# Time discretization of a nonlinear phase field system in general domains

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**Abstract.** This paper deals with the nonlinear phase field system

$$\begin{cases} \partial_t(\theta + \ell\varphi) - \Delta\theta = f & \text{in } \Omega \times (0,T), \\ \partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) = \ell\theta, \ \xi \in \beta(\varphi) & \text{in } \Omega \times (0,T) \end{cases}$$

in a *general* domain $\Omega \subseteq \mathbb{R}^N$. Here $N \in \mathbb{N}$, $T > 0$, $\ell > 0$, $f$ is a source term, $\beta$ is a maximal monotone graph and $\pi$ is a Lipschitz continuous function. We note that in the above system the nonlinearity $\beta + \pi$ replaces the derivative of a potential of double well type. Thus it turns out that the system is a generalization of the Caginalp phase field model and it has been studied by many authors in the case that $\Omega$ is a bounded domain. However, for unbounded domains the analysis of the system seems to be at an early stage. In this paper we study the existence of solutions by employing a time discretization scheme and passing to the limit as the time step $h$ goes to 0. In the limit procedure we face with the difficulty that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is not compact in the case of unbounded domains. Moreover, we can prove an interesting error estimate of order $h^{1/2}$ for the difference between continuous and discrete solutions.

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## 1. Introduction and results

In the present contribution we address a nonlinear phase field system in a *general* domain $\Omega \subseteq \mathbb{R}^N$ and discuss it using a time discretization procedure, which turns out to be useful and efficient for the approximation and the existence proof. Moreover, we show an error estimate of order 1/2 in suitable norms for the difference between continuous and discrete solutions.

Our analysis moves from the consideration of the following simple version of the phase-field system of Caginalp type (cf. [**8, 22**]; one may also see the monographs [**7, 23, 43**]):

$$\partial_t\big(c_0\theta + \ell\phi\big) - \kappa\Delta\theta = f \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\partial_t\phi - \eta\Delta\phi + F'(\phi) = \ell\theta \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

where $\Omega$ is the three-dimensional domain in which the evolution takes place, $T$ is some final time, $\theta$ refers to the relative temperature around some critical value, that is taken to be 0 without loss of generality, and $\phi$ is the order parameter. Moreover, $c_0$, $\ell$, $\kappa$ and $\eta$ are positive constants, $f$ is a source term and $F'$ is the derivative of a smooth double-well potential $F$, whose prototype reads

$$F_{reg}(r) = \frac{1}{4}\,(r^2-1)^2\,, \quad r\in\mathbb{R}. \tag{1.3}$$

In our approach, we aim to consider other kinds of potentials, which are by now widely known and extensively used in the mathematical literature, that are the logarithmic potential and the double obstacle potential:

$$F_{log}(r) = \big((1+r)\ln(1+r) + (1-r)\ln(1-r)\big) - c_1\,r^2 \quad \text{if } r\in(-1,1), \tag{1.4}$$

$$F_{obs}(r) = I(r) - c_2\,r^2\,, \quad r\in\mathbb{R}. \tag{1.5}$$

Here, the coefficient $c_1$ in (1.4) is larger than 1, so that $F_{log}$ admits a double well, and $c_2$ in (1.5) is an arbitrary positive constant, whereas the function $I$ in (1.5) is the indicator function of $[-1,1]$, i.e., it takes the values 0 or $+\infty$ according to whether or not $r$ belongs to $[-1,1]$. Note that $F_{log}$ can be extended by continuity to the closed interval $[-1,1]$, but its derivative $F'_{log}$ turns out to be singular as the variable approaches $-1$ from the right or $+1$ from the left. On the other hand, $F_{obs}$ is even a non-smooth potential and for it it is no longer possible to consider the derivative, but we have to deal with the subdifferential of $I$. Hence, in order to be as general as possible, in our investigation we allow the potential $F$ to be just the sum

$$F = \widehat{\beta} + \widehat{\pi},$$

where $\widehat{\beta}$ is a convex function that is allowed to take the value $+\infty$ somewhere, and $\widehat{\pi}$ is a smooth perturbation which may be concave. In such a case, $\widehat{\beta}$ is supposed to be proper and lower semicontinuous, so that its subdifferential is well defined and can replace the derivative which might not exist. Let us point out that, in the case of a multivalued subdifferential like $\partial I$, equation (1.2) becomes a differential inclusion.

The system (1.1)-(1.2) is of course complemented by initial conditions like $\theta(0) = \theta_0$ and $\phi(0) = \phi_0$, and suitable boundary conditions. Concerning the latter, we set the usual homogeneous Neumann condition for both $\theta$ and $\phi$, that is,

$$\partial_\nu \theta = 0 \quad \text{and} \quad \partial_\nu \phi = 0 \quad \text{on } \partial\Omega \times (0, T),$$

where $\partial_\nu$ denotes differentiation with respect to the outward normal of $\partial\Omega$. Indeed, by these conditions we are assuming that there is no flow exchange with the exterior of $\Omega$.

Equations (1.1)-(1.2) yield a system of phase field type. Such systems have been introduced (cf. [**7, 8, 43**]) in order to include phase dissipation effects in the dynamics of moving interfaces arising in thermally induced phase transitions (the reader may also see [**6, 13, 34, 38, 39**]). In our framework, we are actually considering the following form for the total free energy:

$$G(\theta, \varphi) = \int_\Omega \left( -\frac{c_0}{2}\theta^2 - \ell\theta\varphi + F(\varphi) + \frac{\eta}{2}|\nabla\varphi|^2 \right), \tag{1.6}$$

where $c_0$ and $\ell$ stand for the specific heat and latent heat coefficients, respectively, with a terminology motivated by earlier studies (see [**21**]) on the Stefan problem; let us also refer to the monography [**23**], which deals with phase change models as well. In this connection, it is worth to introduce the enthalpy $e$ by

$$e = -\frac{\delta G}{\delta \theta}, \tag{1.7}$$

where $\frac{\delta G}{\delta \theta}$ denotes the variational derivative of $G$ with respect to $\theta$, so that (1.7) yields $e = c_0\theta + \ell\varphi$. Then the governing balance and phase equations are given by

$$\partial_t e + \operatorname{div} \mathbf{q} = f, \tag{1.8}$$

$$\partial_t \varphi + \frac{\delta G}{\delta \varphi} = 0, \tag{1.9}$$

where $\mathbf{q}$ denotes the thermal flux vector, $f$ represents some heat source and $\frac{\delta G}{\delta \varphi}$ is the variational derivative of $G$ with respect to $\varphi$. Hence (1.9) reduces exactly to (1.2) along with the homogeneous Neumann boundary condition for $\varphi$. Moreover, if we assume the classical Fourier law $\mathbf{q} = -\kappa \nabla\theta$, then (1.8) is nothing but the usual energy balance equation of the Caginalp model. Moreover, (1.1) follows from (1.8) and the Neumann boundary condition for $\theta$ is a consequence of the no-flux condition $\mathbf{q} \cdot \mathbf{n} = 0$ on the boundary. We notice that the above phase field system has received a good deal of attention in the last decades [**1, 9, 12, 15, 19, 35**] and can be deduced as a special gradient-flow problem (cf., e.g., [**41**] and references therein).

Let us point out that questions related to the well-posedness, long-time behaviour of solutions and optimal control problems have been investigated for the Caginalp system (1.1)-(1.2) and for some variation or extension of this phase field system. Without any sake of completeness, we mention the contributions [**2, 10, 11, 14, 18, 20, 22, 25, 29, 36, 37, 40**] for various qualitative analyses and [**5, 16, 17, 27, 28**] for some related control problems.

For the sake of simplicity, in the sequel of the paper we simply let $c_0 = \kappa = \eta = 1$ by observing that our treatment can be easily extended to the case of coefficients different from 1. On the other hand, we keep the parameter $\ell$ in both equations (1.1) and (1.2) since $\ell$ plays a role in the estimates.

The case of unbounded domains has the difficult mathematical point that compactness methods cannot be applied directly (related discussions can be found, e.g., in [**24, 30–33**]). It would be interesting to construct an applicable theory for the case of unbounded domains and to set assumptions for the case of unbounded domains by trying to keep some typical features in previous works, that is, in the case of bounded domains. By considering the case of unbounded domains, it may be possible to make a new finding which is not given or known in the case of bounded domains. Also, the new finding would be useful for other studies of partial differential equations.

