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CONVERGENCE OF SPECTRAL DISCRETIZATIONS OF THE VLASOV-POISSON SYSTEM

G. MANZINI *, D. FUNARO †, AND G. L. DELZANNO ‡

Abstract. We prove the convergence of a spectral discretization of the Vlasov-Poisson system. The velocity term of the Vlasov equation is discretized using either Hermite functions on the infinite domain or Legendre polynomials on a bounded domain. The spatial term of the Vlasov and Poisson equations is discretized using periodic Fourier expansions. Boundary conditions are treated in weak form through a penalty type term, that can be applied also in the Hermite case. As a matter of fact, stability properties of the approximated scheme descend from this added term. The convergence analysis is carried out in details for the $1D$-$1V$ case, but results can be generalized to multidimensional domains, obtained as Cartesian product, in both space and velocity. The error estimates show the spectral convergence, under suitable regularity assumptions on the exact solution.

Key words. Hermite spectral method, Legendre spectral method, Vlasov equation, Vlasov-Poisson system

1. Introduction.

The Vlasov-Maxwell equations, or their electrostatic equivalent Vlasov-Poisson, describe the microscopic dynamics of a collisionless, magnetized plasma combined with Maxwell's equation for the electromagnetic field [11, 13]. These equations are strongly coupled: the plasma provides the sources (density and currents) for the Maxwell equations, while the electromagnetic field moves the plasma particles via the Lorentz force. These equations have an intrinsic complexity, due to the fact that they are defined in a space of six dimensions. They are also extremely multiscale: plasma phenomena span a multitude of spatial and temporal scales, with several orders of magnitude of scale separation between microscopic and system scales. Indeed, the development of methods that can describe the large-scale dynamics of magnetized plasmas while retaining the necessary microscopic physics is the holy grail of computational plasma physics.

There are three major classes of numerical methods for the solution of the Vlasov-Maxwell equations, which differ by how the plasma distribution function (i.e. phase-space density) is treated. In the Particle-In-Cell (PIC) technique the plasma is described by macroparticles that move through a computational mesh [2, 14]. The Eulerian-Vlasov approach introduces a six-dimensional mesh in space and velocity coordinates and defines the distribution function on the mesh [8, 10]. Finally, transform (spectral) methods expand the velocity part of the distribution function in basis functions to obtain a system of differential equations for the coefficients of the expansion [1]. These moment equations are defined in configuration space.

PIC is the method of choice in the plasma physics community because of its relative simplicity, robustness and efficient parallelization on modern computer architectures. It is however a low-order method: reducing the well-known statistical noise associated with the macroparticles can require a prohibitive amount of computational resources. Spectral methods, on the other hand, can be very accurate as one can take advantage of the very high rate of convergence of the method, in presence of regular

*T-5 Applied Mathematics and Plasma Physics Group, Los Alamos National Laboratory, Los Alamos, New Mexico, USA e-mail: gmanzini@lanl.gov

‡Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli Studi di Modena e Reggio-Emilia, Modena, Italy e-mail: daniele.funaro@unimore.it

‡T-5 Applied Mathematics and Plasma Physics Group, Los Alamos National Laboratory, Los Alamos, New Mexico, USA e-mail: delzanno@lanl.gov

solutions. These techniques were popular in the early days of computational plasma physics, where the Hermite or Fourier basis functions where used [1], but have not led to the development of a set of widely used numerical codes for the plasma physics community. Nevertheless, in recent years there has been a renewed interest for spectral methods for Vlasov-based models [3, 4, 9, 17, 18, 20, 21]. This is in part driven by the fact that with a suitable choice of the spectral basis, the low-order moments of the expansion are related to the typical fluid moments (density, momentum, energy, ...) of the plasma. Thus one can describe the plasma macroscopically with a few moments, while the microscopic physics can be retained by adding more moments to the expansion [26]. This can be done only in some parts of the computational domain, as necessary. Thus, from a computational point of view, spectral methods might offer an optimal way to perform large-scale simulations including microscopic physics [27].

With this premise, this paper deals with the numerical analysis of spectral methods for the Vlasov-Poisson equations, proving for the first time the convergence and stability properties of the method when the spectral basis in velocity space consists of either Symmetrically-Weighted (SW) Hermite functions or Legendre polynomials, combined with a Fourier discretization in space. To this end, we consider the *Vlasov-Poisson* system of equations for the electron distribution function $f(\mathbf{x},\mathbf{v},t)$ (with charge q and mass m), and the electric field $\mathbf{E}(\mathbf{x},t)$:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \frac{q}{m}\mathbf{E}\cdot\nabla_{\mathbf{v}} f = 0, \qquad \text{in } \Omega,\ t\in[0,T[, \tag{1.1}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{E} = \rho \ \text{ where } \ \rho = n_i + \frac{q}{\epsilon_0}\int_{\Omega_v} f d\mathbf{v} \qquad \text{in } \Omega_x,\ t\in[0,T[. \tag{1.2}$$

Equations (1.1)-(1.2) are defined on the *six* dimensional phase space and in the time range $t\in[0,T[$, for a given T.

In equation (1.2) the ions are a static neutralizing background of density n_i and ϵ_0 is the vacuum electric permettivity. We normalize these equations without loss of generality by setting $n_i = 1$, $\epsilon_0 = 1$, $m = 1$, $q = -1$. We assume that the phase space domain is *periodic in space*, and that the distribution function is zero at the velocity boundary or is a *rapidly decreasing function* that tends asymptotically to zero as $e^{-|\mathbf{v}|^2/2}$ for $|\mathbf{v}|\to\pm\infty$. To complete the mathematical formulation of the Vlasov-Poisson model we specify the initial distribution function $f_0(\mathbf{x},\mathbf{v})$, and we compute the initial electric field $\mathbf{E}(\mathbf{x},0)$ by solving equation (1.2) at time $t=0$.

For exposition's sake we present the formulation and the convergence analysis of the spectral methods in one-dimension in space and velocity, i.e., the $1D$-$1V$ setting. It should be clear at the end of our exposition that the type of discretizations adopted here can be extended to multidimensional problems with minor modifications. We are basically thinking of situations where the variable $\mathbf{x}$ is periodic in all directions, and the variable $\mathbf{v}$ is defined on a parallelepiped. It is a matter of redefining appropriately the way to treat boundary conditions and use splitting properties of orthogonal projections, but the 1D-1V derivation carries over 3D-3V lenghtly but quite straightforwardly.

In section 2 we introduce the spectral discretizations using Fourier expansion for the spatial term and either Hermite functions or Legendre polynomials for the velocity term. Boundary conditions are handled through a suitable penalty approach in the velocity space. This technique is also applied in the Hermite context. Indeed, in alternative to the standard decay properties of Hermite functions, zero conditions for v may be enforced in weak form at the boundaries of a bounded subset Ω_v of the whole space. The integration of the Hermite spectral method on a finite sized velocity domain is required by the convergence analysis as the constants of the error estimates contain the size of the phase space domain, which would blow up if the size of the velocity domain goes to infinity. In section 3 we reformulate the Vlasov equation and its truncated approximation as a convection equation in a two-dimensional phase space and we prove that both formulations are L^2 stable, thanks also to the role played by the special treatment of boundary conditions. In section 4 we analyze the approximation of the electric field and show that its error is controlled by the approximation error of the distribution function. In section 5 we finally provide a convergence analysis in the L^2 norm.

Further discussion and conclusions are given in section 6.

2. Spectral discretization of the $1D$-$1V$ Vlasov-Poisson system.

2.1. The $1D$-$1V$ Vlasov-Poisson system. We consider the $1D$-$1V$ phase space domain $\Omega = \Omega_x \times \Omega_v$, where the spatial subdomain is $\Omega_x = [0, 2\pi[$ and the velocity subdomain is either $\Omega_v =]v_{\min}, v_{\max}[$ or $\Omega_v = \mathbb{R}$. Equations (1.1)-(1.2) become

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v} = 0 \qquad \text{in } \Omega,\ t \in [0, T[, \tag{2.1}$$

$$\frac{\partial E}{\partial x} = \rho \quad \text{where} \quad \rho = 1 - \int_{\Omega_v} f dv \qquad \text{in } \Omega_x,\ t \in [0, T[. \tag{2.2}$$

Equations (2.1)-(2.2) are defined in the time range $t \in [0, T[$ for some finite time $T \in \mathbb{R}$. We assume that the phase space domain is *periodic*, which implies, in particular, that $f(0, v, t) = f(2\pi, v, t)$ and $E(0, t) = E(2\pi, t)$ for any $t \geq 0$. When $\Omega_v =]v_{\min}, v_{\max}[$, we assume that $f(x, v_{\min}, t) = f(x, v_{\max}, t) = 0$ for every $x \in \Omega_x$ and $t \in [0, T[$, while when $\Omega_v = \mathbb{R}$ we assume that f is a *rapidly decreasing function* in the sense that $f(x, v, t) \to 0$ asymptotically like $e^{-v^2/2}$ for $v \to \pm\infty$. To complete the mathematical formulation of the Vlasov-Poisson model we specify the initial distribution function $f_0(x, v)$, and we compute the initial electric field $E(x, 0)$ by solving equation (2.2) at time $t = 0$.

2.2. Notation and technicalities. Let us introduce the following notation:

- $\Lambda = \Lambda_S \times \Lambda_F = \mathbb{Z}^+ \times \mathbb{Z}$ is the *infinite index range* of velocity and spatial modes. Here, "S" is a generic label that may refer to either "H" (Hermite) or "L" (Legendre), while "F" stands for *Fourier* (periodic type). Throughout the paper, we will specialize when necessary the subindex "S" to refer specifically to the Hermite or the Legendre velocity representation. Indeed, Λ_H or Λ_L will be respectively the *infinite index range* of the spectral decompositions using Hermite functions or Legendre polynomials;
- $N = (N_S, N_F) \in \mathbb{N}^+ \times \mathbb{N}^+$ is the number of modes taken into account in the approximation of velocity and space, respectively;
- $\Lambda^N = \Lambda_S^N \times \Lambda_F^N = [0, N_S - 1] \times [-N_F, N_F]$ is the *finite index range* for the velocity and spatial modes of the *truncated* distribution function. As a consequence, the notation $(n, k) \in \Lambda^N$ means $0 \leq n \leq N_S - 1$ *and* $-N_F \leq k \leq N_F$.
- $|\Omega_x| = 2\pi$, $|\Omega_v| = v_{\max} - v_{\min}$ and $|\Omega| = |\Omega_x||\Omega_v|$ denote the measures of the sets Ω_x, Ω_v and Ω, respectively.

We denote the infinite set of basis functions that are going to be used for the velocity representation by $\{\varphi_n(v)\}_{n\in\Lambda_S}$. These can be either Hermite functions [15] or Legendre polynomials [20]. Both of them satisfy the orthogonality property:

$$\int_{\Omega_v} \varphi_n(v)\varphi_{n'}(v)dv = \delta_{n,n'}. \tag{2.3}$$

Hermite functions are obtained as the product of Hermite polynomials by the exponential $e^{-v^2/2}$. A suitable normalization factor is then introduced in order to have (2.3). This system is generally referred to as *Symmetrically Weighted Hermite*, to distinguish it from the *Asymmetrically Weighted Hermite* system. In the latter, two distinct sets of basis functions are considered in a duality relationships through (2.3) and the exponential e^{-v^2} is, asymmetrically, a multiplier of Hermite polynomials in only one of these sets. We shall work with the symmetric system only, although some of the results of this paper could be adapted to cover the asymmetric case.

