

Quantum thermodynamics of two bosonic systems

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We study the energy exchange between two bosonic systems that interact via bilinear transformations in the mode operators. The first mode is considered as the thermodynamic system, while the second is regarded as the bath. This work finds its roots in a very recent formulation of quantum thermodynamics [1] which allows to consider baths that are not described by the usual Boltzmann-Gibbs canonical form. Baths can possess quantum properties, such as squeezing or coherence, and can be initially correlated with the system, even through entanglement. We focus mainly on the case of Gaussian states, by quantifying the relation between their defining parameters, namely the mean values of the quadratures and the covariance matrix, and relevant thermodynamical quantities such as the heat exchanged and the work performed during the interaction process. We fully solve the case of initially uncorrelated Gaussian states and provide the most general form of the first law of thermodynamics in this case. We also discuss the case of initially correlated states by considering a number of relevant examples, studying how correlations can assist some phenomena, e.g. work extraction or anomalous heat flows. Finally, we present an information-theoretic approach based on the Renyi entropy of order two for clarifying more generally the role of correlations on heat exchanges.

I. INTRODUCTION

In the last years the theory of thermodynamics, that is the study of energy and its changes, which are distinguished as heat and work, has emerged as one of the frameworks where quantum information theory can be fruitfully employed and, conversely, where new insights on the theory can be discovered. A new field of research was born, named quantum thermodynamics [2–5], aiming to extend the classical thermodynamics to systems of sizes well below the thermodynamic limit. In this realm, an extensive use of quantum information tools is applied, ranging from the characterization of heat engines at the nanoscale [6–8] to the study of thermodynamical properties of relativistic and quantum fields [9–12].

Standard formulations of thermodynamics, classical [13] or quantum [2–4, 14, 15], consider systems in interaction with thermal baths that are large, compared to the system dimension, and usually described by states in the Boltzmann-Gibbs canonical form, whose temperatures are not altered during thermodynamic processes. This hypothesis cannot be justified a priori in the quantum regime, since baths may be small in principle and can also possess quantum properties, such as coherence [16, 17] or squeezing [18], and share correlations [19–25], e.g. entanglement, with the system. Energy exchanges between systems and baths will therefore alter the temperature of both parties and the roles of quantum properties must be taken into account.

To this aim, a temperature-independent version of thermodynamics has been recently formulated in [1]. Standard thermodynamics can then be recast as a consequence of information conservation, providing modifications to the laws of thermodynamics consistent with the possible presence of (classical or quantum) correlations between systems and baths, and providing a new definition of the notion of temperature which generalizes the standard one.

According to such a formulation, in this paper we study the thermodynamics of a composite system made of two bosonic modes, where one mode represents the thermodynamical system while the other is treated as the bath, which can be therefore classically or quantum correlated with the system and does not have constant and unchangeable temperature. In particular, we study energy exchanges by considering the first law of thermodynamics in the case of bilinear interactions of modes.

The paper is organized as follows. In Sec. II we review the approach to thermodynamics reported in Ref. [1]. We focus mainly on the first law, thus the definitions of internal energy, heat and work are presented. Then the second law of thermodynamics is briefly discussed, recalling how the Clausius statement can be generalized in the presence of initial correlations between the system and the bath. In Sec. III we discuss how this formulation can be explicitly expressed for two-mode bosonic states. In particular, we focus on Gaussian bipartite states and show how the first law of thermodynamics can be expressed and how the relevant thermodynamical quantities depend on the state parameters, i.e. the initial mean values of the quadratures and the covariance matrix, and on the transformations considered, which are here at most bilinear in the mode operators. In Sec. IV we discuss the case when system and bath are initially uncorrelated; several examples of important classes of two-mode Gaussian states are presented. In Sec. V we study the case when system and bath present initial correlations by means of some illustrative examples, by considering both separable and entangled states. Moreover, we derive an information-theoretic approach based on the 2-Renyi entropy in order to clarify the role of correlations on heat exchanges. This approach allows to link directly the heat flows to the variations of the amount of correlations. Finally, in Sec. VI we consider work extraction schemes, showing how interactions between system and bath can be used to increase the free energy of the system, and thus the amount of extractable work.

II. REVIEW OF QUANTUM THERMODYNAMICS WITH ARBITRARY BATH

We review here the formulation of thermodynamics introduced in Ref. [1], which allows one to go beyond the standard notion of temperature and to consider baths that possess quantum properties in the most general form. This section represents therefore a concise synthesis of the results reported in Ref. [1] that we will use in the next sections of the paper.

A. Intrinsic temperature and complete passivity

In the standard formulation of thermodynamics one usually considers a large thermal bath, whose temperature is well defined and whose state is described by the Boltzmann-Gibbs canonical form. Typically, such a state is not allowed to change during the thermodynamic processes due to its ideally infinite thermal capacity. Hence, for example, after thermalization the temperature of the system coincides with that of the thermal bath. In the quantum regime baths can be small in principle and can also possess quantum properties, such as coherence and squeezing,

or share correlations, e.g. entanglement, with the system. Therefore, baths cannot be described a priori in the Boltzmann-Gibbs canonical form and the concept of temperature must be generalized.

This problem has recently been tackled in [1], where thermodynamics has been viewed as a consequence of information conservation, leading to the notion of intrinsic temperature which can be formulated for any quantum system. The concept of information conservation leads to consider only operations that are globally entropy-preserving. Hence, given the state ρ of an arbitrary system, we consider transformations Λ that are entropy-preserving (EP), namely if $\rho' = \Lambda(\rho)$ is the state after the action of Λ , we have $S(\rho) = S(\rho')$, where $S(X) = -\text{Tr}[X \log X]$ is the von Neumann entropy of X . The central role of entropy can also be seen by the fact that we establish equivalence classes between states that have the same entropy, namely we will say that ρ and σ belong to the same class iff $S(\rho) = S(\sigma)$. When H is the Hamiltonian of the system, then we choose as the representative of each class the state $\gamma(\rho)$ which has the minimum energy within the same class of ρ , namely

$$\gamma(\rho) = \arg \min_{\sigma: S(\sigma)=S(\rho)} E(\sigma), \quad (1)$$

where $E(\sigma) = \text{Tr}[H\sigma]$ is the energy of the state σ .