In this paper we consider the initial-boundary value problem on a *general* domain for the nonlinear phase field system

$$
\text{(P)} \qquad \begin{cases} \partial_t(\theta + \ell\varphi) - \Delta\theta = f & \text{in } \Omega \times (0,T), \\ \partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) = \ell\theta, \ \xi \in \beta(\varphi) & \text{in } \Omega \times (0,T), \\ \partial_\nu\theta = \partial_\nu\varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \theta(0) = \theta_0, \ \varphi(0) = \varphi_0 & \text{in } \Omega, \end{cases}
$$

where $\Omega$ is a *bounded* domain or an *unbounded* domain in $\mathbb{R}^N$ with smooth bounded boundary $\partial\Omega$ (e.g., $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$, where $B(0,R)$ is the open ball with center 0 and radius $R > 0$) or $\Omega = \mathbb{R}^N$ (in this case, no boundary condition should be prescribed) or $\Omega = \mathbb{R}^N_+$. Moreover, we let $\ell > 0$ and deal with the following conditions (A1)-(A4):

(A1) $\beta \subset \mathbb{R} \times \mathbb{R}$ is a maximal monotone graph with effective domain $D(\beta)$ and $\beta(r) = \partial\widehat{\beta}(r)$, where $\partial\widehat{\beta}$ denotes the subdifferential of a proper lower semicontinuous convex function $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\widehat{\beta}(0) = 0$.

(A2) $\pi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $\pi(0) = 0$. Moreover, there exists a function $\widehat{\pi} \in C^1(\mathbb{R})$ such that $\pi = \widehat{\pi}'$.

(A3) $f \in L^2(0, T; L^2(\Omega))$.

(A4) $\theta_0, \varphi_0 \in H^1(\Omega)$ and $\widehat{\beta}(\varphi_0) \in L^1(\Omega)$.

Please note that the three functions in (1.3)-(1.5) actually satisfy (A1) and (A2), indeed we have that

$$\widehat{\beta}(r) = \frac{1}{4}r^4,\ \beta(r) = r^3,\ \widehat{\pi}(r) = \frac{1}{4}(-2r^2+1),\ \pi(r) = -r,\ r \in \mathbb{R}, \quad \text{in (1.3);}$$

$$\widehat{\beta}(r) = \begin{cases} (1+r)\log(1+r) + (1-r)\log(1-r) & \text{if } r \in (-1,1), \\ 2\log 2 & \text{if } r \in \{-1,1\}, \\ +\infty & \text{otherwise,} \end{cases}$$
$$\beta(r) = \log\frac{1+r}{1-r} \quad \text{provided that} \quad r \in (-1,1),$$
$$\widehat{\pi}(r) = -c_1 r^2,\ \pi(r) = -2c_1 r,\ r \in \mathbb{R}, \quad \text{in (1.4);}$$

$$\widehat{\beta}(r) = I(r),\ r \in \mathbb{R},\ \beta(r) = \begin{cases} 0 & \text{if } r \in (-1,1), \\ [0,\infty) & \text{if } r = 1, \\ (-\infty, 0] & \text{if } r = -1, \end{cases}$$
$$\widehat{\pi}(r) = -c_2 r^2,\ \pi(r) = -2c_2 r,\ r \in \mathbb{R}, \quad \text{in (1.5).}$$

Let us define the Hilbert spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega)$$

with inner products

$$(u_1, u_2)_H := \int_\Omega u_1 u_2 \, dx \quad (u_1, u_2 \in H),$$
$$(v_1, v_2)_V := \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx + \int_\Omega v_1 v_2 \, dx \quad (v_1, v_2 \in V),$$

respectively, and with the related Hilbertian norms. Moreover, we use the notation

$$W := \{ z \in H^2(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial\Omega \}.$$

This paper is organized as follows. In Section 2 we introduce a time discretization of (P), set precisely the approximate problem, and state the main theorems. Section 3 contains the proof of the existence for the discrete problem. In Section 4 we establish uniform estimates for the approximate problem and pass to the limit. Section 5 show error estimates between solutions of (P) and solutions of the approximate problem.

## 2. Time discretization and main results

We will prove existence of solutions to (P) by employing a time discretization scheme. More precisely, we will establish existence for (P) by passing to the limit in the problem

$$(\mathrm{P})_n \qquad \begin{cases} \delta_h\theta_n + \ell\delta_h\varphi_n - \Delta\theta_{n+1} = f_{n+1}, \\ \delta_h\varphi_n - \Delta\varphi_{n+1} + \xi_{n+1} + \pi(\varphi_{n+1}) = \ell\theta_n, \ \xi_{n+1} \in \beta(\varphi_{n+1}) \end{cases}$$

for $n = 0, ..., N-1$ as $h \searrow 0$, where $h = \frac{T}{N}$, $N \in \mathbb{N}$,

$$\delta_h\theta_n := \frac{\theta_{n+1} - \theta_n}{h}, \ \delta_h\varphi_n := \frac{\varphi_{n+1} - \varphi_n}{h}, \tag{2.1}$$

and $f_k := \frac{1}{h}\int_{(k-1)h}^{kh} f(s)\,ds$ for $k = 1, ..., N$. Also, putting

$$\widehat{\theta}_h(0) := \theta_0, \ \partial_t\widehat{\theta}_h(t) := \delta_h\theta_n, \quad \widehat{\varphi}_h(0) := \varphi_0, \ \partial_t\widehat{\varphi}_h(t) := \delta_h\varphi_n, \tag{2.2}$$

$$\overline{\theta}_h(t) := \theta_{n+1}, \ \overline{f}_h(t) := f_{n+1}, \ \overline{\varphi}_h(t) := \varphi_{n+1}, \ \overline{\xi}_h(t) := \xi_{n+1}, \ \underline{\theta}_h(t) := \theta_n \tag{2.3}$$

for a.a. $t \in (nh, (n+1)h)$, $n = 0, ..., N-1$, we can rewrite $(\mathrm{P})_n$ as the problem

$$(\mathrm{P})_h \qquad \begin{cases} \partial_t\widehat{\theta}_h + \ell\partial_t\widehat{\varphi}_h - \Delta\overline{\theta}_h = \overline{f}_h & \text{in } \Omega\times(0,T), \\ \partial_t\widehat{\varphi}_h - \Delta\overline{\varphi}_h + \overline{\xi}_h + \pi(\overline{\varphi}_h) = \ell\underline{\theta}_h, \ \overline{\xi}_h \in \beta(\overline{\varphi}_h) & \text{in } \Omega\times(0,T), \\ \widehat{\theta}(0) = \theta_0, \ \widehat{\varphi}(0) = \varphi_0 & \text{in } \Omega. \end{cases}$$

**Remark 2.1.** We have

$$\|\widehat{\theta}_h\|^2_{L^2(0,T;H)} \le h\|\theta_0\|^2_H + 2\|\overline{\theta}_h\|^2_{L^2(0,T;H)}, \tag{2.4}$$

$$\|\widehat{\varphi}_h\|^2_{L^2(0,T;H)} \le h\|\varphi_0\|^2_H + 2\|\overline{\varphi}_h\|^2_{L^2(0,T;H)}, \tag{2.5}$$

$$\|\widehat{\theta}_h\|_{L^\infty(0,T;V)} = \max\{\|\theta_0\|_V, \|\overline{\theta}_h\|_{L^\infty(0,T;V)}\}, \tag{2.6}$$

$$\|\widehat{\varphi}_h\|_{L^\infty(0,T;V)} = \max\{\|\varphi_0\|_V, \|\overline{\varphi}_h\|_{L^\infty(0,T;V)}\}, \tag{2.7}$$

$$\|\overline{\theta}_h - \widehat{\theta}_h\|^2_{L^2(0,T;H)} = \frac{h^2}{3}\|\partial_t\widehat{\theta}_h\|^2_{L^2(0,T;H)}, \tag{2.8}$$

$$\|\overline{\varphi}_h - \widehat{\varphi}_h\|^2_{L^2(0,T;H)} = \frac{h^2}{3}\|\partial_t\widehat{\varphi}_h\|^2_{L^2(0,T;H)}, \tag{2.9}$$

as the reader can check directly by using the definitions (2.2) and (2.3).

We define solutions of (P) as follows.

**Definition 2.1.** A triplet $(\theta, \varphi, \xi)$ with

$$\begin{aligned} &\theta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\ &\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\ &\xi \in L^2(0,T;H) \end{aligned}$$

is called a solution of (P) if $(\theta, \varphi, \xi)$ satisfies

$$\partial_t(\theta + \ell\varphi) - \Delta\theta = f \quad \text{a.e. on } \Omega \times (0,T), \tag{2.10}$$

$$\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) = \ell\theta,\ \xi \in \beta(\varphi) \quad \text{a.e. on } \Omega \times (0,T), \tag{2.11}$$

$$\theta(0) = \theta_0,\ \varphi(0) = \varphi_0 \quad \text{a.e. on } \Omega. \tag{2.12}$$

Now the main results read as follows.