The situation regarding Legendre polynomials is more classical. They are usually defined in the interval $[-1,1]$. Via a suitable affine transformation we map them onto the interval $[v_{\min}, v_{\max}]$, where they are successively normalized in order to recover (2.3).

For the spatial representation we use the Fourier basis functions that are defined and satisfy an orthogonal property as follows:

$$\eta_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}, \qquad \int_0^{2\pi} \eta_k(x)\eta_{-k'}(x)dx = \delta_{k,k'}. \tag{2.4}$$

When not strictly necessary, throughout the paper we ease the notation by dropping out the arguments v and x from $\varphi_n(v)$ and $\eta_k(x)$.

Using the orthogonal system introduced in (2.3) and (2.4), we define the finite dimensional spaces $\mathcal{S}^N = \mathsf{span}\big(\{\varphi_n(v)\}_{n\in\Lambda_S^N}\big)$; $\mathcal{F}^N = \mathsf{span}\big(\{\eta_k(x)\}_{k\in\Lambda_F^N}\big)$; $\mathcal{X}^N = \mathsf{span}\big(\{\eta_k(x)\varphi_n(v)\}_{k\in\Lambda_F^N, n\in\Lambda_S^N}\big)$. Whenever needed, we specify the symbol $\mathcal{S}^N$ to $\mathcal{L}^N$ (Legendre) or $\mathcal{H}^N$ (Hermite). Afterwords, we introduce the orthogonal projection operator $\mathcal{P}_S^N : L^2(\Omega_v) \to \mathcal{S}^N$ such that:

$$\forall\psi \in L^2(\Omega_v) \, : \, \int_\Omega (\psi - \mathcal{P}_S^N\psi)\varphi_n dv = 0, \quad \forall n \in \Lambda_S^N, \tag{2.5}$$

and the orthogonal projection operator $\mathcal{P}_F^N : L^2(\Omega_x) \to \mathcal{F}^N$ such that

$$\forall\phi \in L^2(\Omega_x) \, : \, \int_\Omega (\phi - \mathcal{P}_F^N\phi)\eta_k dx = 0, \quad \forall k \in \Lambda_F^N. \tag{2.6}$$

Their extension to functions on $L^2(\Omega) = L^2(\Omega_x \times \Omega_v)$ is obvious (it is just the matter of freezing one of the two variables), so that we can combine them in order to obtain the orthogonal projection $\mathcal{P}^N : L^2(\Omega) \to \mathcal{X}^N$, which comes from the composition of operators $\mathcal{P}^N := \mathcal{P}_S^N \circ \mathcal{P}_F^N = \mathcal{P}_F^N \circ \mathcal{P}_S^N$.

2.3. Spectral approximation. For any $t \in [0, T[$, the Galerkin formulation for the Vlasov-Poisson system becomes:

$$\int_\Omega \varphi_n\eta_k \left(\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v}\right) dvdx = 0 \qquad \forall(n,k) \in \Lambda, \tag{2.7}$$

$$\int_{\Omega_x} \frac{\partial E}{\partial x}\eta_k \, dx = \int_{\Omega_x} \rho\eta_k \, dx \qquad \forall k \in \Lambda_F, \tag{2.8}$$

with the initial condition $f(\cdot,\cdot,0) = f_0$. The function $\rho(x,t)$ in (2.8) is the right-hand side of (2.2) and represents the total charge density of ions and electrons. The spectral approximation of (2.7)-(2.8) reads as: *Find* $f^N \in \mathcal{X}^N$, $E^N \in \mathcal{F}^N$ *such that*

$$\int_\Omega \varphi_n \eta_k \left(\frac{\partial f^N}{\partial t} + v \frac{\partial f^N}{\partial x} - E^N \frac{\partial f^N}{\partial v} \right) dv dx = \int_\Omega \varphi_n \eta_k R^N \, dv dx \quad \forall (n,k) \in \Lambda^N, \quad (2.9)$$

$$\int_{\Omega_x} \frac{\partial E^N}{\partial x} \eta_k \, dx = \int_{\Omega_x} \rho^N \eta_k \, dx \qquad \forall k \in \Lambda_F^N, \quad (2.10)$$

with the initial condition $f^N(\cdot,\cdot,0) = \mathcal{P}^N f_0$. Here, R^N is a kind of penalty term used to impose weakly boundary conditions in the discrete space. The well-posedness of problem (2.9)-(2.10), i.e., existence and uniqueness of the numerical solutions $f^N \in \mathcal{X}^N$ and $E^N \in \mathcal{F}^N$ can be proved in $]0,T]$ (for any finite final time T) as discussed at the end of appendix C. The term $\rho^N(x,t)$ in equation (2.10) is given by:

$$\rho^N(x,t) = 1 - \int_{\Omega_v} f^N(x,v,t) dv, \quad (2.11)$$

and is an approximation of $\rho(x,t)$ in (2.2). Note that ρ^N does not coincide with the projection $\mathcal{P}_F^N \rho$. In the right-hand side of (2.9), the term R^N allows us to set the boundary conditions at $v_{\min}$ and $v_{\max}$ in weak form. This approach is similar to the penalty strategy proposed in [20]. Term R^N is designed by suitably modifying the boundary term that naturally originates from the integration by parts of the velocity derivative on the finite domain $\Omega_v =]v_{\min}, v_{\max}[$. As it will be clear in Section 3, the special design of R^N ensures the stability of the Legendre-Fourier method. It also guarantees that the Hermite-Fourier method, which is stable on the infinite domain, remains stable when the integration of (2.9) is restricted to the finite domain $\Omega_v =]v_{\min}, v_{\max}[$ (see also Remark 3.1). Term R^N is given by the formula:

$$R^N(x,v,t) = -\frac{1}{2} \mathcal{P}^N E^N(x,t) \left[f^N(x,v_{\max},t) \sum_{n \in \Lambda_S^N} \varphi_n(v_{\max}) \varphi_n(v) - f^N(x,v_{\min},t) \sum_{n \in \Lambda_S^N} \varphi_n(v_{\min}) \varphi_n(v) \right]. \quad (2.12)$$

This way of handling the boundary conditions is valid for both the Legendre and Hermite spectral approximations and makes it possible to develop a full stability and convergence analysis. In the specific case of the Legendre approximation the boundary conditions can be accounted for in several other ways. For instance, one can impose these constraints in strong form. Like in the so-called *tau method* [6] this can be done by projecting the equation to be approximated in a subspace of lower dimension ($N_S - 2$ in place of N_S) and close the system with two additional equations relative to the enforcement of the boundary conditions. An alternative is to encapsulate the boundary constraints directly in the basis functions, but in this fashion one cannot rely on orthogonality properties. Both of these approaches are valid. Nevertheless, their theoretical analysis looks harder, while we do not expect the general performance to improve in comparison to the approach that we are considering in this paper.

For the Hermite discretization, on the other hand, we could choose $R^N = 0$ since the method is defined on $\Omega_v = \mathbb{R}$ and resorts to the rapid decay of the Hermite functions to fulfill the zero boundary conditions at infinity. Nevertheless, the convergence theory for the Hermite method is developed as for the Legendre approximation in Section 5 by assuming that the velocity domain is finite and weakly imposing that f is zero at the velocity boundary through (2.12).

Throughout the paper we will also refer to equations (2.9) and (2.10) as the *truncated Vlasov-Poisson system* in the equivalent form: *Find* $f^N \in \mathcal{X}^N$, $E^N \in \mathcal{F}^N$ *such that*

$$\mathcal{P}^N\left(\frac{\partial f^N}{\partial t} + v\frac{\partial f^N}{\partial x} - E^N\frac{\partial f^N}{\partial v}\right) = R^N, \tag{2.13}$$

$$\mathcal{P}_F^N\left(\frac{\partial E^N}{\partial x} - \rho^N\right) = 0, \tag{2.14}$$

where we recall that $\mathcal{P}^N$ and $\mathcal{P}_F^N$ are the projection operators introduced at the end of subsection 2.2. Actually, in the first equation the action of $\mathcal{P}^N$ can be restricted to the second and third terms since $\mathcal{P}^N\big(\partial f^N/\partial t\big) = \partial f^N/\partial t$. Also, $\mathcal{P}_F^N$ in the second equation can be removed since in the Fourier approximation the differential operator $\partial/\partial x$ commutes with the projector. As this is not true in other approximation systems and in view of possible generalizations we prefer to keep it. The formulation (2.13)-(2.14) is equivalent to a system of ordinary differential equations whose coefficients are provided in Appendix C.

Lemma 2.1. *Let* R^N *be given by* (2.12). *Let* f^N *and* E^N *be the solution of problem* (2.13)-(2.14). *Let* g^N *be a function of* $\mathcal{X}^N$. *Then, it holds that:*

$$\int_\Omega g^N(x,v)R^N(x,v,t)\,dvdx = -\frac{1}{2}\int_{\Omega_x} E^N(x,t)\Big[f^N(x,v_{\max},t)g^N(x,v_{\max}) - f^N(x,v_{\min},t)g^N(x,v_{\min})\Big]\,dx. \tag{2.15}$$

Proof. Since $g^N \in \mathcal{X}^N$, for any $x \in \Omega_x$ we can write:

$$g^N(x,v) = (\mathcal{P}_S^N g^N)(x,v) = \sum_{n\in\Lambda_S^N}\left(\varphi_n(v)\int_{\Omega_v} g^N(x,v')\varphi_n(v')dv'\right). \tag{2.16}$$

Thus, from (2.16) we can derive the following relation:

$$\begin{aligned}
&\int_\Omega g^N(x,v)E^N(x,t)f^N(x,v_{\max},t)\sum_{n\in\Lambda_S^N}\varphi_n(v_{\max})\varphi_n(v)dvdx \\
&\quad= \int_{\Omega_x} E^N(x,t)f^N(x,v_{\max},t)\sum_{n\in\Lambda_S^N}\left(\varphi_n(v_{\max})\int_{\Omega_v} g^N(x,v)\varphi_n(v)dv\right)dx \\
&\quad= \int_{\Omega_x} E^N(x,t)f^N(x,v_{\max},t)g^N(x,v_{\max})dx.
\end{aligned} \tag{2.17}$$

A similar formula for the integral of $E^N(x,t)f^N(x,v_{\min},t)g^N(x,v_{\min})$ is obtained with the same argument. The assertion of the lemma follows by combining these two relations and the definition of R^N provided in (2.12). ∎

In particular, if we take $g^N = f^N(\cdot,\cdot,t)$ for a given $t \geq 0$ we find that:

$$\int_\Omega f^N(x,v,t)R^N(x,v,t)\,dvdx = -\tfrac{1}{2}\int_{\Omega_x} E^N(x,t)[f(x,v_{\max},t)^2 - f(x,v_{\min},t)^2]dx. \tag{2.18}$$

REMARK 2.1. *Finally, it is worth mentioning that an alternative approach (not considered in this work) would be possible by following the dual Petrov-Galerkin formulation for odd-order problems proposed in [19, 23].*

