An analytical technique to construct exact models of the distribution of the potential in electrostatic tripolar spikes

L. Nocera

Theoretical Plasma Physics, IPCF-CNR, Via Moruzzi 1, I-56124 Pisa, Italy

Abstract

In this report, we construct a detailed model for the profile of the electric potential associated with a tripolar spike. Starting from the morphology of the spike, we first give local differential forms of such profile about its two extrema and at infinity. Then we analytically continue each differential form to finite distance from the extrema and, using the uniqueness of the analytical continuation, we finally give a global differential form for the profile, holding over the whole real axis. In the limit of small potential amplitudes, the differential equation associated with this form is then solved, up to second order, in terms of elementary functions. Simple analytical models result which are in excellent agreement with observations.

Key words: plasmas, waves and oscillations, electrostatic tripolar regions, isolated electrostatic structures, solitary waves, double layers, electron and ion holes

PACS: 52.35. g, 52.35.Mw, 52.35.Sb, 52.35.Fp

Email address: `nocera@ipcf.cnr.it` (L. Nocera).

URL: `http://mizar.ipcf.cnr.it/~tpp/` (L. Nocera).

1 Introduction

Electrostatic tripolar spikes are widely observed in fully ionised, collisionless plasmas (cf. e.g. Refs. [1–4]). The spatial waveform of the electric potential within the spike — say $\phi(x)$, conceived as a function of the spatial coordinate x — behaves much in the same way as in solitary waves: it streams in the plasma along the x coordinate at a constant speed, without any apparent distortion of its shape. However, unlike solitary waves, $\phi(x)$ shows a distinctive lack of symmetry. More specifically (cf. Fig. 1): (a) it has two extrema (one absolute minimum and one relative maximum); (b) it displays a distinctive skew about these extrema; (c) it shows an asymptotic, and apparently exponential behaviour as $x \to \pm\infty$; (d) its asymptotic value as $x = -\infty$ in general differs from its asymptotic value as $x = +\infty$; (e) its asymptotic value as $x = -\infty$ in general differs from its values at the extrema; (f) its asymptotic value as $x = +\infty$ in general differs from its values at the extrema. In earlier experiments and observations, structures generally reported as double layers also show the typical three layered distribution of their electric potential waveform (cf. e.g. Refs. [5,6]); these structures may also be described by properties (a)-(f) above.

Analytical models for the waveform of the electric potential associated with asymmetric electron and ion holes and with non monotonic double layers in collisionless plasmas were considered e.g. in Refs. [7–11]. These models use ad hoc velocity distribution functions of the electrons and/or of the ions sustaining the holes or double layers. These distributions usually consist of piecewise combinations of possibly shifted Gaussian velocity distribution functions. The electron and ion electric charge densities resulting from these distributions are then inserted into Poisson's equation. Finally, this equation is solved in favour of the electric potential. These models reproduce some, but not all of the properties of the electric potential waveform of a tripolar spike, as listed in items (*a*)–(*f*) above: specifically, properties (*e*) and (*f*) appear to be a stumbling block for the models so far attempted.

On the other hand, general mathematical models of nonlinear waves occurring in physical systems have been devised, based on non trivial solitary solutions of certain types of classical partial differential equations and on advanced methods to solve them (cf. e.g. Refs. [12,13]). Again, the reported analytical solutions of this type do not succeed in reproducing the observed morphological properties of the potential waveform of the tripolar spike. It is so far unknown whether more elaborated models of this type, worked out with the assistance of computer algebra (cf. e.g. Ref. [12]), might succeed.

It may thus be concluded that the above mentioned approaches do not produce a satisfactory analytical formula giving the electric potential $\phi(x)$ as a function of x, which fits the observed electric potential of a tripolar spike. On the other hand, the reconstruction of such a formula is undoubtedly desirable, both in its own right and also for practical use. The search for such analytical representation of the potential waveform motivates our present undertaking.

Technically, we reduce this task to the construction of a differential equation for the potential $\phi(x)$. However, unlike the above mentioned approaches, we do that without making any reference to the velocity distribution functions of the electrons and of the ions; least of all do we refer to the Vlasov-Poisson system of equations governing these distributions. In so doing, we are also advised by our recent result that the velocity distributions of electrons and ions, which solve these equations and which sustain the tripolar spike, may be singular (cf. Ref. [14]), and they hardly admit a representation in terms of simple functions.

Our approach is to rather construct the desired differential equation for $\phi(x)$ based on the morphological properties (a)–(f) given above and on the sole additional assumption that $\phi(x)$ be an analytic function of position. One obvious advantage of basing our analysis on those morphological properties, rather than on a conjecture on the electron and ion velocity distribution functions, is that those properties are clear observational facts. The qualitative nature of these properties also allows for a fair degree of generality in our treatment, which ensures that the differential equation we work out governs a wide class of potential waveforms. In particular, this equation holds for arbitrary values of the potential amplitude. We show that, when this amplitude is suitably small, the differential equation may be solved by quadrature up to fourth order.

In particular the second order solutions of that equation are described by a simple analytical formula, giving $\phi(x)$ in terms of elementary functions of x. This formula easily reproduces the potential waveforms associated with electron holes, ion holes, monotonic and non monotonic double layers and indeed tripolar spikes. In the latter case, we show that even such simplified solutions are in excellent agreement with observations.

2 Asymptotic differential laws

In tripolar spikes, the average, self-consistent electric potential Φ is a detailed experimental datum emerging from the plasma. In the following, we show that a careful inspection and a judicious functional analysis of its waveform can lead to the full reconstruction of the space distribution of the potential, without the assistance of any specific model.

To do so we assume that the tripolar spike occurs in a fully ionised plasma and that the electric potential Φ of the spike depend on one rectilinear coordinate X only, which ranges from $-\infty$ to $+\infty$. We also assume that, as $X \to \infty$, the electron density and kinetic temperature respectively approach the values $n_{\mathrm{e}\infty}$ and $T_{\mathrm{e}\infty}$. Then, using Gaussian units and energetic units for the kinetic temperature and denoting by $-|e|$ the charge of the electron, we introduce the electron Debye length

$$\lambda_{\mathrm{De}} = \surd\{T_{\mathrm{e}\infty}/[4\pi e^2 n_{\mathrm{e}\infty}]\}, \tag{2.1}$$

the normalised coordinate

$$x = X/\lambda_{\mathrm{De}} \tag{2.2}$$

and the normalised electric potential

$$\phi(x) = |e|\Phi(\lambda_{\mathrm{De}} x)/T_{\mathrm{e}\infty}. \tag{2.3}$$

The morphology of the potential waveform within the tripolar spike, as emerging from observations (cf. e.g. Refs. [1–3]), is characterised as follows:

$$\phi(x) \text{ exponentially approaches } \lim_{x\to+\infty} \phi(x) \text{ for } x \to +\infty. \tag{2.4a}$$
$$\phi(x) \text{ has an absolute minimum at } x = x_{\min}\,, \tag{2.4b}$$
$$\phi(x) \text{ has a relative maximum at } x = x_{\max} < x_{\min}\,, \tag{2.4c}$$
$$\phi(x) \text{ exponentially approaches } \lim_{x\to-\infty} \phi(x) \text{ for } x \to -\infty. \tag{2.4d}$$

In this section, we consider the approximate behaviour of the potential waveform $\phi(x)$ about the two extrema $x = x_{\min}$ and $x = x_{\max}$ and at the lower

($x \to -\infty$) and upper ($x \to +\infty$) boundaries of the tripolar spike, as described by Eq. (2.4). Throughout the remaining part of our analysis, we make the sole additional assumption that the potential waveform $\phi(x)$ be an analytic function of x.

Our first task will be to analyse the potential waveform for $x \to +\infty$. There, we introduce an asymptotic decay scale k^{-1}, and we assume that $\phi(x)$ asymptotically behave as the superposition of a possibly infinite series of decaying exponential functions (cf. Eq. (2.4a)): $\exp(-kx)$, $\exp(-2kx)$, $\exp(-3kx)$, $\ldots$. Specifically, denoting by a " $'$ " differentiation with respect to x and by $p_{+\infty}(\exp(-kx))$ a series of terms whose order, for $x \to +\infty$, is smaller than $\exp(-kx)$, we write

$$\phi(x) = \lim_{x\to+\infty} \phi(x) - \{\lim_{x\to+\infty} [\exp(+kx)\phi'(x)]/k\} \exp(-kx) + p_{+\infty}(\exp(-kx)), \tag{2.5a}$$

$$\phi'(x) = k\{\lim_{x\to+\infty} [\exp(+kx)\phi'(x)]/k\} \exp(-kx) + [p_{+\infty}(\exp(-kx))]', \tag{2.5b}$$

$$\phi''(x) = -k^2\{\lim_{x\to+\infty} [\exp(+kx)\phi'(x)]/k\} \exp(-kx) + [p_{+\infty}(\exp(-kx))]'', \tag{2.5c}$$

$$k > 0, \ \{\lim_{x\to+\infty} [\exp(+kx)\phi'(x)]/k\} > 0. \tag{2.5d}$$

In this way, the behaviour of $\phi(x)$ as $x \to +\infty$, is akin to that of the hyperbolic functions $\tanh(kx/2)$ and $\mathrm{sech}(kx/2)$.

Now, in the domain $x_{\min} < x < +\infty$, where $\phi(x)$ is an analytic and strictly monotonic function of x (cf. Fig. 1), Eq. (2.5a) may be inverted by Lagrange's inversion theorem (cf. e.g. Ref. [15]), thus giving $\exp(-kx)$ as an analytic function of ϕ. Specifically, denoting by $P_{+\infty}([\lim_{x\to+\infty} \phi(x) - \phi])$ a series of terms whose order, for $\phi \simeq \lim_{x\to+\infty} \phi(x)$, is smaller than $|\lim_{x\to+\infty} \phi(x) - \phi|$, we have

$$\exp(-kx) = \{k/\lim_{x\to+\infty}[\exp(kx)\phi'(x)]\}[\lim_{x\to+\infty}\phi(x)-\phi] + P_{+\infty}([\lim_{x\to+\infty}\phi(x)-\phi]). \tag{2.6}$$

Then, we denote by $\phi_x(\phi)$ and $\phi_{xx}(\phi)$, or simply ϕ_x and ϕ_{xx}, the functional relations which result from the substitution of $\exp(-kx)$, given in Eq. (2.6), respectively into Eqs. (2.5b) and (2.5c) and which give ϕ' and ϕ'' as functions of ϕ. Finally, denoting by $R^{(1)}_{+\infty}([\lim_{x\to+\infty}\phi(x)-\phi])$ and $R^{(2)}_{+\infty}([\lim_{x\to+\infty}\phi(x)-\phi])$ the series of all the terms, respectively resulting from these substitutions, and whose order, for $\phi \simeq \lim_{x\to+\infty}\phi(x)$, is smaller than $|\lim_{x\to+\infty}\phi(x)-\phi|$, we respectively rewrite Eqs. (2.5b) and (2.5c) as

$$\phi_x(\phi) = k[\lim_{x\to+\infty}\phi(x)-\phi] + R^{(1)}_{+\infty}([\lim_{x\to+\infty}\phi(x)-\phi]), \tag{2.7a}$$

$$\phi_{xx}(\phi) = -k^2[\lim_{x\to+\infty}\phi(x)-\phi] + R^{(2)}_{+\infty}([\lim_{x\to+\infty}\phi(x)-\phi]), \tag{2.7b}$$

for $\phi \simeq \lim_{x\to+\infty}\phi(x)$ and $x \to +\infty$.