**Theorem 2.1.** *Assume that* (A1)-(A4) *hold. Then for all* $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$ *there exists a unique solution* $(\theta_{n+1}, \varphi_{n+1}, \xi_{n+1})$ *of* $(\mathrm{P})_n$ *satisfying*

$$\theta_{n+1}, \varphi_{n+1} \in W \quad \text{and} \quad \xi_{n+1} \in H \qquad \text{for } n = 0, ..., N-1.$$

**Theorem 2.2.** *Assume that* (A1)-(A4) *hold. Then there exists a unique solution of* (P).

**Theorem 2.3.** *Assume that* (A1)-(A4) *hold. Assume further that* $f \in W^{1,1}(0,T;H)$. *Then there exists* $h_0 \in \left(0, \min\left\{1, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right\}\right)$ *such that for all* $\ell > 0$ *there exists a constant* $M_1 = M_1(\ell, T) > 0$ *such that*

$$\begin{aligned} &\ell\|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;H)} + \ell\|\overline{\varphi}_h - \varphi\|_{L^2(0,T;V)} \\ &+ \|\widehat{\theta}_h - \theta + \ell(\widehat{\varphi}_h - \varphi)\|_{L^\infty(0,T;H)} + \|\overline{\theta}_h - \theta\|_{L^2(0,T;V)} \\ &\le M_1 h^{1/2} \end{aligned} \tag{2.13}$$

*for all* $h \in (0, h_0)$. *In particular, for all* $\ell > 0$ *there exists a constant* $M_2 = M_2(\ell, T) > 0$ *such that*

$$\|\widehat{\theta}_h - \theta\|_{L^\infty(0,T;H)} \le M_2 h^{1/2}$$

*for all* $h \in (0, h_0)$.

## 3. Existence of discrete solution

In this section we will prove Theorem 2.1.

**Lemma 3.1.** *For all* $g \in H$ *and all* $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$ *there exists a unique solution* $(\varphi, \xi)$ *of the equation* $\varphi - h\Delta\varphi + \xi + h\pi(\varphi) = g$, $\xi \in \beta(\varphi)$, *satisfying* $\varphi \in W$ *and* $\xi \in H$.

*Proof.* Let $\varepsilon > 0$ and let $\beta_\varepsilon$ be the Yosida approximation of $\beta$ on $\mathbb{R}$. Then the operator $1 - h\Delta + h\beta_\varepsilon + h\pi : V \to V^*$ is monotone, continuous and coercive for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$. Indeed, we have

$$\begin{aligned} \langle \psi - h\Delta\psi + h\beta_\varepsilon(\psi) + h\pi(\psi), \psi\rangle_{V^*,V} &\ge (1 - h\|\pi'\|_{L^\infty(R)})\|\psi\|_H^2 + h\|\nabla\psi\|_H^2 \\ &\ge \min\{1 - h\|\pi'\|_{L^\infty(R)}, h\}\|\psi\|_V^2 \end{aligned}$$

for all $\psi \in V$. Thus the operator $1 - h\Delta + h\beta_\varepsilon + h\pi : V \to V^*$ is surjective for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$ (see e.g., [**4**, p. 37]) and then the elliptic regularity theory yields that for all $g \in H$ and all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$ there exists a unique solution $\varphi_\varepsilon \in W$ of the equation

$$\varphi_\varepsilon - h\Delta\varphi_\varepsilon + h(\beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon)) = g. \tag{3.1}$$

Here, multiplying (3.1) by $\varphi_\varepsilon$ and integrating over $\Omega$, we obtain the inequality

$$\begin{aligned}
&\|\varphi_\varepsilon\|_H^2 + h\|\nabla\varphi_\varepsilon\|_H^2 + h(\beta_\varepsilon(\varphi_\varepsilon), \varphi_\varepsilon)_H \\
&= (g, \varphi_\varepsilon)_H - h(\pi(\varphi_\varepsilon), \varphi_\varepsilon)_H \\
&\le \frac{1}{2(1 - h\|\pi'\|_{L^\infty(R)})}\|g\|_H^2 + \frac{1 + h\|\pi'\|_{L^\infty(R)}}{2}\|\varphi_\varepsilon\|_H^2
\end{aligned}$$

for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$, and hence for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$ there exists a constant $C_1 = C_1(h) > 0$ such that

$$\|\varphi_\varepsilon\|_V \le C_1 \tag{3.2}$$

for all $\varepsilon > 0$. We see from (3.1) and (3.2) that

$$\begin{aligned}
&\|\beta_\varepsilon(\varphi_\varepsilon)\|_H^2 \\
&= (\beta_\varepsilon(\varphi_\varepsilon), \beta_\varepsilon(\varphi_\varepsilon))_H \\
&= -\frac{1}{h}(\beta_\varepsilon(\varphi_\varepsilon), \varphi_\varepsilon)_H - \int_\Omega \beta_\varepsilon'(\varphi_\varepsilon)|\nabla\varphi_\varepsilon|^2 - (\pi(\varphi_\varepsilon), \beta_\varepsilon(\varphi_\varepsilon))_H + \frac{1}{h}(g, \beta_\varepsilon(\varphi_\varepsilon))_H \\
&\le C_1^2\|\pi'\|_{L^\infty(\mathbb{R})}^2 + \frac{1}{h^2}\|g\|_H^2 + \frac{1}{2}\|\beta_\varepsilon(\varphi_\varepsilon)\|_H^2
\end{aligned}$$

for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$. Hence for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$ there exists a constant $C_2 = C_2(h) > 0$ such that

$$\|\beta_\varepsilon(\varphi_\varepsilon)\|_H \le C_2 \tag{3.3}$$

for all $\varepsilon > 0$. Moreover, (3.1)-(3.3) yield that

$$\begin{aligned}
\|\Delta\varphi_\varepsilon\|_H &= \frac{1}{h}\|\varphi_\varepsilon + h(\beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon)) - g\|_H \\
&\le \frac{C_1}{h} + C_2 + C_1\|\pi'\|_{L^\infty(\mathbb{R})} + \frac{1}{h}\|g\|_H
\end{aligned}$$

for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$. Thus, by the elliptic regularity theory and (3.2), for all $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$ there exists a constant $C_3 = C_3(h) > 0$ such that

$$\|\varphi_\varepsilon\|_W \le C_3 \tag{3.4}$$

for all $\varepsilon > 0$. Therefore we infer from (3.3) and (3.4) that there exist some functions $\varphi \in W$, $\xi \in H$ and a subsequence of $\varepsilon$ such that

$$\varphi_\varepsilon \to \varphi \quad \text{weakly in } W, \tag{3.5}$$

$$\beta_\varepsilon(\varphi_\varepsilon) \to \xi \quad \text{weakly in } H \tag{3.6}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Now we confirm that

$$\varphi - h\Delta\varphi + h\xi + h\pi(\varphi) = g. \tag{3.7}$$

Let $\psi \in C_{\mathrm{c}}^\infty(\overline{\Omega})$. Then there exists a bounded domain $D \subset \Omega$ with smooth boundary such that $\operatorname{supp}\psi \subset D$ and it follows from (3.1) that

$$\begin{aligned} 0 &= \int_\Omega (\varphi_\varepsilon - h\Delta\varphi_\varepsilon + h(\beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon)) - g)\psi \\ &= (\varphi_\varepsilon - h\Delta\varphi_\varepsilon + h\beta_\varepsilon(\varphi_\varepsilon) - g, \psi)_H + h\int_D \pi(\varphi_\varepsilon)\psi. \end{aligned} \tag{3.8}$$

Here, since the embedding $H^1(D) \hookrightarrow L^2(D)$ is compact, we deduce from (3.2) and (3.5) that

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } L^2(D) \tag{3.9}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Thus we derive from (3.5), (3.6), (3.8) and (3.9) that

$$\int_\Omega (\varphi - h\Delta\varphi + h\xi + h\pi(\varphi) - g)\psi = 0$$

for all $\psi \in C_{\mathrm{c}}^\infty(\overline{\Omega})$, which implies (3.7).

Next we show that

$$\xi \in \beta(\varphi) \quad \text{a.e. on } \Omega. \tag{3.10}$$

Let $E \subset \Omega$ be an arbitrary bounded domain with smooth boundary. Then we have

$$1_E\varphi_\varepsilon \to 1_E\varphi \quad \text{strongly in } H \tag{3.11}$$

as $\varepsilon = \varepsilon_j \searrow 0$, where $1_E$ is the characteristic function on $E$. Hence we see from (3.6) and (3.11) that

$$\int_\Omega \beta_\varepsilon(1_E\varphi_\varepsilon) \cdot 1_E\varphi_\varepsilon = (\beta_\varepsilon(\varphi_\varepsilon), 1_E\varphi_\varepsilon)_H \to (\xi, 1_E\varphi)_H = \int_\Omega 1_E\xi \cdot 1_E\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, and consequently it holds that $1_E\xi \in \beta(1_E\varphi)$ a.e. on $\Omega$ (see e.g., [**3**, Lemma 1.3, p. 42]), that is,

$$\xi = 1_E\xi \in \beta(1_E\varphi) = \beta(\varphi) \quad \text{a.e. on } E.$$

Thus, since $E$ is arbitrary, we can obtain (3.10).