The min-energy principle [24, 26, 27] that allows to find the state that minimizes the energy at a fixed entropy and is in some sense complementary to the max-entropy principle [28, 29], indicates that $\gamma(\rho)$ must be thermal. Hence, given an entropic class, the representative thermal state is given by

$$\gamma(\rho) = \frac{1}{Z} e^{-\beta(\rho)H}, \quad (2)$$

where $Z = \text{Tr}[e^{-\beta(\rho)H}]$, and $\beta(\rho)$ represents the intrinsic inverse temperature of a state ρ , which labels the equivalence classes. In this way we are able to consider a generalized notion of temperature $T(\rho)$ for any state ρ , not just for the thermal ones, via the relation

$$T(\rho) = \beta(\rho)^{-1}, \quad (3)$$

where we fixed the Boltzmann constant as $k_B = 1$. Clearly, for thermal states one recovers the standard definition of temperature. The representative state (2) of each class is also a completely passive (CP) state [26, 27], namely no extractable work can be accessed from it, even when multiple copies are available. States of this form are also denoted by $\gamma(H_S, \beta_S)$, where H_S is the Hamiltonian of the system and β_S labels the entropic equivalence classes, or shortly $\gamma(T_S)$. Moreover, these states have also the following property: given a non-interacting Hamiltonian $H_T = \sum_{X=1}^N \mathbb{I}^{\otimes X-1} \otimes H_X \otimes \mathbb{I}^{\otimes N-X}$, the global CP state is given by the tensor product of locally CP states with the same inverse temperature β_T , namely

$$\gamma(H_T, \beta_T) = \otimes_{X=1}^N \gamma(H_X, \beta_T). \quad (4)$$

B. The first law of thermodynamics

The first law of thermodynamics deals with energy conservation and describes the energy distribution in terms of variation of heat and work. We consider a bipartite system described by ρ_{AB} , where A and B represent the thermodynamical system and the bath respectively, that undergoes an EP transformation that results in the final state ρ'_{AB} . In standard thermodynamics the bath and the system are considered initially uncorrelated, with the bath in the Boltzmann-Gibbs canonical form. Heat is then defined as $\Delta Q = E(\rho'_B) - E(\rho_B)$, corresponding to the variation of the internal energy of the bath. However, this definition of heat suffers several issues in the quantum domain. For example, if the temperature of the bath can change and correlations between system and bath are taken into account, it may lead to seeming violations of the second law of thermodynamics [19, 30–33]. Moreover, if we allow the bath to be non-thermal then its internal energy includes also the share of energy that can be extracted and converted into work, since only the thermal states are passive.

In the framework of [1] two relevant forms of energy are distinguished for any given state: the bound and the free energy. The first represents the amount of internal energy that cannot be accessed in form of work. Conversely, the latter is the part of internal energy that can be transformed into work by an EP operation.

The bound energy $B(\rho)$ is defined as

$$B(\rho) = \min_{\sigma: S(\sigma)=S(\rho)} E(\sigma) = E(\gamma(\rho)), \quad (5)$$

and it represents the energy that cannot be extracted further. The min-energy principle identifies the energy of the CP state $\gamma(\rho)$ as the bound energy. Hence, given a state ρ and determined its entropic equivalence class, then the energy of the representative of this class represents the bound energy.

The free energy $F(\rho)$ is given by the difference

$$F(\rho) = E(\rho) - B(\rho). \quad (6)$$

This can also be written as $F(\rho) = T(\rho) D(\rho||\gamma(\rho))$, where $D(\sigma_1||\sigma_2) = \text{Tr}[\sigma_1(\log \sigma_1 - \log \sigma_2)]$ denotes the quantum relative entropy, which measures the distinguishability of two states σ_1 and σ_2 . No privileged a priori temperature is required in the above definition of the free energy. The standard Helmholtz free energy $F_T(\rho) = E(\rho) - TS(\rho)$, where T is fixed a priori by choosing a thermal bath in contact with the system, is recovered in the case of an ideal bath, as was shown in [1], but Eq. (6) fits in a more general framework. In standard thermodynamics the Helmholtz free energy allows to quantify the amount of work W that can be extracted from a system in contact with an ideal large bath at inverse temperature β . Indeed, $W = F_\beta(\rho) - F_\beta(\gamma(\rho))$, where $\gamma(\rho)$ is the thermal equilibrium state of the system after the work extraction. The free energy (6) represents the amount of work that can be extracted by using a bath at the worst possible temperature, namely [1]

$$F(\rho) = \min_{\beta} [F_\beta(\rho) - F_\beta(\gamma(\rho))]. \quad (7)$$

It follows that the inverse temperature that achieves the minimum in Eq. (7) is the intrinsic one $\beta(\rho)$.

More specifically, given an arbitrary system described by ρ , the extractable work from an EP transformation $\rho' = \Lambda(\rho)$, which is $W = E(\rho) - E(\rho')$, is upper bounded by the free energy, that is

$$W \leq F(\rho), \quad (8)$$

where the inequality is saturated iff $\rho' = \gamma(\rho)$. Moreover, if the global system is in the state $\rho = \rho_A \otimes \gamma_B(T_B)$ then the right-hand side becomes the standard Helmholtz free energy, namely in this case $W \leq F_{T_B}(\rho_A) - F_{T_B}(\gamma_A(T_B))$.

We can now define heat and work which allow to formulate the first law of thermodynamics. Heat represents the most degraded form of energy that a system exchanges with the bath during thermodynamical processes. It is therefore natural to define heat as the variation of the bound energy of the bath, namely [1]

$$\Delta Q = B(\rho'_B) - B(\rho_B). \quad (9)$$

We stress that the bath can present initial correlations with the thermodynamical system and its intrinsic temperature can change during the process. If ΔQ is positive the system dissipates energy during the process, thus part of its internal energy is transformed into bound energy of the bath, which contributes to increase its intrinsic temperature: roughly speaking, we can say that the bath is heated up during the transformation. Conversely, if ΔQ is negative the thermodynamical system acquires energy from the bath, whose intrinsic temperature decreases. If the bath is initially in a thermal state $\gamma(T_B)$, heat is bounded as follows

$$T_B \Delta S_B \leq \Delta Q \leq \Delta E_B, \quad (10)$$

and the above three quantities coincide in the limit of ideal large thermal baths. Since the process is entropy-preserving, note that the variation of the von Neumann entropy of the bath ΔS_B is related to that of the system ΔS_A by the identity

$$\Delta S_A + \Delta S_B = \Delta I(A : B), \quad (11)$$

where $\Delta I(A : B)$ denotes the variation of the mutual information $I(A : B) = S_A + S_B - S_{AB}$ between A and B , which measures the amount of both classical and quantum correlations shared by the two parties.