Our next task is to analyse the asymptotic behaviour of the potential waveform in the neighbourhood of $x = x_{\min}$, the position where $\phi(x)$ has a minimum (cf. Fig. 1 and Eq. (2.4b)). There, denoting by $p_{\min}([x-x_{\min}])$ a series of terms whose order, for $x \simeq x_{\min}$, is smaller than $|x-x_{\min}|^3$, we have

$$\phi(x) = \phi(x_{\min}) + [\phi''(x_{\min})/2](x-x_{\min})^2 + [\phi'''(x_{\min})/6](x-x_{\min})^3 + p_{\min}([x-x_{\min}]), \tag{2.8a}$$

$$\phi'(x) = \phi''(x_{\min})(x-x_{\min}) + [\phi'''(x_{\min})/2](x-x_{\min})^2 + [p_{\min}([x-x_{\min}])]', \tag{2.8b}$$

$$\phi''(x) = \phi''(x_{\min}) + \phi'''(x_{\min})(x-x_{\min}) + [p_{\min}([x-x_{\min}])]'', \tag{2.8c}$$

$$\phi''(x_{\min}) > 0. \tag{2.8d}$$

Now, in each of the domains $x_{\rm max} < x < x_{\rm min}$ and $x_{\rm min} < x < +\infty$, where $\phi(x)$ is an analytic and strictly monotonic function of x (cf. Fig. 1), Eq. (2.8a) can be inverted by Lagrange's inversion theorem (cf. e.g. Ref. [15]), thus giving x as an analytic function of ϕ. Specifically, we denote by $\surd\xi \geq 0$ the non-negative arithmetic square root of a non-negative real quantity ξ and we introduce the multi-valued function

$$[\phi - \phi(x_{\rm min})]^{1/2} = +\surd[\phi - \phi(x_{\rm min})], \text{ for } x \geq x_{\rm min}, \tag{2.9a}$$
$$[\phi - \phi(x_{\rm min})]^{1/2} = -\surd[\phi - \phi(x_{\rm min})], \text{for } x < x_{\rm min}. \tag{2.9b}$$

Then, denoting by $P_{\rm min}([\phi - \phi(x_{\rm min})]^{1/2})$ a series of terms whose order, for $\phi \simeq \phi(x_{\rm min})$, is smaller than $\surd|\phi - \phi(x_{\rm min})|$, we have

$$\begin{aligned} & x - x_{\rm min} = \surd[2/\phi''(x_{\rm min})][\phi - \phi(x_{\rm min})]^{1/2} + \\ & P_{\rm min}([\phi - \phi(x_{\rm min})]^{1/2}), \\ & \text{for } \phi \simeq \phi(x_{\rm min}). \end{aligned} \tag{2.10}$$

Finally, introducing the third order skew of the potential waveform at $x = x_{\rm min}$ (cf. Ref. [14])

$$A_0 = \phi'''(x_{\rm min})/\phi''(x_{\rm min}), \tag{2.11}$$

and denoting by $R^{(1)}_{\rm min}([\phi - \phi(x_{\rm min})]^{1/2})$ and $R^{(2)}_{\rm min}([\phi - \phi(x_{\rm min})]^{1/2})$ the series of all the terms, respectively resulting from the substitution of Eq. (2.10) into Eqs. (2.8b) and (2.8c), and whose order, for $\phi \simeq \phi(x_{\rm min})$, is smaller than $\surd|\phi - \phi(x_{\rm min})|$, we respectively rewrite Eq. (2.8b) and (2.8c) as

$$\phi_x(\phi) = \sqrt{[2\phi''(x_{\text{min}})]}[\phi - \phi(x_{\text{min}})]^{1/2} + R^{(1)}_{\text{min}}([\phi - \phi(x_{\text{min}})]^{1/2}), \tag{2.12a}$$

$$\phi_{xx}(\phi) = \phi''(x_{\text{min}}) + A_0\sqrt{[2\phi''(x_{\text{min}})]}[\phi - \phi(x_{\text{min}})]^{1/2} + R^{(2)}_{\text{min}}([\phi - \phi(x_{\text{min}})]^{1/2}), \tag{2.12b}$$

for $\phi \simeq \phi(x_{\text{min}})$.

Our next task is to analyse the asymptotic behaviour of the potential ϕ in the neighbourhood of $x = x_{\text{max}}$, the position where $\phi(x)$ has a relative maximum (cf. Fig. 1 and Eq. (2.4c)). Noticing that $\phi''(x) < 0$ at a maximum, and denoting by $p_{\text{max}}([x - x_{\text{max}}])$ a series of terms whose order, for $x \simeq x_{\text{min}}$, is smaller than $|x - x_{\text{max}}|^3$, we have

$$\phi(x) = \phi(x_{\text{max}}) + [\phi''(x_{\text{max}})/2](x - x_{\text{max}})^2 + [\phi'''(x_{\text{max}})/6](x - x_{\text{max}})^3 + p_{\text{max}}([x - x_{\text{max}}]), \tag{2.13a}$$

$$\phi'(x) = \phi''(x_{\text{max}})(x - x_{\text{max}}) + [\phi'''(x_{\text{max}})/2](x - x_{\text{max}})^2 + [p_{\text{max}}([x - x_{\text{max}}])]', \tag{2.13b}$$

$$\phi''(x) = \phi''(x_{\text{max}}) + \phi'''(x_{\text{max}})(x - x_{\text{max}}) + [p_{\text{max}}([x - x_{\text{max}}])]'', \tag{2.13c}$$

$$\phi''(x_{\text{max}}) < 0.$$

Now, in each of the two domains $-\infty < x < x_{\text{max}}$ and $x_{\text{max}} < x < x_{\text{min}}$, where $\phi(x)$ is an analytic and strictly monotonic function of x (cf. Fig. 1), Eq. (2.13a) may be inverted by Lagrange's inversion theorem (cf. e.g. Ref. [15]), thus giving x as an analytic function of ϕ. Specifically, we introduce the multi-valued function

$$[\phi(x_{\text{max}}) - \phi]^{1/2} = +\sqrt{[\phi(x_{\text{max}}) - \phi]}, \text{ for } x \geq x_{\text{max}}, \tag{2.14a}$$

$$[\phi(x_{\text{max}}) - \phi]^{1/2} = -\sqrt{[\phi(x_{\text{max}}) - \phi]}, \text{ for } x < x_{\text{max}}. \tag{2.14b}$$

Then, denoting by $P_{\text{max}}([\phi(x_{\text{max}}) - \phi]^{1/2})$ a series of terms whose order, for

$\phi(x) \simeq \phi(x_{\max})$, is smaller than $\sqrt{|\phi(x_{\max}) - \phi|}$, we have

$$x - x_{\max} = \sqrt{[-2/\phi''(x_{\max})]}[\phi(x_{\max}) - \phi(x)]^{1/2} + P_{\max}([\phi(x_{\max}) - \phi]^{1/2}),$$
$$\text{for } \phi \simeq \phi(x_{\max}) \text{ and } x \simeq x_{\max}. \quad (2.15)$$

Finally, introducing the third order skew of the potential waveform at $x = x_{\max}$ (cf. Ref. [14])

$$B_0 = \phi'''(x_{\max})/\phi''(x_{\max}), \quad (2.16)$$

and denoting by $R^{(1)}_{\max}([\phi(x_{\max}) - \phi]^{1/2})$ and $R^{(2)}_{\max}([\phi(x_{\max}) - \phi]^{1/2})$ the series of all the terms, respectively resulting from the substitution of Eq. (2.15) into Eq. (2.13b) and Eq. (2.13c), and whose order, for $\phi(x) \simeq \phi(x_{\max})$, is smaller than $\sqrt{|\phi(x_{\max}) - \phi|}$, we respectively rewrite Eq. (2.13b) and (2.13c) as

$$\phi_x(\phi) = -\sqrt{[-2\phi''(x_{\max})]}[\phi(x_{\max}) - \phi]^{1/2} + R^{(1)}_{\max}([\phi(x_{\max}) - \phi]^{1/2}), \quad (2.17a)$$
$$\phi_{xx}(\phi) = \phi_{xx}(x_{\max}) - B_0\sqrt{[-2\phi''(x_{\max})]}[\phi(x_{\max}) - \phi]^{1/2} + R^{(2)}_{\max}([\phi(x_{\max}) - \phi]^{1/2}), \quad (2.17b)$$
$$\text{for } \phi \simeq \phi(x_{\max}), \text{ and } x \simeq x_{\max}.$$

Last, we analyse the asymptotic behaviour of the potential waveform as $x \to -\infty$. There, denoting by k^{-1} the same asymptotic decay scale introduced above for $x \to +\infty$ (cf. Eq. 2.5), we assume that $\phi(x)$ asymptotically behave as the superposition of a possibly infinite series of decaying exponential functions: $\exp(kx)$, $\exp(2kx)$, $\exp(3kx)$, $\ldots$. Specifically, denoting by $p_{-\infty}(\exp(kx))$ a series of terms whose order, for $x \to -\infty$, is smaller than $\exp(kx)$, we write

$$\phi(x) = \lim_{x\to-\infty}\phi(x) + \{\lim_{x\to-\infty}[\exp(-kx)\phi'(x)]/k\}\exp(kx) + p_{-\infty}(\exp(kx)), \tag{2.18a}$$

$$\phi'(x) = k\{\lim_{x\to-\infty}[\exp(-kx)\phi'(x)]/k\}\exp(kx) + [p_{-\infty}(\exp(kx))]', \tag{2.18b}$$

$$\phi''(x) = k^2\{\lim_{x\to-\infty}[\exp(-kx)\phi'(x)]/k\}\exp(kx) + [p_{-\infty}(\exp(kx))]'', \tag{2.18c}$$

$$k > 0,\ \{\lim_{x\to-\infty}[\exp(-kx)\phi'(x)]/k\} > 0. \tag{2.18d}$$

Now, in the domain $-\infty < x < x_{\max}$, where $\phi(x)$ is an analytic and strictly monotonic function of x (cf. Fig. 1), Eq. (2.18a) may be inverted by Lagrange's inversion theorem (cf. e.g. Ref. [15]), thus giving $\exp(kx)$ as an analytic function of ϕ. Specifically, denoting by $P_{-\infty}([\phi - \lim_{x\to-\infty}\phi(x)])$ a series of terms whose order, for $\phi \simeq \lim_{x\to-\infty}\phi(x)$, is smaller than $|\phi - \lim_{x\to+\infty}\phi(x)|$, we have

$$\exp(kx) = \{k/\lim_{x\to-\infty}[\exp(-kx)\phi'(x)]\}[\phi - \lim_{x\to-\infty}\phi(x)] + P_{-\infty}([\phi - \lim_{x\to-\infty}\phi(x)]). \tag{2.19}$$

Finally, denoting by $R^{(1)}_{-\infty}([\phi - \lim_{x\to-\infty}\phi(x)])$ and $R^{(2)}_{-\infty}([\phi - \lim_{x\to-\infty}\phi(x)])$ the series of all the terms, respectively resulting from the substitution of $\exp(kx)$, given by Eq. (2.19), into Eqs. (2.18b) and (2.18c), and whose order, for $\phi \simeq \lim_{x\to-\infty}\phi(x)$, is smaller than $|\phi - \lim_{x\to-\infty}\phi(x)|$, we rewrite Eqs. (2.18b) and (2.18c) as

$$\phi_x(\phi) = k[\phi - \lim_{x\to-\infty}\phi(x)] + R^{(1)}_{-\infty}([\phi - \lim_{x\to-\infty}\phi(x)]), \tag{2.20a}$$

$$\phi_{xx}(\phi) = k^2[\phi - \lim_{x\to-\infty}\phi(x)] + R^{(2)}_{-\infty}([\phi - \lim_{x\to-\infty}\phi(x)]), \tag{2.20b}$$

for $\phi \simeq \lim_{x\to-\infty}\phi(x)$ and $x \to -\infty$.

In the above analysis we produced the desired local differential laws governing the electric potential profile in the neighbourhood of its extrema and at infinity (cf. Eqs. (2.7), (2.12), (2.17) and (2.20)). In Sections 3 and 4, we will extend these laws over the whole real axis.

3 The structure function of the potential

In Section 2, we derived four asymptotic differential laws relating ϕ_x and ϕ_{xx}, respectively the first and second order derivative of the potential waveform, to ϕ in the neighbourhood of each of the four points $x \to +\infty$ (cf. Eqs. (2.7a) and (2.7b)), $x = x_{\rm min}$ (cf. Eqs. (2.12a) and (2.12b)), $x = x_{\rm max}$ (cf. Eqs. (2.17a) and (2.17b)) and $x \to -\infty$ (cf. Eqs. (2.20a) and (2.20b)). These laws were given in terms of power series.

One important property of these series is their radius of convergence, which we now analyse in detail. We first consider the series labelled by the superscript "$^{(1)}$", which are involved in the representation of the first order derivative of the potential ϕ_x. Specifically, the series $R^{(1)}_{+\infty}([\lim_{x\to+\infty}\phi(x)-\phi])$, appearing on the right hand side of Eq. (2.7a), converges over the interval $0 < [\lim_{x\to+\infty}\phi(x) - \phi] < [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})]$, covered by its argument $[\lim_{x\to+\infty}\phi(x) - \phi]$, as x ranges over the domain $x_{\rm min} < x < +\infty$.