3. Stability. We reformulate the Vlasov equation (2.1) as follows:

$$\frac{\partial f}{\partial t} + \mathbf{F} \cdot \nabla f = 0 \qquad \text{where} \qquad \mathbf{F} = \begin{pmatrix} v \\ -E \end{pmatrix}, \tag{3.1}$$

and the truncated Vlasov equation (2.13) as follows:

$$\frac{\partial f^N}{\partial t} + \mathcal{P}^N\big(\mathbf{F}^N \cdot \nabla f^N\big) = R^N \qquad \text{where} \qquad \mathbf{F}^N = \begin{pmatrix} v \\ -E^N \end{pmatrix}, \tag{3.2}$$

where $\nabla = (\partial/\partial x, \partial/\partial v)$. The advective fields $\mathbf{F}$ and $\mathbf{F}^N$ are both divergence-free, i.e., $\mathrm{div}\mathbf{F} = \mathrm{div}\mathbf{F}^N = 0$, since v is an independent variable while E and E^N do not depend on v. This property implies that

$$f\,\mathbf{F} \cdot \nabla f = \mathrm{div}\left(\mathbf{F}\frac{f^2}{2}\right) - \Big(\mathrm{div}\mathbf{F}\Big)\frac{f^2}{2} = \mathrm{div}\left(\mathbf{F}\frac{f^2}{2}\right). \tag{3.3}$$

We now integrate both sides of (3.3) on $\Omega = \Omega_x \times \Omega_v$ and we apply the Divergence Theorem. Furthermore, we recall that $f(\cdot, v_{\max}, \cdot) = f(\cdot, v_{\min}, \cdot) = 0$ if $\Omega_v = [v_{\min}, v_{\max}]$ or that $f(\cdot, v, \cdot) \to 0$ for $v \to \pm\infty$ if $\Omega_v = \mathbb{R}$. We use these boundary conditions and the periodicity along the direction x to obtain:

$$\begin{aligned} \int_\Omega f\,\mathbf{F} \cdot \nabla f dxdv &= \int_\Omega \mathrm{div}\left(\mathbf{F}\frac{f^2}{2}\right) dxdv = \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \cdot \mathbf{F}\frac{f^2}{2} dS \\ &= \int_{\partial\Omega_x \times \Omega_v} \mathbf{n}_{\partial\Omega} \cdot \mathbf{F}\frac{f^2}{2} dv + \int_{\Omega_x \times \partial\Omega_v} \mathbf{n}_{\partial\Omega} \cdot \mathbf{F}\frac{f^2}{2} dx = 0, \end{aligned} \tag{3.4}$$

where $\mathbf{n}_{\partial\Omega}$ is the outward unit vector field normal to $\partial\Omega$. Using (3.1) and (3.4) it is easy to arrive at a stability result in the $L^2(\Omega)$ norm for the continuous Vlasov-Poisson system. As a matter of fact, we have:

$$\frac{d}{dt}\left\|f(\cdot,\cdot,t)\right\|^2_{L^2(\Omega)} = 2\int_\Omega f\frac{\partial f}{\partial t} dxdv = -2\int_\Omega f\,\mathbf{F} \cdot \nabla f dxdv = 0. \tag{3.5}$$

We formally state this result as follows.

THEOREM 3.1. *Let f be the exact solution of* (3.1) *on the domain Ω. Then, it holds that:*

$$\frac{d}{dt}\left\|f(\cdot,\cdot,t)\right\|^2_{L^2(\Omega)} = 0, \tag{3.6}$$

or, equivalently, that $\|f(\cdot,\cdot,t)\|_{L^2(\Omega)} = \|f_0\|_{L^2(\Omega)}$ for every $t \in [0, T[$.

A similar result also holds regarding the truncated problem. In this case a crucial role is played by term R^N.

LEMMA 3.2. *Let g^N be a function in $\mathcal{X}^N$, $\mathbf{F}^N$ the advective field defined in* (3.2)*, f^N the solution of problem* (2.13)-(2.14) *with R^N given by* (2.12)*. Then, it holds that:*

$$\int_\Omega g^N\,\mathbf{F}^N \cdot \nabla f^N\,dvdx + \int_\Omega f^N\,\mathbf{F}^N \cdot \nabla g^N\,dvdx = 2\int_\Omega g^N R^N dxdv. \tag{3.7}$$

Proof. Since $\mathbf{F}^N$ is a zero-divergence field, using Lemma 2.1 and repeating the same calculations as above yields:

$$\begin{aligned}
\int_\Omega g^N \mathbf{F}^N \cdot \nabla f^N dvdx + \int_\Omega f^N \mathbf{F}^N \cdot \nabla g^N dvdx &= \int_\Omega \operatorname{div}\Big(\mathbf{F}^N f^N g^N\Big) dvdx \\
&= \int_{\partial\Omega_x \times \Omega_v} \mathbf{n}_{\partial\Omega} \cdot \mathbf{F}^N f^N g^N dv + \int_{\Omega_x \times \partial\Omega_v} \mathbf{n}_{\partial\Omega} \cdot \mathbf{F}^N f^N g^N dx \\
&= -\int_{\Omega_x} E^N(x,t)\Big(f^N(x, v_{\max}, t) g^N(x, v_{\max}) - f^N(x, v_{\min}, t) g^N(x, v_{\min})\Big) dx \\
&= 2\int_\Omega g^N R^N dxdv,
\end{aligned}$$

where we have used (2.15). This concludes the proof. □

In particular, by taking $g^N = f^N(\cdot,\cdot,t)$ in (3.7) we have that:

$$\int_\Omega f^N \,\mathbf{F}^N \cdot \nabla f^N \, dvdx = \int_\Omega f^N R^N dxdv. \tag{3.8}$$

By putting together (3.2) and (3.8) we easily arrive at:

$$\begin{aligned}
\frac{d}{dt}\left\|f^N(\cdot,\cdot,t)\right\|^2_{L^2(\Omega)} &= 2\int_\Omega f^N \frac{\partial f^N}{\partial t} dxdv = -2\int_\Omega f^N \left(\mathcal{P}^N\big(\mathbf{F}^N \cdot \nabla f^N\big) - R^N\right) dxdv \\
&= -2\int_\Omega f^N \left(\mathbf{F}^N \cdot \nabla f^N - R^N\right) dxdv = 0,
\end{aligned} \tag{3.9}$$

which implies the $L^2(\Omega)$ stability of the discrete solution. We formally state this result as follows.

Theorem 3.3. *Let f^N be the exact solution of the truncated Vlasov-Poisson system* (2.13)-(2.14) *on the domain Ω. Then, it holds that:*

$$\frac{d}{dt}\left\|f^N(\cdot,\cdot,t)\right\|^2_{L^2(\Omega)} = 0, \tag{3.10}$$

or, equivalently, that $\|f^N(\cdot,\cdot,t)\|_{L^2(\Omega)} = \|\mathcal{P}^N f_0\|_{L^2(\Omega)}$ for every $t \in [0,T[$.

Remark 3.1 (Stability of Hermite-Fourier method). *The term R^N can also be used in the framework of Hermite-Fourier approximations. Actually, it is strongly suggested since it provides excellent stabilization properties. In the standard approach it is usually assumed that $R^N = 0$ and the homogeneous boundary conditions in the infinite domain $\Omega_v = \mathbb{R}$ are imposed by taking advantage of the natural decay of the Hermite functions. Nonetheless, once the Hermite-Fourier method has been set up according to* (2.13)-(2.14), *we can restrict the domain to $\Omega_v =]v_{\min}, v_{\max}[$ as in the Legendre-Fourier method and introduce the stabilizing term R^N as a penalty, with the aim of enforcing zero conditions at $v_{\min}$ and $v_{\max}$ (in a weak sense, at least). Of course, this procedure is going to be effective if the size of Ω_v is large enough. Further comments are reported in the concluding section.*

4. Approximation of the electric field. The main result of this section is that the error on the approximation of the electric field is controlled by that on the distribution function.

THEOREM 4.1. *Let f, E be the exact solution of the Vlasov-Poisson problem* (2.7)-(2.8) *and f^N, E^N be the approximations solving Vlasov-Poisson problem* (2.13)-(2.14)*. It holds that:*

$$\|E(\cdot,t) - E^N(\cdot,t)\|_{L^2(\Omega_x)} \le C\,|\Omega_v|^{\frac{1}{2}}\,\|f - f^N\|_{L^2(\Omega)}. \tag{4.1}$$

The proof of this theorem is postponed after a few technical developments that we are going to present right away. In the case of the Hermite-Fourier discretization, the estimate holds by taking a finite velocity domain $\Omega_v =]v_{\min}, v_{\max}[$ according to the observation in Remark 3.1. The path that we shall follow here allows us to generalize the analysis to the multi-dimensional case as we mentioned in the introduction. Indeed, in the 1D-1V case a sharper estimate of error $E(\cdot,t) - E^N(\cdot,t)$ can be obtained by taking the difference between (2.2) and (2.14) and noting that $\mathcal{P}_F^N$ commutes with the differential operator:

$$\|E(\cdot,t) - E^N(\cdot,t)\|_{H^1(\Omega_x)} \le C\,\left\|f - f^N\right\|_{L^2(\Omega)},$$

where C is proportional to $|\Omega_v|^{\frac{1}{2}}$ and indipendent of N. Such a simpler approach is not allowed if we consider the divergence operator acting on the electric field as in (1.2).

The Fourier decomposition of $E(x,t)$ on the spatial domain Ω_x reads as:

$$E(x,t) = \sum_{k\in\Lambda_F} E_k(t)\eta_k(x). \tag{4.2}$$

Using (4.2) in (2.8) we reformulate the k-th Fourier mode of the electric field as follows:

$$E_0(t) = 0, \qquad E_k(t) = \frac{i}{k}\int_{\Omega_x\times\Omega_v} f(x,v,t)\eta_{-k}(x)dxdv \quad \text{for } k \neq 0. \tag{4.3}$$

A similar definition holds for E_k^N (just substitute f with f^N above). The condition that $E_0(t) = E_0^N(t) = 0$ is equivalent to $\int_{\Omega_x} E(x,t)dx = \int_{\Omega_x} E^N(x,t)dx = 0$ and has a physical motivation. Indeed, it plays the role of a normalizing condition for the electric field that indicates that the plasma is neutral at the macroscopic level [20]. The electric field and its approximation can be expressed in integral form by:

$$E(x,t) = \int_\Omega f(x',v',t)K(x,x')dv'dx', \tag{4.4}$$

$$E^N(x,t) = \int_\Omega f^N(x',v',t)K^N(x,x')dv'dx', \tag{4.5}$$

where $K(x,x')$ is the *Poisson kernel*, with the following expression:

$$K(x,x') = -K(x',x) = i\sum_{k\in\Lambda_F\setminus\{0\}} \frac{1}{k}\eta_k(x)\eta_{-k}(x'). \tag{4.6}$$

In the same way, $K^N(x,x')$ is the truncated version given by the formula:

$$K^N(x,x') = -K^N(x',x) = i \sum_{k\in\Lambda_F^N\setminus\{0\}} \frac{1}{k}\eta_k(x)\eta_{-k}(x'). \tag{4.7}$$

Indeed, one has

$$\begin{aligned} E(x,t) &= \sum_{k\in\Lambda_F} E_k(t)\eta_k(x) = \sum_{k\in\Lambda_F\setminus\{0\}} \left(\frac{i}{k}\int_{\Omega_x\times\Omega_v} f(x',v',t)\eta_{-k}(x')dv'dx'\right)\eta_k(x) \\ &= \int_{\Omega_x\times\Omega_v} f(x',v',t)\left(i\sum_{k\in\Lambda_F\setminus\{0\}}\frac{1}{k}\eta_{-k}(x')\eta_k(x)\right)dv'dx' \\ &= \int_{\Omega_x\times\Omega_v} f(x',v',t)K(x,x')dv'dx'. \end{aligned} \tag{4.8}$$

A similar relation holds for E^N (just substitute E_k with E_k^N and f with f^N).

In more dimensions we may introduce the electrostatic potential u and write $\mathbf{E} = \nabla_{\mathbf{x}} u$, so that $\Delta u = \rho$. By expressing u as a function of ρ through the Green's function of the Dirichlet problem (typical references are for instance [16, 25]), the appropriate expression for the kernel follows from taking the gradient of u.