The work performed on the thermodynamical system in the EP transformation is $\Delta W_A = W - \Delta F_B$, where W is the cost needed to implement the EP transformation, namely $W = \Delta E_A + \Delta E_B$, and ΔF_B represents the variation of the free energy of the bath. The first law of thermodynamics is then given by

$$\Delta E_A = \Delta W_A - \Delta Q. \quad (12)$$

C. The second law of thermodynamics

The second law of thermodynamics puts constraints upon the possibility of some thermodynamical processes providing, for instance, limitations to the direction of heat exchange or the impossibility of converting heat into work

completely. For standard thermodynamics it has been formulated in several equivalent ways, such as the Carnot principle, the Clausius statement, the Kelvin-Planck statement, or the Caratheodory principle, just to mention the most known ones. These concepts have been extended to the more general scenario of entropy-preserving operations with arbitrary baths in Ref. [1], where the authors provided generalized statements, which reduce to the usual ones in the regime of large ideal baths. Here we recall the generalized Clausius statement [1]:

Clausius statement. Any iso-entropic process involving two systems A and B in an arbitrary state, with intrinsic temperatures T_A and T_B , respectively, satisfies the following inequality

$$(T_B - T_A) \Delta S_A \geq \Delta F_A + \Delta F_B + T_B \Delta I(A : B) - W, \quad (13)$$

where ΔF_X is the change in the free energy of system X , $\Delta I(A : B)$ is the change of mutual information, and $W = \Delta E_A + \Delta E_B$ is the amount of external work performed on the total system.

In the absence of initial correlations between the two systems, the states being initially thermal and no external work being performed, this implies

$$(T_B - T_A) \Delta S_A \geq 0, \quad (14)$$

so that no iso-entropic equilibration process is possible whose sole result is the transfer of heat from a cooler to a hotter system.

Equation (14) may be then overcome for three reasons: *(i)* external work is provided to the global system, i.e. $W \neq 0$, as in a standard refrigeration cycle; *(ii)* the initial states possess free energy that is consumed; *(iii)* the two systems are initially correlated. Violations of the standard formulation due to correlations have recently gained great attention [19, 30–33], and several physical systems have been proposed to test these violations. We will show that the present framework enables us to study such phenomena for two interacting Bosonic systems.

III. QUANTUM THERMODYNAMICS OF TWO BOSONIC SYSTEMS

A. General description

We address the study of the first law of thermodynamics (12) for two bosonic systems under bilinear interaction (for reviews on the properties of bosonic systems see [34–36]). Each system is described by the mode operators a, a^\dagger and b, b^\dagger , respectively, with the usual commutation relation, and the total free Hamiltonian is given by $H_0 = H_A + H_B = \omega_A (a^\dagger a + \frac{1}{2}) + \omega_B (b^\dagger b + \frac{1}{2})$. The first mode represents the thermodynamical system, while the second is considered as the bath. We note that two bosonic systems are also at the basis of analysis of quantum Otto engines [37] and thermodynamics of quantum fields [9]. We will study the following two global transformations for mode operators

$$a' = \cos \theta a + e^{i\varphi} \sin \theta b, \quad (15)$$

$$b' = \cos \theta b - e^{-i\varphi} \sin \theta a, \quad (16)$$

with $\theta \in [0, \frac{\pi}{2}]$ and $\varphi \in [0, 2\pi]$, and

$$a' = \cosh r a + e^{i\psi} \sinh r b^\dagger, \quad (17)$$

$$b' = \cosh r b + e^{i\psi} \sinh r a^\dagger, \quad (18)$$

with $r \geq 0$ and $\psi \in [0, 2\pi]$.

The Heisenberg transformation in Eqs. (15-16) corresponds to a linear mixing of the modes that can describe a frequency converter for $\omega_A \neq \omega_B$ or a beam splitter for $\omega_A = \omega_B$, and is equivalent to the unitary transformation in the Schroedinger picture [38]

$$U_{FC}(\zeta) = \exp \{ \zeta a^\dagger b - \zeta^* a b^\dagger \}, \quad (19)$$

with $\zeta = \theta e^{i\varphi}$.

The transformation in Eqs. (17-18) can describe non-degenerate parametric amplification (i.e. two-mode squeezing), and is equivalent to the unitary transformation [38]

$$U_{PA}(\xi) = \exp \{ \xi a^\dagger b^\dagger - \xi^* a b \}, \quad (20)$$

with $\xi = r e^{i\psi}$.

The bipartite system undergoes the following process: the initial state ρ_{AB} is transformed into $\rho'_{AB} = U\rho_{AB}U^\dagger$, where $U = U_{FC}(\zeta)$ or $U = U_{PA}(\xi)$, and $S(\rho'_{AB}) = S(\rho_{AB})$ being the process unitary. Without loss of generality, the initial state can be written as follows

$$\rho_{AB} = D(\alpha) \otimes D(\delta) \xi_{AB} D^\dagger(\alpha) \otimes D^\dagger(\delta), \quad (21)$$

where $D(\lambda) = \exp(\lambda a^\dagger - \lambda^* a)$ denotes the displacement operator, and ξ_{AB} has zero-mean field values, namely $\text{Tr}[(a \otimes \mathbb{I}_B) \xi_{AB}] = \text{Tr}[(\mathbb{I}_A \otimes b) \xi_{AB}] = 0$. Notice that the von Neumann entropy of the bath does not depend on the displacement terms, namely $S(\xi_B) = S(\rho_B)$. For the considered transformations, the final state $\rho'_{AB} = U\rho_{AB}U^\dagger$ can also be expressed as

$$\rho'_{AB} = D(\alpha') \otimes D(\delta') \xi'_{AB} D^\dagger(\alpha') \otimes D^\dagger(\delta'), \quad (22)$$

where $\xi'_{AB} = U\xi_{AB}U^\dagger$ satisfies $\text{Tr}[(a \otimes \mathbb{I}_B) \xi'_{AB}] = \text{Tr}[(\mathbb{I}_A \otimes b) \xi'_{AB}] = 0$, and either

$$\alpha' = \alpha \cos \theta + \delta e^{i\varphi} \sin \theta, \quad (23)$$

$$\delta' = \delta \cos \theta - \alpha e^{-i\varphi} \sin \theta, \quad (24)$$

or

$$\alpha' = \alpha \cosh r + \delta^* e^{i\psi} \sinh r, \quad (25)$$

$$\delta' = \delta \cosh r + \alpha^* e^{i\psi} \sinh r, \quad (26)$$

for transformations $U_{FC}(\zeta)$ and $U_{PA}(\xi)$, respectively. As a consequence the von Neumann entropy of the bath in the final state is $S(\rho'_B) = S(\xi'_B) = S(\text{Tr}_A[U\rho_{AB}U^\dagger])$.