The series $R^{(1)}_{\rm min}([\phi - \phi(x_{\rm min})]^{1/2})$, appearing on the right hand side of Eq. (2.12a), has two determinations (cf. Eq. (2.9)): $R^{(1)}_{\rm min}(+\sqrt{[\phi - \phi(x_{\rm min})]})$ for $x > x_{\rm min}$, and $R^{(1)}_{\rm min}(-\sqrt{[\phi - \phi(x_{\rm min})]})$ for $x < x_{\rm min}$. The former converges over the interval $0 < \sqrt{[\phi - \phi(x_{\rm min})]} < \sqrt{[\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})]}$, covered by its argument $\sqrt{[\phi - \phi(x_{\rm min})]}$, as x ranges over the domain $x_{\rm min} < x < +\infty$. The latter converges over the interval $-\sqrt{[\phi(x_{\rm max}) - \phi(x_{\rm min})]} < -\sqrt{[\phi - \phi(x_{\rm min})]} <$

0, covered by its argument $-\surd[\phi-\phi(x_{\rm min})]$, as x ranges over the domain $x_{\rm max} < x < x_{\rm min}$. Thus, in the overall, the series $R^{(1)}_{\rm min}([\phi-\phi(x_{\rm min})]^{1/2})$ converges over the interval $-\surd[\phi(x_{\rm max})-\phi(x_{\rm min})] < [\phi-\phi(x_{\rm min})]^{1/2} < \surd[\lim_{x\to+\infty}\phi(x)-\phi(x_{\rm min})]$, covered by its argument $[\phi-\phi(x_{\rm min})]^{1/2}$, as x ranges over the domain $x_{\rm max} < x < +\infty$.

The series $R^{(1)}_{\rm max}([\phi-\phi(x_{\rm max})]^{1/2})$, appearing on the right hand side of Eq. (2.17a), also has two determinations (cf. Eq. (2.14)): $R^{(1)}_{\rm max}(+\surd[\phi(x_{\rm max})-\phi])$ for $x > x_{\rm max}$, and $R^{(1)}_{\rm max}(-\surd[\phi(x_{\rm max})-\phi])$ for $x < x_{\rm max}$. The former converges over the interval $0 < \surd[\phi(x_{\rm max})-\phi] < \surd[\phi(x_{\rm max})-\phi(x_{\rm min})]$, covered by the argument $\surd[\phi(x_{\rm max})-\phi]$, as x ranges over the domain $x_{\rm max} < x < x_{\rm min}$. The latter converges over the interval $-\surd[\phi(x_{\rm max})-\lim_{x\to-\infty}\phi(x)] < -\surd[\phi(x_{\rm max})-\phi] < 0$, covered by the argument $-\surd[\phi(x_{\rm max})-\phi]$, as x ranges over the domain $-\infty < x < x_{\rm max}$. Thus, in the overall, $R^{(1)}_{\rm max}([\phi(x_{\rm max})-\phi]^{1/2})$ converges over the interval $-\surd[\phi(x_{\rm max})-\lim_{x\to-\infty}\phi(x)] < [\phi(x_{\rm max})-\phi]^{1/2} < \surd[\phi(x_{\rm max})-\phi(x_{\rm min})]$, covered by the argument $[\phi(x_{\rm max})-\phi]^{1/2}$, as x ranges over the domain $-\infty < x < x_{\rm min}$.

Last, the series $R^{(1)}_{-\infty}([\phi-\lim_{x\to-\infty}\phi(x)])$, appearing on the right hand side of Eq. (2.20a), converges over the interval $0 < [\phi-\lim_{x\to+\infty}\phi(x)] < [\phi(x_{\rm max})-\lim_{x\to-\infty}\phi(x)]$, covered by the argument $[\phi-\lim_{x\to-\infty}\phi(x)]$, as x ranges over the domain $-\infty < x < x_{\rm max}$.

The analysis of the radius of convergence of the series which are labelled by the superscript "$^{(2)}$", and which are involved in the representation of the second order derivative ϕ_{xx} in Eqs. (2.7b), (2.12b), (2.17b) and (2.20b), gives the very same radius of convergence of the corresponding series labelled by the superscript "$^{(1)}$" analysed above and needs not be expanded further.

It is now convenient to rearrange the relations appearing in Eqs. (2.7a), (2.7b), (2.12a), (2.12b), (2.17a), (2.17b), (2.20a) and (2.20b). We first consider the relations between ϕ_x and ϕ and between ϕ_{xx} and ϕ in the domain $< x_{\min} < x < +\infty$, as respectively given in Eqs. (2.7a) and (2.7b). We introduce the potential jump (cf. Fig. 1)

$$\Delta = [\phi(x_{\max}) - \phi(x_{\min})], \tag{3.1}$$

and the function

$$s(\phi(x)) = +\surd\{\surd\Delta + \surd[\phi(x) - \phi(x_{\min})]\},$$
$$\text{for } x_{\min} < x < +\infty. \tag{3.2}$$

We notice that, since in the domain $x_{\min} < x < +\infty$, $\phi(x)$ is a monotonically increasing function of x (cf. Fig. 1), then $s(\phi(x))$ monotonically increases there and, introducing its limit value

$$\lim_{x\to+\infty} s(\phi(x)) = Z =$$
$$\surd\{\surd\Delta + \surd[\lim_{x\to+\infty} \phi(x) - \phi(x_{\min})]\} \geq 0, \tag{3.3}$$

we have

$$\sqrt[4]{\Delta} < s(\phi) < Z,$$
$$\text{for } x_{\min} < x < +\infty. \tag{3.4}$$

Next, we consider the series given by the difference

$$Q_{+\infty} = \{\lim_{x\to+\infty} \phi(x) - \phi\} - \{\, 4Z[Z^2 - \surd\Delta][Z - s(\phi)]\}, \tag{3.5}$$

and, denoting by $q_{+\infty}([\lim_{x\to+\infty} \phi(x) - \phi])$ a series of terms whose order, for $\phi \simeq \lim_{x\to+\infty} \phi(x)$, is smaller than $|\lim_{x\to+\infty} \phi(x) - \phi|^2$, we notice that

$$Q_{+\infty} = -\{[3Z^2 - \sqrt{\Delta}]/[8Z^2(Z^2 - \sqrt{\Delta})^2]\}[\lim_{x\to+\infty} \phi(x) - \phi]^2 +$$
$$q_{+\infty}([\lim_{x\to+\infty} \phi(x) - \phi]),$$
$$\text{for } \phi \simeq \lim_{x\to+\infty} \phi(x). \tag{3.6}$$

Therefore, if, in Eq. (2.7a), we replace the quantity $[\lim_{x\to+\infty} \phi(x) - \phi]$ by the expression $[\lim_{x\to+\infty} \phi(x) - \phi] = 4Z[Z^2 - \sqrt{\Delta}][Z - s(\phi)] + Q_{+\infty}$, resulting from Eq. (3.5), the two series $R^{(1)}_{+\infty}([\lim_{x\to+\infty} \phi(x) - \phi])$, appearing on the right hand side of Eq. (2.7a), and $Q_{+\infty}$ may be incorporated in a single series, which we call $S^{(1)}_{+\infty}$, containing terms whose order, for $\phi \simeq \lim_{x\to+\infty} \phi(x)$, is smaller than $|\lim_{x\to+\infty} \phi(x) - \phi|$.

Likewise, if the same replacement of $[\lim_{x\to+\infty} \phi(x) - \phi]$ is made in Eq. (2.7b), the two series $R^{(2)}_{+\infty}([\lim_{x\to+\infty} \phi(x) - \phi])$, appearing on the right hand side of Eq. (2.7b), and $Q_{+\infty}$ may be incorporated in a single series, which we call $S^{(2)}_{+\infty}$, also containing terms whose order, for $\phi \simeq \lim_{x\to+\infty} \phi(x)$, is smaller than $|\lim_{x\to+\infty} \phi(x) - \phi|$.

Furthermore, since, by Eqs. (3.2) and (3.3),

$$[\lim_{x\to+\infty} \phi(x) - \phi] = [Z - s(\phi)]\{2Z - [Z - s(\phi)]\} \times$$
$$\{2[Z^2 - \sqrt{\Delta}] - 2Z[Z - s(\phi)] + [Z - s(\phi)]^2\}, \tag{3.7}$$

both $Q_{+\infty}$, in Eq. (3.5), $R^{(1)}_{+\infty}([\lim_{x\to+\infty} \phi(x) - \phi])$, in Eq. (2.7a), and $R^{(2)}_{+\infty}([\lim_{x\to+\infty} \phi(x) - \phi])$, in Eq. (2.7b) may be arranged as functions of the variable $[Z - s(\phi)]$. Thus, the above mentioned series $S^{(1)}_{+\infty}$ and $S^{(2)}_{+\infty}$ may be respectively conceived as $S^{(1)}_{+\infty}([Z - s(\phi)])$ and $S^{(2)}_{+\infty}([Z - s(\phi)])$, i.e. series containing powers of the variable $[Z - s(\phi)]$, whose order, for $s(\phi) \simeq Z$, is smaller than $|Z - s(\phi)|$.

Finally, taking into account the above considerations, we respectively write the desired rearrangements of Eqs. (2.7a) and (2.7b) as

$$\phi_x(\phi) = 4kZ[Z^2 - \sqrt{}\Delta][Z - s(\phi)] + S^{(1)}_{+\infty}([Z - s(\phi)]), \tag{3.8a}$$

$$\phi_{xx}(\phi) = -4k^2 Z[Z^2 - \sqrt{}\Delta][Z - s(\phi)] + S^{(2)}_{+\infty}([Z - s(\phi)]), \tag{3.8b}$$

for $s(\phi) \simeq Z$.

The procedure described above will now be applied to rearranging the relations between ϕ_x and ϕ and between ϕ_{xx} and ϕ in the domain $< x_{\min} < x < +\infty$, as respectively given in Eqs. (2.12a) and (2.12b). To do so, we consider the series given by the difference

$$Q_{\min} = \sqrt{}[\phi - \phi(x_{\min})] - 2\sqrt[4]{}\Delta[s(\phi) - \sqrt[4]{}\Delta], \tag{3.9}$$

and, denoting by $q_{\min}([\phi - \phi(x_{\min})]^{1/2})$ a series of terms whose order, for $\phi \simeq \phi(x_{\min})$, is smaller than $\{\sqrt{}|\phi - \phi(x_{\min})|\}^2$, we notice that

$$Q_{\min} = [\phi - \phi(x_{\min})]/(4\sqrt{}\Delta) + q_{\min}([\phi - \phi(x_{\min})]^{1/2}),$$
$$\text{for } \phi \simeq \phi(x_{\min}). \tag{3.10}$$

Therefore, if, in Eq. (2.12a), we replace the quantity $\sqrt{}[\phi - \phi(x_{\min})]$ by the expression $\sqrt{}[\phi - \phi(x_{\min})] = 2\sqrt[4]{}\Delta[\sqrt{}\Delta - s(\phi)] + Q_{\min}$, resulting from Eq. (3.9), the two series $R^{(1)}_{\min}(\sqrt{}[\phi - \phi(x_{\min})])$, appearing on the right hand side of Eq. (2.12a), and $Q_{\min}$ may be incorporated in a single series, which we call $S^{(1)}_{\min}$, containing terms whose order, for $\phi \simeq \phi(x_{\min})$, is smaller than $\sqrt{}|\phi - \phi(x_{\min})|$.

Likewise, if the same replacement of $\sqrt{}[\phi - \phi(x_{\min})]$ is made in Eq. (2.12b), the two series $R^{(2)}_{\min}(\sqrt{}[\phi - \phi(x_{\min})])$, appearing on the right hand side of Eq. (2.12b), and $Q_{\min}$ may be incorporated in a single series, which we call $S^{(2)}_{\min}$, also containing terms whose order, for $\phi \simeq \phi(x_{\min})$, is smaller than $\sqrt{}|\phi - \phi(x_{\min})|$.

Furthermore, since, by Eq. (3.2),

$$\surd[\phi - \phi(x_{\rm min})] = [s(\phi) - \sqrt[4]{\Delta}]\{2\sqrt[4]{\Delta} + [\, s(\phi) - \sqrt[4]{\Delta}]\}, \tag{3.11}$$

both $Q_{\rm min}$, in Eq. (3.9), $R^{(1)}_{\rm min}(\surd[\phi - \phi(x_{\rm min})])$, in Eq. (2.12a), and $R^{(2)}_{\rm min}(\surd[\phi - \phi(x_{\rm min})])$, in Eq. (2.12b) may be rearranged as functions of the variable $[s(\phi) - \sqrt[4]{\Delta}]$. Thus, the above mentioned series $S^{(1)}_{\rm min}$ and $S^{(2)}_{\rm min}$ may be conceived as $S_{\rm min}([s(\phi) - \sqrt[4]{\Delta}])$ and $S^{(2)}_{\rm min}([s(\phi) - \sqrt[4]{\Delta}])$, i.e. series containing powers of the variable $[s(\phi) - \sqrt[4]{\Delta}]$ whose order, for $s(\phi) \simeq \sqrt[4]{\Delta}$ is smaller than $|s(\phi) - \sqrt[4]{\Delta}|$.