Moreover, both kernels $K(x,x')$ and $K^N(x,x')$ are real-valued functions. Indeed, since the complex conjugate of the Fourier basis function is $\overline{\eta(x)}_k = \eta_{-k}(x)$, by swapping the summation index from k to $-k$, we note that the complex conjugate of $K(x,x')$ is given by

$$\overline{K(x,x')} = -i\sum_{k\in\Lambda_F\setminus\{0\}}\frac{1}{k}\eta_{-k}(x)\eta_k(x') = -i\sum_{k\in\Lambda_F\setminus\{0\}}\frac{1}{-k}\eta_k(x)\eta_{-k}(x') = K(x,x'),$$

and the same holds for $K^N(x,x')$. These properties imply that $|K(x,x')|^2 = \big(K(x,x')\big)^2$ and $\left|K^N(x,x')\right|^2 = \big(K^N(x,x')\big)^2$. We are now ready to prove the following estimates.

Lemma 4.2.

$$\|K\|^2_{L^2(\Omega_x\times\Omega_x)} = \frac{\pi^2}{3}, \tag{4.9}$$

$$\|K-K^N\|^2_{L^2(\Omega_x\times\Omega_x)} \le 2\frac{1}{N_F}. \tag{4.10}$$

Proof. Since K is a real-valued function, by swapping index l to $-l$, the first inequality of the lemma is proven through the following development:

$$\|K\|^2_{L^2(\Omega_x\times\Omega_x)} = \int_{\Omega_x\times\Omega_x} \big(K(x,x')\big)^2 dxdx'$$
$$= -\int_{\Omega_x\times\Omega_x} \sum_{k,l\in\Lambda_F\setminus\{0\}} \frac{1}{kl}\eta_k(x)\eta_{-k}(x')\eta_l(x)\eta_{-l}(x')dxdx'$$
$$= -\sum_{k,l\in\Lambda_F\setminus\{0\}} \frac{1}{k(-l)} \int_{\Omega_x\times\Omega_x} \eta_k(x)\eta_{-k}(x')\eta_{-l}(x)\eta_l(x')dxdx'$$
$$= \sum_{k,l\in\Lambda_F\setminus\{0\}} \frac{1}{kl}\left[\int_{\Omega_x}\eta_k(x)\eta_{-l}(x)dx\right]\left[\int_{\Omega_x}\eta_{-k}(x')\eta_l(x')dx'\right]$$
$$= \sum_{k,l\in\Lambda_F\setminus\{0\}} \frac{1}{kl}\left[\int_{\Omega_x}\eta_k(x)\eta_{-l}(x)dx\right]\delta_{k,l}$$
$$= \sum_{k\in\Lambda_F\setminus\{0\}} \frac{1}{k^2} = 2\sum_{k=1}^{\infty}\frac{1}{k^2} = 2\frac{\pi^2}{6} = \frac{\pi^2}{3}. \tag{4.11}$$

As far as the second inequality is concerned, noting that also $K - K^N$ is a real-valued function and swapping index l to $-l$, one has:

$$\left\|K - K^N\right\|^2_{L^2(\Omega_x\times\Omega_x)} = \int_{\Omega_x\times\Omega_x}\big(K(x,x') - K^N(x,x')\big)^2 dxdx'$$
$$= \int_{\Omega_x\times\Omega_x}\left(i\sum_{k\notin\Lambda_F^N\setminus\{0\}}\frac{1}{k}\eta_k(x)\eta_{-k}(x')\right)^2 dxdx'$$
$$= -\int_{\Omega_x\times\Omega_x}\sum_{k,l\notin\Lambda_F^N\setminus\{0\}}\frac{1}{kl}\eta_k(x)\eta_{-k}(x')\eta_l(x)\eta_{-l}(x')dxdx'$$
$$= -\int_{\Omega_x\times\Omega_x}\sum_{k,l\notin\Lambda_F^N\setminus\{0\}}\frac{1}{k(-l)}\eta_k(x)\eta_{-k}(x')\eta_{-l}(x)\eta_l(x')dxdx'$$
$$= \sum_{k\notin\Lambda_F^N\setminus\{0\}}\frac{1}{k^2} \le 2(N_F)^{-1}. \tag{4.12}$$

The very last inequality comes from estimating the remainder of a convergent series. □

Additional estimates are reported below.

Lemma 4.3.

$$\left|K^N(x,x')\right| \le C\ln N_F \qquad \forall x,x'\in\Omega_x, \tag{4.13}$$

$$\left\|K^N\right\|^2_{L^2(\Omega_x\times\Omega_x)} \le \frac{\pi^2}{3}(1+N_F^{-1}), \tag{4.14}$$

where C in the first inequality is independent of N_F.

Proof. The first part is proven by noting that:

$$\left|K^N(x,x')\right| = \left|\sum_{k\in\Lambda_F^N\setminus\{0\}} \frac{1}{k}\eta_k(x)\eta_{-k}(x')\right| \leq \frac{1}{2\pi}\sum_{k\in\Lambda_F^N}\left|\frac{1}{k}\right| \leq C\ln N_F, \tag{4.15}$$

where the final inequality follows from bounding the partial sum of the harmonic series. The second inequality follows by using (4.9) and (4.10):

$$\begin{aligned}\left\|K^N\right\|^2_{L^2(\Omega_x\times\Omega_x)} &\leq \|K\|^2_{L^2(\Omega_x\times\Omega_x)} + \left\|K^N - K\right\|^2_{L^2(\Omega_x\times\Omega_x)} \\ &\leq \frac{\pi^2}{3} + 2N_F^{-1} \leq \frac{\pi^2}{3}\big(1+N_F^{-1}\big).\end{aligned} \tag{4.16}$$

□

We are now capable of proving theorem 4.1.

Proof of Theorem 4.1. Consider the approximation E^N of the electric field as suggested in (4.5). The evaluation of the $L^2(\Omega_x)$ norm of the error requires a further integration:

$$\begin{aligned}&\|E(\cdot,t) - E^N(\cdot,t)\|^2_{L^2(\Omega_x)} \\ &= \int_{\Omega_x}\left|\int_\Omega \big(f(x',v',t)K(x,x') - f^N(x',v',t)K^N(x,x')\big)dx'dv'\right|^2 dx.\end{aligned} \tag{4.17}$$

Since f^N is orthogonal to the difference $K(x,\cdot) - K^N(x,\cdot)$ it holds that

$$\int_\Omega f^N(x',v',t)\big(K(x,x') - K^N(x,x')\big)dx'dv' = 0, \tag{4.18}$$

and we transform the inner integral of (4.17) according to the following algebra:

$$\begin{aligned}&\int_\Omega \big(f(x',v',t)K(x,x') - f^N(x',v',t)K^N(x,x')\big)dx'dv' \\ &= \int_\Omega \Big(f(x',v',t)\big(K(x,x') - K^N(x,x')\big) + \big(f(x',v',t) - f^N(x',v',t)\big)K^N(x,x')\Big)dx'dv' \\ &= \int_{\Omega_x\times\Omega_v} \big(f(x',v',t) - f^N(x',v',t)\big)\big(K(x,x') - K^N(x,x')\big)dx'dv' \\ &\quad + \int_\Omega \big(f(x',v',t) - f^N(x',v',t)\big)K^N(x,x')dx'dv'.\end{aligned} \tag{4.19}$$

Therefore, by using (4.19) in (4.17) and the standard inequality $|a+b|^2 \leq 2a^2 + 2b^2$, we obtain:

$$\begin{aligned}&\|E(\cdot,t) - E^N(\cdot,t)\|^2_{L^2(\Omega_x)} \\ &\leq 2\int_{\Omega_x}\left|\int_\Omega \big(f(x',v',t) - f^N(x',v',t)\big)\big(K(x,x') - K^N(x,x')\big)dx'dv'\right|^2 dx \\ &\quad + 2\int_{\Omega_x}\left|\int_\Omega \big(f(x',v',t) - f^N(x',v',t)\big)K^N(x,x')dx'dv'\right|^2 dx.\end{aligned} \tag{4.20}$$

At this point, we further bound the error with the help of the Cauchy-Schwarz inequality:

$$\|E(\cdot,t)-E^N(\cdot,t)\|^2_{L^2(\Omega_x)} \le 2\left\|f-f^N\right\|^2_{L^2(\Omega)}|\Omega_v|\int_{\Omega_x}\left\|K(x,\cdot)-K^N(x,\cdot)\right\|^2_{L^2(\Omega_x)}dx$$

$$+\,2\left\|f-f^N\right\|^2_{L^2(\Omega)}|\Omega_v|\int_{\Omega_x}\left\|K^N(x,\cdot)\right\|^2_{L^2(\Omega_x)}dx$$

$$\le 2\left\|f-f^N\right\|^2_{L^2(\Omega)}|\Omega_v|\left(\left\|K-K^N\right\|^2_{L^2(\Omega_x)\times L^2(\Omega_x)}+\left\|K^N\right\|^2_{L^2(\Omega_x)\times L^2(\Omega_x)}\right) \quad (4.21)$$

(we recall that for the Hermite-Fourier method we consider the finite domain $\Omega =]v_{\min}, v_{\max}[$). Using the estimates of Lemma 4.2, we finally get:

$$\|E(\cdot,t)-E^N(\cdot,t)\|_{L^2(\Omega_x)} \le C\,|\Omega_v|^{\frac{1}{2}}\,\|f-f^N\|_{L^2(\Omega)}, \quad (4.22)$$

where $C = 2\pi(\sqrt{6}/3)$. □

We end this section with a technical lemma that provides an estimate of the $L^2(\Omega)$ norm of the electric fields E and E^N. This result follows immediately from the estimate of the kernels K and K^N and the stability of f and f^N. Such bounds will be used in the convergence analysis of the next section.

Lemma 4.4.

$$\|E(\cdot,t)\|_{L^2(\Omega)} \le C'\;|\Omega_v|^{\frac{1}{2}}\;\|f_0\|_{L^2(\Omega)}, \quad (4.23)$$

$$\|E^N(\cdot,t)\|_{L^2(\Omega)} \le C'\;|\Omega_v|^{\frac{1}{2}}\;\left(1+N_F^{-1}\right)^{\frac{1}{2}}\|\mathcal{P}^N f_0\|_{L^2(\Omega)}, \quad (4.24)$$

where $C' = \sqrt{3}\pi/3$.