Our primary aim is now to quantify and discuss the first law $\Delta W_A = \Delta E_A + \Delta Q$, with particular focus on heat flows. In the present scheme the first law dictates the distribution of energy between work and heat due to the EP (indeed unitary) interaction. Let us first focus on the heat $\Delta Q = B(\rho'_B) - B(\rho_B)$. For a thermal state $\gamma(\rho_B)$, the von Neumann entropy is given by $S(\gamma(\rho_B)) = g(N_B)$, where $N_B = \langle b^\dagger b \rangle_{\gamma(\rho_B)}$, and

$$g(x) = (x+1) \ln(x+1) - x \ln x. \quad (27)$$

Hence, being $g(x)$ an increasing invertible function, one has $N_B = g^{-1}[S(\gamma(\rho_B))]$ and $E(\gamma(\rho_B)) = \omega_B(N_B + \frac{1}{2})$, which represents the bound energy $B(\rho_B)$ for all ρ_B such that $S(\rho_B) = S(\gamma(\rho_B))$. It follows that ΔQ can be expressed as

$$\Delta Q = \omega_B [g^{-1}(S(\rho'_B)) - g^{-1}(S(\rho_B))]. \quad (28)$$

Let us notice that for infinitesimal transformations Eq. (28) provides the customary Clausius relation $\delta Q = T_B dS_B$, where T_B is the intrinsic temperature of the bath. Explicitly, one has

$$\delta Q = \omega_B \frac{\partial}{\partial S_B} [g^{-1}(S_B)] dS_B = \omega_B \frac{1}{g'(g^{-1}(S_B))} dS_B = \frac{\omega_B}{\ln \frac{N_B+1}{N_B}} dS_B = T_B dS_B, \quad (29)$$

where we used $g'(x) = \ln \frac{x+1}{x}$ and the identity for the bosonic Gibbs state $\frac{N_B}{N_B+1} = e^{-\omega/T_B}$. Moreover, in the present scenario where total EP transformations are considered, through Eqs. (11) and (29), the usual separation of the infinitesimal system entropy variation $dS_A = dS_{rev} + dS_{irr}$ [39–41] in terms of exchange (or reversible) entropy $dS_{rev} = -\frac{\delta Q_B}{T_B}$ and irreversible production of entropy dS_{irr} allows us to identify the last term as the variation of mutual information, namely

$$dS_{irr} = dS_A + \frac{\delta Q_B}{T_B} = dS_A + dS_B = dI(A : B). \quad (30)$$

Hence, the formulation of the second principle of classical thermodynamics in terms of the statement $dS_{irr} \geq 0$ can be violated when $dI(A : B) < 0$.

The variation of the internal energy in Eq. (12) is given by

$$\Delta E_A = \text{Tr}_{AB} [(H_A \otimes \mathbb{I}_B) (\rho'_{AB} - \rho_{AB})] = \omega_A \left(\langle a^\dagger a \rangle_{\rho'_A} - \langle a^\dagger a \rangle_{\rho_A} \right). \quad (31)$$

The above relation can be refined by using a phase-space description of bosonic states. Let us introduce the vector of quadrature operators $\mathbf{R} = (R_A, R_B)^T = (q_A, p_A, q_B, p_B)^T$, where $q_A = \frac{1}{\sqrt{2}}(a + a^\dagger)$, $p_A = \frac{1}{i\sqrt{2}}(a - a^\dagger)$, $q_B = \frac{1}{\sqrt{2}}(b + b^\dagger)$, and $p_B = \frac{1}{i\sqrt{2}}(b - b^\dagger)$. The components of \mathbf{R} satisfy

$$[R_k, R_l] = i\Omega_{kl}, \quad (32)$$

where Ω_{kl} denotes the element of the symplectic matrix

$$\Omega = \oplus_{AB}\omega, \quad (33)$$

with

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (34)$$

For a two-mode bipartite state, the covariance matrix σ_{AB} , whose elements are $\sigma_{kl} = \frac{1}{2} \langle \{R_k, R_l\} \rangle - \langle R_k \rangle \langle R_l \rangle$, can be written as

$$\sigma_{AB} = \begin{pmatrix} \sigma_A & \epsilon \\ \epsilon^T & \sigma_B \end{pmatrix}, \quad (35)$$

where σ_A and σ_B are the covariance matrices of the two subsystems and ϵ describes their correlations. The expectation $\langle \mathbf{R} \rangle_{\rho_{AB}}$ and σ_{AB} cannot fully characterize any two-mode state, since higher-order moments are generally needed, but they allow to express Eq. (31) as:

$$\Delta E_A = \frac{\omega_A}{2} \left(\text{Tr}[\sigma'_A] - \text{Tr}[\sigma_A] + \|\langle R'_A \rangle\|^2 - \|\langle R_A \rangle\|^2 \right), \quad (36)$$

where σ'_A and R'_A denote the covariance matrix and the vector of quadrature operators for the thermodynamical system at the end of the EP transformation, respectively.

B. Two-mode Gaussian states

States that are fully characterized just by $\langle \mathbf{R} \rangle_{\rho_{AB}}$ and σ_{AB} are called Gaussian. These states, named for the Gaussian character of their Wigner function, can be prepared by applying unitary operators that are at most bilinear in the mode operators to thermal states. This feature makes them suitable to be easily prepared and manipulated in laboratories. Nowadays, several applications are indeed designed exclusively for Gaussian states in different areas [34–36, 42–44]. The Gaussian character of a state is preserved under linear and bilinear interaction of the modes. If a two-mode Gaussian state ρ_{AB} is changed in $\rho'_{AB} = U\rho_{AB}U^\dagger$ by a unitary U as our $U_{FC}(\zeta)$ or $U_{PA}(\xi)$ in Eqs. (19) and (20), then there exists a symplectic matrix Γ_U that transforms \mathbf{R} and σ_{AB} in $\mathbf{R}' = \Gamma_U \mathbf{R}$ and $\sigma'_{AB} = \Gamma_U \sigma_{AB} \Gamma_U^T$, respectively. In this case the symplectic matrix is a 4×4 invertible real matrix satisfying

$$\Gamma \Omega \Gamma^T = \Omega, \quad (37)$$

where Ω is defined in (33). The symplectic matrix Γ can be decomposed into a block-form as

$$\Gamma = \begin{pmatrix} A & D \\ C & B \end{pmatrix}, \quad (38)$$

where A, B, C and D are 2×2 matrices such that: $A^T B - C^T D = \mathbb{I}_2$, $A^T C = C^T A$, and $B^T D = D^T B$.

Equations (28) and (36) can be further specified when ξ_{AB} in Eq. (21) is a Gaussian state and by exploiting the decomposition (38). In fact, any covariance matrix for n modes can be diagonalized through a symplectic transformation [45], namely σ can be written as

$$\sigma = \mathbf{S}^T \mathbf{W} \mathbf{S}, \quad (39)$$

where \mathbf{S} is a $2n \times 2n$ symplectic matrix, $\mathbf{W} = \oplus_{k=1}^n d_k \mathbb{I}_2$, and the elements d_k are the symplectic eigenvalues of σ . Heisenberg-Robertson's uncertainty relation imposes physical constraints on the admissible covariance matrices, which can be simply expressed in terms of the symplectic eigenvalues as

$$d_k \geq \frac{1}{2}, \quad (40)$$

that must hold for any k . The symplectic diagonalization (39) implies that any Gaussian state ρ can be obtained from a thermal state ν by applying a Gaussian unitary transformation U_S , associated to \mathbf{S} , namely

$$\rho = U_S \nu U_S^\dagger, \quad (41)$$

where $\nu = \otimes \nu_k$ is a product of thermal states ν_k .