Finally, taking into account the above considerations, and recalling the definition of A_0, the skew of the potential waveform at $x = x_{\rm min}$ (cf. Eq. (2.11)), we respectively write the desired rearrangement of Eqs. (2.12a) and (2.12b) as

$$\begin{aligned} &\phi_x(\phi) = 2\sqrt[4]{\Delta}\surd[2\phi''(x_{\rm min})][s(\phi) - \sqrt[4]{\Delta}] + \\ &S^{(1)}_{\rm min}([s(\phi) - \sqrt[4]{\Delta}]), && (3.12a) \\ &\phi_{xx}(\phi) = \phi''(x_{\rm min}) + 2A_0\sqrt[4]{\Delta}\surd[2\phi''(x_{\rm min})][s(\phi) - \sqrt[4]{\Delta}] + \\ &S^{(2)}_{\rm min}([s(\phi) - \sqrt[4]{\Delta}]), && (3.12b) \\ &\text{for } s(\phi) \simeq \sqrt[4]{\Delta}. \end{aligned}$$

This relation may be extended into the domain $x_{\rm max} < x < x_{\rm min}$ by extending the function $s(\phi)$ into that domain, i.e. by changing the sign of $\surd[\phi - \phi(x_{\rm min})]$ in Eq. (3.2), according to the prescription given in Eq. (2.9):

$$\begin{aligned} &s(\phi) = +\surd\{\surd\Delta - \surd[\phi - \phi(x_{\rm min})]\}, \\ &\text{for } x_{\rm max} < x < x_{\rm min}\,. \end{aligned} \tag{3.13}$$

We notice that, since in the domain $x_{\rm max} < x < x_{\rm min}$, $\phi(x)$ is a monotonically decreasing function of x (cf. Fig. 1), then $s(\phi(x))$ monotonically increases there, and we have

$$\begin{aligned} &0 < s(\phi) < \sqrt[4]{\Delta}, \\ &\text{for } x_{\rm max} < x < x_{\rm min}\,. \end{aligned} \tag{3.14}$$

Our next task will be to rearrange the relations between ϕ_x and ϕ and between ϕ_x and ϕ in the domain $x_{\max} < x < x_{\min}$, as respectively given in Eqs. (2.17a) and (2.17b). To do so, we consider the series given by the difference

$$Q_{\max} = \surd[\phi(x_{\max}) - \phi] - \surd[2\surd\Delta]s(\phi), \tag{3.15}$$

and, denoting by $q_{\max}([\phi(x_{\max}) - \phi]^{1/2})$ a series of terms whose order, for $\phi \simeq \phi(x_{\max})$, is smaller than $\{\surd|\phi(x_{\max}) - \phi|\}^{3/2}$, we notice that

$$Q_{\max} = [\phi(x_{\max}) - \phi]^{3/2}/(8\Delta) + q_{\max}([\phi(x_{\max}) - \phi]^{1/2}),$$
$$\text{for } \phi \simeq \phi(x_{\max}). \tag{3.16}$$

Therefore, if, in Eq. (2.17a), we replace the quantity $\surd[\phi(x_{\max}) - \phi]$ by the expression $\surd[\phi(x_{\max}) - \phi] = \surd[2\surd\Delta]s(\phi) + Q_{\max}$, resulting from Eq. (3.15), the two series $R^{(1)}_{\max}(\surd[\phi(x_{\max}) - \phi])$, appearing on the right hand side of Eq. (2.17a), and $Q_{\max}$ may be incorporated in a single series, which we call $S^{(1)}_{\max}$, containing terms whose order, for $\phi \simeq \phi(x_{\max})$, is smaller than $\surd|\phi(x_{\max}) - \phi|$.

Likewise, if the same replacement of $\surd[\phi(x_{\max}) - \phi]$ is made in Eq. (2.17b) the two series $R^{(2)}_{\max}(\surd[\phi - \phi(x_{\min})])$, appearing on the right hand side of Eq. (2.17b), and $Q_{\max}$ may be incorporated in a single series, which we call $S^{(2)}_{\max}$, also containing terms whose order, for $\phi \simeq \phi(x_{\max})$, is smaller than $\surd|\phi(x_{\max}) - \phi|$.

Furthermore, since, from Eq. (3.13),

$$\surd[\phi(x_{\max}) - \phi] = s(\phi)\surd[2\surd\Delta - s^2(\phi)],$$
$$\text{for } x_{\max} < x < x_{\min}, \tag{3.17}$$

both $Q_{\max}$, in Eq. (3.15), $R^{(1)}_{\max}(\surd[\phi(x_{\max}) - \phi])$, in Eq. (2.17a) and $R^{(2)}_{\max}(\surd[\phi(x_{\max}) - \phi])$, in Eq. (2.17b), may be rearranged as functions of the variable $s(\phi)$, whose order, for $s(\phi) \simeq 0$, is smaller than $|s(\phi)|$. Thus, the above mentioned series $S^{(1)}_{\max}$ and $S^{(2)}_{\max}$ may be conceived as $S^{(1)}_{\max}(s(\phi))$ and $S^{(2)}_{\max}(s(\phi))$, i.e. series containing powers of the variable $s(\phi)$, whose order, for $s(\phi) \simeq 0$, is smaller than $|s(\phi)|$.

Finally, taking into account the above considerations, and recalling the definition of B_0, the skew of the potential waveform at $x = x_{\max}$ (cf. Eq. (2.16)), we respectively write the desired rearrangement of Eqs. (2.17a) and (2.17b) as

$$\phi_x(\phi) = -\surd[2\surd\Delta]\surd[-2\phi''(x_{\max})]s(\phi) + S^{(1)}_{\max}(s(\phi)), \tag{3.18a}$$

$$\phi_{xx}(\phi) = \phi''(x_{\max}) - \surd 2B_0\sqrt[4]{\Delta}\surd[-2\phi''(x_{\max})]s(\phi) + S^{(2)}_{\max}(s(\phi)), \tag{3.18b}$$

for $s(\phi) \simeq 0$.

The relations given in Eqs. (3.18a) and (3.18b) may be extended into the domain $-\infty < x < x_{\max}$ by extending the function $s(\phi)$ into that domain, i.e. by changing the sign of $\surd\{\surd\Delta - \surd[\phi - \phi(x_{\min})]\}$ in Eq. (3.13), according to the prescription given in Eq. (2.14):

$$s(\phi) = -\surd\{\surd\Delta - \surd[\phi - \phi(x_{\min})]\}, \quad \text{for } -\infty < x < x_{\max}. \tag{3.19}$$

We notice that, since in the domain $-\infty < x < x_{\max}$, $\phi(x)$ is a monotonically increasing function of x, the function $s(\phi(x))$ monotonically increases there and, introducing its limit value

$$\lim_{x\to-\infty} s(\phi(x)) = -z = -\surd\{\surd\Delta - \surd[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})]\} \leq 0, \tag{3.20}$$

we have

$$-z < s(\phi) < 0,$$
$$\text{for } -\infty < x < x_{\max}. \tag{3.21}$$

Our last task will be to rearrange the relations between ϕ_x and ϕ and between ϕ_{xx} and ϕ in the domain $-\infty < x < x_{\max}$, as respectively given in Eqs. (2.20a) and (2.20b). To do so, we consider the series given by the difference

$$Q_{-\infty} = \{\phi - \lim_{x\to-\infty} \phi(x)\} - \{\, 4z[\sqrt{\Delta} - z^2][z + s(\phi)]\}, \tag{3.22}$$

and, denoting by $q_{-\infty}([\phi - \lim_{x\to-\infty} \phi(x)])$ a series of terms whose order, for $\phi \simeq \lim_{x\to-\infty} \phi(x)$, is smaller than $|\phi - \lim_{x\to-\infty} \phi(x)|^2$, we notice that

$$Q_{-\infty} = \{[3z^2 + \sqrt{\Delta}]/[8z^2(z^2 + \sqrt{\Delta})^2]\}[\phi - \lim_{x\to-\infty} \phi(x)]^2 +$$
$$q_{-\infty}([\phi - \lim_{x\to-\infty} \phi(x)]),$$
$$\text{for } \phi \simeq \lim_{x\to-\infty} \phi(x). \tag{3.23}$$

Therefore, if, in Eq. (2.20a), we replace the quantity $[\phi - \lim_{x\to-\infty} \phi(x)]$ by the expression $[\phi - \lim_{x\to-\infty} \phi(x)] = 4z[\sqrt{\Delta} - z^2][z + s(\phi)] + Q_{-\infty}$, resulting from Eq. (3.22), the two series $R^{(1)}_{-\infty}([\phi - \lim_{x\to-\infty} \phi(x)])$, appearing on the right hand side of Eq. (2.20a), and $Q_{-\infty}$ may be incorporated in a single series, which we call $S^{(1)}_{-\infty}$, containing terms whose order, for $\phi \simeq \lim_{x\to-\infty} \phi(x)$, is smaller than $|\phi - \lim_{x\to-\infty} \phi(x)|$.

Likewise, if the same replacement of $[\phi - \lim_{x\to-\infty} \phi(x)]$ is made in Eq. (2.20b) the two series $R^{(2)}_{-\infty}([\phi - \lim_{x\to-\infty} \phi(x)])$, appearing on the right hand side of Eq. (2.17b), and $Q_{\min}$ may be incorporated in a single series, which we call $S^{(2)}_{-\infty}$, also containing terms whose order, for $\phi \simeq \lim_{x\to-\infty} \phi(x)$, is smaller than $|\phi - \lim_{x\to-\infty} \phi(x)|$.

Furthermore, since, by Eqs. (3.19) and (3.20),

$$[\phi - \lim_{x\to-\infty} \phi(x)] = [z + s(\phi)]\{2z - [z + s(\phi)]\} \times$$
$$\{2[\sqrt{\Delta} - z^2] + 2z[z + s(\phi)] - [z + s(\phi)]^2\}, \tag{3.24}$$

both $Q_{-\infty}$, in Eq. (3.22), $R^{(1)}_{-\infty}([\phi - \lim_{x\to+\infty} \phi(x)])$, in Eq. (2.20a) and $R^{(2)}_{-\infty}([\phi - \lim_{x\to+\infty} \phi(x)])$, in Eq. (2.20b), may be arranged as functions of the variable $[z + s(\phi)]$. Thus, the above mentioned series $S^{(1)}_{-\infty}$ and $S^{(2)}_{-\infty}$ may be conceived as $S^{(1)}_{-\infty}([z + s(\phi)])$ and $S^{(2)}_{-\infty}([z + s(\phi)])$, i.e. series containing powers of the variable $[z + s(\phi)]$, whose order, for $s(\phi) \simeq -z$, is smaller than $|z + s(\phi)|$.

Finally, taking into account the above considerations, we respectively write the desired rearrangement of Eqs. (2.20a) and (2.20b) as

$$\phi_x(\phi) = 4kz[\sqrt{\Delta} - z^2][z + s(\phi)] + S^{(1)}_{-\infty}([z + s(\phi)]), \tag{3.25a}$$
$$\phi_{xx}(\phi) = 4k^2z[\sqrt{\Delta} - z^2][z + s(\phi)] + S^{(2)}_{-\infty}([z + s(\phi)]), \tag{3.25b}$$
$$\text{for } \ s(\phi) \simeq -z.$$

In the above analysis we produced the desired rearrangements of the relations between ϕ_x, ϕ_{xx}, respectively the first and second derivative of the potential waveform, and ϕ. Starting from their primaeval forms, given in Eqs. (2.7), (2.12), (2.17) and (2.20), the rearranged relations are given in terms of power series of the function $s(\phi)$, respectively in Eqs.(3.8), (3.12), (3.18) and (3.25).