Proof. In order to get inequality (4.23) we use (4.4), the definition of K given in (4.6), and we apply the Cauchy-Schwarz inequality:

$$\|E(\cdot,t)\|^2_{L^2(\Omega)} = \int_{\Omega_x}\left|\int_{\Omega} f(x',v',t)K(x,x')dv'dx'\right|^2 dx$$

$$\le \|f(\cdot,\cdot,t)\|^2_{L^2(\Omega)}\,|\Omega_v|\int_{\Omega_x}\|K(x,\cdot)\|^2_{L^2(\Omega_x)}dx$$

$$\le \|f_0\|^2_{L^2(\Omega)}\;|\Omega_v|\;\|K\|^2_{L^2(\Omega_x\times\Omega_x)} \le \|f_0\|^2_{L^2(\Omega)}\,|\Omega_v|\,\left(\frac{\pi^2}{3}\right).$$

Note that $\|f(\cdot,\cdot,t)\|_{L^2(\Omega)} = \|f_0\|_{L^2(\Omega)}$ follows from the stability of f, while the estimate of the quantity $\|K\|_{L^2(\Omega_x\times\Omega_x)}$ has been proven in Lemma (4.2). The proof of inequality (4.24) follows a similar pattern. In fact, we use (4.5), the definition of K^N given in (4.7), we apply the Cauchy-Schwarz inequality, we note that $\|f^N(\cdot,\cdot,t)\|_{L^2(\Omega)} = \|\mathcal{P}^N f_0\|_{L^2(\Omega)}$ from the stability of f^N and we use the estimate for

$\|K^N\|_{L^2(\Omega_x\times\Omega_x)}$ proven in Lemma (4.3):

$$\|E^N(\cdot,t)\|^2_{L^2(\Omega)} = \int_{\Omega_x}\left|\int_{\Omega} f^N(x',v',t)K^N(x,x')dv'dx'\right|^2 dx$$

$$\leq \|f^N(\cdot,\cdot,t)\|^2_{L^2(\Omega)}\,|\Omega_v|\int_{\Omega_x}\|K^N(x,\cdot)\|^2_{L^2(\Omega_x)}dx$$

$$\leq \|\mathcal{P}^N f_0\|^2_{L^2(\Omega)}\;|\Omega_v|\;\|K^N\|^2_{L^2(\Omega_x\times\Omega_x)}$$

$$\leq \|\mathcal{P}^N f_0\|^2_{L^2(\Omega)}\;|\Omega_v|\left(\frac{\pi^2}{3}\right)\left(1+N_F^{-1}\right).$$

This concludes the proof of the lemma. □

An immediate consequence of the previous lemma is an $L^2(\Omega)$ estimate of the convective fields $\mathbf{F}$ and $\mathbf{F}^N$, as stated by the following result.

Lemma 4.5.

$$\|\mathbf{F}\|^2_{L^2(\Omega)} \leq V^2\,|\Omega| + C''\frac{\pi^2}{3}, \tag{4.25}$$

$$\left\|\mathbf{F}^N\right\|^2_{L^2(\Omega)} \leq V^2\,|\Omega| + C''\frac{\pi^2}{3}(1+N_F^{-1}), \tag{4.26}$$

with $C''=|\Omega_v|\,\|f_0\|^2_{L^2(\Omega)}$.

Proof. Inequality (4.25) follows as a consequence of (4.23) and by noting that $|v|$ can be bounded by $V:=\max\{|v_{\min}|,|v_{\max}|\}$. Likewise, we derive inequality (4.26) by using (4.24). □

5. Convergence analysis. In this section, we derive a general convergence result valid for both Hermite-Fourier and Legendre-Fourier approximations.

Theorem 5.1. *Let f be the solution of the Galerkin formulation* (2.7)-(2.8) *of the Vlasov-Poisson system on the domain $\Omega=\Omega_x\times\Omega_v$ (where Ω_v may be either $]v_{\min},v_{\max}[$ or $\mathbb{R}$). Let f belong to the Sobolev space $H^{m_F}(\Omega_x)\times H^{m_S}(\Omega_v)$ for some positive numbers m_F and m_S. Moreover, let f^N be the solution of the truncated Vlasov-Poisson system* (2.13)-(2.14) *on the domain $\Omega_x\times]v_{\min},v_{\max}[$. Then, for any $\epsilon>0$, we have the error estimates:*

- *for the* Legendre-Fourier *method with m_F, $m_S\geq 2+\epsilon$:*

$$\|f(\cdot,\cdot,t)-f^N(\cdot,\cdot,t)\|_{L^2(\Omega)} \leq (N_F+N_L^2)(C_1N_F^{1-m_F+\epsilon}+C_2N_L^{3/2-m_S+2\epsilon}); \tag{5.1}$$

- *for the* Hermite-Fourier *method with m_F, $m_S\geq 2+\epsilon$:*

$$\|f(\cdot,\cdot,t)-f^N(\cdot,\cdot,t)\|_{L^2(\Omega)} \leq (N_F+\sqrt{N_H})(C_1N_F^{1-m_F+\epsilon}+C_2N_H^{(1-m_S+\epsilon)/2}). \tag{5.2}$$

In both cases, constants C_1 and C_2 are independent of N ($N=(N_L,N_F)$ for Legendre-Fourier and $N=(N_H,N_F)$ for Hermite-Fourier), but may depend on T, $v_{\min}$ and $v_{\max}$. According to the projection estimates in Appendix B, C_1 and C_2 are proportional to the Sobolev norms of f.

Proof. In view of the stability of f and f^N and using (3.1) and (3.2) we find that:

$$\begin{aligned}\frac{d}{dt}\|f-f^N\|^2_{L^2(\Omega)} &= \frac{d}{dt}\left(\|f\|^2_{L^2(\Omega)}+\|f^N\|^2_{L^2(\Omega)}-2\int_\Omega ff^N\,dvdx\right)\\ &= -2\frac{d}{dt}\int_\Omega ff^N\,dvdx = -2\int_\Omega f^N\frac{\partial f}{\partial t}dvdx - 2\int_\Omega f\frac{\partial f^N}{\partial t}dvdx\\ &= 2\int_\Omega f^N\big(\mathbf{F}\cdot\nabla f\big)dvdx + 2\int_\Omega f\Big(\mathcal{P}^N\big(\mathbf{F}^N\cdot\nabla f^N\big)-R^N\Big)dvdx. \qquad (5.3)\end{aligned}$$

Noting that R^N belongs to $\mathcal{X}^N$ and using the result of Lemma 3.2 with $g^N = \mathcal{P}^N f(\cdot,\cdot,t)$, we can transform the last integral in (5.3) as follows:

$$\begin{aligned}\int_\Omega f\big(\mathcal{P}^N\big(\mathbf{F}^N\cdot\nabla f^N\big)-R^N\big)dvdx &= \int_\Omega \big(\mathcal{P}^N f\big)\,\big(\mathbf{F}^N\cdot\nabla f^N - R^N\big)dvdx\\ &= \frac{1}{2}\int_\Omega\Big(\big(\mathcal{P}^N f\big)\,\mathbf{F}^N\cdot\nabla f^N - f^N\mathbf{F}^N\cdot\nabla(\mathcal{P}^N f)\Big)dvdx. \qquad (5.4)\end{aligned}$$

Now, with little algebraic manipulation we can get the identity:

$$\begin{aligned}&f^N\big(\mathbf{F}\cdot\nabla f\big)+\frac{1}{2}\Big(\big(\mathcal{P}^N f\big)\mathbf{F}^N\cdot\nabla f^N - f^N\mathbf{F}^N\cdot\nabla(\mathcal{P}^N f)\Big)\\ &= \frac{1}{2}\Big(\big(\mathcal{P}^N f - f\big)\mathbf{F}^N\cdot\nabla f^N + f^N\mathbf{F}^N\cdot\nabla\big(f-\mathcal{P}^N f\big)\Big)\\ &\quad + \big(f^N - f\big)\big(\mathbf{F}-\mathbf{F}^N\big)\cdot\nabla f + \frac{1}{2}\mathrm{div}\Big(\big(\mathbf{F}-\mathbf{F}^N\big)f^2 + \mathbf{F}^N f^N f\Big). \qquad (5.5)\end{aligned}$$

We substitute (5.4) in (5.3) and use (5.5). Since the integral of the divergence term is zero because of the boundary conditions on f, we reformulate (5.3) as follows:

$$\frac{d}{dt}\left\|f-f^N\right\|^2_{L^2(\Omega)} = \mathcal{E}_{\mathrm{proj}}(t)+\mathcal{E}_{\mathrm{appr}}(t) \le \left|\mathcal{E}_{\mathrm{proj}}(t)\right| + \left|\mathcal{E}_{\mathrm{appr}}(t)\right|, \qquad (5.6)$$

where

$$\mathcal{E}_{\mathrm{proj}}(t) = \int_\Omega\Big(\big(\mathcal{P}^N f - f\big)\mathbf{F}^N\cdot\nabla f^N + f^N\mathbf{F}^N\cdot\nabla\big(f-\mathcal{P}^N f\big)\Big)dvdx, \qquad (5.7)$$

$$\mathcal{E}_{\mathrm{appr}}(t) = 2\int_\Omega \big(f^N - f\big)\big(\mathbf{F}-\mathbf{F}^N\big)\cdot\nabla f dvdx. \qquad (5.8)$$

Term $\mathcal{E}_{\mathrm{proj}}(t)$ depends on the *projection* error $\mathcal{P}^N f - f$ and its gradient; term $\mathcal{E}_{\mathrm{appr}}(t)$ depends on the *approximation* errors $f^N - f$ and $E^N - E$. In the next subsections, we will prove that:

$$\left|\mathcal{E}_{\mathrm{proj}}(t)\right| \le \alpha(t;N), \qquad (5.9)$$

$$\left|\mathcal{E}_{\mathrm{appr}}(t)\right| \le \beta(t)\left\|f-f^N\right\|^2_{L^2(\Omega)}, \qquad (5.10)$$

where $\alpha(t;N)\to 0$ for $|N|\to\infty$ and $\beta(t)>0$ is independent of N. The specific form of these functions depends on the choice of the spectral discretization and is detailed in the following subsections for the Legendre-Fourier method and the Hermite-Fourier method. Substituting (5.7) and (5.8) in (5.6) yields

$$\frac{d}{dt}\|f-f^N\|^2_{L^2(\Omega)} \leq \alpha(t;N) + \beta(t)\left\|f-f^N\right\|^2_{L^2(\Omega)}. \tag{5.11}$$

The assertion of the theorem follows by applying the Gronwall inequality and the estimates established in the next subsections.
□

5.1. Estimates of the projection error. To estimate the first term of the projection error (5.7), we first extract the supremum of $\left|\mathcal{P}^N f - f\right|$ from the integral, we apply the Cauchy-Schwarz inequality and we note that $\|\mathbf{F}^N\|_{L^2(\Omega)}$ can be bounded by a positive constant that is independent of N but depends on $|\Omega_v|$ in view of (4.26). We obtain:

$$\begin{aligned}\left|\int_\Omega \big(\mathcal{P}^N f - f\big)\mathbf{F}^N \nabla f^N\, dvdx\right| &\leq C_1\Big(\sup_{(x,v)\in\Omega}\left|\mathcal{P}^N f - f\right|\Big)\|\mathbf{F}^N\|_{L^2(\Omega)}\|\nabla f^N\|_{L^2(\Omega)} \\ &\leq C_2\,\|f-\mathcal{P}^N f\|_{H^{1+\epsilon}(\Omega)}\|\nabla f^N\|_{L^2(\Omega)},\end{aligned} \tag{5.12}$$

where both constants C_1 and C_2 are strictly positive and independent of N (they may however depend on f_0 and $|\Omega_v|$, cf. Lemma 4.5). Note that $H^{1+\epsilon}(\Omega)$ with $\epsilon > 0$ is included in $L^\infty(\Omega)$, in order to justify the last inequality.