Therefore combining (3.7) and (3.10) leads to the equation $\varphi - h\Delta\varphi + h(\xi + \pi(\varphi)) = g$, with $\xi \in \beta(\varphi)$. Moreover, the solution $(\varphi, \xi)$ of this problem is unique. Indeed, letting $(\varphi_j, \xi_j)$, $j = 1, 2$, be two solutions, we infer that

$$\begin{aligned} 0 &= \langle \varphi_1 - \varphi_2 - h\Delta(\varphi_1 - \varphi_2) + h(\xi_1 - \xi_2) + h(\pi(\varphi_1) - \pi(\varphi_2)), \varphi_1 - \varphi_2 \rangle_{V^*,V} \\ &\geq (1 - h\|\pi'\|_{L^\infty(R)})\|\varphi_1 - \varphi_2\|_H^2 + h\|\nabla(\varphi_1 - \varphi_2)\|_H^2 \\ &\geq \min\{1 - h\|\pi'\|_{L^\infty(R)}, h\}\|\varphi_1 - \varphi_2\|_V^2, \end{aligned}$$

which means that $\varphi_1 = \varphi_2$. Then the identity $\xi_1 = \xi_2$ holds by comparing the equations for $(\varphi_1, \xi_1)$ and $(\varphi_2, \xi_2)$. □

**Proof of Theorem 2.1.** The problem $(\mathrm{P})_n$ can be written as

$$(\mathrm{Q})_n \qquad \begin{cases} \theta_{n+1} - h\Delta\theta_{n+1} = hf_{n+1} + \ell\varphi_n - \ell\varphi_{n+1} + \theta_n, \\ \varphi_{n+1} - h\Delta\varphi_{n+1} + h(\xi_{n+1} + \pi(\varphi_{n+1})) = \varphi_n + h\ell\theta_n, \ \xi_{n+1} \in \beta(\varphi_{n+1}). \end{cases}$$

To prove Theorem 2.1 it suffices to establish existence and uniqueness of solutions to $(\mathrm{Q})_n$ in the case that $n = 0$. Let $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$. Then, by Lemma 3.1, there exists a unique solution $(\varphi_1, \xi_1)$ of

$$\varphi_1 - h\Delta\varphi_1 + h(\xi_1 + \pi(\varphi_1)) = \varphi_0 + \ell h\theta_0, \ \xi_1 \in \beta(\varphi_1),$$

satisfying $\varphi_1 \in W$ and $\xi_1 \in H$. Also, for this function $\varphi_1$ there exists a unique solution $\theta_1 \in W$ of the equation

$$\theta_1 - h\Delta\theta_1 = hf_1 + \ell\varphi_0 - \ell\varphi_1 + \theta_0.$$

Thus we conclude that there exists a unique solution $(\theta_{n+1}, \varphi_{n+1}, \xi_{n+1})$ of $(\mathrm{P})_n$ satisfying $\theta_{n+1}, \varphi_{n+1} \in W$ and $\xi_{n+1} \in H$ for $n = 0, ..., N-1$.

## 4. Uniform estimates and passage to the limit

This section will prove Theorem 2.2. We will derive a priori estimates for $(\mathrm{P})_h$ to establish existence for (P) by passing to the limit in $(\mathrm{P})_h$.

**Lemma 4.1.** *There exists $h_1 \in \left(0, \min\left\{1, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right\}\right)$ such that for all $\ell > 0$ there is a constant $C = C(\ell, T) > 0$ such that*

$$\begin{aligned} &\|\overline{\theta}_h\|_{L^\infty(0,T;H)}^2 + \ell^2\|\overline{\varphi}_h\|_{L^\infty(0,T;V)}^2 + h\|\partial_t\widehat{\theta}_h\|_{L^2(0,T;H)}^2 \\ &\quad + \ell^2\|\partial_t\widehat{\varphi}_h\|_{L^2(0,T;H)}^2 + \|\overline{\theta}_h\|_{L^2(0,T;V)}^2 + \ell^2\|\widehat{\beta}(\overline{\varphi}_h)\|_{L^\infty(0,T;L^1(\Omega))} \leq C \end{aligned}$$

*for all $h \in (0, h_1)$.*

*Proof.* Please note that, provided $h \in \left(0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right)$, then the solution $(\theta_{n+1}, \varphi_{n+1}, \xi_{n+1})$ to $(\mathrm{P})_n$ is well-defined due to Theorem 2.1. Multiplying the first equation in $(\mathrm{P})_n$ by $h\theta_{n+1}$ and integrating over $\Omega$, we have

$$
\begin{aligned}
(4.1)\qquad & \frac{1}{2}\int_\Omega |\theta_{n+1}|^2 - \frac{1}{2}\int_\Omega |\theta_n|^2 + \frac{1}{2}\int_\Omega |\theta_{n+1}-\theta_n|^2 + h\int_\Omega |\nabla\theta_{n+1}|^2 \\
&= h\int_\Omega f_{n+1}\theta_{n+1} + \ell h \int_\Omega \frac{\varphi_n - \varphi_{n+1}}{h}\theta_{n+1},
\end{aligned}
$$

where the identity $(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$ $(a, b \in \mathbb{R})$ was applied. Multiplying the second equation in $(\mathrm{P})_n$ by $\ell^2(\varphi_{n+1} - \varphi_n)$ and integrating over $\Omega$ lead to

$$
\begin{aligned}
(4.2)\qquad & \ell^2 h \int_\Omega \left|\frac{\varphi_{n+1}-\varphi_n}{h}\right|^2 + \ell^2(\varphi_{n+1}, \varphi_{n+1}-\varphi_n)_V \\
&+ \ell^2\int_\Omega (\xi_{n+1} + \pi(\varphi_{n+1}) - \varphi_{n+1})(\varphi_{n+1}-\varphi_n) = \ell^3 h\int_\Omega \theta_n \frac{\varphi_{n+1}-\varphi_n}{h}.
\end{aligned}
$$

Thus it follows from (4.1), (4.2) and the Young inequality that

$$
\begin{aligned}
(4.3)\qquad & \frac{1}{2}\int_\Omega |\theta_{n+1}|^2 - \frac{1}{2}\int_\Omega |\theta_n|^2 + \frac{1}{2}\int_\Omega |\theta_{n+1}-\theta_n|^2 + h\int_\Omega |\nabla\theta_{n+1}|^2 \\
&+ \ell^2 h \int_\Omega \left|\frac{\varphi_{n+1}-\varphi_n}{h}\right|^2 + \ell^2(\varphi_{n+1}, \varphi_{n+1}-\varphi_n)_V \\
&+ \ell^2\int_\Omega (\xi_{n+1} + \pi(\varphi_{n+1}) - \varphi_{n+1})(\varphi_{n+1}-\varphi_n) \\
&= h\int_\Omega f_{n+1}\theta_{n+1} + \ell h\int_\Omega \frac{\varphi_n-\varphi_{n+1}}{h}\theta_{n+1} + \ell^3 h\int_\Omega \theta_n\frac{\varphi_{n+1}-\varphi_n}{h} \\
&\le \frac{h}{2}\int_\Omega |f_{n+1}|^2 + \frac{3}{2}h\int_\Omega |\theta_{n+1}|^2 + \frac{\ell^2 h}{2}\int_\Omega \left|\frac{\varphi_{n+1}-\varphi_n}{h}\right|^2 + h\ell^4\int_\Omega |\theta_n|^2.
\end{aligned}
$$

Here we point out the identity

$$
(4.4)\qquad \ell^2(\varphi_{n+1}, \varphi_{n+1}-\varphi_n)_V = \frac{\ell^2}{2}\|\varphi_{n+1}\|_V^2 - \frac{\ell^2}{2}\|\varphi_n\|_V^2 + \frac{\ell^2}{2}\|\varphi_{n+1}-\varphi_n\|_V^2
$$

and recall the inclusion $\xi_{n+1} \in \beta(\varphi_{n+1})$, (A1), the definition of the subdifferential, and the Young inequality in order to infer that

$$\begin{aligned}
(4.5)\qquad &\ell^2\int_\Omega(\xi_{n+1}+\pi(\varphi_{n+1})-\varphi_{n+1})(\varphi_{n+1}-\varphi_n)\\
&=\ell^2\int_\Omega\xi_{n+1}(\varphi_{n+1}-\varphi_n)+\ell^2\int_\Omega\pi(\varphi_{n+1})(\varphi_{n+1}-\varphi_n)-\ell^2\int_\Omega\varphi_{n+1}(\varphi_{n+1}-\varphi_n)\\
&\geq \ell^2\int_\Omega\widehat{\beta}(\varphi_{n+1})-\ell^2\int_\Omega\widehat{\beta}(\varphi_n)-2(\|\pi'\|^2_{L^\infty(\mathbb{R})}+1)\ell^2h\|\varphi_{n+1}\|^2_V\\
&\quad-\frac{\ell^2h}{4}\int_\Omega\left|\frac{\varphi_{n+1}-\varphi_n}{h}\right|^2.
\end{aligned}$$

