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Review

Entropy as a Metric Generator of Dissipation in Complete Metriplectic Systems

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Abstract: This lecture is a short review on the role entropy plays in those classical dissipative systems whose equations of motion may be expressed via a *Leibniz Bracket Algebra* (LBA). This means that the time derivative of any physical observable f of the system is calculated by putting this f in a "bracket" together with a "special observable" F, referred to as a Leibniz generator of the dynamics. While conservative dynamics is given an LBA formulation in the Hamiltonian framework, so that F is the Hamiltonian H of the system that generates the motion via classical Poisson brackets or quantum commutation brackets, an LBA formulation can be given to classical dissipative dynamics through the *Metriplectic Bracket Algebra* (MBA): the conservative component of the dynamics is still generated via Poisson algebra by the total energy H, while S, the entropy of the degrees of freedom statistically encoded in friction, generates dissipation via a metric bracket. The motivation of expressing through a bracket algebra and a motion-generating function F is to endow the theory of the system at hand with all the powerful machinery of Hamiltonian systems in terms of symmetries that become evident and readable. Here a (necessarily partial) overview of the types of systems subject to MBA formulation is presented, and the physical meaning of the quantity S involved in each is discussed. Here the aim is to review the different MBAs for isolated systems in a synoptic way. At the end of this collection of examples, the fact that dissipative dynamics may be constructed also in the absence of friction with microscopic degrees of freedom is stressed. This reasoning is a hint to introduce dissipation at a more fundamental level.

Keywords: dissipative systems; metriplectic dynamics; Leibniz algebra

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1. Introduction

Entropy is known to play the role of a thermodynamic state function to measure the degradation of the energy transferred, as well as the spread of it through microscopic degrees of freedom, i.e., dynamical variables that evolve much more quickly than other ones [1]. Quantities defined similar to entropy, as understood in information theory [2], are employed to quantify ignorance or unpredictability, and even to track causality in the complex relationships of interacting systems [3–5].

In the context of the classical dissipative dynamical systems that will be examined in this review, entropy plays the same role of energy, when the latter wears the costume of the Hamiltonian H: entropy contributes to generating the motion of the system, even if through a mechanism algebraically different from the Hamiltonian one [6,7]. In this way entropy plays directly and vividly the role of "$\varepsilon\nu\ \tau\rho o\pi\eta$", a Greek phrase translatable as "the inner transformer", to which the physical function owes its name. Impressively, while Hamiltonian and symplectic machinery just produce *time-reversible* changes, what entropy does with the extended version of symplectic algebra we will examine is to generate *irreversible*

transformations. The extension of symplectic algebra making entropy play such a role is referred to as *metriplectic algebra*. The way the latter works is extremely intriguing and possibly related to the long-standing question of the origin of irreversibility in physics. "Possibly related to" does mean "resolving"; however, researchers aiming at clarifying the origin of irreversibility should have some knowledge of metriplectic formalism.

The framework of metriplectic bracket algebra (MBA) clarifies definitely the dynamical relationship between dissipation and entropy from a formal point of view. Even if entropy increase was known to take place in the context of dissipative processes, e.g., via the $dQ = TdS$ equation of Equilibrium Thermodynamics, through MBA formalism the entropy–dissipation relationship is given a fundamental role, where entropy becomes the generator of dissipative motion. In the same way one sees the analytical expression of a Hamiltonian determine the nature of non-dissipative dynamics, one will expect the characteristics of a dissipative motion to descend from how entropy depends on the dynamical variables of the system.

In Hamiltonian systems, motion is a one-parameter (*time*) group of transformations through M obtained by exponentiating the symplectic product of H with the quantity to be involved:

$$Motion = \exp\left(\{\ldots, H\}\right).$$

In systems described by the MBA formalism, the evolution of a classical system with dissipation is a 1-parameter (again, *time*) semi-group resulting from the exponentiation of the sum of the *symplectic bracket* $\{\ldots, H\}$ ruled by H, plus another kind of bracket, referred to as *metric bracket* $(\ldots, S)$ ruled by entropy S:

$$Motion = \exp\left(\{\ldots, H\} + (\ldots, S)\right).$$

Such a semi-group is designed in order to describe dissipative relaxation of non-Hamiltonian systems.

In this lecture dissipative *complete systems* are treated, i.e., pieces of the universe that conserve their energy while increasing their entropy along their evolution. These are also referred to as isolated systems; non-isolated systems treated through metriplectic systems are not discussed here. Non-complete metriplectic systems are reviewed in [8], while important applications/examples of them may be found in [9,10]. Although non-complete metriplectic dynamics has very important applications, here we restrict ourselves to complete systems because in the latter the different roles played by the Hamiltonian versus entropy, and by the Poisson brackets versus the metric ones, appear clearer; from a didactic point of view this seems to be the correct introductory choice. The student will find it easier to work on non-complete systems once complete systems are understood.

This review is articulated as follows.

In Section 2 the concept of dynamics algebrization through Leibniz brackets is reported, and the usefulness of representing dynamics via bracket algebra highlighted. Hamiltonian systems and metric dissipative systems are recognized among these dynamical theories.

Section 3 is the core of the keynote lecture: the concept of complete metriplectic system is described, and the relationship between dissipation and friction is discussed. In particular, it is stressed how it is possible to construct dissipative dynamics even in the absence of "explicit friction" via metriplectic algebra. Section 3 is rich with examples, taken from relatively elementary physics, to get acquainted with metriplectic formalism applied to isolated systems.

Section 4 is devoted to some final considerations on the potentiality and meaning of metriplectic formalism.

2. Leibniz Bracket Algebra and Dynamics

Suppose we have a physical system described by a certain set of dynamical variables $\mathbf{x} = \left(x^1, x^2, \ldots, x^N\right)$, so that its evolution is a continuous flow throughout a manifold M referred to as *phase space*. Suppose that the equations of motion of the system are expressed as:

$$\dot{x}^k = L^{kj}(\mathbf{x})\,\partial_j F(\mathbf{x}), \tag{1}$$

where $L^{kj}(\mathbf{x})$ is a tensor field on M and $F(\mathbf{x})$ a scalar field, and the symbol ∂_j stands for the derivative $\frac{\partial}{\partial x^j}$. These equations are generally a set of non-linear, coupled ordinary differential equations (ODEs). If the state of the system $\mathbf{x}$ undergoes (1), it is easy to show that any function $f \in C^\infty(M, \mathbf{R})$ depending on $\mathbf{x}$ evolves along the system motion according to:

$$\dot{f} = L^{ij}\partial_i f \partial_j F. \tag{2}$$

While analytically solving initial value problems with Equation (1) may be generally impossible, some global and local properties of their solutions are suggested by the nature of L^{ij} and F, e.g., their symmetries, boundness, and spectral properties. For instance, any quantity $f(\mathbf{x})$ whose gradient has the property $L^{ij}\partial_i f = 0$ for all j is necessarily conserved along the motion of the system: this is an effect of the properties of L^{ij}; the same thing happens to any $f(\mathbf{x})$ whose gradient is orthogonal to the vector of components $L^{ij}\partial_j F$, which is a property of L^{ij} and F together. A complete study of $L^{kj}(\mathbf{x})$ and $F(\mathbf{x})$ throughout the space phase M could in principle reveal which are the quantities invariant along the motions of the dynamical system (1).

It is also worth mentioning that, assuming L^{kj} to be a proper tensor map on M, and F a proper scalar, Equation (1) results in a generally covariant form under diffeomorphisms on the phase space: one can invertibly and smoothly rearrange the variables $\mathbf{x}$, ending up with a new description of the system $\mathbf{z}$ evolving as $\dot{z}^k = L'^{kh}\partial'_h F$, with ∂'_h the derivative with respect to z^h and L'^{kh} the suitably tensor-transformed version of L^{ij}.

Mathematicians have defined a suitable differential-algebraic structure to endow with a phase spaces M, in order to construct dynamics on it in which a kind of generalization of the system (1) may be constructed. This structure is referred to as *Leibniz algebra* [11]. Such algebra is defined by two elements: a manifold M, on which the space of (real) smooth functions $C^\infty(M, \mathbf{R})$ is defined, and a $C^\infty(M, \mathbf{R}) \times C^\infty(M, \mathbf{R}) \mapsto C^\infty(M, \mathbf{R})$ map, namely *the bracket* $(.,.)_{\mathrm{L}}$, that associates to two functions f and g in $C^\infty(M, \mathbf{R})$ a third smooth function

$$\ell = (f, g)_{\mathrm{L}}$$

Contact is made with Equation (1) just defining the particular case:

$$(f, g)_{\mathrm{L}} \equiv L^{ij}\partial_i f \partial_j g. \tag{3}$$

The "bracket" of the two functions $(f, g)_{\mathrm{L}}$ has the following properties [12,13]:

$$\begin{cases} \left(\sum_i \lambda_i f_i, \sum_j \mu_j g_j\right)_{\mathrm{L}} = \sum_{i,j} \lambda_i \mu_j \left(f_i, g_j\right)_{\mathrm{L}}, \\ (f_1 f_2, g)_{\mathrm{L}} = f_1 (f_2, g)_{\mathrm{L}} + f_2 (f_1, g)_{\mathrm{L}}, \\ (f, g_1 g_2)_{\mathrm{L}} = g_1 (f, g_2)_{\mathrm{L}} + g_2 (f, g_1)_{\mathrm{L}}, \end{cases} \tag{4}$$

where λ_i and μ_j are arbitrary real coefficients. The properties of (4) make the Leibniz bracket *derivative in both its arguments*, as the expression $L^{ij}\partial_i f \partial_j g$ is indeed with respect to both f and g. Clearly, Equation (4) generalizes the properties of the expression $L^{ij}\partial_i f \partial_j g$.

The application $(., F)_{\mathrm{L}}$ maps $C^\infty(M, \mathbf{R})$ onto vector fields on M: indeed the map

$$X_F f = (f, F)_{\mathrm{L}} \tag{5}$$

is a vector field. About this, the relationship $\dot{x}^i = \left(x^i, F\right)_{\mathrm{L}}$, i.e., the algebraic version of (1), indicates that the velocity of the system transforms as a proper vector. The vector field X_F in Equation (5) defines a flux throughout M, and then a dynamics of any variable: the function $F \in C^\infty(M, \mathbf{R})$ is referred

to as the Leibniz generator of the dynamics given by X_F. In a more physical way, if Equation (3) is considered, one has Equations (1) and (2).