To estimate the second term of the projection error (5.7) we argue in a similar way, obtaining:

$$\begin{aligned}\left|\int_\Omega f^N\mathbf{F}^N\nabla\big(\mathcal{P}^N f - f\big)\, dvdx\right| &\leq \Big(\sup_{(x,v)\in\Omega}\left|\nabla(\mathcal{P}^N f - f)\right|\Big)\|\mathbf{F}^N\|_{L^2(\Omega)}\|f^N\|_{L^2(\Omega)} \\ &\leq C_3\|f-\mathcal{P}^N f\|_{H^{2+\epsilon}(\Omega)},\end{aligned} \tag{5.13}$$

where the $L^2(\Omega)$-norm of $\mathbf{F}^N$ is absorbed by constant C_3, which is independent of N, but may still depend on f_0 and $|\Omega_v|$. Putting together (5.12) and (5.13) yields:

$$|\mathcal{E}_{\text{proj}}(t)| \leq C_4\left(\|\nabla f^N\|_{L^2(\Omega)}\|f-\mathcal{P}^N f\|_{H^{1+\epsilon}(\Omega)} + \|f-\mathcal{P}^N f\|_{H^{2+\epsilon}(\Omega)}\right), \tag{5.14}$$

where C_4 absorbs the previous constants and does not depend on N. Using standard inverse inequalities of spectral approximations, see section A in appendix, and recalling that $\|f^N(\cdot,\cdot,t)\|_{L^2(\Omega)} = \|f^N(\cdot,\cdot,0)\|_{L^2(\Omega)} = \|\mathcal{P}^N f_0\|_{L^2(\Omega)}$, $\forall t\in[0,T[$, we obtain:

$$\|\nabla f^N\|_{L^2(\Omega)} \leq \xi^N\|\mathcal{P}^N f_0\|_{L^2(\Omega)}, \tag{5.15}$$

where we introduced the auxiliary coefficient:

$$\xi^N = \begin{cases} N_F + N_L^2 & \textit{(Legendre-Fourier method)}, \\ N_F + \sqrt{N_H} & \textit{(Hermite-Fourier method)}. \end{cases} \tag{5.16}$$

Using (5.15) in (5.14) yields

$$|\mathcal{E}_{\text{proj}}(t)| \leq C_5\left(\xi^N\|f-\mathcal{P}^N f\|_{H^{1+\epsilon}(\Omega)} + \|f-\mathcal{P}^N f\|_{H^{2+\epsilon}(\Omega)}\right), \tag{5.17}$$

where the constant C_5 absorbs the $L^2(\Omega)$-norm of $\mathcal{P}^N f_0$ and the previous constants.

The estimation of the bound of $\mathcal{E}_{\rm proj}$ is concluded by applying the estimates for the projection error onto the functional spaces $\mathcal{F}^N$, $\mathcal{L}^N$, $\mathcal{H}^N$ (see section B in appendix). From these estimates we may note that the error in $H^{1+\epsilon}(\Omega)$ decays faster than that in $H^{2+\epsilon}(\Omega)$, however the last one is multiplied by ξ^N, so that the terms on the right-hand side of (5.17) are well balanced. The two estimates can be merged to obtain:

$$|\mathcal{E}_{\rm proj}(t)| \leq \xi^N \times \begin{cases} \big(C_5 N_F^{1-m_F+\epsilon} + C_6 N_L^{3/2-m_S+2\epsilon}\big) & \textit{(Legendre-Fourier)}, \\ \big(C_5 N_F^{1-m_F+\epsilon} + C_6 N_H^{(1-m_S+\epsilon)/2}\big) & \textit{(Hermite-Fourier)}, \end{cases} \tag{5.18}$$

where the positive constants C_5 and C_6 are independent of N, but depend on the regularity of f through its higher-order Sobolev norms. To this regard, let us note that in the Hermite-Fourier case we have $\Omega = [0, 2\pi[\times]v_{\rm min}, v_{\rm max}[$ in (5.2). Nevertheless, the norms of f on the right-hand side are evaluated in $\Omega = [0, 2\pi[\times\mathbb{R}$.

Remark 5.1. *These estimates are, perhaps, not optimal due to the use of the inverse inequality in* (5.15). *We recall that the boundary conditions are imposed in the discrete space through the penalty term R^N and are only satisfied in weak form. Thus, whenever one tries to modify the integrals with an integration by parts, some boundary terms are produced that are difficult to estimate in optimal way. This is indeed the case of $\mathcal{E}_{proj}$ in* (5.7) *if we try to transform the gradient operator of f^N into the divergence of $(\mathcal{P}^N f - f)\mathbf{F}^N$, in order to avoid the inverse inequality in* (5.15).

5.2. Estimate of the approximation error. As far as the approximation error $\mathcal{E}_{\rm appr}$ is concerned, we first note that:

$$\mathbf{F}(x, v, t) - \mathbf{F}^N(x, v, t) = \begin{pmatrix} 0 \\ -\big(E(x,t) - E^N(x,t)\big) \end{pmatrix}. \tag{5.19}$$

We plug this relation into the definition of the approximation error (5.8). Afterwords, we proceed with a few standard inequalities and we apply the result of Theorem 4.1:

$$\begin{aligned} |\mathcal{E}_{\rm appr}| &\leq 2 \int_\Omega \big|f^N - f\big| \, \big|E - E^N\big| \, \left|\frac{\partial f}{\partial v}\right| \, dvdx \\ &\leq 2 \sup_{(x,v)\in\Omega} \left|\frac{\partial f}{\partial v}\right| \|f^N - f\|_{L^2(\Omega)} \, \|E - E^N\|_{L^2(\Omega)} \leq C_7 \|f^N - f\|^2_{L^2(\Omega)}, \end{aligned} \tag{5.20}$$

where the positive constant C_7 is independent of N, but may depend on the regularity of f and $|\Omega_v|$. If C_7 were dependent on N we would be in trouble as $\beta(t)$ in (5.10) could grow with N. In fact, after applying the Gronwall inequality to (5.11), we get an estimate containing the term $\exp\Big(\int_0^T \beta(t)dt\Big)$ in the right-hand side. This expression might become huge, thus providing a meaningless final error estimate if β were unbounded with respect to N.

5.3. Further remarks on Hermite-Fourier approximations. The Hermite-Fourier method deserves some further comments. Although Hermite-based approximations are usually stated on $\Omega_v = \mathbb{R}$, the estimate of Theorem 5.1 is derived in $L^2(\Omega_x \times \Omega_v)$ with $\Omega_x = [0,\, 2\pi[$, $\Omega_v =]v_{\rm min}, v_{\rm max}[$, and assuming that $|\Omega_v| = v_{\rm max} - v_{\rm min}$ is finite and hopefully not too big. In fact, the constants of this estimate depend on the size of Ω and blow up for $|\Omega_v|$ tending to infinity. There are some critical issues here that we want to point it out. First, Hermite functions are substantially different from zero on a support that grows as $\sqrt{N_H}$. Second, even if the exact solution f has compact support, its approximation by the Hermite functions may require a larger support as N_H grows. Assuming that the size of Ω_v depends on N_H may lead us to serious drawbacks for the reasons detailed at the end of the previous section.

Note that Reference [22], where a similar analysis was carried out for $R^N = 0$ in the 2D-2V case, did not address these issues. There, estimates were given on a finite domain (only depending on time t) without imposing artificial conditions and assuming (with too much optimism, maybe) that the discretized solutions were remaining with good approximation within the support of the exact solution independently of N_H. The above considerations teach us that domain Ω_v must be chosen "wisely" depending on the behavior manifested by the exact solution. In particular, Ω_v should be large enough so that imposing zero boundary constraints weakly is not too stringent; at the same time, Ω_v must not be too large to avoid the negative influence mentioned above on the error estimates.

6. Conclusions. In this paper we provided a convergence theory for the approximation of the Vlasov-Poisson system by the symmetrically-weighted Hermite-Fourier spectral method (restricted to a finite sized velocity domain) and the Legendre-Fourier spectral method.

A modified weak form of the boundary conditions at the extrema of the velocity domain made it possible to prove the stability of both approximations. It is well-known that the symmetrically-weighted Hermite-based approximation is stable when the integration is on the infinite velocity domain. Therefore, what we proved here is that the stability remains preserved in our formulation also when the Hermite-Fourier method integrates the Vlasov-Poisson system on a finite velocity domain.

Finally, we note that the error estimates are weak, since they are obtained in the $L^2(\Omega)$ norm. For first-order nonlinear problems such as the one we are dealing with, developing a better convergence theory could be hard. Note that the situation in the Hermite case with $R^N = 0$ (usually employed in many applications) is even worse, since the subset consisting of rapidly decaying functions is not closed in the $L^2(\mathbb{R})$ metric. Nevertheless, this paper provides a solid theoretical foundation to spectral methods applied to Vlasov-Poisson systems. In addition, the penalty term R^N offers a promising strategy of handling joining conditions in multi-domain spectral approximations, which will be the topic of further research.

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References.

[1] T. P. Armstrong, R. C. Harding, G. Knorr, and D. Montgomery. Solution of Vlasov's equation by transform methods. *Methods Comput. Phys.*, 9:29, 1970.

[2] C. K. Birdsall and A. B. Langdon. *Plasma Physics Via Computer Simulation.* Taylor & Francis, 2004.

[3] E. Camporeale, G. L. Delzanno, B. K. Bergen, and J. D. Moulton. On the velocity space discretization for the Vlasov-Poisson system: comparison between implicit Hermite spectral and Particle-in-Cell methods. *Computer Physics Communications*, 198:47–58, 2015.

[4] E. Camporeale, G. L. Delzanno, G. Lapenta, and W. Daughton. New approach for the study of linear Vlasov stability of inhomogeneous systems. *Physics of Plasmas*, 13(9):092110, 2006.

[5] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. *Spectral Methods in Fluid Dynamics*. Springer Series in Computational Physics. Springer-Verlag, Berlin, Heidelberg, 1988.

[6] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. *Spectral Methods: Fundamentals in Single Domains*. Scientific Computation. Springer, Berlin, Heidelberg, 2010.

[7] C. Canuto and A. Quarteroni. Approximation results for orthogonal polynomials in Sobolev spaces. *Mathematics of Computation*, 38:67–86, 1982.

[8] C. Z. Cheng and G. Knorr. The integration of the Vlasov equation in configuration space. *Journal of Computational Physics*, 22(3):330–351, 1976.

[9] G. L. Delzanno. Multi-dimensional, fully-implicit, spectral method for the Vlasov-Maxwell equations with exact conservation laws in discrete form. *Journal of Computational Physics*, 301:338–356, 2015.

[10] F. Filbet, E. Sonnendrücker, and P. Bertrand. Conservative numerical schemes for the Vlasov equation. *Journal of Computational Physics*, 172(1):166–187, 2001.

[11] R. Fitzpatrick. *Plasma Physics*. CRC Press, 2014.

[12] D. Funaro. *Polynomial Approximation of Differential Equations*. Lecture Notes in Physics Monographs. Springer Berlin Heidelberg, 2008.

[13] R. J. Goldston and P. H. Rutherford. *Introduction to Plasma Physics*. Plasma Physics Series. Institute of Physics Publications, 1995.

[14] R. Hockney and J. Eastwood. *Computer Simulation Using Particles*. Taylor & Francis, 1988.

[15] J. P. Holloway. Spectral velocity discretizations for the Vlasov-Maxwell equations. *Transport Theory and Statistical Physics*, 25(1):1–32, 1996.

[16] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer, Berlin, 1983.

[17] N. F. Loureiro, W. Dorland, L. Fazendeiro, A. Kanekar, A. Mallet, M. S. Vilelas, and A. Zocco. Viriato: a Fourier-Hermite spectral code for strongly magnetised fluid-kinetic plasma dynamics. *arXiv:1505.02649*, 2015.

[18] N. F. Loureiro, A. A. Schekochihin, and A. Zocco. Fast collisionless reconnection and electron heating in strongly magnetized plasmas. *Physical Review Letters*, 111:025002, Jul 2013.

[19] H. Ma and W. Sun. A Legendre-Petrov-Galerkin and Chebyshev collocation method for third-order differential equations. *SIAM Journal on Numerical Analysis*, 38(5):1425–1438, 2000.