Thus from (4.3)-(4.5) we have

$$\begin{aligned}
(4.6)\qquad &\frac{1}{2}\int_\Omega|\theta_{n+1}|^2-\frac{1}{2}\int_\Omega|\theta_n|^2+\frac{1}{2}\int_\Omega|\theta_{n+1}-\theta_n|^2+h\int_\Omega|\nabla\theta_{n+1}|^2\\
&+\frac{\ell^2h}{4}\int_\Omega\left|\frac{\varphi_{n+1}-\varphi_n}{h}\right|^2+\frac{\ell^2}{2}\|\varphi_{n+1}\|^2_V-\frac{\ell^2}{2}\|\varphi_n\|^2_V\\
&+\frac{\ell^2}{2}\|\varphi_{n+1}-\varphi_n\|^2_V+\ell^2\int_\Omega\widehat{\beta}(\varphi_{n+1})-\ell^2\int_\Omega\widehat{\beta}(\varphi_n)\\
&\leq\frac{h}{2}\int_\Omega|f_{n+1}|^2+\frac{3}{2}h\int_\Omega|\theta_{n+1}|^2+h\ell^4\int_\Omega|\theta_n|^2+2(\|\pi'\|^2_{L^\infty(\mathbb{R})}+1)\ell^2h\|\varphi_{n+1}\|^2_V.
\end{aligned}$$

Therefore, summing (4.6) over $n=0,...,m-1$ with $1\leq m\leq N$, we obtain the inequality

$$\begin{aligned}
&\frac{1}{2}(1-3h)\int_\Omega|\theta_m|^2-\frac{1}{2}\int_\Omega|\theta_0|^2+\frac{1}{2}\sum_{n=0}^{m-1}\int_\Omega|\theta_{n+1}-\theta_n|^2+h\sum_{n=0}^{m-1}\int_\Omega|\nabla\theta_{n+1}|^2\\
&+\frac{\ell^2h}{4}\sum_{n=0}^{m-1}\int_\Omega\left|\frac{\varphi_{n+1}-\varphi_n}{h}\right|^2+\frac{\ell^2}{2}(1-4(\|\pi'\|^2_{L^\infty(\mathbb{R})}+1)h)\|\varphi_m\|^2_V-\frac{\ell^2}{2}\|\varphi_0\|^2_V\\
&+\frac{\ell^2}{2}\sum_{n=0}^{m-1}\|\varphi_{n+1}-\varphi_n\|^2_V+\ell^2\int_\Omega\widehat{\beta}(\varphi_m)-\ell^2\int_\Omega\widehat{\beta}(\varphi_0)\\
&\leq\frac{h}{2}\sum_{n=0}^{m-1}\int_\Omega|f_{n+1}|^2+\frac{3}{2}h\sum_{n=0}^{m-2}\int_\Omega|\theta_{n+1}|^2\\
&\quad+h\ell^4\sum_{n=0}^{m-1}\int_\Omega|\theta_n|^2+2(\|\pi'\|^2_{L^\infty(\mathbb{R})}+1)\ell^2h\sum_{n=0}^{m-2}\|\varphi_{n+1}\|^2_V.
\end{aligned}$$

Then there exists $h_1\in\left(0,\min\left\{1,\frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}}\right\}\right)$ (e.g., $h_1=\frac{1}{4(\|\pi'\|^2_{L^\infty(\mathbb{R})}+1)}$) such that for all

$\ell > 0$ there exists a constant $C_1 = C_1(\ell, T) > 0$ such that

$$\int_\Omega |\theta_m|^2 + h^2 \sum_{n=0}^{m-1} \int_\Omega |\delta\theta_n|^2 + h \sum_{n=0}^{m-1} \int_\Omega |\nabla\theta_{n+1}|^2 + \ell^2 h \sum_{n=0}^{m-1} \int_\Omega |\delta\varphi_n|^2 + \ell^2 \|\varphi_m\|_V^2$$
$$+ \ell^2 \int_\Omega \widehat{\beta}(\varphi_m) \le C_1 + C_1 h \sum_{n=0}^{m-1} \int_\Omega |\theta_j|^2 + C_1 h \sum_{n=0}^{m-1} \|\varphi_j\|_V^2$$

for all $h \in (0, h_1)$ and $m = 1, ..., N$. Hence we see from the discrete Gronwall lemma (see e.g., [**26**, Prop. 2.2.1]) that for all $\ell > 0$ there exists a constant $C_2 = C_2(\ell, T) > 0$ such that

$$\int_\Omega |\theta_m|^2 + h^2 \sum_{n=0}^{m-1} \int_\Omega |\delta\theta_n|^2 + h \sum_{n=0}^{m-1} \int_\Omega |\nabla\theta_{n+1}|^2$$
$$+ \ell^2 h \sum_{n=0}^{m-1} \int_\Omega |\delta\varphi_n|^2 + \ell^2 \|\varphi_m\|_V^2 + \ell^2 \int_\Omega \widehat{\beta}(\varphi_m) \le C_2$$

for all $h \in (0, h_1)$ and $m = 1, ..., N$. □

**Lemma 4.2.** *Let $h_1$ be as in Lemma 4.1. Then for all $\ell > 0$ there exists a constant $C = C(\ell, T) > 0$ such that*

$$\|\partial_t \widehat{\theta}_h\|_{L^2(0,T;H)}^2 + \|\overline{\theta}_h\|_{L^\infty(0,T;V)}^2 \le C$$

*for all $h \in (0, h_1)$.*

*Proof.* Multiplying the first equation in $(\mathrm{P})_n$ by $h\delta_h\theta_n$ and integrating over $\Omega$, we have

$$h \int_\Omega |\delta_h\theta_n|^2 + \ell h \int_\Omega \delta_h\varphi_n \cdot \delta_h\theta_n + h \int_\Omega \nabla\theta_{n+1} \cdot \nabla\delta_h\theta_n = h \int_\Omega f_{n+1}\delta_h\theta_n. \tag{4.7}$$

Here it holds that

$$h \int_\Omega \nabla\theta_{n+1} \cdot \nabla\delta_h\theta_n = \frac{1}{2} \int_\Omega |\nabla\theta_{n+1}|^2 - \frac{1}{2} \int_\Omega |\nabla\theta_n|^2 + \frac{1}{2} \int_\Omega |\nabla(\theta_{n+1} - \theta_n)|^2. \tag{4.8}$$

Thus we see from (4.7), (4.8) and the Young inequality that

$$\begin{aligned} h \int_\Omega |\delta_h\theta_n|^2 &+ \frac{1}{2} \int_\Omega |\nabla\theta_{n+1}|^2 - \frac{1}{2} \int_\Omega |\nabla\theta_n|^2 + \frac{1}{2} \int_\Omega |\nabla(\theta_{n+1} - \theta_n)|^2 \\ &= h \int_\Omega f_{n+1}\delta_h\theta_n - \ell h \int_\Omega \delta_h\varphi_n \cdot \delta_h\theta_n \\ &\le h \int_\Omega |f_{n+1}|^2 + \frac{h}{2} \int_\Omega |\delta_h\theta_n|^2 + \ell^2 h \int_\Omega |\delta_h\varphi_n|^2. \end{aligned} \tag{4.9}$$

Therefore, by summing (4.9) over $n = 0, ..., m-1$ with $1 \le m \le N$ and Lemma 4.1, we can prove Lemma 4.2. □

**Lemma 4.3.** *Let $h_1$ be as in Lemma 4.1. Then for all $\ell > 0$ there exists a constant $C = C(\ell, T) > 0$ such that*

$$\|\overline{\xi}_h\|^2_{L^2(0,T;H)} \le C$$

*for all $h \in (0, h_1)$.*

*Proof.* We formally derive the estimate by assuming that $\beta$ is Lipschitz continuous (as for the Yosida approximation) and observing that the estimate can be extended to the general case by lower semicontinuity. We test the second equation in $(\mathrm{P})_n$ by $h\xi_{n+1}$. Since $(-\Delta\varphi_{n+1}, \xi_{n+1})_H = \int_\Omega \beta'(\varphi_{n+1})|\nabla\varphi_{n+1}|^2 \ge 0$, then by the Young inequality we obtain

$$\begin{aligned}
&h\int_\Omega |\xi_{n+1}|^2 \\
&\le h\ell\int_\Omega \theta_n\xi_{n+1} - h\int_\Omega \delta_h\varphi_n \cdot \xi_{n+1} - h\int_\Omega \pi(\varphi_{n+1})\xi_{n+1} \\
&\le \frac{3}{2}h\ell^2\int_\Omega |\theta_n|^2 + \frac{3}{2}h\int_\Omega |\delta_h\varphi_n|^2 + \frac{3}{2}\|\pi'\|^2_{L^\infty(\mathbb{R})}h\int_\Omega |\varphi_{n+1}|^2 + \frac{1}{2}h\int_\Omega |\xi_{n+1}|^2.
\end{aligned}$$

Thus, by summing over $n = 0, ..., m-1$ with $1 \le m \le N$, and recalling Lemmas 4.1 and 4.2, we conclude that Lemma 4.3 holds. □