Let us now determine the intrinsic temperature of a single-mode Gaussian state. Any such state ρ can be written as

$$\rho = U_S \nu_{N_{th}} U_S^\dagger = D(\alpha) S(\zeta) \nu_{N_{th}} S(\zeta)^\dagger D(\alpha)^\dagger, \quad (42)$$

where $S(\zeta) = \exp\left[\frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2\right]$ denotes the single-mode squeezing operator and $\nu_{N_{th}}$ is a thermal state with $\langle a^\dagger a \rangle_{\nu_{N_{th}}} = N_{th} = (e^{\beta\omega_A} - 1)^{-1}$. The unitary operator U_S does not affect the purity $\mu_\rho = \text{Tr}[\rho^2]$ and the von Neumann entropy of ρ , which therefore depend only on the thermal seed $\nu_{N_{th}}$ and are given by

$$\mu_\rho = \frac{1}{2N_{th} + 1}, \quad (43)$$

and

$$S(\rho) = g(N_{th}), \quad (44)$$

respectively. Moreover, the purity is related to the determinant of the covariance matrix of ρ by the relation

$$\mu_\rho = \frac{1}{2\sqrt{\det(\sigma_\rho)}}, \quad (45)$$

and hence

$$N_{th} = \sqrt{\det(\sigma_\rho) - \frac{1}{2}}. \quad (46)$$

As a consequence, the von Neumann entropy depends only on the determinant of its covariance matrix σ_ρ , and one has

$$S(\rho) = g\left(\sqrt{\det(\sigma_\rho) - \frac{1}{2}}\right). \quad (47)$$

Clearly, the thermal state that represents the entropic equivalence class of ρ is the one that appears in the decomposition (42), which thus identifies the intrinsic temperature of ρ as the following increasing function of N_{th}

$$T(\rho) = T(\nu_{N_{th}}) = \omega_A \left[\log\left(\frac{1 + N_{th}}{N_{th}}\right) \right]^{-1}. \quad (48)$$

Notice that for $N_{th} \gg 1$ one has $T \simeq \omega_A N_{th}$. Equation (46) can be used to evaluate the bound energy of the bath which is in the state $\rho_B = \text{Tr}_A[\rho_{AB}]$. The thermal state $\gamma(\rho_B)$ representing the entropic class of ρ_B has energy $\omega_B(N_{B,th} + \frac{1}{2})$, and hence

$$B(\rho_B) = \omega_B \left(N_{B,th} + \frac{1}{2} \right) = \frac{\omega_B}{2} \coth\left(\frac{\beta\omega_B}{2}\right) = \omega_B \sqrt{\det(\sigma_B)}. \quad (49)$$

Notice that for $\beta\omega_B \ll 1$ one has $B(\rho_B) \simeq \beta^{-1}$, as in the classical equipartition theorem. The heat absorbed by the bath is then given by

$$\Delta Q = B(\rho'_B) - B(\rho_B) = \omega_B \left(\sqrt{\det(\sigma'_B)} - \sqrt{\det(\sigma_B)} \right), \quad (50)$$

where σ'_B is the covariance matrix at the end of the transformation, which from Eq. (38) can be expressed as:

$$\sigma'_B = B\sigma_B B^T + C\sigma_A C^T + C\epsilon B^T + B\epsilon^T C^T. \quad (51)$$

The direction of the heat flow is determined just by the sign of $\Delta_B = \det(\sigma'_B) - \det(\sigma_B)$. We also remind that σ_B and σ'_B can be equivalently referred to (ρ_B, ξ_B) and (ρ'_B, ξ'_B) , respectively, since for the property of Eqs. (21) and (22) they do not depend on the displacement terms.

Let us now consider the variation of the internal energy of the system ΔE_A . By using Eqs. (36) and (38) one has

$$\Delta E_A = \frac{\omega_A}{2} (\text{Tr}[A\sigma_A A^T + D\sigma_B D^T + A\epsilon D^T + D\epsilon^T A^T]) + \frac{\omega_A}{2} \left(\|\langle AR_A + DR_B \rangle\|^2 - \|\langle R_A \rangle\|^2 - \text{Tr}[\sigma_A] \right). \quad (52)$$

We will show in the following how Eqs. (50) and (52) can be explicitly evaluated for the transformations in Eqs. (19) and (20).

1. Frequency converter/beam splitter

The symplectic matrix Γ_ζ corresponding to the transformation $U_{FC}(\zeta)$ in Eq. (19) is given by

$$\Gamma_\zeta = \begin{pmatrix} \cos \theta \mathbb{I}_2 & \sin \theta R_\varphi \\ -\sin \theta R_\varphi^T & \cos \theta \mathbb{I}_2 \end{pmatrix}, \quad (53)$$

where R_φ is the rotation operator

$$R_\varphi = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}. \quad (54)$$

The block-matrix decomposition of Γ_ζ is then provided by: $A = B = \cos \theta \mathbb{I}_2$, $D = \sin \theta R_\varphi$, and $C = -\sin \theta R_\varphi^T$. Hence, the variation of the internal energy (52) for the initial state ρ_{AB} in Eq. (21) reads

$$\Delta E_A = \omega_A \sin^2 \theta \left[\frac{1}{2} (\text{Tr}[\sigma_B] - \text{Tr}[\sigma_A]) + |\delta|^2 - |\alpha|^2 \right] + \omega_A \sin 2\theta \left[\frac{1}{2} \text{Tr}[R_\varphi^T \epsilon] + \text{Re}(\alpha \delta^* e^{-i\varphi}) \right]. \quad (55)$$

Correspondingly, from Eq. (51), the covariance matrix of the bath evolves as

$$\sigma'_B = \cos^2 \theta \sigma_B + \sin^2 \theta R_\varphi^T \sigma_A R_\varphi - \frac{1}{2} \sin 2\theta (\epsilon^T R_\varphi + R_\varphi^T \epsilon). \quad (56)$$

Thus, ΔQ can be computed according to Eq. (50).