[20] G. Manzini, G. L. Delzanno, J. Vencels, and S. Markidis. A Legendre-Fourier spectral method with exact conservation laws for the Vlasov-Poisson system. *Journal of Computation Physics*, 317:82–107, 2016. Online 3 May 2016, doi: 10.1016/j.jcp.2016.03.069.

[21] J. T. Parker and P. J. Dellar. Fourier-Hermite spectral representation for the Vlasov-Poisson system in the weakly collisional limit. *Journal of Plasma Physics*, 81, 4 2015.

[22] M. Pulvirenti and J. Wick. Convergence of Galerkin approximation for two-dimensional Vlasov-Poisson equation. *Zeitschrift für angewandte Mathematik und Physik (ZAMP)*, 35(6):790–801, 1984.

[23] J. Shen. A new dual Petrov-Galerkin method for third and higher odd-order differential equations: application to the KDV equation. *SIAM Journal on Numerical Analysis*, 41(5):1595–1619, 2003.

[24] J. Shen, T. Tang, and L.-L. Wang. *Spectral Methods, Algorithms, Analysis and Applications*. Springer, 2011.

[25] S. L. Sobolev. *Partial Differential Equations of Mathematical Physics*. Dover Publications, New York, 1964.

[26] J. Vencels, G. L. Delzanno, A. Johnson, I. B. Peng, E. Laure, and S. Markidis. Spectral solver for multi-scale plasma physics simulations with dynamically adaptive number of moments. *Procedia Computer Science*, 51:1148–1157, 2015.

[27] J. Vencels, G. L. Delzanno, G. Manzini, S. Markidis, I. Bo Peng, and V. Roytershteyn. SpectralPlasmaSolver: a spectral code for multiscale simulations of collisionless, magnetized plasmas. *Journal of Physics: Conference Series*, 719:012022, 2016.

Appendix A. Inverse inequalities.

In this appendix and the next one, we list a series of well-known results, limiting the exposition to the simplest case where the indices m and r are integer numbers. More general results that cover the case where m and r are non integer numbers are available from the literature. These results could be used to obtain sharper estimates but would also increase the technicality of the exposition.

Let $\Omega_x = [0, 2\pi[$, $\Omega_v =]v_{\min}, v_{\max}[$. Let $\mathcal{L}^N := \mathcal{S}^N$ in the Legendre case and $\mathcal{H}^N := \mathcal{S}^N$ in the Hermite case. Moreover, let us denote by $H^m(\Omega_v)$ the standard Sobolev space of L^2-integrable functions whose derivatives are also L^2-integrable up to order m. Similarly, we have that $H^m_p(\Omega_x) = H^m_p(0, 2\pi)$ is the corresponding Sobolev space in the case of periodic functions. As usual: $H^0(\Omega_v) = L^2(\Omega_v)$ and $H^0_p(0, 2\pi) = L^2(0, 2\pi)$. We also consider the space $L^2_w(\mathbb{R})$ of functions that are square integrable with respect to the positive weight function $w(v) = e^{-v^2}$. Of course, the corresponding norm is:

$$\|\psi\|_{L^2_w(\mathbb{R})} = \left(\int_{\mathbb{R}} |\psi(v)|^2 \, e^{-v^2} dv \right)^{1/2}. \tag{A.1}$$

Sobolev type functional spaces for a non integer $m \geq 0$ are obtained through standard interpolation techniques. Then, in the finite dimensional spaces, one has the following inverse inequalities.

- Periodic Fourier: for all numbers m and r such that $0 \leq r \leq m$ it holds that

$$\|\phi\|_{H^r_p(\Omega_x)} \leq C N_F^{r-m} \|\phi\|_{H^m_p(\Omega_x)} \qquad \forall \phi \in \mathcal{F}^{N_F}, \tag{A.2}$$

  where C is independent of N_F; see [6, Section 5.8.1]
- Legendre polynomials: for all $r \geq 1$ it holds that

$$\left\| \frac{\partial^r \phi}{\partial v^r} \right\|_{L^2(\Omega_v)} \leq C N_L^{2r} \|\phi\|_{L^2(\Omega_v)} \qquad \forall \phi \in \mathcal{L}^{N_L}, \tag{A.3}$$

  where the constant C is independent of N_L but depends on $|\Omega_v|$; see [5, Section 9.4.1]
- Hermite polynomials:

$$\left\| \frac{\partial \phi}{\partial v} \right\|_{L^2_w(\mathbb{R})} \leq C \sqrt{N_H} \|\phi\|_{L^2_w(\mathbb{R})}, \tag{A.4}$$

  for all polynomials of degree at most N_H, where the constant C is independent of N_H. A similar result holds in the case of Hermite functions ($\phi \in \mathcal{H}^{N_H}$). In this case the weight function is $w(v) = e^{v^2}$. One can pass from a case to the other by virtue of [12, Lemma 6.7.4].

Appendix B. Orthogonal projections.

- Periodic Fourier: consider the operator $\mathcal{P}_F^{N_F}$, which projects $L^2(0, 2\pi)$ onto $\text{span}\{\eta_k\}_{k \in \Lambda_F^N}$. We have the following estimate for the projection error:

$$\|\psi - \mathcal{P}_F^{N_F} \psi\|_{H^r_p(0,2\pi)} \leq C N_F^{r-m} \left\| \frac{\partial^m \psi}{\partial x^m} \right\|_{L^2(0,2\pi)}, \tag{B.1}$$

  which holds for every $0 \leq r \leq m$. More details are found in [6, Section 5.1.2].

- Legendre polynomials: consider the operator $\mathcal{P}_L^{N_L}$, which projects $L^2(\Omega_v)$ onto the space of polynomials of degree at most N_L. We have the following estimate:

$$\|\psi - \mathcal{P}_L^{N_L}\psi\|_{L^2(\Omega_v)} \le CN_L^{-m}\|\psi\|_{H^m(\Omega_v)}, \tag{B.2}$$

  where $m \ge 0$ and the constant C is independent of N_L but depends on $|\Omega_v|$. An extension, where at the left-hand side we find higher Sobolev norms is available:

$$\|\psi - \mathcal{P}_L^{N_L}\psi\|_{H^r(\Omega_v)} \le CN_L^{2r-1/2-m}\|\psi\|_{H^m(\Omega_v)}, \tag{B.3}$$

  where $m \ge r \ge 1$. More details are found in [6, Section 5.4.2].
- Hermite polynomials: we have the following estimate for the projection error in the space of polynomials of degree at most N_H:

$$\|\psi - \mathcal{P}_H^{N_H}\psi\|_{L^2_w(\mathbb{R})} \le CN_H^{-m/2}\left\|\frac{\partial^m\psi}{\partial v^m}\right\|_{L^2_w(\mathbb{R})}, \tag{B.4}$$

  which holds for any $\psi \in H^m_w(\mathbb{R})$, $m \ge 0$; see [12, Theorem 6.2.6]. This can be also generalized when higher-order norms are present on the left-hand side (see [24], Theorem 7.13, p.270):

$$\|\psi - \mathcal{P}_H^{N_H}\psi\|_{H^r_w(\mathbb{R})} \le CN_H^{(r-m)/2}\left\|\frac{\partial^m\psi}{\partial v^m}\right\|_{L^2_w(\mathbb{R})}, \tag{B.5}$$

  Similar results hold in the case of Hermite functions where the weight is $w(v) = e^{v^2}$.

We are ready to provide an estimate to the projection operator for functions of both variables x and v. We begin by observing that for $f \in L^2(\Omega)$ one has:

$$(I - \mathcal{P}^N)f = (I - \mathcal{P}_F^{N_F})f + \mathcal{P}_F^{N_F}(I - \mathcal{P}_S^{N_S})f, \tag{B.6}$$

By virtue of this equality we get:

$$\begin{aligned}\|f - \mathcal{P}^N f\|_{L^2(\Omega)} &\le \|f - \mathcal{P}_F^{N_F}f\|_{L^2(\Omega)} + \|\mathcal{P}_F^{N_F}\|_{\mathcal{L}\left(L^2(\Omega),L^2(\Omega)\right)}\|f - \mathcal{P}_S^{N_S}f\|_{L^2(\Omega)}\\ &\le \|f - \mathcal{P}_F^{N_F}f\|_{L^2(\Omega)} + \|f - \mathcal{P}_S^{N_S}f\|_{L^2(\Omega)}\end{aligned} \tag{B.7}$$

since the norm of the projector $\mathcal{P}_F^{N_F}$ in $\mathscr{L}\big(L^2(\Omega), L^2(\Omega)\big)$ is less than one.

The bound of the last term can be specialized according to the method adopted. This gives the final estimates:

$$\|f - \mathcal{P}^N f\|_{L^2(\Omega)} \le \begin{cases} C_1N_F^{-m_F} + C_2N_L^{-m_S} & m_F, m_S \ge 0 \quad \textit{(Leg.-Fou.)},\\ C_1N_F^{-m_F} + C_2N_H^{-m_S/2} & m_F, m_S \ge 0 \quad \textit{(Her.-Fou.)}.\end{cases} \tag{B.8}$$

In the Hermite case one has $\Omega_v = \mathbb{R}$. As before, the constants C_1 and C_2 do not depend on the discretization parameters. They depend however on Sobolev type norms type norms of the given function $f \in H^{m_F}(\Omega_x) \times H^{m_S}(\Omega_v)$. When the Sobolev space on the left-hand side is $H^r(\Omega)$, with $r \ge 1$, we have:

$$\|f - \mathcal{P}^N f\|_{H^r(\Omega)} \le \begin{cases} C_1N_F^{r-m_F} + C_2N_L^{2r-1/2-m_L} & m_F, m_L \ge r \quad \textit{(Leg.-Fou.)},\\ C_1N_F^{r-m_F} + C_2N_H^{(r-m_H)/2} & m_F, m_H \ge r \quad \textit{(Her.-Fou.)}.\end{cases} \tag{B.9}$$

A way to prove the above result is to differentiate the equality in (B.6) and note that the Fourier projector commutes with derivatives. Successively, one makes use of (B.3) and (B.4). Regarding these techniques, we refer to the original paper [7] for more insight.

Appendix C. Implementation.

C.1. Spectral decomposition of the Vlasov equation. The spectral decomposition of $f(x,v,t)$ on the space-velocity domain $(x,v) \in \Omega_x \times \Omega_v$ and $t \in [0,T[$ reads as:

$$f(x,v,t) = \sum_{(n,k)\in\Lambda} C_{n,k}(t)\varphi_n(v)\eta_k(x). \tag{C.1}$$

By substituting (C.1) in (2.7) we obtain the following infinite non-linear system of ordinary differential equations for the coefficients $C_{n,k}(t)$:

$$\frac{dC_{n,k}}{dt} + \sum_{(n',k')\in\Lambda} A_{(n,k),(n',k')} C_{n',k'} + \sum_{(n',k'),(n'',k'')\in\Lambda} B_{(n,k),(n',k'),(n'',k'')} C_{n',k'} C_{n'',k''} = 0 \quad \forall (n,k) \in \Lambda, \tag{C.2}$$

with the initial conditions $C_{n,k}(0) = C^0_{n,k}$, $\forall (n,k) \in \Lambda$, which are obtained from the spectral expansion of the initial solution $f_0(x,v)$. This writing is not used in the theoretical analysis; it is important however for the implementation of the algorithm, since the coefficients $A_{(n,k),(n',k')}$ and $B_{(n,k),(n',k'),(n'',k'')}$ are the same as those of the numerical approximation. We are now going to show how to compute them. The *linear term* in (C.2) is such that:

$$\sum_{(n',k')\in\Lambda} A_{(n,k),(n',k')} C_{n',k'} = \int_\Omega \varphi_n \eta_k \, v \frac{\partial f}{\partial x} \, dv dx \tag{C.3}$$

with

$$A_{(n,k),(n',k')} = \int_\Omega \varphi_n \eta_k \, v \frac{\partial}{\partial x} \big(\varphi_{n'} \eta_{k'} \big) \, dv dx. \tag{C.4}$$

Note that $A_{(n,k),(n',k')}$ does not depend on t because such a dependence is clearly expressed through the coefficient $C_{n',k'}(t)$. The coefficient in (C.4) can be recovered using the orthogonality properties of Hermite or Legendre polynomials, and in particular by the respective three-term recursion formulas [3, 9, 20].