**Lemma 4.4.** *Let $h_1$ be as in Lemma 4.1. Then for all $\ell > 0$ there exist constants $C = C(\ell, T) > 0$ and $\widetilde{C} = \widetilde{C}(\ell, T) > 0$ such that*

$$\|\Delta\overline{\theta}_h\|^2_{L^2(0,T;H)} + \|\Delta\overline{\varphi}_h\|^2_{L^2(0,T;H)} \le C,$$

$$\|\overline{\theta}_h\|^2_{L^2(0,T;W)} + \|\overline{\varphi}_h\|^2_{L^2(0,T;W)} \le \widetilde{C} \tag{4.10}$$

*for all $h \in (0, h_1)$.*

*Proof.* From the first equation in $(\mathrm{P})_n$ we have

$$\begin{aligned}
h^{1/2}\|\Delta\theta_{n+1}\|_H &= h^{1/2}\|\delta_h\theta_n + \ell\delta_h\varphi_n - f_{n+1}\|_H \\
&\le h^{1/2}\|\delta_h\theta_n\|_H + h^{1/2}\ell\|\delta_h\varphi_n\|_H + h^{1/2}\|f_{n+1}\|_H.
\end{aligned}$$

Thus, by Lemmas 4.1 and 4.2, there exists a constant $C_1 = C_1(T) > 0$ such that

$$\|\Delta\overline{\theta}_h\|^2_{L^2(0,T;H)} \le C_1. \tag{4.11}$$

It follows from the second equation in $(\mathrm{P})_n$ that

$$\begin{aligned}
h^{1/2}\|\Delta\varphi_{n+1}\|_H &= h^{1/2}\|\delta_h\varphi_n + \xi_{n+1} + \pi(\varphi_{n+1}) - \ell\theta_n\|_H \\
&\le h^{1/2}\|\delta_h\varphi_n\|_H + h^{1/2}\|\xi_{n+1}\|_H + h^{1/2}\|\pi'\|_{L^\infty(\mathbb{R})}\|\varphi_{n+1}\|_H + h^{1/2}\ell\|\theta_n\|_H.
\end{aligned}$$

Hence Lemmas 4.1-4.3 yield that there exists a constant $C_2 = C_2(T) > 0$ such that

$$\|\Delta\overline{\varphi}_h\|^2_{L^2(0,T;H)} \le C_2. \tag{4.12}$$

Now we recall Lemma 4.1 once more and invoke the elliptic regularity theory to infer the estimate (4.10). Therefore, by virtue of (4.11) and (4.12), Lemma 4.4 is completely proved. □

**Lemma 4.5.** *Let $h_1$ be as in Lemma 4.1. Then for all $\ell > 0$ there exists a constant $C = C(\ell, T) > 0$ such that*

$$\|\widehat{\theta}_h\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\widehat{\varphi}_h\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C$$

*for all $h \in (0, h_1)$.*

*Proof.* Lemmas 4.1 and 4.2, along with (2.4)-(2.7), lead to Lemma 4.5. □

**Proof of Theorem 2.2.** By Lemmas 4.1-4.5, (2.1)-(2.3), (2.8) and (2.9) there exist some functions $\theta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$, $\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ and $\xi \in L^2(0,T;H)$ and a subsequence of $h$ such that

$$\widehat{\theta}_h \to \theta \quad \text{weakly in } H^1(0,T;H), \tag{4.13}$$

$$\widehat{\varphi}_h \to \varphi \quad \text{weakly in } H^1(0,T;H), \tag{4.14}$$

$$\overline{\theta}_h \to \theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;V),$$

$$\overline{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } L^\infty(0,T;V),$$

$$\overline{\theta}_h \to \theta \quad \text{weakly in } L^2(0,T;W), \tag{4.15}$$

$$\overline{\varphi}_h \to \varphi \quad \text{weakly in } L^2(0,T;W), \tag{4.16}$$

$$\underline{\theta}_h = \overline{\theta}_h - h\partial_t\widehat{\theta}_h \to \theta \quad \text{weakly in } L^2(0,T;H), \tag{4.17}$$

$$\overline{\xi}_h \to \xi \quad \text{weakly in } L^2(0,T;H) \tag{4.18}$$

as $h = h_j \searrow 0$. Combining (4.13)-(4.15), observing that $\overline{f}_h \to f$ strongly in $L^2(0,T;H)$ as $h \searrow 0$ (cf. Remark 5.1) and passing to the limit in the first equation in $(\mathrm{P})_h$ lead to (2.10). Now we show that

$$\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) = \ell\theta. \tag{4.19}$$

Let $\psi \in C_c^\infty([0,T] \times \overline{\Omega})$. Then there exists a bounded domain $D \subset \Omega$ with smooth boundary such that $\operatorname{supp}\psi \subset (0,T) \times D$ and we see from the second equation in $(\mathrm{P})_h$ that

$$\begin{aligned} 0 = &\int_0^T (\partial_t\widehat{\varphi}_h(t) - \Delta\overline{\varphi}_h(t) + \overline{\xi}_h(t) - \ell\underline{\theta}_h(t), \psi(t))_H \, dt \\ &+ \int_0^T \left( \int_D \pi(\overline{\varphi}_h(t))\psi(t) \right) dt. \end{aligned} \tag{4.20}$$

Here, since the embedding $H^1(D) \hookrightarrow L^2(D)$ is compact, we derive from Lemma 4.5 and (4.14) that

$$\widehat{\varphi}_h \to \varphi \quad \text{strongly in } C([0,T];L^2(D)) \tag{4.21}$$

as $h = h_j \searrow 0$ (see e.g., [**42**, Section 8, Corollary 4]). Also, we infer from (2.9), Lemma 4.1 and (4.21) that

$$\begin{aligned}\|\overline{\varphi}_h - \varphi\|_{L^2(0,T;L^2(D))} &\le \|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^2(0,T;H)} + \|\widehat{\varphi}_h - \varphi\|_{L^2(0,T;L^2(D))} \\ &= \frac{h}{\sqrt{3}}\|\partial_t\widehat{\varphi}_h\|_{L^2(0,T;H)} + \|\widehat{\varphi}_h - \varphi\|_{L^2(0,T;L^2(D))} \to 0\end{aligned}$$

as $h = h_j \searrow 0$, whence it holds that

$$\overline{\varphi}_h \to \varphi \quad \text{strongly in } L^2(0,T;L^2(D)) \tag{4.22}$$

as $h = h_j \searrow 0$. Thus it follows from (4.14), (4.16)-(4.18), (4.20) and (4.22) that

$$\int_0^T \left(\int_\Omega (\partial_t\varphi(t) - \Delta\varphi(t) + \xi(t) + \pi(\varphi(t)) - \ell\theta(t))\psi(t)\right) dt = 0$$

for all $\psi \in C_{\mathrm{c}}^\infty([0,T] \times \overline{\Omega})$, which implies (4.19).

Next we prove that

$$\xi \in \beta(\varphi) \quad \text{a.e. on } \Omega \times (0,T). \tag{4.23}$$

Let $E \subset \Omega$ be an arbitrary bounded domain with smooth boundary. Then we can verify that

$$1_E\overline{\varphi}_h \to 1_E\varphi \quad \text{strongly in } L^2(0,T;H) \tag{4.24}$$

as $h = h_j \searrow 0$, where $1_E$ is the characteristic function on $E$. Hence from (4.18) and (4.24) we deduce that

$$\begin{aligned}\int_0^T (1_E\overline{\xi}_h(t), 1_E\overline{\varphi}_h(t))_H\,dt = \int_0^T (\overline{\xi}_h(t), 1_E\overline{\varphi}_h(t))_H\,dt &\to \int_0^T (\xi(t), 1_E\overline{\varphi}_h(t))_H\,dt \\ &= \int_0^T (1_E\xi(t), 1_E\overline{\varphi}_h(t))_H\,dt\end{aligned}$$

as $h = h_j \searrow 0$. Then the inclusion $1_E\xi \in \beta(1_E\varphi)$ holds a.e. on $\Omega \times (0,T)$ (see e.g., [**3**, Lemma 1.3, p. 42]), that is,

$$\xi = 1_E\xi \in \beta(1_E\varphi) = \beta(\varphi) \quad \text{a.e. on } E \times (0,T).$$

Thus, since $E \subset \Omega$ is arbitrary, we conclude that (4.23) holds.

Therefore combining (4.19) and (4.23) leads to (2.11). Next we confirm (2.12). Let $E \subset \Omega$ be an arbitrary bounded domain with smooth boundary. Then we see that

$$\widehat{\theta}_h \to \theta,\ \widehat{\varphi}_h \to \varphi \quad \text{strongly in } C([0,T];L^2(E)).$$

In particular, it follows from (2.2) that

$$\|\theta_0 - \theta(0)\|_{L^2(E)} = \|\widehat{\theta}_h(0) - \theta(0)\|_{L^2(E)} \to 0,$$
$$\|\varphi_0 - \varphi(0)\|_{L^2(E)} = \|\widehat{\varphi}_h(0) - \varphi(0)\|_{L^2(E)} \to 0$$

as $h = h_j \searrow 0$. Hence we can show that

$$\theta(0) = \theta_0,\ \varphi(0) = \varphi_0 \quad \text{a.e. on } E.$$

Hence, since $E \subset \Omega$ is arbitrary, we can obtain (2.12).