2. Parametric amplifier

The symplectic matrix Γ_ξ corresponding to the transformation $U_{PA}(\xi)$ in Eq. (20) is given by

$$\Gamma_\xi = \begin{pmatrix} \cosh r \mathbb{I}_2 & \sinh r \tilde{R}_\psi \\ \sinh r \tilde{R}_\psi & \cosh r \mathbb{I}_2 \end{pmatrix}, \quad (57)$$

with

$$\tilde{R}_\psi = \begin{pmatrix} \cos \psi & \sin \psi \\ \sin \psi & -\cos \psi \end{pmatrix}. \quad (58)$$

The block-form of Γ_ξ is expressed by $A = B = \cosh r \mathbb{I}_2$ and $C = D = \sinh r \tilde{R}_\psi$. Hence, the variation of the internal energy (52) for the initial state (21) is

$$\Delta E_A = \omega_A \sinh^2 r \left[\frac{1}{2} (\text{Tr}[\sigma_B] + \text{Tr}[\sigma_A]) + |\delta|^2 + |\alpha|^2 \right] + \omega_A \sinh 2r \left[\frac{1}{2} \text{Tr}[\tilde{R}_\psi \epsilon] + \text{Re}(\alpha \delta e^{-i\psi}) \right]. \quad (59)$$

From Eq. (51), the covariance matrix of the bath after the transformation is

$$\sigma'_B = \cosh^2 r \sigma_B + \sinh^2 r \tilde{R}_\psi \sigma_A \tilde{R}_\psi + \frac{1}{2} \sinh 2r (\epsilon^T \tilde{R}_\psi + \tilde{R}_\psi \epsilon). \quad (60)$$

IV. UNCORRELATED SYSTEM AND BATH

We consider first the case where the thermodynamical system and the bath are initially uncorrelated, namely the initial state is $\rho_{AB} = \rho_A \otimes \rho_B$, being ρ_A and ρ_B single-mode Gaussian states of the general form

$$\rho_A = D(\alpha) S(\zeta_A) \nu_{N_A} S(\zeta_A)^\dagger D(\alpha)^\dagger, \quad (61)$$

$$\rho_B = D(\delta) S(\zeta_B) \nu_{N_B} S(\zeta_B)^\dagger D(\delta)^\dagger, \quad (62)$$

where $\alpha, \delta \in \mathbb{C}$ and $\zeta_A = r_A e^{i\theta_A}$, $\zeta_B = r_B e^{i\theta_B}$, with $r_A, r_B \geq 0$ and $\theta_A, \theta_B \in [0, 2\pi]$. The respective covariance matrices can be represented in terms of their elements as follows [34]

$$\begin{aligned}\sigma_{X,11} &= \frac{2N_X + 1}{2} (\cosh 2r_X + \cos \theta_X \sinh 2r_X), \\ \sigma_{X,22} &= \frac{2N_X + 1}{2} (\cosh 2r_X - \cos \theta_X \sinh 2r_X), \\ \sigma_{X,12} = \sigma_{X,21} &= -\frac{2N_X + 1}{2} \sin \theta_X \sinh 2r_X,\end{aligned}\tag{63}$$

where $X = A, B$ labels the mode. The intrinsic temperatures depend only on N_X and are obtained by Eq. (48).

Clearly, excluding initial correlations between the modes simplifies the problem and precludes possible interesting features. On the other hand, it allows to analyze and emphasize some quantum properties of the bath, such as the presence of squeezing that cannot be found in the standard treatments. For both the considered bilinear transformations, we will discuss the first law in general. Particular emphasis will be given to the sign of the heat, i.e. the direction of heat flow.

A. Frequency converter/beam splitter

For the transformation in Eq. (19) the variation of the internal energy (55) for a factorized state $\rho_{AB} = \rho_A \otimes \rho_B$, with ρ_A and ρ_B given by Eqs. (61) and (62), can be expressed as

$$\Delta E_A = \omega_A \sin^2 \theta \left(\frac{2N_B + 1}{2} \cosh 2r_B - \frac{2N_A + 1}{2} \cosh 2r_A + |\delta|^2 - |\alpha|^2 \right) + \omega_A \sin 2\theta \operatorname{Re}(\alpha \delta^* e^{-i\varphi}),\tag{64}$$

since the traces of the initial covariance matrices are $\operatorname{Tr}[\sigma_A] = (2N_A + 1) \cosh 2r_A$ and $\operatorname{Tr}[\sigma_B] = (2N_B + 1) \cosh 2r_B$. The internal energy of the system increases the more the bath is squeezed while, conversely, squeezing in the initial state of the system decreases the internal energy. The same consideration holds for the thermal part: the hotter is the bath, namely the higher is its intrinsic temperature, the more the internal energy increases, while the opposite holds for the system. The second term in Eq. (64) is a phase-sensitive contribution due to the coherence interference.

From Eq. (56) one has

$$\sigma'_B = \cos^2 \theta \sigma_B + \sin^2 \theta \sigma_A^\varphi,\tag{65}$$

with $\sigma_A^\varphi = R_\varphi^T \sigma_A R_\varphi$. The determinant of σ'_B can be expressed as [46]

$$\det(\sigma'_B) = \cos^4 \theta \det \sigma_B + \sin^4 \theta \det \sigma_A^\varphi + \sin^2 \theta \cos^2 \theta \left(\sigma_{A,11}^\varphi \sigma_{B,22} + \sigma_{A,22}^\varphi \sigma_{B,11} - 2\sigma_{A,12}^\varphi \sigma_{B,12} \right).\tag{66}$$

Explicitly, one obtains

$$\det(\sigma'_B) = \sin^4 \theta \left(N_A + \frac{1}{2} \right)^2 + \cos^4 \theta \left(N_B + \frac{1}{2} \right)^2 + 2 \sin^2 \theta \cos^2 \theta \left(N_A + \frac{1}{2} \right) \left(N_B + \frac{1}{2} \right) F_S,\tag{67}$$

where

$$F_S = \cosh 2r_A \cosh 2r_B - \sinh 2r_A \sinh 2r_B \cos(\theta_{AB} - 2\varphi),\tag{68}$$

with $\theta_{AB} = (\theta_A - \theta_B)$, describes how the final temperature of the bath depends on the squeezing terms. When no squeezing is present $F_S = 1$, which is also its lower bound. Conversely, there is no upper bound. Notice also that the relative direction of squeezing for the modes deeply contributes to the final temperature, and then to the heat exchanged.

The heat exchanged ΔQ can be obtained using Eqs. (50), (67), and (68), along with the relation $\sqrt{\det(\sigma_B)} = N_B + \frac{1}{2}$. Since $F_S \geq 1$, then $\sqrt{\det(\sigma'_B)} \geq \sin^2 \theta (N_A + \frac{1}{2}) + \cos^2 \theta (N_B + \frac{1}{2})$, and hence

$$\Delta Q \geq \omega_B \sin^2 \theta (N_A - N_B).\tag{69}$$

In particular, for $\omega_A = \omega_B$ then $W = 0$ and no anomalous heat flows can occur if system and bath are initially uncorrelated, namely $\Delta Q > 0$ iff $T_A > T_B$.