Before moving on, it is important to stress that the dynamics expressed as in Equation (2) reinterprets the evolution as a transformation of functions on M, since infinitesimal variations of an observable is understood as $\delta f = (f, F)_{\mathrm{L}}\ \delta t$: this is the exquisite idea of *algebrizing dynamics*, i.e., reducing the evolution (with time) to a set of algebraic transformations [14]. Next to this, there comes the taxonomy of algebraic structures describing "chains of transformations", in particular *groups and semigroups* [15]. For physical systems, here and throughout this paper, the functions $f \in C^{\infty}(M, \mathbf{R})$ are intended as physical observables: for example, energy, mass, fluxes, densities, and linear or angular momenta can be cited.

Since, in general, all the observables with zero bracket with F are constant,

$$(f, F)_{\mathrm{L}} = 0 \quad \Rightarrow \quad \dot{f} = 0, \tag{6}$$

any f with gradient that is a null vector of L^{ij} has zero Leibniz bracket with any other function and is conserved:

$$L^{ij}\partial_i f = 0 \quad \Leftrightarrow \quad (f, g)_L = 0 \quad \forall \quad g \in C^{\infty}(M, \mathbf{R}) \quad \Rightarrow \quad \dot{f} = 0. \tag{7}$$

About the generating function F, its mathematical aspect determines *the possible steady states* of the system: indeed, thanks to (1) and (2), one may state

$$\partial_j F(\mathbf{x}_0) = 0 \quad \Rightarrow \quad \dot{x}^k(\mathbf{x}_0) = 0, \quad \dot{f}(\mathbf{x}_0) = 0. \tag{8}$$

If the system is *abandoned precisely at a point* $\mathbf{x}_0$ where the gradient of F vanishes, then it does not move away from there (of course, the stability of such a stationary point is another matter). Note that Equation (8) does not exclude the possibility that other points are stationary configurations of the system in M; instead, points with a null gradient of F represent the steady points in M of the system, if the tensor L^{ij} is non-singular.

The generator F undergoes the same Leibniz dynamics as in Equation (2): the relationship

$$\dot{F} = L^{ij}\partial_i F \partial_j F = (F, F)_L$$

may yield conservation, decrease, or increase of F, depending on the algebraic local characteristics of the tensor L, or even non-monotonic variability. Where the tensor is semidefinitely positive or negative, then F will increase or decrease—$\dot{F} \geq 0$ or $\dot{F} \leq 0$, respectively.

2.1. Hamiltonian Systems and Poisson Algebra

Classical Analytical Mechanics and then Quantum Mechanics teach that *fundamental systems in physics are considered Hamiltonian systems*, for which a Hamiltonian observable H is written, and the dynamics is generated through *Poisson brackets*. Poisson brackets are an example of Leibniz algebra [16]. When it comes to quantum laws, Poisson brackets are replaced by *commutators* [17] that show the same essential properties.

Poisson brackets, indicated here as $\{.,.\}$, show all the properties of relationships (4); moreover, they are anti-symmetric and satisfy the Jacobi identity:

$$\begin{cases} \{f, g\} = -\{g, f\} \quad \forall \quad f, g, h \in C^{\infty}(M, \mathbf{R}), \\ \{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0. \end{cases} \tag{9}$$

Equation (3) is written for Poisson brackets as

$$\{f(\mathbf{x}), g(\mathbf{x})\} = J^{ij}(\mathbf{x})\,\partial_i f(\mathbf{x})\,\partial_j g(\mathbf{x}): \tag{10}$$

one then assigns suitable properties to the tensor J so that the relationships (9) are satisfied. In particular, its anti-symmetric nature $J^{ij} = -J^{ji}$ and a differential relationship corresponding to the Jacobi identity $J^{ih}\partial_h J^{jk} + J^{kh}\partial_h J^{ij} + J^{jh}\partial_h J^{ki} = 0$ are required for J to define a Poisson bracket [18,19].

From the relationship

$$\dot{f} = \{f, H\},$$

reading $\dot{x}^i = J^{ij}\partial_j H$ for the dynamical variables describing the state of the system, and due to the anti-symmetric property of $\{.,.\}$, one has

$$\dot{H} = 0. \tag{11}$$

Equation (11) is simply *energy conservation*: the dynamics generator of a Hamiltonian system is conserved throughout the motion. Examples of a Hamiltonian system are not provided here, but surveyed in Section 3 (they are also very abundant in the literature).

Observables in Hamiltonian systems fulfilling the condition (7), written here as

$$J^{kj}\partial_k f = 0 \quad \Rightarrow \quad \{f, g\} = 0 \quad \forall \quad g \in C^\infty(M, \mathbf{R}),$$

are referred to as *Casimir observables* [19]. This name unveils a relationship between Poisson bracket algebra and the world of *Lie groups*: indeed, a Casimir is a fundamental algebraic invariant of the group algebra, depending on the algebra elements themselves. In the context of Hamiltonian systems, if the Leibniz brackets at hand satisfy Equation (9), then they show the same structure as a Lie algebra, hence the name of functions having null Poisson bracket with any other observable. By the way, for suitable expressions of the tensor J^{ij}, Equation (10) may reproduce the generating algebra of an otherwise known Lie group (in this case one speaks about *Poisson–Lie algebra*) [19]: this is the case with the system reported in Section 3.6, where the Lie algebra is that of the group of three-dimensional rotations.

Due to the properties of Poisson brackets, the trajectories of Hamiltonian systems in M cannot converge to an attracting point, or become confined to an attractor; instead, trajectories of Hamiltonian systems are either unbounded, or they "eternally" turn, regularly or not, without ever "stopping" at a point. In particular, one may state that *Hamiltonian systems do not admit asymptotically stable equilibria*. This is why Hamiltonian dynamics is perfect to describe the evolution of "immutable" systems: they may just redistribute energy among their degrees of freedom *in a reversible way*, as in the case of an ideal pendulum that will turn its gravitational energy into kinetic one and then backwards forever.

2.2. Dissipative Systems and Metric Algebra

Now, suppose the tensor L in (6) to be a *semimetric tensor* G, i.e., a symmetric, semi-definite (e.g., positive) tensor:

$$(f, g) = G^{ij}\partial_i f \partial_j g \quad / \quad G^{ij} = G^{ji}, \quad G^{ij}\partial_i f \partial_j f \geq 0 \quad \forall \quad f, g \in C^\infty(M, \mathbf{R}). \tag{12}$$

No Jacobi identity is satisfied by such a bracket. The bracket $(.,.)$, referred to as *(semi)metric bracket*, is a type of Leibniz bracket quite different from Poisson algebra [6,20]. Once a generating function $Q(\mathbf{x})$ is defined so that the dynamics of the system is governed by

$$\dot{x}^i = G^{ij}\partial_j Q, \quad \dot{f} = (f, Q), \tag{13}$$

it is possible to make some statements about this particular kind of system.

The essential fact to pick up in this discussion is that the function Q generating the dynamics is not constant; in particular, if G is a semi-definite positive tensor, then Q tends to grow monotonically along the evolution

$$\dot{Q} = (Q, Q) \geq 0: \tag{14}$$

it is possible to show that *isolated maxima of Q are asymptotically stable equilibrium points*. This is readily proved (see [20]) by realizing that, thanks to (14), Q is a *Lyapunov quantity*, so that points $\mathbf{x}_0$ such that $\partial_i Q(\mathbf{x}_0) = 0$ are places towards which the motion (13) will converge. If G were a semi-definite negative tensor, equally some Q' could be defined to play the opposite role, i.e., that of a monotonically decreasing quantity, and the asymptotically stable equilibrium points would then be its minima.

It is very important to have asymptotically stable equilibria into which trajectories converge, because then the system may be used to represent a dissipative process: dissipation drives systems in different states of motion to converge to a steady state. This steady state is characterized by either a minimum of non-thermal energy (for open systems, the energy of which is drained by friction) or a maximum of entropy (for closed "complete" systems obtained including the degrees of freedom responsible for dissipation, see Section 3): in both cases one may use a quantity monotonic in time to define a Leibniz dynamics representing the system. As a consequence, metric dynamics is perfect to mimic systems *evolving in an irreversible way*, for which the energy is transferred from one form into another and cannot come back by virtue of the same equations of motion. In other words, metric systems evolve irreversibly and age (their only feasible history is that of making Q grow, or decrease, forever, according to the sign of det$G$). While Hamiltonian systems have the conservation of energy as their pivoting principle, metric systems have the *increase of entropy* (or decrease of the free energy, see below) as their guiding law.

3. Complete Metriplectic Systems

In general, the structure described in Section 2 has an L that is neither symmetric nor antisymmetric, and is composed of these two parts:

$$L = J + G \quad / \quad J^{ab} = -J^{ba}, \quad G^{ab} = G^{ba}.$$

If the anti-symmetric part J is a Poisson tensor satisfying the Jacobi identity

$$J^{ih}\partial_h J^{jk} + J^{kh}\partial_h J^{ij} + J^{jh}\partial_h J^{ki} = 0,$$

and if G is semi-definite (for instance, positive semi-definite)

$$G^{ij}\partial_i f \partial_j f \geq 0 \quad \forall \quad f \in C^\infty(M, \mathbf{R}), \tag{15}$$

then the Leibniz system with bracket

$$\langle\langle f(\mathbf{x}), g(\mathbf{x})\rangle\rangle = \left(J^{ij}(\mathbf{x}) + G^{ij}(\mathbf{x})\right)\partial_i f(\mathbf{x})\,\partial_j g(\mathbf{x}) \tag{16}$$

is referred to as a *metriplectic system*.