Next, we should prove the uniqueness of the solution, which is of course known. However, for the sake of completeness we detail here a uniqueness proof for the reader. Let $(\theta_j, \varphi_j, \xi_j)$, $j = 1, 2$, be two solutions. Then the identities

$$\partial_t(\theta_1 - \theta_2) + \ell\partial_t(\varphi_1 - \varphi_2) - \Delta(\theta_1 - \theta_2) = 0, \tag{4.25}$$
$$\partial_t(\varphi_1 - \varphi_2) - \Delta(\varphi_1 - \varphi_2) = \ell(\theta_1 - \theta_2) - \xi_1 + \xi_2 - \pi(\varphi_1) + \pi(\varphi_2) \tag{4.26}$$

hold a.e. on $\Omega \times (0, T)$. Integrating (4.25) over $(0, t)$, where $t \in [0, T]$, multiplying by $\theta_1 - \theta_2$ and integrating over $\Omega$, we have

$$\begin{aligned} &\|\theta_1(t) - \theta_2(t)\|_H^2 + \ell(\varphi_1(t) - \varphi_2(t), \theta_1(t) - \theta_2(t))_H \\ &+ \frac{1}{2}\frac{d}{dt}\left\|\nabla \int_0^t (\theta_1(s) - \theta_2(s))\, ds\right\|_H^2 = 0. \end{aligned} \tag{4.27}$$

On the other hand, multiplying (4.26) by $\varphi_1 - \varphi_2$ leads to the identity

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\varphi_1(t) - \varphi_2(t)\|_H^2 + \|\nabla(\varphi_1(t) - \varphi_2(t))\|_H^2 \\ &= \ell(\theta_1(t) - \theta_2(t), \varphi_1(t) - \varphi_2(t))_H - (\xi_1(t) - \xi_2(t), \varphi_1(t) - \varphi_2(t))_H \\ &\quad - (\pi(\varphi_1(t)) - \pi(\varphi_2(t)), \varphi_1(t) - \varphi_2(t))_H. \end{aligned} \tag{4.28}$$

Thus it follows from (4.27), (4.28) and the monotonicity of $\beta$ that

$$\begin{aligned} &\|\theta_1(t) - \theta_2(t)\|_H^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla \int_0^t (\theta_1(s) - \theta_2(s))\, ds\right\|_H^2 \\ &+ \frac{1}{2}\frac{d}{dt}\|\varphi_1(t) - \varphi_2(t)\|_H^2 + \|\nabla(\varphi_1(t) - \varphi_2(t))\|_H^2 \\ &= -(\xi_1(t) - \xi_2(t), \varphi_1(t) - \varphi_2(t))_H - (\pi(\varphi_1(t)) - \pi(\varphi_2(t)), \varphi_1(t) - \varphi_2(t))_H \\ &\le \|\pi'\|_{L^\infty(\mathbb{R})}\|\varphi_1(t) - \varphi_2(t)\|_H^2. \end{aligned} \tag{4.29}$$

Owing to the initial conditions (2.12) satisfied by both $(\theta_j, \varphi_j)$, $j = 1, 2$, the integration of (4.29) over $(0, t)$, where $t \in [0, T]$, yields that

$$\begin{aligned}
&\int_0^t \|\theta_1(s) - \theta_2(s)\|_H^2 \, ds + \frac{1}{2} \left\| \nabla \int_0^t (\theta_1(s) - \theta_2(s)) \, ds \right\|_H^2 \\
&+ \frac{1}{2} \|\varphi_1(t) - \varphi_2(t)\|_H^2 + \int_0^t \|\nabla(\varphi_1(s) - \varphi_2(s))\|_H^2 \, ds \\
&\le \|\pi'\|_{L^\infty(\mathbb{R})} \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_H^2 \, ds.
\end{aligned}$$

Therefore, applying the Gronwall lemma, we see that $\theta_1 = \theta_2$ and $\varphi_1 = \varphi_2$. Then the identity $\xi_1 = \xi_2$ holds by (2.11). □

## 5. Error estimates

In this section we will prove Theorem 2.3.

**Lemma 5.1.** *Let $h_1$ be as in Lemma 4.1. Then for all $\ell > 0$ there exists a constant $M_1 = M_1(\ell, T) > 0$ such that*

$$\begin{aligned}
&\ell \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;H)} + \ell \|\overline{\varphi}_h - \varphi\|_{L^2(0,T;V)} \\
&+ \|\widehat{\theta}_h - \theta + \ell(\widehat{\varphi}_h - \varphi)\|_{L^\infty(0,T;H)} + \|\overline{\theta}_h - \theta\|_{L^2(0,T;V)} \\
&\le M_1 h^{1/2} + M_1 \|\overline{f}_h - f\|_{L^2(0,T;H)}
\end{aligned} \tag{5.1}$$

*for all $h \in (0, h_1)$. In particular, for all $\ell > 0$ there exists a constant $M_2 = M_2(\ell, T) > 0$ such that*

$$\|\widehat{\theta}_h - \theta\|_{L^\infty(0,T;H)} \le M_2 h^{1/2} + M_2 \|\overline{f}_h - f\|_{L^2(0,T;H)}$$

*for all $h \in (0, h_1)$.*

*Proof.* Let $h \in (0, h_1)$. Then we infer from the second equation in $(\mathrm{P})_h$, (2.1)-(2.3) and (2.11) that

$$\begin{aligned}
&\frac{\ell^2}{2} \frac{d}{dt} \|\widehat{\varphi}_h(t) - \varphi(t)\|_H^2 \\
&= -\ell^2 \|\nabla(\overline{\varphi}_h(t) - \varphi(t))\|_H^2 - \ell^2(-\Delta(\overline{\varphi}_h(t) - \varphi(t)), \widehat{\varphi}_h(t) - \overline{\varphi}_h(t))_H \\
&\quad - \ell^2(\overline{\xi}_h(t) + \pi(\overline{\varphi}_h(t)) - \xi(t) - \pi(\varphi(t)), \overline{\varphi}_h(t) - \varphi(t))_H \\
&\quad - \ell^2(\overline{\xi}_h(t) + \pi(\overline{\varphi}_h(t)) - \xi(t) - \pi(\varphi(t)), \widehat{\varphi}_h(t) - \overline{\varphi}_h(t))_H \\
&\quad + \ell^3(\widehat{\theta}_h(t) - \theta(t) + \ell(\widehat{\varphi}_h(t) - \varphi(t)), \widehat{\varphi}_h(t) - \varphi(t))_H - \ell^4 \|\widehat{\varphi}_h(t) - \varphi(t)\|_H^2 \\
&\quad + \ell^3(\overline{\theta}_h(t) - \widehat{\theta}_h(t), \widehat{\varphi}_h(t) - \varphi(t))_H - \ell^3 h(\partial_t \widehat{\theta}_h(t), \widehat{\varphi}_h(t) - \varphi(t))_H.
\end{aligned}$$

Hence, integrating over $(0,t)$, where $t \in [0,T]$, the Schwarz and Young inequalities help us to deduce that

$$
\begin{aligned}
(5.2)\quad & \frac{\ell^2}{2}\|\widehat{\varphi}_h(t)-\varphi(t)\|_H^2 + \ell^2\int_0^t \|\nabla(\overline{\varphi}_h(s)-\varphi(s))\|_H^2\,ds \\
& \le \ell^2\|\Delta(\overline{\varphi}_h-\varphi)\|_{L^2(0,T;H)}\|\widehat{\varphi}_h-\overline{\varphi}_h\|_{L^2(0,T;H)} \\
& \quad - \ell^2\int_0^t (\overline{\xi}_h(s)+\pi(\overline{\varphi}_h(s))-\xi(s)-\pi(\varphi(s)),\overline{\varphi}_h(s)-\varphi(s))_H\,ds \\
& \quad + \ell^2\|\overline{\xi}_h+\pi(\overline{\varphi}_h)-\xi-\pi(\varphi)\|_{L^2(0,T;H)}\|\widehat{\varphi}_h-\overline{\varphi}_h\|_{L^2(0,T;H)} \\
& \quad + \frac{3}{2}\ell^2\int_0^t \|\widehat{\theta}_h(s)-\theta(s)+\ell(\widehat{\varphi}_h(s)-\varphi(s))\|_H^2\,ds \\
& \quad - \frac{2}{3}\ell^4\int_0^t \|\widehat{\varphi}_h(s)-\varphi(s)\|_H^2\,ds + \frac{3}{2}\ell^2\|\overline{\theta}_h-\widehat{\theta}_h\|_{L^2(0,T;H)}^2 + \frac{3}{2}\ell^2h^2\|\partial_t\widehat{\theta}_h\|_{L^2(0,T;H)}^2.
\end{aligned}
$$