For simplicity, let us consider $\varphi = 0$ and analyze how F_S depends on θ_{AB} , namely on the relative squeezing direction of the input states. If the modes are squeezed in the same direction, i.e. $\theta_A = \theta_B$, then $F_S = \cosh(2r_A - 2r_B)$

depends just on the difference between the squeezing strengths, and it may give a small contribution if $r_A \simeq r_B$, even if $r_A, r_B \gg 1$. If the modes are squeezed in orthogonal directions, i.e. $\theta_{AB} = \pi$, then $F_S = \cosh(2r_A + 2r_B)$, namely the squeezing strongly increases the final temperature of the bath, since the two effects add up. For fixed values of r_A and r_B , the strongest contribution can be achieved when the modes are squeezed in orthogonal directions. For arbitrary phase squeezing θ_A and θ_B one can always tune the phase φ of the transformation in order to achieve one of the two above opposite effects.

By combining ΔE_A given in Eq. (64) and $\Delta Q = \omega_B \left(\sqrt{\det(\sigma'_B)} - \sqrt{\det(\sigma_B)} \right)$, we can express the work performed on the system as

$$\Delta W_A = \Delta E_A + \Delta Q. \quad (70)$$

Equation (70) provides the most general formulation of the first law of thermodynamics between two uncorrelated Gaussian modes which undergo a frequency converter/beam splitter transformation.

We now discuss some illustrative examples to show how Eq. (70) can be used to study heat flows and the balance between the different forms of energy.

1. Coherent thermal states

We consider local thermal states with coherent signal α and δ . The variation of the internal energy of the system is given by

$$\Delta E_A = \omega_A \sin^2 \theta \left[(N_B - N_A) + |\delta|^2 - |\alpha|^2 \right] + \omega_A \sin 2\theta \operatorname{Re}(\alpha \delta^* e^{-i\varphi}), \quad (71)$$

whereas the heat ΔQ reads

$$\Delta Q = \omega_B \sin^2 \theta (N_A - N_B). \quad (72)$$

The process is then dissipative, i.e. $\Delta Q > 0$, iff $N_A > N_B$. The work performed on the system is

$$\Delta W_A = (\omega_A - \omega_B) \sin^2 \theta (N_B - N_A) + \omega_A \sin^2 \theta \left(|\delta|^2 - |\alpha|^2 \right) + \omega_A \sin 2\theta \operatorname{Re}(\alpha \delta^* e^{-i\varphi}).$$

Note that for a beam splitter, i.e. $\omega = \omega_A = \omega_B$, the work performed on the system does not depend on the temperature, and one has

$$\Delta W_A = \omega \sin^2 \theta \left(|\delta|^2 - |\alpha|^2 \right) + \omega \sin 2\theta \operatorname{Re}(\alpha \delta^* e^{-i\varphi}). \quad (73)$$

Moreover, no anomalous heat flows can occur, i.e. $\Delta Q > 0$ iff $T_A > T_B$. Note also that even if $|\alpha| = |\delta|$ we may have $\Delta W_A \neq 0$ for the interference contribution of the last term in Eq. (73).

2. Squeezed states under a balanced frequency converter

We consider here initial states with $r_A = r_B$, $\alpha = \delta = 0$, and $N_A \neq 0$, $N_B \neq 0$, transformed by balanced frequency conversion with no phase shift, i.e. $\theta = \frac{\pi}{4}$ and $\varphi = 0$. From Eq. (55) the variation of the internal energy of the system reads

$$\Delta E_A = \frac{\omega_A}{2} [(N_B - N_A) \cosh 2r_A]. \quad (74)$$

At the end of the transformation, from Eq. (67), one has

$$\det \sigma'_B = \frac{1}{4} \left(N_A + \frac{1}{2} \right)^2 + \frac{1}{4} \left(N_B + \frac{1}{2} \right)^2 + \frac{1}{2} \left(N_A + \frac{1}{2} \right) \left(N_B + \frac{1}{2} \right) F_S, \quad (75)$$

where $F_S = \cosh^2 2r_A - \cos \theta_{AB} \sinh^2 2r_A$. The heat flows from the bath to the system iff $\det \sigma'_B - \det \sigma_B < 0$. Since $\det \sigma_B = \left(\frac{2N_B + 1}{2} \right)^2$, a negative heat can be achieved iff

$$N_B > \frac{1}{6} (F_S + 2N_A F_S - 3) + \frac{1}{6} \sqrt{(3 + F_S^2)(1 + 2N_A)^2}. \quad (76)$$

Since $F_S \geq 1$, notice that Eq. (76) implies $N_B \geq N_A$, and hence heat flow from the bath to the system is not possible if $N_A > N_B$.

3. Squeezed states with phase compensation

Let us consider the case $N_A \neq 0$, $N_B \neq 0$, $\theta_{AB} = 2\varphi$, and $r_A = r_B$, namely squeezed thermal initial states with the relative direction of equal squeezing that matches the phase of the transformation. The variation of the internal energy of the system reads as

$$\Delta E_A = \omega_A \sin^2 \theta \left[(N_B - N_A) \cosh 2r_A + |\delta|^2 - |\alpha|^2 \right] + \omega_A \sin 2\theta \operatorname{Re} (\alpha \delta^* e^{-i\varphi}) . \quad (77)$$

The heat exchanged in the process [see Eq. (67) to compute the determinant in the final state] is

$$\Delta Q = \omega_B \sin^2 \theta (N_A - N_B) , \quad (78)$$

and the inequality (69) is saturated since $F_S = 1$. Then, the direction of the heat flow is governed just by the condition $N_A \leq N_B$, as the states were just thermal. This fact can also be understood from the identity

$$[U_{FC}(\theta e^{i\varphi}), S(re^{i\varphi_A}) \otimes S(re^{i\varphi_B})] = 0 \quad (79)$$

for $2\varphi = \theta_A - \theta_B$.

The work performed on the system is given by

$$\Delta W_A = (\omega_A \cosh 2r_A - \omega_B) \sin^2 \theta (N_B - N_A) + \omega_A \sin^2 \theta (|\delta|^2 - |\alpha|^2) + \omega_A \sin 2\theta \operatorname{Re} (\alpha \delta^* e^{-i\varphi}) . \quad (80)$$

Let us analyze the role of the coherent signal α of the system in the sign of (80). For the sake of simplicity let us put $\delta = 0$. Then Eq. (80) rewrites

$$\Delta W_A = (\omega_A \cosh 2r_A - \omega_B) \sin^2 \theta (N_B - N_A) - \omega_A \sin^2 \theta |\alpha|^2 . \quad (81)$$

The more the system is displaced, the less work is needed to perform the process. No work is performed on the system for

$$|\alpha|^2 = \frac{(\omega_A \cosh 2r_A - \omega_B) (N_B - N_A)}{\omega_A} , \quad (82)$$

which holds for any θ . Clearly, when $\delta \neq 0$ a trade-off relation between α and δ emerges, along with interference effects.