Now, the function $s(\phi)$ was piece-wise defined in each of the three intervals $-\infty < x < x_{\max}$ (cf. Eq. (3.2)), $x_{\min} < x < x_{\max}$ (cf. Eq. (3.13)) and $x_{\max} < x < +\infty$ (cf. Eq. (3.19)). Using the prescriptions given in Eqs. (2.9) and (2.14), $s(\phi)$ may more conveniently be defined over the whole domain $-\infty < x < +\infty$ as a multi-valued function:

$$s(\phi) = \{\sqrt{\Delta} + [\,\phi - \phi(x_{\min})]^{1/2}\}^{1/2}, \tag{3.26a}$$

$$\text{where } \Delta = [\phi(x_{\max}) - \phi(x_{\min})]. \tag{3.26b}$$

This function has a number of remarkable properties and, for the importance it will have in the following analysis, it will be called the "structure function" of the potential. As shown above, $s(\phi(x))$ is a monotonically increasing function of x over the three open domains $-\infty < x < x_{\max}$, $x_{\max} < x < x_{\min}$, $x_{\min} < x < +\infty$ and, being obviously continuous at $x = x_{\max}$ and $x = x_{\min}$, it is a monotonically increasing function of x over the whole domain $-\infty < x < +\infty$ and its bounds are

$$\lim_{x\to-\infty} s(\phi(x)) = -z < s(\phi) < Z = \lim_{x\to+\infty} s(\phi(x)). \tag{3.27}$$

Here, $Z \geq 0$ (cf. Eq. (3.3)) and $z \geq 0$ (cf. (3.20)), and thus $s(\phi(x))$ has exactly one simple zero, this latter indeed occurring at $x = x_{\max}$.

It is also easily verified that $\mathrm{d}s(\phi(x))/\mathrm{d}x$ is a continuous function of x over the whole domain $-\infty < x < +\infty$, and in particular at $x = x_{\min}$ and $x = x_{\max}$. Last, since $\phi(x)$ was assumed to be analytic and strictly monotonic over the three domains $-\infty < x < x_{\max}$, $x_{\max} < x < x_{\min}$ and $x_{\min} < x < +\infty$, so is $s(\phi(x))$ there.

4 The shape factor of the potential

One of the anticipated advantages in rearranging the relations between ϕ_x and ϕ and between ϕ_{xx} and ϕ in terms of the structure function $s(\phi)$ lies in the simplicity of their analytic extensions. Indeed, we notice that both $\phi'(x)$, $\phi''(x)$ and $s(\phi(x))$ are analytic function of x over the three open domains

$-\infty < x < x_{\max}$, $x_{\max} < x < x_{\min}$ and $x_{\min} < x < +\infty$. Therefore, each of the relations given in Eqs. (3.8), (3.12), (3.18) and (3.25) define $\phi_x(\phi)$ and $\phi_{xx}(\phi)$ as analytic functions of the variable $s(\phi(x))$, which may be extended by analytic continuation.

Specifically, the relations in Eq. (3.8) can be extended over the interval $\sqrt[4]{\Delta} < s(\phi) < Z$, which $s(\phi(x))$ covers as x ranges over the domain $x_{\min} < x < +\infty$. The relations in Eq. (3.12) can be extended over the interval $\sqrt[4]{\Delta} < s(\phi) < Z$, which $s(\phi(x))$ covers as x ranges over the domain $x_{\min} < x < +\infty$, and over the interval $0 < s(\phi) < \sqrt[4]{\Delta}$, which $s(\phi(x))$ covers as x ranges over the domain $x_{\max} < x < x_{\min}$. The relations in Eq. (3.18) can be extended over the two intervals $-z < s(\phi) < 0$ and $0 < s(\phi) < \sqrt[4]{\Delta}$, which $s(\phi(x))$ covers as x respectively ranges over the domains $-\infty < x < x_{\max}$ and $x_{\max} < x < x_{\min}$. Last, the relations in Eq. (3.25) can be extended over the interval $-z < s(\phi) < 0$, which $s(\phi(x))$ covers as x ranges over the domain $-\infty < x < x_{\max}$.

In this way, three pairs of analytic extensions of the function $\phi_x(\phi)$ and three pairs of analytic extensions of the function $\phi_{xx}(\phi)$ are produced, each pair holding in one of the three domains $-z < s(\phi) < 0$, $0 < s(\phi) < \sqrt[4]{\Delta}$ and $\sqrt[4]{\Delta} < s(\phi) < Z$. Specifically, the pair of extensions of $\phi_x(\phi)$ originating from Eqs. [(3.8a), (3.12a)] and the pair of extensions of $\phi_{xx}(\phi)$ originating from Eqs. [(3.8b), (3.12b)] hold over the domain $\sqrt[4]{\Delta} < s(\phi) < Z$, respectively starting from its right and left bound. Likewise, the pair of extensions of $\phi_x(\phi)$ originating from Eqs. [(3.12a), (3.18a)] and the pair of extensions of $\phi_{xx}(\phi)$ originating from Eqs. [(3.12b), (3.18b)] hold in the domain $0 < s(\phi) < \sqrt[4]{\Delta}$. Last, the pair of extensions of $\phi_x(\phi)$ originating from Eqs. [(3.18a),(3.25a)] and the pair of extensions of $\phi_{xx}(\phi)$ originating from Eqs. [(3.18b),(3.25b)] hold in the domain $-z < s(\phi) < 0$. Notice in particular that the extensions of $\phi_x(\phi)$ given in Eqs. (3.12a) and (3.18a) and the extensions of $\phi_{xx}(\phi)$ given in Eqs. (3.12b) and (3.18b) belong, at the same time, to two pairs of analytic extensions.

Now, because of the uniqueness of analytic continuation, the two extensions in each of the above mentioned pairs should coincide. One important consequence of this uniqueness is that, although ϕ_x and ϕ_{xx} are not an analytic function of ϕ, their relation to $s(\phi)$ is in fact analytic over the whole s-domain $-z < s < Z$. Indeed we have already established that these relations are analytic over the three open intervals $-z < s < 0$, $0 < s < \sqrt[4]{\Delta}$ and $\sqrt[4]{\Delta} < s < Z$. To produce the analytic Taylor expansions of ϕ_x and ϕ_{xx} at $s = 0$, we respectively use the series defined by the right hand side of Eqs. (3.18a) and (3.18b), both of which hold for $-z < s < \sqrt[4]{\Delta}$. Likewise, to produce the analytic Taylor expansions of ϕ_x and ϕ_{xx} at $s = \sqrt[4]{\Delta}$, we respectively use the series defined by the right hand side of Eqs. (3.12a) and (3.12b), both of which hold for $0 < s < Z$.

One second consequence of the uniqueness of the analytic continuation is that, once the coefficients of one of the four analytic extensions of ϕ_x and ϕ_{xx} are given, the coefficients of the other three extensions are uniquely determined. We shall use this result to work out the analytic representation of $\phi_x(\phi)$ holding over the whole x-domain.

To do so, we fix the extension in Eq. (3.8a) (the one holding at the right boundary of the tripolar spike as $x \to +\infty$) and compare the two extensions on the right hand sides of Eqs. (3.8a) and (3.12a), which belong to the same pair of extensions of ϕ_x: in this way, we see that the right hand side of Eq. (3.12a) should vanish at $s(\phi) = Z$. In other words, in Eq. (3.12a), the quantity

ϕ_x not only has $[s(\phi) - \sqrt[4]{\Delta}]$ as its obvious factor but also $[Z - s(\phi)]$. Having established this, we proceed by comparing the extensions given in Eqs. (3.12a) and (3.18a) and, by the same argument, we see that the quantity ϕ_x in Eq. (3.18a) not only has $s(\phi)$ as its factor, but also $[s(\phi) - \sqrt[4]{\Delta}]$ and $[Z - s(\phi)]$. Finally, by comparing Eqs. (3.18a) and (3.25a), we prove that ϕ_x admits the four factors $[z + s(\phi)]$, $s(\phi)$, $[s(\phi) - \sqrt[4]{\Delta}]$ and $[Z - s(\phi)]$.

In conclusion, the sought relation between ϕ_x and ϕ, holding over the whole x domain, will be given by factoring all of these four factors and a fifth extra factor which, without loss of generality, we write as $[4k/(Z + z)][s(\phi) + \sqrt[4]{\Delta}]S(s(\phi))$:

$$\begin{aligned}\phi_x(\phi) = s(\phi)[s^2(\phi) - \sqrt{\Delta}][z + s(\phi)][Z - s(\phi)] \times \\ [4k/(Z + z)]S(s(\phi)).\end{aligned} \tag{4.1}$$

In practice, the factor $S(s)$ accounts for the fine adjustments of the potential waveform, whose main morphological properties are accommodated by the first four factors appearing on the right hand side of Eq. (4.1). In the following, the quantity $S(s)$ will be known as the "shape factor" of the potential waveform.

Eq. (4.1) might be regarded as a differential equation for the potential $\phi(x)$, provided the shape factor $S(s)$ were related to ϕ by way of some physical argument. Now, through Poisson's law and the obvious relation $\mathrm{d}^2\phi(x)/\mathrm{d}x^2 = \mathrm{d}\{[\phi_x(\psi)]^2/2\}/\mathrm{d}\psi|_{\psi=\phi(x)}$, the quantity $S^2(\phi)$ is related to the electric charge density in the plasma. However, this latter quantity is generally not known at any arbitrarily given value of x. Therefore, its relation to ϕ — and hence the relation between $S^2(\phi)$ and ϕ — cannot be generally worked out on experimental grounds. The remaining part of this section will be devoted to the reconstruction of the relation between $S^2(\phi)$ and ϕ, based on the functional properties of the structure function $s(\phi)$.

To this end, we notice that the first four factors on the right hand side of Eq. (4.1) are obviously analytic functions of $s(\phi)$ over the whole s-domain $-z < s < Z$, covered by $s(\phi(x))$ as x ranges over the domain $-\infty < x < +\infty$ (cf. Eq. (3.27)). On the other hand, the quantity $\phi_x(\phi)$ was also shown to be an analytic function of $s(\phi)$ over that domain. Thus $S(s)$ is itself an analytic function of $s(\phi)$ and it may be represented by a Taylor series. It is in fact convenient to write the Taylor series for $S^2(s)$, obviously an analytic function too:

$$S^2(s) = \sum_{n=0}^{+\infty} S_n s^n. \qquad (4.2)$$

A second property of the shape factor $S(s)$ is determined by the fact that, since, by construction, the first four factors on the right hand side of Eq. (4.1) account for all the zeroes of ϕ_x, then $S(s)$ must not vanish anywhere. Also, given the choice of the sign of $s(\phi)$ (cf. Eqs. (3.2), (3.13) and (3.19)), Eq. (4.1) reproduces the slope of the potential waveform shown in Fig. 1 only if

$$S(s) > 0. \qquad (4.3)$$

A third set of properties of the shape factor $S(s)$ comes by taking the values of $\phi_x(\phi)$ in Eq. (4.1) for $\phi \simeq \lim_{x\to+\infty}(\phi(x))$ (i.e. for $s \simeq Z$, cf. Eq. (3.3)) and $\phi \simeq \lim_{x\to-\infty}(\phi(x))$ (i.e. for $s \simeq -z$, cf. Eq. (3.20)), and by comparing these values respectively with those given by Eqs. (3.8a), and (3.25a). In this way we find

$$S(Z) = 1, \tag{4.4a}$$

$$S(-z) = 1. \tag{4.4b}$$

A fourth set of properties of the shape factor $S(s)$ comes by taking the values of $\phi_{xx}(\phi)$, calculated from Eq. (4.1) as $\phi_{xx}(\phi) = \mathrm{d}\{[\phi_x(\phi)]^2/2\}/\mathrm{d}\phi$, for $\phi \simeq \phi(x_{\min})$ (i.e. for $s(\phi) \simeq \sqrt[4]{\Delta}$, cf. Eq. (3.13)) and $\phi \simeq \phi(x_{\max})$ (i.e. for $s(\phi) \simeq 0$, cf. Eq. (3.19)), and by comparing these values respectively with those given by Eqs. (3.12b) and (3.18b). In this way, the skews at the potential minimum and maximum are related to the shape factor S as

$$A_0 = 3k\frac{\sqrt[4]{\Delta}}{Z+z}\left\{1 + \frac{(Z - 2\sqrt[4]{\Delta})(z + 2\sqrt[4]{\Delta})}{\sqrt{\Delta}} + \right.$$
$$\left. \frac{(Z - \sqrt[4]{\Delta})(z + \sqrt[4]{\Delta})}{\sqrt{\Delta}} \frac{[\sqrt[4]{\Delta}][\mathrm{d}S/\mathrm{d}s]_{s=\sqrt[4]{\Delta}}}{S(\sqrt[4]{\Delta})}\right\} S(\sqrt[4]{\Delta}), \tag{4.5a}$$

$$B_0 = -3k\frac{\sqrt[4]{\Delta}}{Z+z}\left\{\frac{z - Z}{\sqrt[4]{\Delta}} - \frac{Zz}{\sqrt{\Delta}}\frac{[\sqrt[4]{\Delta}][\mathrm{d}S/\mathrm{d}s]_{s=0}}{S(0)}\right\} S(0). \tag{4.5b}$$

The above established properties will be used to uniquely determine the shape factor S in the limit of small potential amplitudes. This task will be carried out in Section 5.