A metriplectic bracket algebra (MBA) can be turned into a dynamic system once a generating function $F \in C^\infty(M, \mathbf{R})$ is adopted, and the prescriptions

$$\dot{x}^i = \langle\langle x^i, F\rangle\rangle, \quad \dot{f} = \langle\langle f, F\rangle\rangle \quad \forall \quad f \in C^\infty(M, \mathbf{R}) \tag{17}$$

are made. If Equation (16) is used, then the expression for $\dot{f}$ reads

$$\dot{f} = J^{ij}\partial_i f \partial_j F + G^{ij}\partial_i f \partial_j F : \tag{18}$$

in particular, the behavior along the motion of the generating function of a metriplectic system is entirely provided by the character of G, as the term $J^{ij}\partial_i F \partial_j F$ vanishes identically due to the anti-symmetric property of J. One then has

$$\dot{F} = G^{ij}\partial_i F \partial_j F,$$

so that, under Equation (15), this will be monotonically increasing: $\dot{F} \geq 0$. The generating function F that "evolves the system" through the algebra $\langle\langle ., F \rangle\rangle$ in Equation (18) is hence a Lyapunov quantity of the theory itself, as described in Section 2.2: in general, it will contain a part corresponding to the symplectic component of $\langle\langle ., . \rangle\rangle$ in Equation (16), and a part referring to metric algebra. Physical reasoning drives the definition of this and that in Section 3.1 below.

3.1. Energy Conservation, Entropy Increase

In [20] a complete system is referred to as a system that conserves its energy, but redistributes it in an irreversible way: this "irreversible redistribution" is named *dissipation*. Complete systems described via an MBA are indicated as *complete metriplectic systems* (CMS).

In everyday life, dissipation takes place due to the interaction of "macroscopic" degrees of freedom with "microscopic" ones, and this interaction mode is also named *friction*. When friction is at work, mechanical or electromagnetic energy is dissipated, i.e., irreversibly transformed, into kinetic energy of the microscopic constituents of the system, the degrees of freedom of which are, however, included in the system. These degrees of freedom are referred to as *Microscopic Statistically Treated Degrees of Freedom* (μSTDoF) [14]; here, "statistically treated" means that what describes these degrees of freedom in the phase space of the whole system is some collective quantities referring to them, more precisely their *thermodynamic coordinates*. The need to treat these degrees of freedom statistically comes from their timescales of evolution, much faster than those of "non-microscopic" ones.

3.1.1. In Case of Friction

When friction is the pathway to dissipation, the "standard" way to construct a complete system is to consider a Hamiltonian system, with its energy H_0 and dynamical variables $\mathbf{y}$, without dissipation, and add dissipation by making the system interact with μSTDoF, which converts the "ordered" energy of the "macroscopic, deterministic" degrees of freedom into thermal agitation: these degrees of freedom are included in the system in order to keep track of the energy that abandons the Hamiltonian part for dissipation. Once the μSTDoF are included, the system dynamical variables are enlarged as $\mathbf{x} = (\mathbf{y}, \mathbf{\Sigma})$, with the vector $\mathbf{\Sigma}$ collecting the thermodynamic representation of the μSTDoF. The total energy of the complete system is represented by the sum

$$H(\mathbf{y}, \mathbf{\Sigma}) = H_0(\mathbf{y}) + U(\mathbf{y}, \mathbf{\Sigma}): \tag{19}$$

the addendum $U(\mathbf{x})$ in general includes both a purely μSTDoF term, what we would refer to as *internal energy*, and an interaction term depending on the whole configuration $\mathbf{x}$; for simplicity, U may be supposed to depend only on the μSTDoF, and Equation (19) is rewritten as

$$H(\mathbf{y}, \mathbf{\Sigma}) = H_0(\mathbf{y}) + U(\mathbf{\Sigma}) \tag{20}$$

(in this case one says that the Hamiltonian system of variables $\mathbf{y}$ and the μSTDoF of variables $\mathbf{\Sigma}$ are assumed to be *separable*).

Even if the formulation with Hamiltonian Equation (19), or its simplified version, Equation (20), includes all the dynamical variables of the system, spanning the phase space M of complete configurations $\mathbf{x} = (\mathbf{y}, \mathbf{\Sigma})$, as long as it remains purely Hamiltonian, no hope exists of seeing the system converge to an asymptotic equilibrium, as required instead for any isolated system relaxing. This is why one needs to move ahead by including properly "dissipative forces" into the formulation of Hamiltonian $H(\mathbf{y}, \mathbf{\Sigma})$, introducing a metric component so that a metriplectic scheme is obtained. As metric systems are moved by a Lyapunov quantity, such an attribute of the system, monotonic with time due to dissipation, must be used. For isolated systems with dissipation, Equilibrium Thermodynamics predicts that dissipation is accompanied by the increase of entropy, a quantity that measures how underdetermined the microscopic configuration is once the macroscopic one is assigned [21]: the system of variables $\mathbf{x} = (\mathbf{y}, \mathbf{\Sigma})$ must have a proper entropy, hence, and is expected to grow monotonically. Actually the entropy is only attributed to the μSTDoF, which is the only part to

be treated statistically; the total entropy S of the complete system will simply be the thermodynamic entropy of its μSTDoF. One then has:

$$S = S(\mathbf{\Sigma}).$$

Now, in order to define the metric part of the MBA, this S is of course the most obvious candidate to play the Lyapunov quantity Q in Section 2.2: the entropy of the μSTDoF must enter F as the metric contribution to the metriplectic generator. The construction of the MBA describing complete systems is then performed by considering as generating function a combination of H and S named free energy

$$F(\mathbf{y}, \mathbf{\Sigma}) = H(\mathbf{y}, \mathbf{\Sigma}) + \alpha S(\mathbf{\Sigma}), \tag{21}$$

where α is a parameter to be adjusted suitably. Since everything is chosen in order for $H(\mathbf{y}, \mathbf{\Sigma})$ to be constant, and for $S(\mathbf{\Sigma})$ to grow with time, the behavior of F with time depends on the sign of α:

$$\dot{F} = \alpha \dot{S} \quad \Rightarrow \quad \text{sign}\left(\dot{F}\right) = \text{sign}(\alpha). \tag{22}$$

Equation (21) turns Equation (18) into:

$$\dot{f} = J^{ij}\partial_i f \partial_j H + \alpha J^{ij}\partial_i f \partial_j S + G^{ij}\partial_i f \partial_j H + \alpha G^{ij}\partial_i f \partial_j S. \tag{23}$$

The scheme is completed by choosing J and G in Equation (23) according to physics. In general, the interaction between the original Hamiltonian system and the μSTDoF is tuned by some constant η, so that when $\eta \to 0$ the subsystems decouple and dissipation disappears. In this limit, clearly, the bracket $\langle\langle .,. \rangle\rangle$ must reduce to the original Poisson bracket $\{.,.\}$ moving $\mathbf{y}$ via $\{., H_0\}$, in which only derivatives with respect to $\mathbf{y}$ appear: the internal energy $U(\mathbf{\Sigma})$ is not affected by non-dissipative dynamics, so that one has $\{., H_0\} = \{., H\}$. All in all, the limit $\lim_{\eta \to 0} \langle\langle .,. \rangle\rangle = \{.,.\}$ must hold, so that, on the one hand the metric tensor G in (18) has to vanish for $\eta \to 0$

$$\lim_{\eta \to 0} G^{ij} = 0,$$

and on the other hand the tensor J is simply the one forming the Poisson bracket of the Hamiltonian system with dynamical variables $\mathbf{y}$ and the Hamiltonian H_0 we started from: the components pertaining to the sub-manifold described by $\mathbf{\Sigma}$ are zero.

When the metriplectic dynamics is enforced as $\dot{f} = J^{ij}\partial_i f \partial_j F + G^{ij}\partial_i f \partial_j F$, two facts must hold:

$$\dot{H}(\mathbf{y}, \mathbf{\Sigma}) = 0, \quad \dot{S}(\mathbf{\Sigma}) \geq 0.$$

The requirement of H to be constant with time is inserted into Equation (23), giving rise to

$$0 = \alpha J^{ij}\partial_i H \partial_j S + G^{ij}\partial_i H \partial_j H + \alpha G^{ij}\partial_i H \partial_j S,$$

where the anti-symmetry of J has already been taken into account. On the other hand, the first addendum $\alpha J^{ij}\partial_i H \partial_j S$ is equal to $-\alpha J^{ab}\partial_a S \partial_b H = -\alpha \{S, H\}$, $\{S, H\}$ being the variability with time of S under the mere Hamiltonian part of the motion: since no change in the μSTDoF entropy is expected due to the "conservative forces" represented by $\{., H\}$, one expects to have $\alpha J^{ij}\partial_i H \partial_j S = 0$. This is a precise request for the relationship between S and the symplectic part of $\langle\langle .,. \rangle\rangle$, and we will discuss it soon; for the time being, let us simply consider

$$\dot{H} = 0 \quad \Rightarrow \quad 0 = G^{ij}\partial_i H \left(\partial_j H + \alpha \partial_j S\right).$$

In order for the factor $G^{ij}\partial_i H \left(\partial_j H + \alpha \partial_j S\right)$ to be zero, the simplest possible assumption is

$$G^{ij}\partial_i H = 0 \quad \Rightarrow \quad (H, f) = 0 \quad \forall \quad f \in C^\infty(M, \mathbf{R}), \tag{24}$$

i.e., the metric tensor G has the gradient of H among its null vectors, so the Hamiltonian has a null metric bracket with any other element of $C^\infty(M, \mathbf{R})$. Equation (24) is what remains of the request that the total energy of the complete system is conserved.

About the monotonic increase of S, instead, while Equation (24) holds, one may apply Equation (18) and obtain:

$$\dot{S} = J^{ij}\partial_i S \partial_j H + \alpha G^{ij}\partial_i S \partial_j S.$$

Now, if a formal hypothesis

$$\{f, S\} = 0 \quad \forall \quad f \in C^\infty(M, \mathbf{R}),$$

is formulated, meaning that S is a Casimir of the Poisson bracket at hand, one simply gains:

$$\dot{S} = \alpha G^{ij}\partial_i S \partial_j S. \tag{25}$$

Then, choosing the sign of $\alpha \det G$ as positive, the condition $\dot{S} \geq 0$ is immediately enforced (here $\det G$ is the determinant of G).