Also, the first equation in $(\mathrm{P})_h$ and (2.10) yield that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\widehat{\theta}_h(t)-\theta(t)+\ell(\widehat{\varphi}_h(t)-\varphi(t))\|_H^2 \\
& = -\|\nabla(\overline{\theta}_h(t)-\theta(t))\|_H^2 - (-\Delta(\overline{\theta}_h(t)-\theta(t)),\widehat{\theta}_h(t)-\overline{\theta}_h(t))_H \\
& \quad - \ell(-\Delta(\overline{\theta}_h(t)-\theta(t)),\widehat{\varphi}_h(t)-\overline{\varphi}_h(t))_H - \ell(\nabla(\overline{\theta}_h(t)-\theta(t)),\nabla(\overline{\varphi}_h(t)-\varphi(t)))_H \\
& \quad + (\overline{f}_h(t)-f(t),\widehat{\theta}_h(t)-\theta(t)+\ell(\widehat{\varphi}_h(t)-\varphi(t)))_H.
\end{aligned}
$$

Thus, from the integration over $(0,t)$, where $t \in [0,T]$, and the Schwarz and Young inequalities we see that

$$
\begin{aligned}
(5.3)\quad & \frac{1}{2}\|\widehat{\theta}_h(t)-\theta(t)+\ell(\widehat{\varphi}_h(t)-\varphi(t))\|_H^2 + \frac{1}{2}\int_0^t \|\nabla(\overline{\theta}_h(s)-\theta(s))\|_H^2\,ds \\
& \le \|\Delta(\overline{\theta}_h-\theta)\|_{L^2(0,T;H)}\|\widehat{\theta}_h-\overline{\theta}_h\|_{L^2(0,T;H)} \\
& \quad + \ell\|\Delta(\overline{\theta}_h-\theta)\|_{L^2(0,T;H)}\|\widehat{\varphi}_h-\overline{\varphi}_h\|_{L^2(0,T;H)} + \frac{\ell^2}{2}\int_0^t \|\nabla(\overline{\varphi}_h(s)-\varphi(s))\|_H^2\,ds \\
& \quad + \frac{1}{2}\|\overline{f}_h-f\|_{L^2(0,T;H)}^2 + \frac{1}{2}\int_0^t \|\widehat{\theta}_h(s)-\theta(s)+\ell(\widehat{\varphi}_h(s)-\varphi(s))\|_H^2\,ds.
\end{aligned}
$$

Here, thanks to the inclusions $\overline{\xi}_h(t) \in \beta(\overline{\varphi}_h(t))$, $\xi(t) \in \beta(\varphi(t))$ and the monotonicity of $\beta$, it holds that

$$
\begin{aligned}
(5.4)\quad & -\ell^2(\overline{\xi}_h(t)+\pi(\overline{\varphi}_h(t))-\xi(t)-\pi(\varphi(t),\overline{\varphi}_h(t)-\varphi(t))_H \\
& = -\ell^2(\overline{\xi}_h(t)-\xi(t),\overline{\varphi}_h(t)-\varphi(t))_H - \ell^2(\pi(\overline{\varphi}_h(t))-\pi(\varphi(t)),\overline{\varphi}_h(t)-\varphi(t))_H \\
& \le \ell^2\|\pi'\|_{L^\infty(\mathbb{R})}\|\overline{\varphi}_h(t)-\varphi(t)\|_H^2 \\
& \le 2\ell^2\|\pi'\|_{L^\infty(\mathbb{R})}\|\overline{\varphi}_h(t)-\widehat{\varphi}_h(t)\|_H^2 + 2\ell^2\|\pi'\|_{L^\infty(\mathbb{R})}\|\widehat{\varphi}_h(t)-\varphi(t)\|_H^2.
\end{aligned}
$$

Hence, in view of (2.2), (2.8), (2.9), (2.12) and Lemmas 4.1-4.4, we collect (5.2)-(5.4) and deduce that there exists a constant $C_1 = C_1(\ell, T) > 0$ such that

$$\begin{aligned}
&\frac{\ell^2}{2}\|\widehat{\varphi}_h(t) - \varphi(t)\|_H^2 + \frac{1}{2}\|\widehat{\theta}_h(t) - \theta(t) + \ell(\widehat{\varphi}_h(t) - \varphi(t))\|_H^2 \\
&+ \frac{\ell^2}{2}\int_0^t \|\nabla(\overline{\varphi}_h(s) - \varphi(s))\|_H^2\, ds + \frac{1}{2}\int_0^t \|\nabla(\overline{\theta}_h(s) - \theta(s))\|_H^2\, ds \\
&\le C_1 h + 2\ell^2\|\pi'\|_{L^\infty(\mathbb{R})}\int_0^t \|\widehat{\varphi}_h(s) - \varphi(s)\|_H^2\, ds \\
&\quad + \left(\frac{3}{2}\ell^2 + \frac{1}{2}\right)\int_0^t \|\widehat{\theta}_h(s) - \theta(s) + \ell(\widehat{\varphi}_h(s) - \varphi(s))\|_H^2\, ds + \frac{1}{2}\|\overline{f}_h - f\|_{L^2(0,T;H)}^2.
\end{aligned}$$

Therefore, by applying the Gronwall lemma, we can obtain (5.1). □

**Proof of Theorem 2.3.** Since $W^{1,1}(0,T;H) \subset L^\infty(0,T;H)$, we have

$$\begin{aligned}
\|\overline{f}_h - f\|_{L^2(0,T;H)}^2 &= \frac{1}{h^2}\sum_{k=1}^N \int_{(k-1)h}^{kh} \left\| \int_{(k-1)h}^{kh} (f(s) - f(t))\, ds \right\|_H^2 dt \\
&\le \frac{1}{h^2}\sum_{k=1}^N \int_{(k-1)h}^{kh} \left( \int_{(k-1)h}^{kh} \|f(s) - f(t)\|_H\, ds \right)^2 dt \\
&\le \frac{2\|f\|_{L^\infty(0,T;H)}}{h^2}\sum_{k=1}^N \int_{(k-1)h}^{kh} \left( \int_{(k-1)h}^{kh} \|f(s) - f(t)\|_H^{1/2}\, ds \right)^2 dt \\
&\le \frac{2\|f\|_{L^\infty(0,T;H)}}{h}\sum_{k=1}^N \int_{(k-1)h}^{kh} \left( \int_{(k-1)h}^{kh} \|f(s) - f(t)\|_H\, ds \right) dt.
\end{aligned} \tag{5.5}$$

Here it holds that

$$\begin{aligned}
&\frac{2\|f\|_{L^\infty(0,T;H)}}{h}\sum_{k=1}^N \int_{(k-1)h}^{kh} \left( \int_{(k-1)h}^{kh} \|f(s) - f(t)\|_H\, ds \right) dt \\
&= \frac{2\|f\|_{L^\infty(0,T;H)}}{h}\sum_{k=1}^N \int_{(k-1)h}^{kh} \left( \int_{(k-1)h}^{kh} \left\| \int_t^s \partial_t f(r)\, dr \right\|_H ds \right) dt \\
&\le \frac{2\|f\|_{L^\infty(0,T;H)}}{h}\sum_{k=1}^N \int_{(k-1)h}^{kh} \left( \int_{(k-1)h}^{kh} \left( \int_t^s \|\partial_t f(r)\|_H\, dr \right) ds \right) dt \\
&\le 2\|f\|_{L^\infty(0,T;H)}\|\partial_t f\|_{L^1(0,T;H)} h.
\end{aligned} \tag{5.6}$$

Therefore, combining Lemma 5.1, (5.5) and (5.6), we can prove Theorem 2.3. □

**Remark 5.1.** The above argument can be used also to the check that

$$\overline{f}_h \to f \quad \text{strongly in } L^2(0,T;H) \tag{5.7}$$

as $h \searrow 0$, in the case when $f$ is just in $L^2(0,T;H)$. Indeed, by density, for all $\varepsilon > 0$ there exists a function $g \in W^{1,1}(0,T;H)$ such that

$$\|f - g\|_{L^2(0,T;H)} < \varepsilon.$$

By fixing $g$ and introducing $\overline{g}_h$ as in (2.3), with

$$g_k := \frac{1}{h} \int_{(k-1)h}^{kh} g(s)\, ds$$

for $k = 1, ..., N$, the reader can easily verify that

$$\|\overline{f}_h - \overline{g}_h\|_{L^2(0,T;H)} < \varepsilon$$

as well. Then there exists $\widehat{h} > 0$ sufficiently small such that

$$\|g - \overline{g}_h\|_{L^2(0,T;H)} < \varepsilon$$

for all $h \in (0, \widehat{h})$ (cf. (5.5) and (5.6)). Hence this gives a proof of (5.7).

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