5 The potential waveform

In Section 4 we established the differential equation governing the potential waveform ϕ by relating ϕ_x, the derivative of ϕ, to ϕ itself. Such relation involves five factors (cf. Eq. (4.1)): the first four factors give the basic morphological properties of the potential, i.e. its extrema and its asymptotic behaviour as $|x| \to \infty$; the fifth factor — the shape factor $S(s)$ — accounts for the fine adjustments of the waveform. Although the relation between ϕ_x and ϕ is generally not analytic, we showed that a function exists — the structure function of the potential $s(\phi)$ — such that ϕ_x is an analytic function of s.

It in now convenient to introduce the "rescaled structure function"

$$r(\phi) = [2s(\phi) - (Z - z)]/(Z + z), \tag{5.1a}$$

$$\lim_{x\to-\infty} r(\phi(x)) = -1 < r(\phi) < \lim_{x\to+\infty} r(\phi(x)) = 1, \tag{5.1b}$$

whose boundary values, given in Eq. (5.1b), come from the corresponding values of the structure function $s(\phi)$ (cf. Eq. (3.27)). Then, using Newton's binomial formula, Eq. (4.2) is transformed into

$$S^2(s(\phi)) = \sum_{n=0}^{+\infty} R_n[r(\phi)]^n, \tag{5.2a}$$

$$R_n = [(Z + z)/(Z - z)]^n \sum_{m=n}^{+\infty} \binom{m}{n} S_m[(Z - z)/2]^m. \tag{5.2b}$$

Correspondingly, the constraints set in Eqs. (4.4) and (4.5) are transformed into constraints for the coefficients R_n. In particular, considering that $r(\lim_{x\to\pm\infty} \phi(x)) = \pm 1$, (cf. Eq. (5.1b)), the substitution of Eq. (5.2a) into Eqs. (4.4a) and (4.4b) respectively gives

$$\sum_{n=0}^{+\infty} R_n = 1, \tag{5.3a}$$

$$\sum_{n=0}^{+\infty} (-1)^n R_n = 1. \tag{5.3b}$$

We are now ready to solve the differential equation Eq. (4.1) in favour of $\phi(x)$. We first notice that, since $S(s) > 0$ (cf. Eqs. (4.3)), in substituting the shape factor $S(s)$ given in Eq. (5.2a) into Eq. (4.1) we must set $S(s) = +\sqrt{[S^2(s)]}$. Then, denoting by a " $'$ " differentiation with respect to x, and using the definition of the structure function $s(\phi)$ (cf. Eq. (3.26a)), we finally work Eq. (4.1) into a differential equation for the rescaled structure function $r(\phi)$ defined in Eq. (5.1a):

$$[r(\phi(x))]' = \frac{k}{2}[1 - r^2(\phi(x))]\sqrt{\sum_{n=0}^{+\infty} R_n[r(\phi(x))]^n}. \tag{5.4}$$

This equation, subject to the constraints of Eq. (5.3), is the general nonlinear differential equation for the potential waveform ϕ, which is related to the rescaled structure function $r(\phi)$ by means of Eqs. (5.1a) and (3.26a).

Eq. (5.4) is not integrable in its general form. However, provided the amplitude of the electric potential ϕ is sufficiently small (values of $\epsilon \simeq 10^{-4}$ were observed in Ref. [1]), it is amenable to quadrature. Indeed, if

$$|\phi| = O(\epsilon), \ \epsilon \ll 1. \tag{5.5}$$

then, the quantities Z (cf. Eq. (3.3)) and z (cf. Eq. (3.20)) are $O(\sqrt[4]{\epsilon})$. Therefore, the orders of the coefficients R_n in Eq. (5.2b) are

$$R_0 = O(1), \ R_0 \geq 0, \tag{5.6a}$$

$$R_n = O([\sqrt[4]{\epsilon}]^n), \quad n = 1, 2, \ldots . \tag{5.6b}$$

The inequality in Eq. (5.6a) ensures, to lowest order, the non-negativity of the quantity S^2 in Eq. (5.2a).

In practice, the quadrature of Eq. (5.4) is affordable only up to the fourth order approximation. In this case, only the terms containing R_0, R_1, R_2, R_3 and R_4 would be retained in Eq. (5.4) and this equation would be solved in terms of elliptic functions.

This task is outside the scope of the present work, where we rather adopt a second order assumption in which $R_n = 0$ for $n > 2$. We shall see that even the results of such simpler approach are in excellent agreement with observations. In this approximation, the constraints on the coefficients R_n given in Eqs. (5.3a) and (5.3b) reduce to

$$R_0 > 0, \tag{5.7a}$$

$$R_1 = 0, \tag{5.7b}$$

$$R_2 = 1 - R_0,\ |R_2| \ll 1, \tag{5.7c}$$

and Eq. (5.4) reads

$$[r(\phi(x))]' = +\frac{k}{2}\{1 - [r(\phi(x))]^2\} \times \sqrt{}\{R_0 + (1 - R_0)[r(\phi(x)]^2\}. \tag{5.8}$$

The quadrature formula for this equation is easily found to be

$$\frac{kx}{2} = \tanh^{-1}\left(\frac{r(\phi(x))}{\sqrt{}\{R_0 + (1 - R_0)[r(\phi(x))]^2\}}\right) \tag{5.9}$$

and it leads to the solution

$$r(\phi(x)) = \frac{[\sqrt{}R_0]\tanh(kx/2)}{\sqrt{}[1 - (1 - R_0)\tanh^2(kx/2)]}. \tag{5.10}$$

The fact that $R_0 > 0$ (cf. Eq. (5.7a)), ensures that, in Eq. (5.10), $r(\phi(x))$ has no singularities for any value of x. The rescaled structure function $r(\phi(x))$ is drawn in Fig. 3 for $R_0 = 0.7$.

Finally, Eqs. (3.26a) and (5.1a) provide the potential waveform

$$[\phi(x) - \phi(x_{\min})]/\Delta = \{[(Z + z)r(\phi(x)) + (Z - z)]^2/(4\sqrt{}\Delta) - 1\}^2. \tag{5.11}$$

We remark that the model waveform provided by Eq. (5.11) has one free parameter only. Indeed, we see that the right hand side of Eq. (5.11) actually depends on the two rescaled parameters (cf. Eqs. (3.1) and (3.3))

$$Z/\sqrt[4]{\Delta} = \sqrt{\left\{1 + \frac{\sqrt{[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})]}}{\sqrt{[\phi(x_{\max}) - \phi(x_{\min})]}}\right\}} \tag{5.12}$$

and (cf. Eqs. (3.1) and (3.20))

$$z/\sqrt[4]{\Delta} = \sqrt{\left\{1 - \frac{\sqrt{[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})]}}{\sqrt{[\phi(x_{\max}) - \phi(x_{\min})]}}\right\}}. \tag{5.13}$$

These parameters are uniquely determined, once the two potential jumps as $x \to +\infty$ and as $x \to -\infty$ are assigned relative to the reference potential jump $[\phi(x_{\max}) - \phi(x_{\min})]$. Also, of the two parameters brought into Eq. (5.11) by the function r (cf. Eq. (5.10)), k is given by the decay rate of the potential waveform as $x \to \pm\infty$ (cf. Eqs. (2.5a) and (2.18a)), whereas R_0 remains undetermined. Therefore all of these parameters are provided by observation, except R_0, which remains the only free parameter of our second order model, and which we may adjust to best fit the observed waveforms. Despite this limitation, the solid line curve in Fig. 4 indicates that the waveforms given by Eq. (5.11) are in excellent agreement with observations (cf. [1–3]).

We conclude this section by considering some special cases of the general solution given in Eq. (5.11). The first two cases come by considering the constraints given in Eq. (4.5) when the amplitude of the electric potential ϕ is sufficiently small (cf. Eq. (5.5)). In this approximation, the quantity $\sqrt[4]{\Delta}$ (cf. Eq. (3.1)), the structure function $s(\phi)$ (cf. Eq. 3.26a)), the quantities Z (cf. Eq. (3.3)) and z (cf. Eq. (3.20)) are $O(\sqrt[4]{\epsilon})$, and the shape factor reduces to (cf. Eq. (5.2a)) $S = \sqrt{\sum_{n=0}^{+\infty} R_n[r(\phi)]^n} \simeq \sqrt{R_0}$. Therefore, to leading order, Eqs. (4.5a) and (4.5b) respectively reduce to

$$A_0 = 3k\frac{\sqrt[4]{\Delta}}{Z+z}\left\{1+\frac{(Z-2\sqrt[4]{\Delta})(z+2\sqrt[4]{\Delta})}{\sqrt{\Delta}}\right\}\sqrt{R_0}, \tag{5.14a}$$

$$B_0 = 3k\frac{\sqrt[4]{\Delta}}{Z+z}\frac{Z-z}{\sqrt[4]{\Delta}}\sqrt{R_0}. \tag{5.14b}$$

We note that, since $k > 0$ (cf. Eq.(2.5d)) and $Z > z$ (cf. Eqs. (3.3) and (3.20)), B_0, the skew of the potential waveform about the potential maximum, is positive in Eq. (5.14b), in agreement with observations (cf. e.g. Refs. [1–3]) and with our earlier results of Ref. [14].

On the other hand, observations of some weak tripolar spikes (cf. e.g. Ref. [1]) indicate that the potential waveform is nearly symmetric about its minimum, i.e. that, in Eq. (2.11), the skew A_0 must be very small. Through Eq. (5.14a), this condition implies that the quantities Z and z must lie on the hyperbola $(Z-2\sqrt[4]{\Delta})(z+2\sqrt[4]{\Delta}) = -\sqrt{\Delta}$. The further obvious constraint $0 \le z \le \sqrt[4]{\Delta}$ (cf. Eq. (3.20)) requires that only the upper branch of that hyperbola be considered and only for $[3/2]\sqrt[4]{\Delta} \le Z \le [5/3]\sqrt[4]{\Delta}$. Finally, the definition of Z (cf. Eq. (3.3)) casts these considerations in the following form (cf. Eq. 2):

$$\frac{[\lim_{x\to-\infty}\phi(x)-\phi(x_{\min})]}{\Delta} = \frac{[Z-\sqrt[4]{\Delta}]^2[3Z-5\sqrt[4]{\Delta}]^2}{[Z-2\sqrt[4]{\Delta}]^4}, \tag{5.15a}$$

$$\text{for}\quad [5/4]^2 \le \frac{[\lim_{x\to+\infty}\phi(x)-\phi(x_{\min})]}{\Delta} < [16/9]^2, \tag{5.15b}$$

$$\text{where}\quad Z = \sqrt{\{\sqrt{\Delta}+\sqrt{[\lim_{x\to+\infty}\phi(x)-\phi(x_{\min})]}\}}.$$

In particular, when Z is at its lower bound, $Z = [3/2]\sqrt[4]{\Delta}$, then $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = [5/4]^2\Delta$; in this case, $\lim_{x\to-\infty}\phi(x) = \phi(x_{\max})$, the position at which the potential waveform has a maximum $x_{\max}$ shifts to $-\infty$, and the tripolar spike approaches a non monotonic double layer (cf. the long-dashed curve in Fig. 4 and e.g. Ref. [10]). In the opposite situation, when Z is at its upper bound, $Z = [5/3]\sqrt[4]{\Delta}$, then $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = [16/9]^2\Delta$: in this case, $\lim_{x\to-\infty}\phi(x) = \phi(x_{\min})$. The corresponding potential waveform has the maximum potential gain $[\lim_{x\to+\infty}\phi(x) - \lim_{x\to-\infty}\phi(x)] = [16/9]^2\Delta \simeq 3.16\Delta$ (cf. the short-dashed curve in Fig. 4).

Three further degenerate cases of the solution given in Eq. (5.11) are worth notice: all have $R_0 = 1$, which gives $r(\phi(x)) = \tanh(kx/2)$ in Eq. (5.10). One case arises when $\lim_{x\to-\infty}\phi(x) = \lim_{x\to+\infty}\phi(x) = \phi(x_{\max})$: from Eqs. (3.3) and (3.20), we find $z = 0$ and $Z = \sqrt{[2\sqrt{\Delta}]}$, and Eq. (5.11) reduces to

$$[\phi(x) - \phi(x_{\min})]/\Delta = \{[\tanh(kx/2) + 1]^2/2 - 1\}^2, \tag{5.16}$$

which depicts an *asymmetrical* waveform of the type associated to ion holes, with a single trough at $x = x_{\min} = \ln(1+\sqrt{2})/k$; the position of the maximum of this waveform moves to infinity: $|x_{\max}| \to +\infty$ (cf. the dash single dotted curve in Fig. 4).

