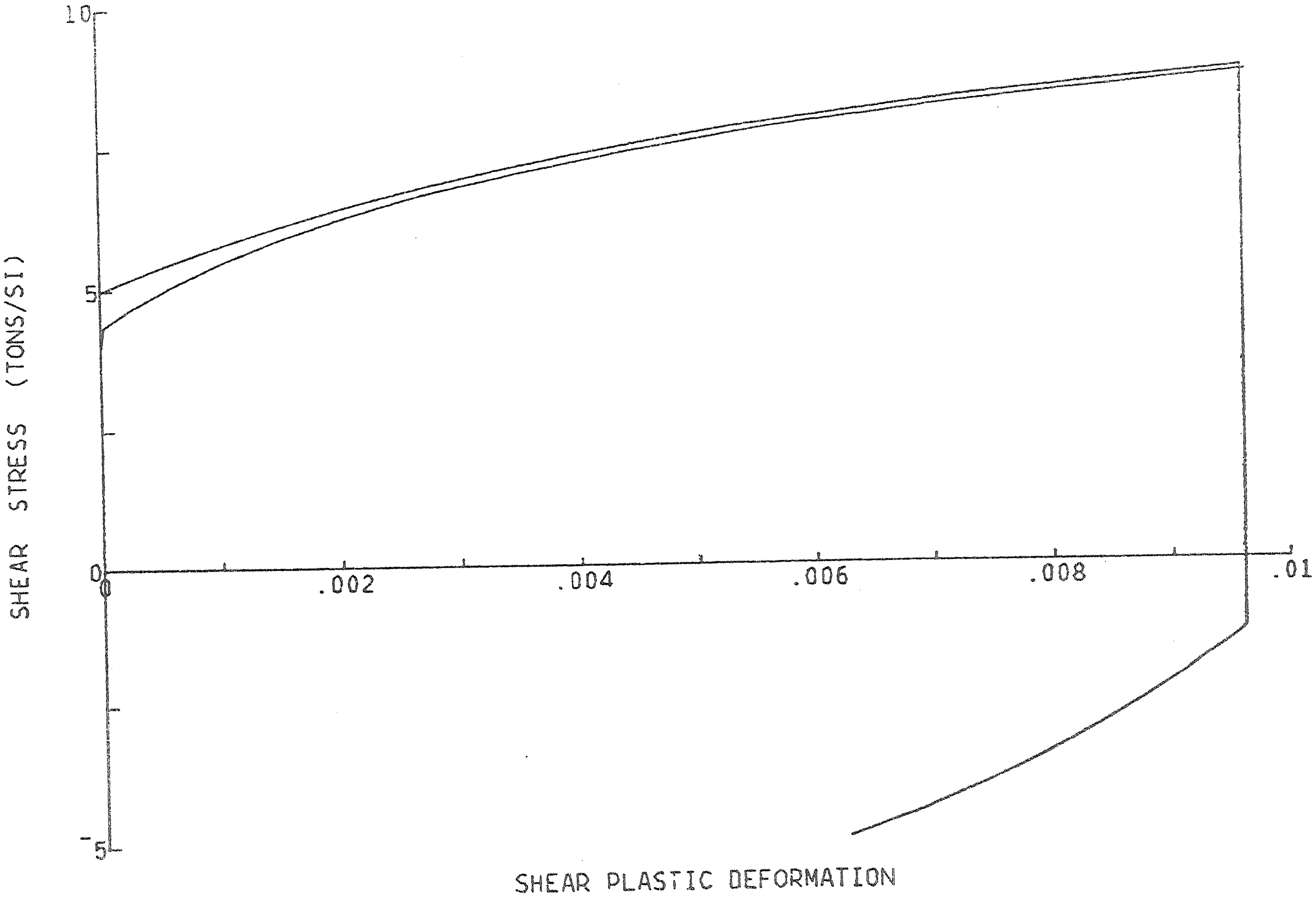

10
5
0
-5
SHEAR STRESS (TONS/SI)
.002
.004
.006
.008
.01
SHEAR PLASTIC DEFORMATION

Fig. 7.7

REFERENCES

1. GALLAGHER R. H., "Time Indipendent Inelastic Analysis", Course on Advances Topics in Finite Element Analysis, S. Margherita, Italy, June 1974.

2. VALANIS K. C., "Fundamental Consequences of a New Intrinsic Time Measure. Plasticity as a Limit of the Endochronic Theory", The University of Iowa, Report n. G-224/DME-78-01, April 1978

3. ZIENKIEWICZ O. C., "The Finite Element Method", Mc Graw Hill, 1977.

4. KACHANOV L. M., "Fundamentals of the Theory of Plasticity", MIR Publishers, Moscow, 1974.

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