Considering $(., H) = 0$ and $\{., S\} = 0$ identically, the evolution of the system, and of any observable along the system trajectory, reads:

$$\dot{x}^i = \{x^i, H\} + \alpha(x^i, S), \quad \dot{f} = \{f, H\} + \alpha(f, S). \tag{26}$$

Thus, we end up with a CMS.

3.1.2. Frictionless Dissipation

The intervention of μSTDoF draining energy from deterministic variables $\mathbf{y}$ of a Hamiltonian system through friction may be not necessary in order for a CMS to admit asymptotically stable equilibria: there exist CMS the dynamics of which still undergoes Equations (15)–(17) as:

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \{\mathbf{x}, H(\mathbf{x})\} + \alpha(\mathbf{x}, Q(\mathbf{x})), \\ \dot{Q} \geq 0, \\ \{Q, f\} = 0, \quad (H, f) = 0 \quad \forall \quad f, \end{cases}$$

in which the entropy-like observable Q and the Hamiltonian H depend on the same variables $\mathbf{x}$. These CMS may be indicated as "frictionless" because μSTDoF cannot be singled out, as in the ones treated in Sections 3.5 and 3.6 below.

The "big difference" between CMS with friction and frictionless ones is that in the first case a granular nature of the material system must be supposed, so that dissipation transfers energy from "macroscopic" to "microscopic" scales: hence, a CMS with friction must be non-elementary in a sense, because the deterministic variable $\mathbf{y}$, as in those in Equation (19), describes the macroscopic world, averaging away fluctuations of microscopic constituents of matter that will be encoded in the thermodynamic coordinates $\mathbf{\Sigma}$ of the μSTDoF. The μSTDoF are the "really elementary" degrees of freedom, even if treated statistically.

In a sense, frictionless CMS implement dissipation at a fundamental level, via the $\alpha(., Q)$ component, which does not require any "smaller" constituent to exist. Simply, frictionless CMS seem to teach that irreversibility do not require the "coarse graining of micro-things", but rather the appearance of the semi-metric component of $\langle\langle ., . \rangle\rangle$, yielding a Lyapunov Q growing with time flow. The contribution of frictionless CMS at a fundamental level is two-fold: on the one hand, they generalize the idea of entropy $S(\Sigma)$ to Lyapunov dynamics generators $Q(\mathbf{x})$; on the other hand, their quantization might give hints to implement dissipation-irreversibility in terms of the fundamental "microscopic" laws of physics [22].

3.2. Stationary Points of CMS

The role of free energy F is, as in any Leibniz system, to provide the stationary points of the dynamics at hand in terms of its extrema.

In the case of CMS with friction, due to Equation (21) and the nature of the phase space coordinates $\mathbf{x} = (\mathbf{y}, \mathbf{\Sigma})$, the condition $\partial_j F(\mathbf{x}_0) = 0$ corresponds to this collection of conditions:

$$\begin{cases} \dfrac{\partial H}{\partial \mathbf{y}} = 0, \\ \dfrac{\partial H}{\partial \mathbf{\Sigma}} + \alpha \dfrac{\partial S}{\partial \mathbf{\Sigma}} = 0. \end{cases}$$

More precisely, considering the decomposition of the Hamiltonian in Equation (23), one rather writes:

$$\frac{\partial F}{\partial \mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{\partial H_0}{\partial \mathbf{y}} = 0, \\ \dfrac{\partial U}{\partial \mathbf{\Sigma}} + \alpha \dfrac{\partial S}{\partial \mathbf{\Sigma}} = 0. \end{cases} \tag{27}$$

The relationship $\frac{\partial H_0}{\partial \mathbf{y}} = 0$ prescribes the mechanical equilibrium (in radiation systems this could also be a "radiative" equilibrium, in which case $\mathbf{y}$ is a field variable), while the relationship $\frac{\partial U}{\partial \mathbf{\Sigma}} + \alpha \frac{\partial S}{\partial \mathbf{\Sigma}} = 0$ is the thermodynamic equilibrium request. The latter will fix α in a physically sensible way, typically minus the temperature of the μSTDoF.

About the nature of maxima or minima of the stationary points (Equation (27)), it is sensible to establish α by considering the physical sense of Equation (27), and then assessing whether $\dot{F}$ has a positive or negative sign. The fact that α is tuned so to match a particular "temperature", i.e., a particular equilibrium state, stresses the whole construction of the CMS as *the local description of a complex system in the neighborhood of an equilibrium point towards which it is known to relax*.

As far as the frictionless CMS are concerned, their dynamical generator F is still subdivided into a Hamiltonian "plus some Lyapunov":

$$F(\mathbf{x}) = H(\mathbf{x}) + \alpha Q(\mathbf{x});$$

however, the steady state prescription

$$\frac{\partial F}{\partial \mathbf{x}} = 0 \tag{28}$$

will have a different interpretation.

3.3. Two Simple Examples with Friction

Two very simple examples can be given, taken from Newtonian mechanics, and are very useful to start getting acquainted with CMS. In the first example, in Section 3.3.1, the sub-systems represented by the variables $\mathbf{y}$ and Σ are separable, which is not the case in the example in Section 3.3.2, where the system shows an internal energy $U(\mathbf{y}, \mathbf{\Sigma})$.

3.3.1. The Point Particle in the Viscous Medium

In the first example the complete system is formed by a point particle of mass m moving in three-dimensional space through an infinite viscous medium [14], so that its equations of motion read

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}, \quad \frac{d\vec{p}}{dt} = -\frac{\partial V}{\partial \vec{x}} - \frac{\lambda}{m}\vec{p}. \tag{29}$$

The foregoing equations would reduce to a Hamiltonian system $\frac{d}{dt}\vec{x} = m^{-1}\vec{p}$ and $\frac{d}{dt}\vec{p} = -\partial_{\vec{x}}V$ in the non-dissipative limit $\lambda \to 0$, the dynamical variables of which would be $\mathbf{y} = \left(\vec{x}, \vec{p}\right)$, and with Hamiltonian $H_0(\mathbf{y}) = \frac{1}{2m}p^2 + V\left(\vec{x}\right)$. In order to recognize a metriplectic complete system one has to include the viscous medium variables, introducing a very simple collection of (thermo)dynamical coordinates, namely the medium entropy only $\mathbf{\Sigma} = (S)$: hence the whole Hamiltonian reads

$$H = H_0(\mathbf{y}) + U(S)$$

(separable variables), with internal energy of the viscous medium $U(S)$. The complete set of dynamical variables is $\mathbf{x} = \left(\vec{x}, \vec{p}, S\right)$.

Note that, as the viscous medium is considered infinitely extended, S is rigorously an infinite quantity. This creates no problem for our work, since $\dot{S}$ is always finite (and positive), while infinitesimal dS can be considered. The cartoon of the system is that of Figure 1.

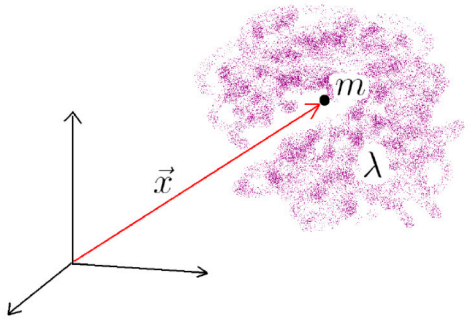

Figure 1. The point particle of mass m and position $\vec{x}$ moving through the medium of viscous constant λ, described as a metriplectic system in Section 3.3.1.

In order to construct the MBA for Equation (29) it is necessary to know the ODE for the entropy, which is obtained by the classical relationship $dQ = TdS$, with T the temperature of the medium and dQ the amount of thermal energy transferred from the mechanical degrees of freedom $\mathbf{y}$ to the viscous medium. Actually, since the energy must be conserved, dQ is simply minus the power of the dissipative force $-m^{-1}\lambda\vec{p}$ exerted by the medium on the point particle, times the infinitesimal time interval dt: this power being $w_\lambda = -m^{-1}\lambda\vec{p}\cdot m^{-1}\vec{p}$ and $dQ = -w_\lambda dt$, one has $dQ = m^{-2}\lambda p^2 dt$, and considering $dS = \dot{S}dt$ one ends up with:

$$\dot{S} = \frac{\lambda p^2}{m^2 T} \tag{30}$$

(obviously, the temperature of the medium is supposed to remain constant, while friction transfers such a small amount of energy from the pointlike particle to the infinite medium: as commented before, the CMS is constructed to mimic the relaxation to an equilibrium with this assigned temperature). Equation (30) completes the ODEs of the complete system we are looking for, together with Equation (29).

According to Equation (21), the free energy of the system reads

$$F\left(\vec{x}, \vec{p}, S\right) = \frac{p^2}{2m} + V\left(\vec{x}\right) + U(S) + \alpha S: \tag{31}$$

the parameter α may be determined by imposing that the extrema of F represent steady states. The steady state $\mathbf{x}_0$ is found as the solution of

$$\frac{\partial F}{\partial \vec{x}} = 0, \quad \frac{\partial F}{\partial \vec{p}} = 0, \quad \frac{\partial F}{\partial S} = 0. \tag{32}$$

If Equation (31) is put into Equation (32) one finds

$$\frac{\partial V}{\partial \vec{x}}\left(\vec{x}_0\right)=0,\quad \vec{p}_0=0,\quad \frac{\partial U}{\partial S}\left(S_0\right)+\alpha=0:$$

the first and second equations mean that the point particle will stop at an extremum of the mechanical potential, with null momentum (velocity); the third equation, better re-written as $\alpha=-\frac{\partial U}{\partial S}\left(S_0\right)$, instead determines α as minus $\frac{\partial U}{\partial S}\left(S_0\right)$, which is clearly the equilibrium temperature of the viscous medium T, after the relationship $\frac{\partial U}{\partial S}=T$ due to Equilibrium Thermodynamics [1]. Equilibrium equations

$$\frac{\partial V}{\partial \vec{x}}\left(\vec{x}_0\right)=0,\quad \vec{p}_0=0,\quad \alpha=-T$$

allow us to re-write the free energy (Equation (31)) in the following way:

$$F\left(\vec{x},\vec{p},S\right)=\frac{p^2}{2m}+V\left(\vec{x}\right)+U\left(S\right)-TS. \tag{33}$$

By the way, due to the relationship $\alpha=-T$, the tensor G in (13) has to be negative semi-definite. This is due to the construction of F as $F=H+\alpha S$, hence to the negative value of $\alpha=-T$: when Equation (13) is applied one obtains $\dot{F}=\alpha\left(F,F\right)$, and since this has to be semi-positive, the form $(.,.)$, i.e., G, must be semi-negative.