A second degenerate case arises when $\lim_{x\to-\infty}\phi(x) = \lim_{x\to+\infty}\phi(x) = \phi(x_{\min})$: from Eqs. (3.3) and (3.20), we find $Z = z = \sqrt[4]{\Delta}$ and Eq. (5.11) reduces to

$$[\phi(x) - \phi(x_{\min})]/\Delta = \operatorname{sech}^4(kx/2), \tag{5.17}$$

which obviously depicts a symmetrical waveform of the type associated with electron holes, with a single hump at $x = x_{\max} = 0$; the position of the minimum of this waveform moves to infinity: $|x_{\min}| \to +\infty$ (cf. the dash double dotted curve in Fig. 4).

A last degenerate case arises when $\lim_{x\to-\infty}\phi(x) = \phi(x_{\max})$ and $\lim_{x\to+\infty}\phi(x) =$

$\phi(x_{\min})$: from Eqs. (3.3) and (3.20), we find $z = 0$ and $Z = \sqrt[4]{\Delta}$, and Eq. (5.11) reduces to

$$[\phi(x) - \phi(x_{\min})]/\Delta = \{[\tanh(kx/2) + 1]^2/4 - 1\}^2, \tag{5.18}$$

which depicts the waveform of a monotonically decreasing double layer without any extrema (cf. the dash triple dotted curve in Fig. 4).

In conclusion, the solution given in Eq. (5.11) may reproduce the electric potential of electron holes, ion holes, monotonic and non monotonic double layers and indeed tripolar spikes, in remarkable agreement with observations (cf. e.g. Refs. [1–3]).

6 Summary and discussion

The object of the present work is the reproduction, by analytical means, of the distinctively asymmetric waveforms of the electric potential of tripolar spikes in plasmas (cf. e.g Refs. [2,3] and Fig. 1). Currently, analytical models of similar plasma structures, such as asymmetric ion and electron holes and non monotonic double layers are available. These models are based on the specification of the velocity distribution functions of the electrons and of the ions sustaining the spikes, on the calculation of the electron and ion charge densities, and on the subsequent solution of Poisson's equation in favour of the electric potential.

This equation usually takes the form of a steady state equation of the modified Korteweg de Vries family (cf. e.g. Refs. [7–11]). The proposed classical solution of these equations, and even the elaborated solutions of more complicated equations of this and other families by means of advanced techniques (cf. e.g. Refs. [12,13]) fail to reproduce all of the peculiar and essential features of the potential waveform of tripolar spikes.

We propose a model for these waveforms based on a novel technique of waveform reconstruction. We do so by simply considering the morphological properties of the waveform $\phi(x)$ (i.e. maxima, minima and asymptotic behaviour) and making the sole additional assumption that the waveform be an analytic function of the position x. We assume no knowledge of the velocity distribution functions of the electrons and of the ions sustaining the tripolar spike.

The proposed technique aims at constructing a general nonlinear differential equation for the potential waveform of the tripolar spike. We show that the quantity $\mathrm{d}\phi(x)/\mathrm{d}x$, conceived as a function of ϕ — a function which we call ϕ_x — is a piecewise analytic function in each of the open domains of the tripolar spike where $\phi(x)$ is a monotonic function of x (cf. Fig. 1). This property is proved by simply invoking the analytic nature of $\phi(x)$ as a function of x, rather than by relying on models of the charge density which are themselves analytic (cf. e.g. Ref. [10]).

Next, we show that the piecewise nature of this analytic property may be taken into account by introducing a suitable multi-valued function of ϕ, which we call the structure function of the potential and we denote by $s(\phi)$ (cf. Eq. (3.26a)), and by requiring that ϕ_x depend on ϕ, through $s(\phi)$, rather than through ϕ itself: we show in fact that, in this way, ϕ_x is everywhere an analytic function of s.

This allows us to immediately construct four Taylor series for ϕ_x in terms of powers of $s(\phi)$, each based at one of the two extrema of the potential waveform

$x = x_{\min}$ and $x = x_{\max}$, and at the lower ($x \to -\infty$) and upper ($x \to +\infty$) boundaries of the tripolar spike. We proceed by noting that these points are pair-wise adjacent (cf. Fig. 1): $-\infty$ is adjacent to $x_{\max}$, $x_{\max}$ is adjacent to $x_{\min}$, $x_{\min}$ is adjacent to $+\infty$. We show that the radius of convergence of the series originating from two adjacent points overlap and thus, because of the uniqueness of the analytic continuation, that the two Taylor series based at these points must coincide.

We use this result to further show that the Taylor series of ϕ_x contains five factors: four of them vanish at the appropriate values of $s(\phi)$ corresponding to the two extrema of the potential waveform $x = x_{\min}$ and $x = x_{\max}$, and at the lower ($x \to -\infty$) and upper ($x \to +\infty$) of the tripolar spike, as it should indeed be expected. A fifth factor, called the shape factor S accounts for the fine adjustments of the potential waveform. We show that S is an analytic function of s and we use the functional properties of the structure function $s(\phi)$ to work out the Taylor series for $S(s)$. The Taylor expansion for ϕ_x thus found precisely gives the desired differential equation for the electric potential waveform (cf. Eq. (4.1)). This nonlinear differential equation may be solved by quadrature up to fourth order in the amplitude of the potential, in terms of elliptic functions, to finally produce the desired waveform $\phi(x)$.

The ordering for the amplitude of the potential (cf. (5.5)) deserves special notice. From the definitions of the structure function $s(\phi)$ (cf. Eq. (3.26a)), we see that, if $\phi = O(\epsilon)$, $\epsilon \ll 1$, then $s(\phi) = O(\epsilon^{1/4})$. This ordering shows that the structure function — and hence the leading nonlinear terms in the differential equation governing the potential waveform (cf. Eq. (4.1)) — are $O(\epsilon^{1/4})$, rather than $O(\epsilon)$.

This fact explains, in a simple way, why nonlinear effects should be so important in tripolar spikes, even when the amplitude of the potential waveform, normalised to the electron temperature (cf. Eq. (2.3)), is very small (values of $\epsilon \simeq 10^{-4}$ were observed in Ref. [1]). It also suggests that our approach, based on the structure function formulation, properly captures the physical processes behind the nonlinear behaviour of tripolar spikes.

This conclusion is more quantitatively corroborated by even a second order model, which we explore in detail, and which allows for one free parameter only. Despite this limitation, this model is able to reproduce the potential waveforms associated with electron holes, ion holes, monotonic and non monotonic double layers and indeed tripolar spikes (compare the solid curve in Fig. 4 and e.g. Ref. [3]). In particular, this model predicts that, in all tripolar spikes, the potential waveform should have a positive skew about the position of the potential maximum (cf. Eq. (2.11)), in agreement with observations and in compliance with our earlier results of Ref. [14].

In conclusion, based on a careful inspection and a judicious functional analysis of the observed potential waveforms, our model reproduces, in a simple analytical formula, many of the morphological properties of both classical and novel nonlinear waves occurring in plasmas.

Acknowledgements

The author is grateful to A. Mangeney, J.S. Pickett and P.J. Kellogg for stimulating discussions about the observations of tripolar spikes.

References

[1] A. Mangeney, C. Salem, C. Lacombe, J.-L. Bougeret, C. Perche, R. Manning, P. Kellogg, K. Goetz, S. Monson, J.-M. Bosqued, WIND observations of coherent electrostatic waves in the solar wind, Ann. Geophys. 17 (1999) 307–320.

[2] J. Pickett, S. Kahler, L.-J. Chen, R. Huff, O. Santolík, Y. Khotyaintsev, P. Décréau, D. Winningham, M. Frahm, M. Goldstein, G. Lakhina, B. Tsurutani, B. Lavraud, D. Gurnett, M. André, A. Fazakerley, A. Balogh, H. Réme, Solitary waves observed in the auroral zone: the Cluster multi-spacecraft perspective, Nonlinear Proc. Geoph. 11 (2004) 183–196.

[3] Y. Hobara, S. Walker, M. Balikhin, O. Pokhotelov, M. Gedalin, V. Krasnoselskikh, M. Hayakawa, M. André, M. Dunlop, H. Réme, A. Fazakerley, Cluster observations of electrostatic solitary waves near the Earth's bow shock, J. Geophys. Res. 113 (2008) A05211.

[4] S. Li, X. Deng, M. Zhou, R. Tang, K. Liu, H. Kojima, H. Matsumoto, Statistical study of electrostatic solitary waves associated with reconnection: Geotail observations, Advances in Space Res. 43 (2009) 394–400.

[5] B. Quon, A. Wong, Formation of potential double layers in plasmas, Phys. Rev. Lett. 37 (1976) 1393–1396.

[6] M. Temerin, K. Cerny, W. Lotko, F. Mozer, Observation of double layers and solitary waves in the auroral plasma, Phys. Rev. Lett. 48 (1982) 1175–1179.

[7] A. Hasegawa, T. Sato, Existence of negative potential solitary wave structure and formation of a double layer, Phys. Fluids 25 (1982) 631–634.

[8] H. Schamel, Electron holes, ion holes and double layers, Phys. Rep. 140 (1986) 162–191.

[9] G. Chanteur, M. Raadu, Formation of shocklike modified Korteweg-deVries solitons: Application to double layers, Phys. Fluids 30 (1987) 2708–2719.

[10] K. Kim, Theory of nonmonotonic double layers, Phys. Fluids 30 (1987) 3686–3694.

[11] T. Kim, K. Kim, Modified K-dV theory of non-monotonic double layer in a weak relativistic plasma, Phys. Lett. A 286 (2001) 180–184.

[12] E. Parkes, Z. Zhou, B. Duffy, H. Huang, Sech-polynomial travelling solitary-wave solutions of odd-order generalized KdV equations, Phys. Lett. A 248 (1998) 219–224.

[13] Y.-Z. Peng, Exact solutions for some nonlinear partial differential equations, Phys. Lett. A 314 (2003) 401–408.

[14] L. Nocera, The morphology of electrostatic tripolar regions, Ann. Phys. 323 (2008) 2482-2504.

[15] E. Whittaker, G. Watson, A Course in Modern Analysis, Cambridge University Press, Cambridge, 1996.

Figure captions

(1) The typical waveform of the electric potential $\phi(x)$ in a tripolar spike drawn according to the morphological properties given Eq. (2.4). $x_{\min}$ and $x_{\max}$ are the extrema at which the potential respectively has an absolute minimum and a relative maximum. Δ is the reference potential jump given in Eq. (3.1).

(2) The compatibility relation (solid curve, left axis) between the boundary values of the potential waveform for a weak tripolar spike symmetrical about the potential minimum and drawn according to Eq. (5.15a). The corresponding value of the net potential gain is also shown (dashed curve, right axis). The reference potential jump Δ is given in Eq. (3.1) and Fig. 1.

(3) The rescaled structure function $r(\phi(x))$ of the potential drawn according to Eq. (5.10) for $R_0 = 0.7$.

(4) The waveforms of the electric potential $\phi(x)$ drawn according to Eqs. (5.10)–(5.13). Solid curve: generic tripolar spike, $R_0 = 1.3$, $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = 1.8\Delta$, $[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})] = 0.5\Delta$. Long dash: weak non monotonic double layer symmetric about the position of its minimum, $R_0 = 1.3$, $[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})] = \Delta$, $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = [5/4]^2\Delta$. Short dash: weak tripolar spike, symmetric about the position of its minimum and having a maximal potential gain, $R_0 = 1.3$, $[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})] = 0$, $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = [16/9]^2\Delta$. Dash single dot: asymmetric solitary ion hole (cf. Eq. (5.16)), $R_0 = 1$, $[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})] = \Delta$, $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = \Delta$. Dash double dot: symmetric solitary electron hole (cf. Eq. (5.17)), $R_0 = 1$, $[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})] = 0$, $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = 0$. Dash triple

dot: monotonically decreasing double layer (cf. Eq. (5.18)), $R_0 = 1$, $[\lim_{x\to-\infty}\phi(x) - \phi(x_{\min})] = \Delta$, $[\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] = 0$. The reference potential jump Δ is given in Eq. (3.1) and Fig. 1.