In a similar manner, the *non-linear term* in (C.2) is such that:

$$\sum_{(n',k'),(n'',k'')\in\Lambda} B_{(n,k),(n',k'),(n'',k'')} C_{n',k'} C_{n'',k''} = -\int_\Omega \varphi_n \eta_k \, E \frac{\partial f}{\partial v} \, dv dx. \tag{C.5}$$

This time, to derive the expression of the coefficients $B_{(n,k),(n',k'),(n'',k'')}$ we first need to write the electric field E in terms of the coefficients $C_{n,k}$ in (C.5). For this purpose, we consider the decomposition:

$$E(x,t) = \sum_{(n,k)\in\Lambda} \widehat{E}_{n,k}(x) \, C_{n,k}(t), \tag{C.6}$$

basically corresponding to the Fourier expansion of E and that will be discussed in the next subsection (in particular, $\widehat{E}_{n,k}$ is given by formula (C.11)). Combining (C.1) and (C.6), from (C.5) we find that:

$$B_{(n,k),(n',k'),(n'',k'')} = -\int_{\Omega} \varphi_n \eta_k \, \widehat{E}_{n',k'} \frac{\partial}{\partial v}\big(\varphi_{n''}\eta_{k''}\big) dv dx. \tag{C.7}$$

As already noted for $A_{(n,k),(n',k')}$, also the coefficient $B_{(n,k),(n',k'),(n'',k'')}$ does not depend on time.

C.2. Spectral decomposition of the Poisson equation. Consider the Fourier decomposition of $E(x,t)$ on the spatial domain Ω_x given by (4.2). By substituting (C.1) and (4.2) in (2.8) we obtain:

$$\sum_{k\in\Lambda_F} (ik) E_k \eta_k = \sqrt{2\pi}\eta_0 - \sum_{(n,k)\in\Lambda} C_{n,k}\eta_k \int_{\Omega_v} \varphi_n dv. \tag{C.8}$$

We multiply (C.8) by $\eta_{-k'}$, integrate on Ω_x and use the orthogonality property (2.4) to obtain (changing the summation index back to k):

$$E_k(t) = \sum_{n\in\Lambda_S} \gamma_{n,k} C_{n,k}(t) \quad \forall k \in \Lambda_F, \tag{C.9}$$

where for all $n \in \Lambda_S$ we take:

$$\gamma_{n,0} = 0 \qquad \text{and} \qquad \gamma_{n,k} = \frac{i}{k}\int_{\Omega_v} \varphi_n(v) dv \qquad k \in \Lambda_F \backslash \{0\}. \tag{C.10}$$

From the orthogonality properties of the Symmetrically Weighted Hermite functions it holds that $\gamma_{n,k} = 0$ for every k and odd n. Instead, for Legendre polynomials it holds that $\gamma_{n,k} = 0$ for every $n > 0$. From $\gamma_{n,0} = 0$ for all n it follows that $E_0(t) = 0$, which is equivalent to $\int_{\Omega_x} E(x,t)dx = 0$.

Using the definition of E_k in (C.9), by comparing (4.2) and (C.6) we immediately find that:

$$\widehat{E}_{n,k}(x) = \gamma_{n,k}\eta_k(x) = \frac{i}{k}\left(\int_{\Omega_v} \varphi_n(v) dv\right)\eta_k(x) \quad k \in \Lambda_F \backslash \{0\}. \tag{C.11}$$

C.3. Global discretization. Consider the *approximated distribution function*:

$$f^N(x,v,t) = \sum_{(n,k)\in\Lambda^N} C^N_{n,k}(t)\varphi_n(v)\eta_k(x), \tag{C.12}$$

which approximates the function $f(x,v,t)$ in $\mathcal{X}^N$, as well as the *approximated electric field*:

$$E^N(x,t) = \sum_{(n,k)\in\Lambda^N} \gamma_{n,k} C^N_{n,k}(t)\eta_k(x), \tag{C.13}$$

which approximates $E(x,t)$ in $\mathcal{F}^N$. Clearly, f^N does not coincide with projection $\mathcal{P}^N f$ and E^N with projection $\mathcal{P}^N_F E$. The coefficients $C^N_{n,k}$ are determined by imposing that

f^N and E^N are the solution of the *truncated Vlasov-Poisson system* given by (2.13)-(2.14). From (2.13)-(2.14), a straightforward calculation yields that the coefficients $C^N_{n,k}(t)$ are the solution of a system of ordinary differential equations, namely:

$$\frac{dC^N_{n,k}}{dt} + \sum_{(n',k')\in\Lambda^N} A_{(n,k),(n',k')} C^N_{n',k'} + \sum_{(n',k'),(n'',k'')\in\Lambda^N} \Big(B_{(n,k),(n',k'),(n'',k'')} - \widetilde{B}_{(n,k),(n',k'),(n'',k'')} \Big) C^N_{n',k'} C^N_{n'',k''} = 0 \qquad \forall (n,k)\in\Lambda^N, \tag{C.14}$$

and at $t = 0$ we set $C^N_{n,k}(0) = C^0_{n,k}$, $\forall (n,k) \in \Lambda^N$ using the same initial conditions of problem (C.2). The coefficients A and B are the same as in (C.4) and (C.7), respectively. Instead, the coefficients denoted by $\widetilde{B}$ are obtained from integration of term R^N and using the result of Lemma 2.1 to derive their explicit formula. In fact, with the special choice $g^N(x,t) = \eta_k(x)\varphi_n(v)$ in (2.15) we find that:

$$\int_\Omega \eta_k(x)\varphi_n(v) R^N(x,v,t)dvdx = -\frac{1}{2}\int_{\Omega_x} E^N(x,t)\eta_k(x)\Big[f^N(x,v_{\max},t)\varphi_n(v_{\max}) - f^N(x,v_{\min},t)\varphi_n(v_{\min})\Big] dx.$$

Using the expansions of $f^N(x,v_{\max},t)$ and $f^N(x,v_{\min},t)$, cf. (C.1), and the expression of the electric field shown in (C.6), one gets:

$$\begin{aligned}\int_\Omega \eta_k(x)\varphi_n(v) R^N(x,v,t)dvdx &= \sum_{(n',k'),(n'',k'')} C^N_{n',k'}(t) C^N_{n'',k''}(t) \Big[\varphi_n(v_{\max})\varphi_{n'}(v_{\max}) \\ &\quad - \varphi_n(v_{\min})\varphi_{n'}(v_{\min})\Big] \int_{\Omega_x} \widehat{E}_{(n'',k'')}(x)\eta_k(x)\eta_{k'}(x)dx \\ &= \sum_{(n',k'),(n'',k'')} C^N_{n',k'} C^N_{n'',k''} \widetilde{B}_{(n,k),(n',k'),(n'',k'')},\end{aligned}$$

from which one finally recovers:

$$\widetilde{B}_{(n,k),(n',k'),(n'',k'')} = \Big[\varphi_n(v_{\max})\varphi_{n'}(v_{\max}) - \varphi_n(v_{\min})\varphi_{n'}(v_{\min})\Big] \int_{\Omega_x} \widehat{E}_{(n'',k'')}\eta_k\eta_{k'} dx.$$

Since coefficients $A_{(n,k),(n',k')}$, $B_{(n,k),(n',k'),(n'',k'')}$, and $\widetilde{B}_{(n,k),(n',k'),(n'',k'')}$ do not depend on t, the well-posedness of the present system of ordinary differential equations follows from classical results. Indeed, the forcing term is the sum of a linear and a quadratic part. Such a term is locally Lipschitz, so that existence and uniqueness in a suitable interval $[0,t_N[\subset [0,T[$ may be recovered through a contraction theorem. By the way, this result can be extended to the whole interval $[0,T[$ (and, possibly, to the entire semi-axis $[0,+\infty[$). It is enough to recall equality (3.6), from which we deduce that the quantity $\sum_{(n,k)\in\Lambda^N} |C^N_{n,k}|^2$ is bounded by a constant κ^N that is independent of t. This prevents the blow up of the solution in a finite time. More in detail, we put equation (C.14) in vector form:

$$\frac{dC^N_{n,k}}{dt} + \mathcal{A}^N_{(n,k)}\underline{C}^N + \mathcal{B}^N_{(n,k)}(\underline{C}^N,\underline{C}^N) = 0 \qquad \forall (n,k)\in\Lambda^N \tag{C.15}$$

where $\mathcal{A}^N_{(n,k)}$ represents the linear part, $\mathcal{B}^N_{(n,k)}$ the quadratic one (including the penalty term $\widetilde{B}_{(n,k),(n',k'),(n'',k'')}$), and vector $\underline{C}^N = \{C^N_{n,k}\}_{(n,k)\in\Lambda^N}$ collects all the spectral modes. We also define the closed subspace $\mathcal{Y}^N$ of $\mathcal{X}^N$ consisting of all functions bounded in $L^2(\Omega)$ by a given constant κ^N, independent of t. For two vectors $\underline{C}^N_1$ and $\underline{C}^N_2$, it holds that, for any $(n,k)\in\Lambda^N$, $\mathcal{B}^N_{(n,k)}$ is Lipschitz in $\mathcal{Y}^N\times\mathcal{Y}^N$ by virtue of the following inequalities:

$$
\begin{aligned}
&\left|\mathcal{B}^N_{(n,k)}(\underline{C}^N_1,\underline{C}^N_1)-\mathcal{B}^N_{(n,k)}(\underline{C}^N_2,\underline{C}^N_2)\right| \\
&\quad\le \left|\mathcal{B}^N_{(n,k)}(\underline{C}^N_1-\underline{C}^N_2,\underline{C}^N_1)\right|+\left|\mathcal{B}^N_{(n,k)}(\underline{C}^N_2,\underline{C}^N_1-\underline{C}^N_2)\right|\le\lambda^N\|\underline{C}^N_1-\underline{C}^N_2\|_N,
\end{aligned}
$$

where $\|\cdot\|_N$ denotes the classical norm in finite dimension and $\lambda^N>0$ is the Lipschitz constant. To obtain the above estimate, we used the Schwarz inequality and the boundedness in $\mathcal{Y}^N$, i.e., $\|\underline{C}^N_i\|_N\le\kappa^N$ for $i=1,2$. By observing that the coefficients of $\mathcal{B}^N_{(n,k)}$ do not depend on time we obtain that the Lipschitz constant λ^N also does not depend on time. We can actually fix λ^N in such a way that it does not depend on the indices (n,k). These last remarks allow for the prolongation of the discrete solution to the additional interval $[t_N,2t_N[$, which is of the same size of the initial one. Such a procedure can be repeated as many times is necessary to cover the interval $[0,T[$ (and beyond).

The solution of the system of ordinary differential equations is infinitely times differentiable in space and time. However the existence proof provided above is naturally posed in the space $C^0(0,T;L^2(\Omega))$, which is where we prove the error estimates.