In order to complete the MBA one has to define suitable J and G. The tensor J is simply the symplectic tensor in the $\left(\vec{x},\vec{p}\right)$ space extended to the seven-dimensional manifold of $\left(\vec{x},\vec{p},S\right)$. In order to correctly define G, instead, one has to look for an $\mathbf{R}^{7,7}$ matrix matching the conditions of Equations (24) and (25) for H and S, and give rise (together with the already established J) to the ODEs (Equation (29)). The same arguments lead to the expression in Equation (39); see below.

The tensors will be written with respect to the components $\vec{x}$, $\vec{p}$ and S, so they will look like:

$$J=\begin{pmatrix}\mathbf{0}_{3,3} & \mathbf{1}_{3,3} & \mathbf{0}_{3,1}\\ -\mathbf{1}_{3,3} & \mathbf{0}_{3,3} & \mathbf{0}_{3,1}\\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0\end{pmatrix},\quad G=\frac{1}{\alpha}\begin{pmatrix}\frac{\left|\partial_{\vec{x}}V\right|^2\mathbf{1}_{3,3}-\partial_{\vec{x}}V\otimes\partial_{\vec{x}}V}{\left|\partial_{\vec{x}}V\right|^2} & \mathbf{0}_{3,3} & \mathbf{0}_{3,1}\\ \mathbf{0}_{3,3} & \lambda T\mathbf{1}_{3,3} & -\lambda m^{-1}\vec{p}\\ \mathbf{0}_{1,3} & -\lambda m^{-1}\vec{p}^{\,T} & \frac{\lambda p^2}{m^2T}\end{pmatrix}. \tag{34}$$

In Equation (34), note that G has the same definition as the sign of α, since the matrix multiplying α^{-1} has either positive or null eigenvector. With the matrices defined in (34) the relationships

$$\{f,S\}=0,\quad (f,H)=0$$

for any f are satisfied, moreover one obtains

$$\left\{\vec{x},H\right\}+\alpha\left(\vec{x},S\right)=\frac{\vec{p}}{m},\quad \left\{\vec{p},H\right\}+\alpha\left(\vec{p},S\right)=-\frac{\partial V}{\partial\vec{x}}-\frac{\lambda}{m}\vec{p},\quad \alpha\left(S,S\right)=\frac{\lambda p^2}{m^2T},$$

meaning that we are in the presence of the correct tensors (Equation (34)) to define the MBA reproducing Equations (29) and (30). All in all, with the tensors in Equation (34) and the generating function F in Equation (33), we can state $\dot{f}=\langle\langle f,F\rangle\rangle$ for any observable f of the simple system.

3.3.2. The Piston and the Spring

The second example of a metriplectic system taken from everyday physics is a piston of mass m and surface A, moving along a horizontal guide and pushed by a spring of constant k. This piston makes work against a viscous gas of mass M, the pressure of which is indicated as P. This system is depicted as in Figure 2.

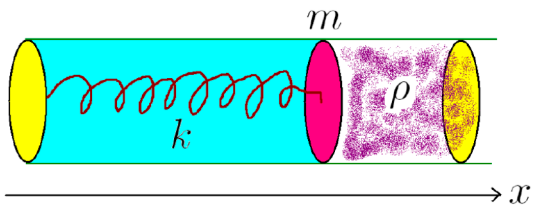

Figure 2. The piston and the spring, described in Section 3.3.2 as a metriplectic complete system. The density of the gas is indicated as ρ.

If no viscosity were present, the system would be conservative, i.e., Hamiltonian: the necessary, independent dynamical variables would just be x and p of the piston; when energy is irreversibly transferred between the mechanical degrees of freedom and the μSTDoF of the gas, due to viscosity, some thermodynamic coordinate of the medium must be included: the entropy S of it is the simplest candidate, so that in this case the dynamical variables of the complete system are collected in the vector $\mathbf{x} = (x, p, S)$. The ODEs of $\mathbf{x}$ are written as [14]:

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -k\,(x - \ell) - PA - \lambda \frac{p}{m}, \quad \dot{S} = \frac{\lambda p^2}{m^2 T} \tag{35}$$

(the parameter ℓ is the equilibrium length of the spring, while T is the temperature of the gas, and the same reasoning about $dQ = TdS$ as in Section 3.3.1 has been applied). As far as the thermodynamics of the gas is concerned, this case is more interesting than the one discussed above, where the point particle could produce no variation in the infinite medium, apart from increasing its entropy. In general, indeed, one may expect that the thermodynamic coordinates of the gas should be S and, e.g., the mass density ρ, so that its internal energy reads $U(\rho, S)$, and the pressure P in (35) is defined as

$$PV = \rho \frac{\partial U}{\partial \rho} \quad \Rightarrow \quad P = \frac{\rho^2}{M} \frac{\partial U}{\partial \rho}. \tag{36}$$

Here the only fixed things of the gas are the area A of the piston and the mass M of the medium: in general, its volume reads $V(x) = A\,(L_0 - x)$, so that the density will depend on the position of the piston as

$$\rho(x) = \frac{M}{A\,(L_0 - x)} \quad \Rightarrow \quad \frac{\partial x}{\partial \rho} = \frac{M}{A\rho^2}, \quad \frac{\partial \rho}{\partial x} = \frac{A\rho^2}{M}. \tag{37}$$

This x-dependence in the density implies that the coordinates of the complete system are (x, p, S) instead of the redundant set (x, p, ρ, S); it also yields the dependence $U(\rho(x), S)$, so that the subsystems "piston-attached-to-the-spring" and "gas" *are not separable*.

The whole Hamiltonian of the system includes the kinetic energy of the piston, the elastic energy of the spring, and the internal energy of the viscous gas $U(\rho(x), S)$. Once it is written, it is very easy to also write the free energy F:

$$\begin{aligned} H(x, p, S) &= \frac{p^2}{2m} + \frac{k}{2}(x - \ell)^2 + U(\rho(x), S), \\ F(x, p, S) &= \frac{p^2}{2m} + \frac{k}{2}(x - \ell)^2 + U(\rho(x), S) + \alpha S. \end{aligned} \tag{38}$$

In order for the generator $F(x, p, S)$ to produce the correct ODEs (35), one just has to choose the two tensors J and G as follows:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda T & -\lambda m^{-1} p \\ 0 & -\lambda m^{-1} p & \dfrac{\lambda p^2}{m^2 T} \end{pmatrix}, \tag{39}$$

written with respect to the dynamical variables (x, p, S). Last but not least, one may find out the equilibria of the system by checking the extrema of the function F as:

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial p} = 0, \quad \frac{\partial F}{\partial S} = 0.$$

With the specific form in Equation (39), and considering the relationship among U, P, x, and ρ in Equations (36) and (37), one obtains

$$x_0 = \ell - \frac{PA}{k}, \quad p_0 = 0, \quad \alpha = -T:$$

respectively, these are the balance between the gas and the spring forces, the zero velocity of the piston and the correspondence of the parameter α to minus the temperature of the gas. Of course, equilibrium x_0 and p_0 could have been calculated directly from Equation (36); the exercise presented here is just intended to explain the metriplectic formalism in the very simple case under examination.

3.4. Classical Fluids

In fluid theories, in the absence of dissipation the energy transfer would take place among degrees of freedom making sense at a macroscopic scale, without involving the granular nature of matter; however, as "friction" is turned on by considering finite viscosity and thermal conduction, the degrees of freedom of the microscopic particles forming the continuum play the role of the μSTDoF discussed before.

About this, a remark is necessary, to stress the difference between these systems and those discussed in Section 3.3. In the examples in Section 3.3, the complete system was subdivided into two subsystems "materially" separated: in Section 3.3.1 there was a point particle treated deterministically and a viscous fluid with μSTDoF, while in Section 3.3.2 the deterministic degrees of freedom were those of the piston, while μSTDoF were attributed to the viscous gas against which the piston was working. Dissipative continua do not show a "material separation" between the deterministic, Hamiltonian part of the complete system and the μSTDoF draining energy irreversible and, hence, giving rise to dissipation. Consider a "macroscopic infinitesimal" parcel, i.e., a portion of fluid containing a thermodynamic number of particles and still being so small that the continuum field variables are constant within it: the motion of the particles within the parcel represent the μSTDoF, while the motion of the parcel's center-of-mass (CoM) throughout the space represent the degrees of freedom $\mathbf{y}$ in Section 3.1, with a deterministic dynamics. In a fluid context, the energy pertaining to the center-of-mass of a given parcel δC is irreversibly converted by dissipation into the energy of the μSTDoF relative to another parcel $\delta C'$ (δC and $\delta C'$ are *different parcels*, otherwise a mechanical system would be able to alter its own CoM motion, against Newton's Principles: this is why dissipative terms appear in the equations of motion of fluid dynamics with space-derivative terms, taking into account the different CoM velocities of nearby parcels [23]. With the dissipation due to currents the thing is slightly subtler, because the energy irreversibly converted there comes from the magnetic degrees of freedom [7,24]). The energy dissipated may then be transported, still irreversibly, by heat conduction against temperature gradients.

Dissipative fluids may be understood as complete metriplectic systems in which the $\mathbf{y}$ and the $\mathbf{\Sigma}$ just describe different degrees of freedom of the same material system: as explained before, these variables describe the system at different space- and time-scales.

In the Lagrangian representation of dissipative fluids, the dynamical variables are the CoM position $\vec{\zeta}\left(\vec{a}\right)$ and momentum $\vec{\pi}\left(\vec{a}\right)$ of each fluid parcel, plus the variables relative to the CoM of the same parcels, thermodynamically described through the entropy density $s\left(\vec{a}\right)$. The 3D continuous index $\vec{a} \in \mathbf{D}_0$ materially labels the fluid parcels [25], and may refer to the initial position of the parcel

that labels as $\vec{a} = \vec{\zeta}\left(\vec{a}, 0\right)$, if $\vec{\zeta}\left(\vec{a}, t\right)$ is the position of the $\vec{a}$-th parcel at time t. The domain $\mathbf{D}_0$ is the initial volume occupied by the continuous matter.