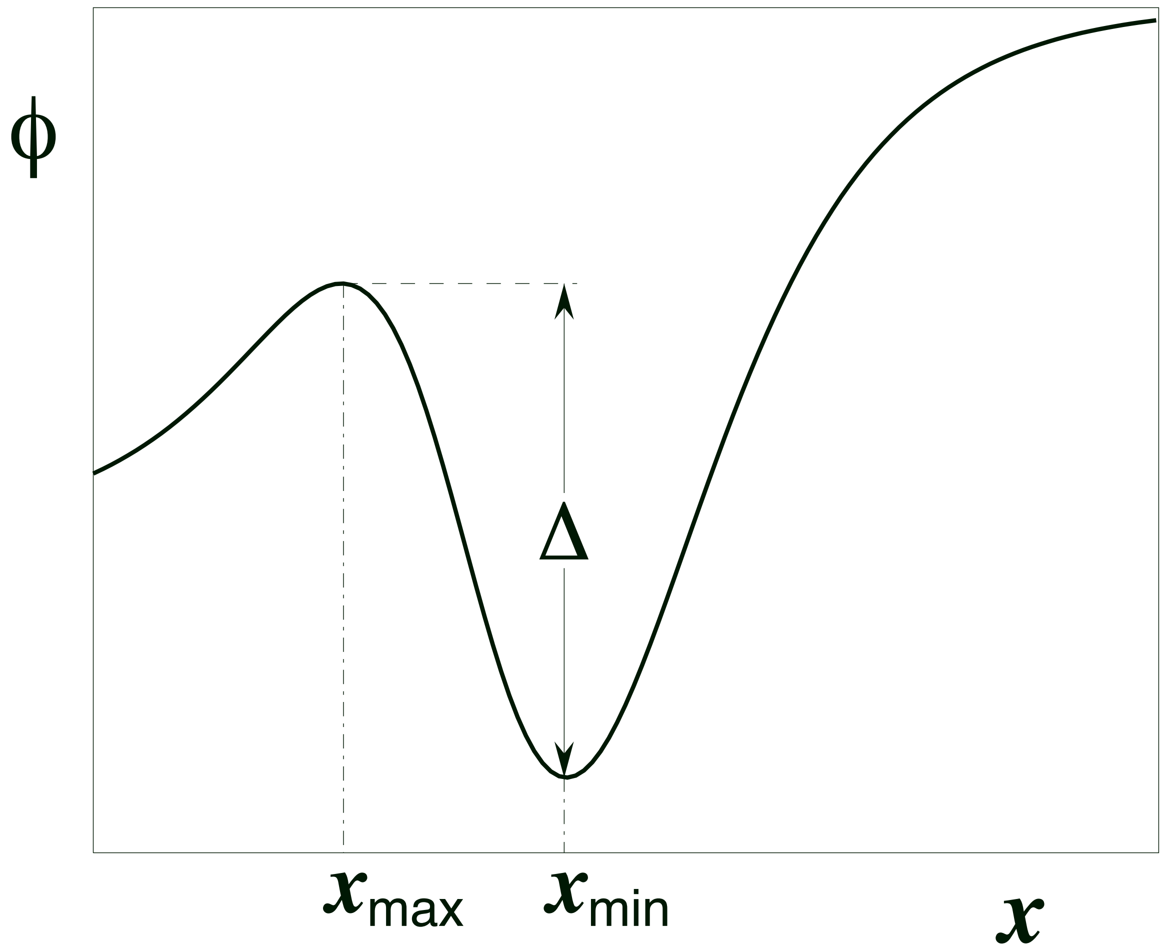

Fig. 1.

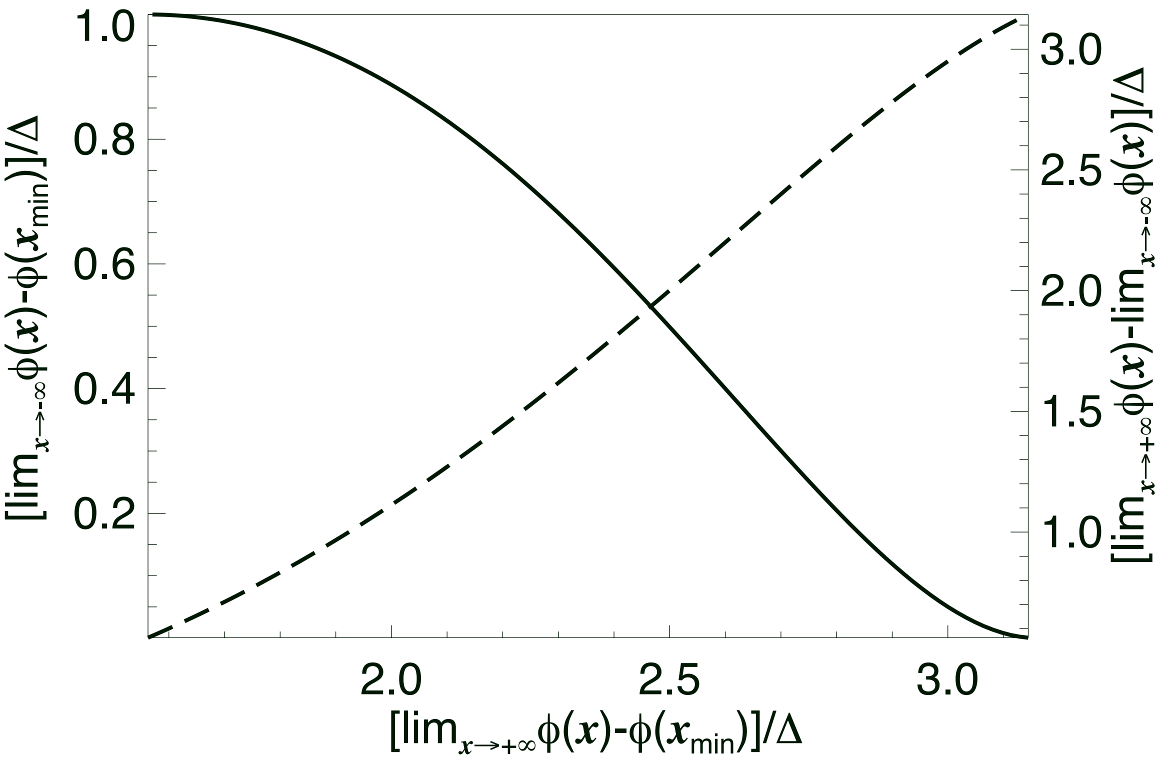
1.0
0.8
0.6
0.4
0.2
[lim$_{x\to-\infty}$φ(x)-φ(x$_{min}$)]/Δ
3.0
2.5
2.0
1.5
1.0
[lim$_{x\to+\infty}$φ(x)-lim$_{x\to-\infty}$φ(x)]/Δ
2.0
2.5
3.0
[lim$_{x\to+\infty}$φ(x)-φ(x$_{min}$)]/Δ

Fig. 2.

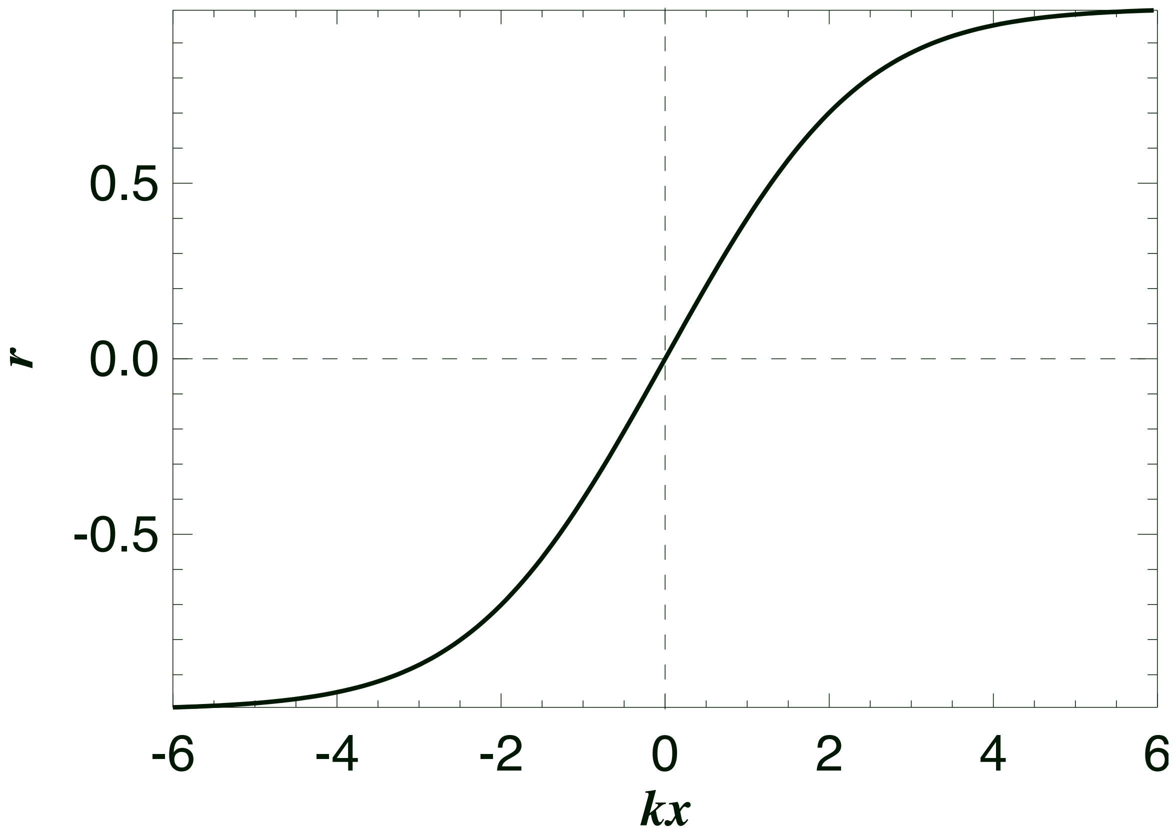

Fig. 3.

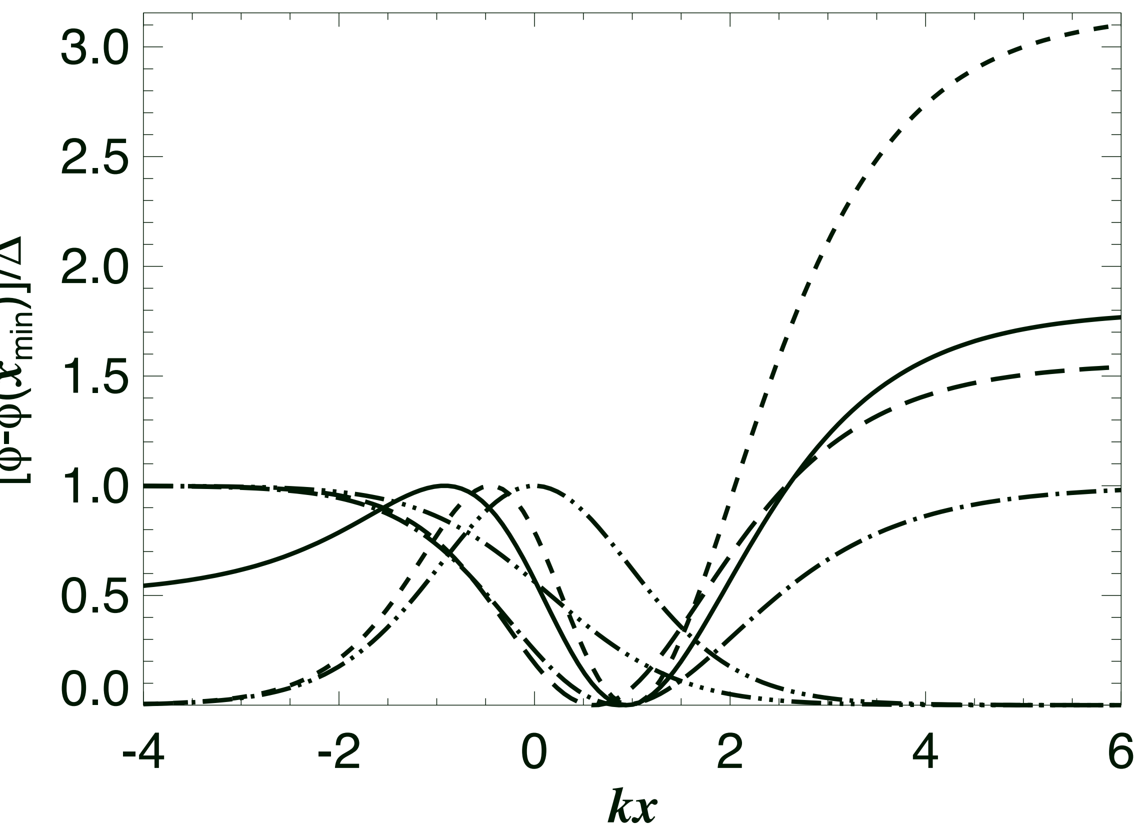

Fig. 4.