The subdivision between "Hamiltonian" variables $\mathbf{y}$ and μSTDoF variables $\mathbf{\Sigma}$ is simple, $\mathbf{y} = \left(\vec{\zeta}\left(\vec{a}\right), \vec{\pi}\left(\vec{a}\right)\right)$, $\mathbf{\Sigma} = \left(s\left(\vec{a}\right)\right)$, while the mass $\rho\left(\vec{a}\right)$ of the density $\vec{a}$-th parcel may be expressed in terms of the initial mass density $\rho_0\left(\vec{a}\right)$ that characterizes the mass geometry of the continuum and is assigned once and forever, and the Jacobian determinant $\mathrm{J}\left(\vec{a}\right) = \det||\frac{\partial \vec{\zeta}}{\partial \vec{a}}||$, as:

$$\rho\left(\vec{a}\right) = \frac{\rho_0\left(\vec{a}\right)}{\mathrm{J}\left(\vec{a}\right)}$$

$\rho\left(\vec{a}\right)$ would be redundant, as a dynamical variable, to the complete configuration $\mathbf{x} = \left(\vec{\zeta}\left(\vec{a}\right), \vec{\pi}\left(\vec{a}\right), s\left(\vec{a}\right)\right)$.

Excluding dissipative forces, the infinitesimal parcel described by the configuration $\mathbf{x}$ can be attributed a Hamiltonian dynamics by considering its energy determined by the sum of a kinetic part, a potential part giving rise to the pressure forces exerted by the surrounding parcels, and a part due to external "conservative" forces. The total Hamiltonian of the fluid reads:

$$H\left[\vec{\zeta}, \vec{\pi}, s\right] = \int_{\mathbf{D}_0} d^3a \left[\frac{\pi^2\left(\vec{a}\right)}{2\rho_0\left(\vec{a}\right)} + \rho_0\left(\vec{a}\right) U\left(\frac{\rho_0\left(\vec{a}\right)}{\mathrm{J}\left(\vec{a}\right)}, s\left(\vec{a}\right)\right) + \rho_0\left(\vec{a}\right) V\left(\vec{\zeta}\left(\vec{a}\right)\right)\right]. \tag{40}$$

The dependence of U on CoM variables $\vec{\zeta}$ through the Jacobian determinant qualifies this system as a non-separable CMS.

The dissipation free motion of the fluid is generated by the foregoing Hamiltonian and the Poisson bracket:

$$\{f, g\} = \int_{\mathbf{D}_0} d^3a \left[\frac{\delta f}{\delta \vec{\zeta}\left(\vec{a}\right)} \cdot \frac{\delta g}{\delta \vec{\pi}\left(\vec{a}\right)} - \frac{\delta g}{\delta \vec{\zeta}\left(\vec{a}\right)} \cdot \frac{\delta f}{\delta \vec{\pi}\left(\vec{a}\right)}\right]. \tag{41}$$

The Hamiltonian limit of the fluid dynamics is then:

$$\dot{f}\left[\vec{\zeta}, \vec{\pi}, s\right] = \left\{f\left[\vec{\zeta}, \vec{\pi}, s\right], H\left[\vec{\zeta}, \vec{\pi}, s\right]\right\}. \tag{42}$$

The symplectic bracket (41) is a canonical one, not different from those of ordinary point particles of Newtonian Physics [16]. In particular, it does not contain any derivative with respect to the entropy density $s\left(\vec{a}\right)$, so it does not involve any statistical proxy of the μSTDoF: this is a benefit brought by the Lagrangian representation, while in the Eulerian one derivatives with respect to ρ and s appear [24]. Of course, microscopic degrees of freedom of the parcel are present in $H\left[\vec{\zeta}, \vec{\pi}, s\right]$, in particular through the internal potential energy $U\left(\frac{\rho_0\left(\vec{a}\right)}{\mathrm{J}\left(\vec{a}\right)}, s\left(\vec{a}\right)\right)$: still, due to the form of Equation (41), the entropy density $s\left(\vec{a}\right)$ is perfectly conserved

$$\dot{s}\left(\vec{a}\right) = \left\{s\left(\vec{a}\right), H\left[\vec{\zeta}, \vec{\pi}, s\right]\right\} = 0:$$

as the ideal, dissipation-free motion of the fluid takes place, the amount $s\left(\vec{a}\right)$, encoding the complexity of the particle motions internal to the macroscopic parcel, remains frozen to its original value. One could well consider $s\left(\vec{a},t\right)=s\left(\vec{a},0\right)\equiv s_0\left(\vec{a}\right)$ a parameter rigidly assigned as the density $\rho_0\left(\vec{a}\right)$), so that the Hamiltonian Equation (40) would become a quantity as the H_0 in Equation (20). In our case, the "free Hamiltonian" $H_0\left(\mathbf{y}\right)$ would rather read

$$H_0\left[\vec{\zeta},\vec{\pi}\right]=\int\limits_{\mathbf{D}_0}d^3a\left[\frac{\pi^2\left(\vec{a}\right)}{2\rho_0\left(\vec{a}\right)}+\rho_0\left(\vec{a}\right)U\left(\frac{\rho_0\left(\vec{a}\right)}{\mathrm{J}\left(\vec{a}\right)},s_0\left(\vec{a}\right)\right)+\rho_0\left(\vec{a}\right)V\left(\vec{\zeta}\left(\vec{a}\right)\right)\right].$$

In the foregoing formula, H_0 just depends on the CoM variables because the other ones are frozen and "do not exist" as dynamical variables. Considering Equation (42), the fluid dynamics in Lagrangian variables is written as:

$$\begin{cases}\dot{\zeta}_\alpha=\pi_\alpha,\\ \dot{\pi}_\alpha=-\rho_0\dfrac{\partial V}{\partial\zeta^\alpha}+A^i_\alpha\dfrac{\partial}{\partial a^i}\left(\rho_0\dfrac{\partial U}{\partial\mathrm{J}}\right),\quad A^i_\alpha=\dfrac{\varepsilon_{\alpha\kappa\lambda}\varepsilon^{imn}}{2}\dfrac{\partial\zeta^\kappa}{\partial a^m}\dfrac{\partial\zeta^\lambda}{\partial a^n},\\ \dot{s}=0\end{cases}\tag{43}$$

(Greek indices refer to the vectors $\vec{\zeta}$ and $\vec{\pi}$, Latin ones to the 3D parcel-index $\vec{a}$). The entropy density s has zero Poisson bracket "with anything":

$$\left\{f,s\left(\vec{a}\right)\right\}=0\quad\forall\quad f,\tag{44}$$

so does the total entropy of the fluid defined in Equation (46); see below: this renders it a Casimir of (41).

Including the interaction with the μSTDoF means simply unfreezing the thermodynamic quantity $s\left(\vec{a}\right)$: thanks to dissipative "forces", it may now change its value. Considering the granular nature of matter (i.e., including viscosity and thermal resistivity) will then turn Equation (43) into the following ones:

$$\begin{cases}\dot{\zeta}_\alpha=\pi_\alpha,\\ \dot{\pi}_\alpha=-\rho_0\dfrac{\partial V}{\partial\zeta^\alpha}+A^i_\alpha\dfrac{\partial}{\partial a^i}\left(\rho_0\dfrac{\partial U}{\partial\mathrm{J}}\right)+\Lambda_{\alpha\beta\gamma\delta}\mathrm{J}\nabla^\beta\nabla^\gamma\left(\dfrac{\pi^\delta}{\rho_0}\right),\\ \dot{s}=\dfrac{\mathrm{J}}{\rho_0T}\Lambda_{\alpha\beta\gamma\delta}\nabla^\alpha\left(\dfrac{\pi^\beta}{\rho_0}\right)\nabla^\gamma\left(\dfrac{\pi^\delta}{\rho_0}\right)+\dfrac{\mathrm{J}\kappa}{\rho_0T}\nabla^\alpha\nabla_\alpha T.\end{cases}\tag{45}$$

The convention $\vec{\nabla}=\frac{\partial}{\partial\vec{\zeta}}$ is intended. In Equation (45) the coefficient κ is the thermal conductivity, while T is the temperature of the μSTDoF within the parcel, defined as always: $T\left(\vec{a}\right)=\frac{\partial U}{\partial s\left(\vec{a}\right)}$. As far as the symbol $\Lambda_{\alpha\beta\gamma\delta}$ is concerned, we simply state

$$\Lambda_{\alpha\beta\gamma\delta}=\eta\left(\delta_{\delta\alpha}\delta_{\gamma\beta}+\delta_{\delta\beta}\delta_{\gamma\alpha}-\frac{2}{3}\delta_{\alpha\beta}\delta_{\gamma\delta}\right)+\zeta\delta_{\alpha\beta}\delta_{\gamma\delta}:$$

this is the suitable tensor mimicking viscosity. In Equation (45) the entropy variation is due to the non-ideal stress tensor: hence, to viscosity

$$\frac{\mathrm{J}}{\rho_0T}\Lambda_{\alpha\beta\gamma\delta}\nabla^\alpha\left(\frac{\pi^\beta}{\rho_0}\right)\nabla^\gamma\left(\frac{\pi^\delta}{\rho_0}\right),$$

and to thermal conduction

$$\frac{\mathrm{J}\kappa}{\rho_0 T}\nabla^\alpha \nabla_\alpha T,$$

i.e., to the two irreversible processes taking place. These are the processes draining energy out of the parcels' CoM degrees of freedom $\mathbf{y} = \left(\vec{\zeta}\left(\vec{a}\right), \vec{\pi}\left(\vec{a}\right)\right)$, smoothing the velocity difference between nearby parcels (i.e., killing the gradients $\partial_{\vec{a}}\vec{\pi}$) and homogenizing temperature T via thermal diffusion.

It is possible to construct a functional derivative semi-metric bracket that defines an MBA and, together with the Poisson bracket in Equation (41), can reproduce Equation (45). As usual, the entropy of the μSTDoF responsible for dissipation must be introduced:

$$S[s] = \int\limits_{\mathbf{D}_0} d^3 a \rho_0 \left(\vec{a}\right) s\left(\vec{a}\right), \tag{46}$$

which is a Casimir of Equation (41), as its integrand is (see Equation (44)), and composes the free energy together with the Hamiltonian:

$$F\left[\vec{\zeta}, \vec{\pi}, s\right] = H\left[\vec{\zeta}, \vec{\pi}, s\right] + \alpha S[s]. \tag{47}$$

The symmetric bracket completing the MBA to give rise to Equation (45) reads [23]:

$$(f,g) = \frac{1}{\alpha}\int\limits_{\mathbf{D}_0} \mathrm{J} d^3 a \left\{ T\Lambda_{\alpha\beta\gamma\delta}\left[\nabla^\alpha\left(\frac{\delta f}{\delta \pi_\beta}\right) - \frac{1}{\rho_0 T}\nabla^\alpha\left(\frac{\pi^\beta}{\rho_0}\right)\frac{\delta f}{\delta s}\right]\left[\nabla^\gamma\left(\frac{\delta g}{\delta \pi_\delta}\right) - \frac{1}{\rho_0 T}\nabla^\gamma\left(\frac{\pi^\delta}{\rho_0}\right)\frac{\delta g}{\delta \sigma}\right] + \right.$$
$$\left. + \kappa T^2 \nabla^\alpha\left(\frac{1}{\rho_0 T}\frac{\delta f}{\delta s}\right)\nabla_\alpha\left(\frac{1}{\rho_0 T}\frac{\delta g}{\delta s}\right)\right\}.$$

The Hamiltonian Equation (40) has null semi-metric bracket with anything, while the increase of entropy is given by putting together Equation (46) and the third formula in Equation (45).

With this semi-metric bracket (which has the same definition of α, in terms of the sign) Equation (45) is reproduced by assigning the usual metriplectic dynamics $\dot{f} = \{f, H\} + \alpha(f, S)$ for any physical functional $f\left[\vec{\zeta}, \vec{\pi}, s\right]$.

3.5. Kinetic Theory

The CMS described until now are essentially formed by a Hamiltonian system of variables $\mathbf{y}$ plus a "thermal bath" of statistical variables $\mathbf{\Sigma}$, indicated as μSTDoF. The interaction between the two sub-manifolds of the phase space M is described by the metric part of the Leibniz bracket $\langle\langle .,. \rangle\rangle$ generating dynamics. In this section, instead, we will make the example of a dynamical system that has only one dynamical variable and, still, may admit either a Hamiltonian description or a metriplectic one depending on whether dissipative, time-asymmetric interactions are included or not: together with the example of Section 3.6, this case permits us to introduce a subtler distinction between the Hamiltonian and the non-Hamiltonian part of a MBA, namely the time-reversible and the time-irreversible one, crucial for the conclusions in Section 4.

The example we are going to treat here is that of *kinetic theories*.

Kinetic theories represent a system of many identical particles through the so called *Boltzmann distribution* in the μ-space [26], i.e., a single particle space $\mathbf{R}^6$ in which $f = f\left(\vec{x}, \vec{v}, t\right)$ is the probability density that a particle of the system picked at random is at the position $\vec{x}$ in the 3D space, with velocity $\vec{v}$ at time t. We are discussing the so called Vlasov–Poisson system: a gas of electrically charged particles is coupled to an electrostatic field represented by its scalar potential $\phi\left(\vec{x}\right)$ acting on the

charged particles themselves; then, the equation of motion of the μ-space distribution $f\left(\vec{x},\vec{v},t\right)$ is the Vlasov equation coupled with the Poisson equation for the electrostatic field [20]:

$$\partial_t f + \frac{\partial f}{\partial \vec{x}} \cdot \vec{v} - \frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial \phi}{\partial \vec{x}} [f] = W_{\text{coll}}[f]. \tag{48}$$

In Equation (48) the term $-\partial_{\vec{x}}\phi[f]$ represents the electrostatic field that, at any time, depends on the distribution f itself through the charge distribution in the space, while $W_{\text{coll}}[f]$ is referred to as the collision term representing the time variation of $f\left(\vec{x},\vec{v},t\right)$ due to two-particle collisions at that given point $\vec{x}$ at that time. The term $\phi[f]$ is constructed as a functional of f, rendering Equation (48) an integro-differential equation:

$$\phi\left(\vec{x},t\right) = \int d^3v' \int d^3x \prime V\left(\left|\vec{x}-\vec{x}'\right|\right) f\left(\vec{x}',\vec{v}',t\right).$$

The kernel $V\left(\left|\vec{x}-\vec{x}'\right|\right)$ is simply the electrostatic potential in a point $\vec{x}$ due to the presence of the point particle at the position $\vec{x}'$: this determines the forces through which particles sense each other.

Kinetic theory states that equations such as Equation (48) are non-time-reversible due to the term $W_{\text{coll}}[f]$, which in practice represents the dissipative term giving rise to the increase of entropy (the so-called Boltzmann's H-Theorem): the beautiful thing is that, in the collisionless limit $W_{\text{coll}}[f] \to 0$, what remains of Equation (48), i.e.,

$$\partial_t f + \frac{\partial f}{\partial \vec{x}} \cdot \vec{v} - \frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial \phi}{\partial \vec{x}} [f] = 0, \tag{49}$$

is a Hamiltonian dynamical system [7], while the collisional Vlasov–Poisson system (48) is represented as a CMS. In practice, the dissipative term in Equation (48), i.e., the two-particle collision term, does determine the non-Hamiltonian, semi-metric contribution to the dynamical system and, needless to say, is moved via a symmetric bracket by an entropic functional.

Consider, first of all, the functional

$$H[f] = \int d^3x \int d^3v f\left(\vec{x},\vec{v}\right) \frac{mv^2}{2} + \frac{1}{2}\int d^3x \int d^3v \phi\left(\vec{x},t\right) f\left(\vec{x},\vec{v}\right) \tag{50}$$

and the Leibniz functional bracket

$$\{A[f],B[f]\} = \int d^3x \int d^3v f\left(\vec{x},\vec{v}\right)\left(\partial_{\vec{x}}\frac{\delta A}{\delta f\left(\vec{x},\vec{v}\right)} \cdot \partial_{\vec{v}}\frac{\delta B}{\delta f\left(\vec{x},\vec{v}\right)} - \partial_{\vec{x}}\frac{\delta B}{\delta f\left(\vec{x},\vec{v}\right)} \cdot \partial_{\vec{v}}\frac{\delta A}{\delta f\left(\vec{x},\vec{v}\right)}\right): \tag{51}$$

not only is $\{.,.\}$ in Equation (51) a Poisson bracket satisfying the Leibniz property, anti-symmetry, and Jacobi identity; it also generates the collisionless Equation (49) once the Hamiltonian functional $H[f]$ defined in Equation (50) is made use of:

$$\partial_t f\left(\vec{x},\vec{v}\right) = \left\{f\left(\vec{x},\vec{v}\right), H[f]\right\}.$$

In order to turn on collisions, and then obtain the Equation (48), one resorts Boltzmann's entropy

$$S[f] = -k\int d^3x \int d^3v f\left(\vec{x},\vec{v}\right) \ln f\left(\vec{x},\vec{v}\right): \tag{52}$$

as the symmetric, semi-metric functional bracket

$$\begin{aligned}&(A[f],B[f])=\\&=-\tfrac{L}{2\alpha}\int d^3x\int d^3v\int d^3x'\int d^3v'\left(\tfrac{\partial}{\partial v^i}\tfrac{\delta A}{\delta f\left(\vec{x},\vec{v}\right)}-\tfrac{\partial}{\partial v'^i}\tfrac{\delta A}{\delta f\left(\vec{x}',\vec{v}'\right)}\right)*\\&\left(\tfrac{\partial}{\partial v^i}\tfrac{\delta B}{\delta f\left(\vec{x},\vec{v}\right)}-\tfrac{\partial}{\partial v'^i}\tfrac{\delta B}{\delta f\left(\vec{x}',\vec{v}'\right)}\right)f\left(\vec{x},\vec{v}\right)f\left(\vec{x},\vec{v}'\right)\left[\tfrac{\delta^{ij}}{\left|\vec{v}-\vec{v}'\right|}-\tfrac{\left(v^i-v'^i\right)\left(v^j-v'^j\right)}{\left|\vec{v}-\vec{v}'\right|^3}\right]\delta^3\left(\vec{x}-\vec{x}'\right)\end{aligned}\tag{53}$$

is defined, the collisional term may be obtained as

$$W_{\text{coll}}\left[f;\vec{x},\vec{v}\right)=\alpha\left(f\left(\vec{x},\vec{v}\right),S[f]\right),$$

provided the collisional term $W_{\text{coll}}\left[f;\vec{x},\vec{v}\right)$ is assumed to have the form [20]:

$$\begin{aligned}&W_{\text{coll}}\left[f;\vec{x},\vec{v}\right)=\\&=L\tfrac{\partial}{\partial v^i}\left\{\int d^3v'\left[\tfrac{\delta^{ij}}{\left|\vec{v}-\vec{v}'\right|}-\tfrac{\left(v^i-v'^i\right)\left(v^j-v'^j\right)}{\left|\vec{v}-\vec{v}'\right|^3}\right]\left[f\left(\vec{x},\vec{v}\right)\tfrac{\partial f\left(\vec{x},\vec{v}'\right)}{\partial v'^j}-f\left(\vec{x},\vec{v}'\right)\tfrac{\partial f\left(\vec{x},\vec{v}\right)}{\partial v^j}\right]\right\}.\end{aligned}$$

Considering the full free energy functional and the MBA constructed through Equations (51) and (53)

$$F[f]=H[f]+\alpha S[f]\ ,\quad\langle\langle A,B\rangle\rangle=\{A,B\}+(A,B),\tag{54}$$

Equation (48) is finally reproduced:

$$\partial_t f\left(\vec{x},\vec{v}\right)=\left\langle\left\langle f\left(\vec{x},\vec{v}\right),F[f]\right\rangle\right\rangle\tag{55}$$

(consider that the Hamiltonian is a null mode of the semi-metric component $(H[f],.)=0$, and the entropy is a Casimir of the Poisson component $\{S[f],.\}=0$).

Something more should be stressed about Equation (55), to hint at concluding remarks. As cleverly shown in [20], Equation (55) may be adapted to make the system relax into different equilibrium field configurations $f_0\left(\vec{x},\vec{v}\right)$, and the "tailoring" must be done on the entropic functional S in Equation (52): since any functional of the form $S[f]=\int d^3x\int s\left(f\left(\vec{x},\vec{v}\right)\right)d^3v$ is a Casimir of the Poisson bracket in Equation (51), to each function $s\in C^\infty(\mathbf{R},\mathbf{R})$ there will correspond an MBA, relaxing to a suitable $f_0\left(\vec{x},\vec{v}\right)$ thanks to the action of S through $\alpha(.,S[f])_s$; $(.,.)_s$ is a semi-metric functional bracket "tailored" on function s. For instance, the S in Equation (52) leads to the Boltzmann equilibrium with absolute temperature $T=-\alpha$, while if the function $s(f)=-k[f\ln f+(1-f)\ln(1-f)]$ were used, with a suitably adapted bracket $(.,.)_s$ (see [20] for details), then $f_0\left(\vec{x},\vec{v}\right)$ would be a Fermi–Dirac one, still with absolute temperature given by the constant α appearing in (54) as $T=-\alpha$. This observation may lead to the argument that the space of Boltzmann distributions $L_1(\mu)$ contains several possible equilibria, around each one the general dynamics of f is well approximated by the proper MBA with the proper s.

The other observation to be made on Equations (54) and (55) is that the CMS at hand has one field variable only, i.e., the distribution f, so one could wonder where the "Hamiltonian" sub-system (i.e., $\mathbf{y}$) and the μSTDoF (i.e., $\mathbf{\Sigma}$) are. The answer, contained in $\dot{A}[f]=\{A[f],H[f]\}+\alpha(A,S[f])$, is that indeed there does not exist any frictionless sub-system and any μSTDoF draining "ordered" energy from it, there are not fundamental sub-systems: rather, there exist fundamental algebraic sub-structures, i.e., components of the algebraic structure of dynamics, namely $\{.,.\}$ and $(.,.)$, apparently giving rise to

deeply different behaviors with respect to time flow [27], i.e., "eternal perfect conservative Hamiltonian" and "ageing time-irreversible entropic" evolutions.

3.6. Morrison's Rotator

What is referred to as Morrison's rotator here is a toy model very similar, in many aspects, to the kinetic metriplectic theory described in Section 3.5; because the system has only one dynamic variable, namely the angular momentum $\vec{L} \in \mathbf{R}^3$ of a rigid body of the Newtonian physics, let us refer to the phase of the system as $\mathbf{R}^3_L$.

The free rigid rotator has equations of motion, written in terms of the components of $\vec{L}$ in the principal inertial coordinate system, given by:

$$\dot{L}^{\,i} = \frac{\varepsilon^{i}{}_{jk} L_j L_k}{I^{(i)}}, \tag{56}$$

where $I^{(a)}$ is the momentum of inertia with respect to the a-th principal axis, along which the component of $\vec{L}$ reads L^a. The symbol $\varepsilon^{i}{}_{jk}$ is the partially contravariant form of Ricci $SO(3)$-tensor. The ODEs in Equation (56) are straightforwardly reproduced by the Poisson bracket,

$$\{f, g\} = -\varepsilon_i{}^{jk} L^i \frac{\partial f}{\partial L^j} \frac{\partial g}{\partial L^k}, \tag{57}$$

provided that the mechanical energy of the rigid rotator $\frac{1}{2} \vec{L} \cdot \vec{\omega}$ is used as a Hamiltonian, being $\vec{\omega}$ the angular velocity of the system, with $L^a = I^{(a)} \omega^a$. If the diagonal tensor of inertia $\sigma^{ab} = I^{(a)} \delta^{ab}$ is used, the Hamiltonian giving Equation (56) thanks to Equation (57) reads:

$$H\left(\vec{L}\right) = \tfrac{1}{2} \left(\sigma^{-1}\right)_{ab} L^a L^b, \quad \dot{L}^{\,i} = \left\{L^i, H\right\}. \tag{58}$$

In order to produce a CMS based on the Hamiltonian model in Equation (58), of course physical friction could be added, for instance considering the rotation energy dissipation due to the μSTDoF of a viscous fluid through which the rigid body is rotating. However, another way is possible, presented originally in [20] as far as the author knows, and henceforward referred to as Morrison's rotator, from the name of the author of [20]; indeed, since any function $W\left(L^2\right)$, being $L^2 \equiv \vec{L} \cdot \vec{L}$, is a Casimir of the Poisson bracket defined in Equation (57), hence any function

$$F\left(\vec{L}\right) = H\left(\vec{L}\right) + \alpha W\left(L^2\right) \tag{59}$$

may be used as free energy to build up an MBA, once the suitable semi-metric tensor G_{ij}, and a consequent semi-metric bracket $(.,.)$ are constructed:

$$G_{ij}\left(\vec{L}\right) = \frac{1}{\alpha}\left(\left|\partial_{\vec{L}} H\right|^2 \delta_{ij} - \frac{\partial H}{\partial L^i}\frac{\partial H}{\partial L^j}\right), \quad (f, g) = G^{ij} \frac{\partial f}{\partial L^i} \frac{\partial g}{\partial L^j}. \tag{60}$$

In Equation (60) the projector orthogonal to the $\vec{L}$-gradient of H has been used to construct G, so it is clear that $(f, H) = 0$ for any f. All in all, the right brackets to prepare a MBA with Equation (59) as the free energy are ready:

$$\langle\langle f, g \rangle\rangle = \{f, g\} + (f, g), \quad \dot{f}\left(\vec{L}\right) = \left\langle\left\langle f\left(\vec{L}\right), F\left(\vec{L}\right)\right\rangle\right\rangle \quad \forall \quad f \in C^\infty\left(\mathbf{R}^3, \mathbf{R}\right). \tag{61}$$

The CMS of dynamics in Equation (61) represents a free rigid rotator that relaxes to become aligned with one of its axes of inertia: it may be shown that the steady points of the system with free energy in Equation (59) are the configurations with $\vec{L}$ aligned along one principal axis of inertia, i.e.,

with only one component: these are actually configurations in which the rigid rotator spins freely, without dissipation. Again, as was already happening for the kinetic theory in Section 3.5, there exist no macroscopic degrees of freedom the ordered energy of which is swallowed by the disordered microscopic degrees of freedom giving friction: rather, Equation (61) represents an *algebrized dynamical system* with a "time-reversible" symplectic part and an "ageing" metric part. It is pretty clear that this MBA works in the same way as a CMS with friction; of course, the delicate point is to understand what the function $W\left(L^2\right)$ of Equation (59) has to do physically with the entropies seen until now. Morrison underlines how on the one hand constant $W\left(L^2\right)$ manifolds in $\mathbf{R}^3_L$ are symplectic leaves as the S-constant manifolds were in the other MBAs; on the other hand, how L^2 "counts the degeneracy" of the spinning states, grossly as entropy does [28].

4. Conclusions

In the review an attempt is made to present in a didactic way how, extending symplectic brackets suitably, one is able to describe systems that relax to asymptotic equilibria. The reader interested in the subject can read further in the references reported here, in particular [12,13] and [8–10]. The stress in this paper is rather put on the physics–algebra interplay that apparently changes example by example, while highlighting the common features of all the systems examined, possibly pointing towards new interpretations of dissipative processes.

After having gone through this review, the reader should retain a few important facts, listed here.

1. (At least some) classical dynamical systems with dissipation may be formulated in terms of bracket algebra, through a generalization of the symplectic brackets of Hamiltonian systems: this is, the metriplectic bracket formalism. In bracket algebra formalism the symmetry properties are much more under control and may be exploited, as we explained in some detail in Section 2.
2. The bracket algebra associated with a classical dissipative complete system is composed of the Poisson bracket and the Hamiltonian, which describe "what the system is made like" (i.e., what are its degrees of freedom and dynamical variables, what are its fundamental time scales, the core of its phase space structure), and the metric brackets and the entropy, which describe "what the system ages like", relaxing to an asymptotically stable state. This kind of representation is referred to as a complete metriplectic system. From the point of view exposed in [17], the symplectic part of an MBA describes *the being* of the system, while the metric one encodes *its becoming*.
3. The metric component giving rise to dissipation is crucially symmetric and positive (semi-) definite: this fact does allow irreversible motion to take place, and entropy to grow.
4. CMS may have their dissipative component either originated from the interaction of a "macroscopic" Hamiltonian system with microscopic degrees of freedom (usually treated statistically), giving rise to "friction" (examples in Sections 3.3 and 3.4); or from a "postulated" non-Hamiltonian interaction among dynamical variables already involved in the Hamiltonian component (examples in Sections 3.5 and 3.6).
5. In the cases studied in Sections 3.3 and 3.4 (dissipation with friction) and in Section 3.5 (dissipation due to particle collisions), the observable responsible for the non-Hamiltonian, metric part of the dynamical algebra is easily interpreted as the entropy of the system. In the system discussed in Section 3.6, the interpretation of the same quantity is not completely clear, at least to the author.

The author is sure of the fact that, continuing with the development and application of MBA, the role of dissipation in fundamental physics, even at an elementary particle level, will be clarified; moreover, the algebraic nature of CMS is a blatant invitation to match what we learn at a classical level with the quantum world. In this regard it is intuitive to stress the analogy between Equation (26) and the Lindblad–Kossakowski equations of open quantum systems [15], where part of the motion is determined by the Hamiltonian operator through the anti-symmetric commutator algebra, and part by the Lindbladian operators, showing a symmetric structure. Symmetry and semi-definiteness are also what appear to mimic the non-conservative part of quantum dynamics: as a final note, let us underline

how these classical–quantum analogies appear to show up clearly after the algebrization of dynamics, but would remain "hidden" without this algebraic reformulation.

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