4. Equal initial purity

We consider initial states with $N_A = N_B$ (i.e. $T_A/T_B = \omega_A/\omega_B$). The variation of the internal energy reads

$$\Delta E_A = \omega_A \sin^2 \theta \left[\left(N_A + \frac{1}{2} \right) (\cosh 2r_B - \cosh 2r_A) + |\delta|^2 - |\alpha|^2 \right] + \omega_A \sin 2\theta \operatorname{Re} (\alpha \delta^* e^{-i\varphi}) , \quad (83)$$

whereas the determinant of the final covariance matrix of the bath is given by

$$\det(\sigma'_B) = \left(N_A + \frac{1}{2} \right)^2 \left[1 + 2 \sin^2 \theta \cos^2 \theta (F_S - 1) \right] . \quad (84)$$

Since $F_S \geq 1$, the process always heats up the bath or at most $\Delta Q = 0$, when $r_A = r_B$ and $\theta_{AB} = 2\varphi$. Note that in this last case the work ΔW_A will depend only on the coherent signals. As we will show in Sec. V, the presence of initial correlations changes drastically the results of this example. For $r_A = r_B$ and $2\varphi = \theta_{AB} + \pi$, when $\theta = \pi/4$ all the free energy due to the squeezing of both the signal and the bath is consumed to generate entanglement. This fact can also be understood by means of the following algebraic identity [47]

$$U_{FC} \left(\frac{\pi}{4} e^{i\varphi} \right) (S(re^{i\theta_A}) \otimes S(re^{i\theta_B})) U_{FC}^\dagger \left(\frac{\pi}{4} e^{i\varphi} \right) = U_{PA} \left(ire^{\frac{i}{2}(\theta_A + \theta_B)} \right) , \quad (85)$$

for $2\varphi = \theta_A - \theta_B + \pi$.

B. Parametric amplifier

We now consider the parametric amplifier transformation of Eq. (20) for initial product state $\rho_{AB} = \rho_A \otimes \rho_B$, with ρ_A and ρ_B given by Eqs. (61) and (62), respectively. The variation of the internal energy of the system can be written as

$$\Delta E_A = \omega_A \sinh^2 r \left[\left(\frac{2N_B + 1}{2} \right) \cosh 2r_B + \left(\frac{2N_A + 1}{2} \right) \cosh 2r_A + |\delta|^2 + |\alpha|^2 \right] + \omega_A \sinh 2r \operatorname{Re} (\alpha \delta e^{-i\psi}). \quad (86)$$

By increasing the initial squeezing ΔE_A increases, and the same occurs by raising the temperature of both the system and the bath.

From Eq. (60), the final covariance matrix of the bath is given by

$$\sigma'_B = \cosh^2 r \sigma_B + \sinh^2 r \tilde{R}_\psi \sigma_A \tilde{R}_\psi, \quad (87)$$

and its determinant can be expressed as

$$\det(\sigma'_B) = \sinh^4 r \left(\frac{2N_A + 1}{2} \right)^2 + \cosh^4 r \left(\frac{2N_B + 1}{2} \right)^2 + 2 \sinh^2 r \cosh^2 r \left(\frac{2N_A + 1}{2} \right) \left(\frac{2N_B + 1}{2} \right) G_S, \quad (88)$$

where, similarly to Eq. (68) for F_S , one has

$$G_S = \cosh 2r_A \cosh 2r_B - \sinh 2r_A \sinh 2r_B \cos(\theta_{AB} - 2\psi). \quad (89)$$

Since $G_S \geq 1$, then $\sqrt{\det \sigma'_B} \geq \sinh^2 r (N_A + \frac{1}{2}) + \cosh^2 r (N_B + \frac{1}{2})$, and hence

$$\Delta Q \geq \omega_B \sinh^2 r (N_A + N_B + 1). \quad (90)$$

Then, for system and bath initially uncorrelated, parametric amplification always increases the intrinsic temperature of the bath (and, for symmetry also of the system). This fact highlights the deep difference between parametric amplification and frequency conversion.

Let us consider explicitly the following example.

1. Squeezed states with phase compensation

We consider here initial thermal states that are squeezed with equal strength $r_A = r_B$ and relative direction matched with the phase of the transformation as $\theta_{AB} = 2\psi$. The heat is then given by

$$\Delta Q = \omega_B \sinh^2 r (N_A + N_B + 1), \quad (91)$$

and inequality (90) is saturated. Similarly to the case of frequency conversion, when the transformation achieves phase compensation the heat is the same as the modes were in thermal states. This fact can also be understood from the identity

$$[U_{PA}(r e^{i\psi}), S(r' e^{i\theta_A}) \otimes S(r' e^{i\theta_B})] = 0 \quad (92)$$

for $2\psi = \theta_A - \theta_B$, which can be compared with Eq. (79).

The variation of the internal energy reads

$$\Delta E_A = \omega_A \sinh^2 r (N_A + N_B + 1) \cosh 2r_A + \omega_A \sinh^2 r (|\delta|^2 + |\alpha|^2) + \omega_A \sinh 2r \operatorname{Re} (\alpha \delta e^{-i\psi}), \quad (93)$$

and hence the work performed on the system is given by

$$\Delta W_A = \sinh^2 r (N_A + N_B + 1) (\omega_A \cosh 2r_A + \omega_B) + \omega_A \sinh^2 r (|\delta|^2 + |\alpha|^2) + \omega_A \sinh 2r \operatorname{Re} (\alpha \delta e^{-i\psi}). \quad (94)$$

We notice that for increasing values of the initial squeezing an increasing work is performed on the system.

V. CORRELATED SYSTEM AND BATH

We now examine the case when system and bath are initially correlated. First, we review the main properties of the correlation matrices for bipartite Gaussian states. As we will see a complete description of the correlation matrices is far to be simple and a full treatment of the thermodynamics in the most general case is beyond the scope of this paper. Therefore we will consider only some relevant classes of correlated Gaussian states in order to show how the presence of correlations affects the results we derived in the previous Section. In particular, we will see how Eqs. (69) and (90) can be violated. Finally, the problem of heat exchanges in the presence of correlations is discussed from an information-theoretic perspective by using the Renyi entropy of order 2.

A. Correlations for bipartite Gaussian states

The covariance matrix of two bosonic modes is generally given as in Eq. (35). The constraints in Eq. (40) can be written as

$$d_{\pm} \geq \frac{1}{2}, \quad (95)$$

where d_{\pm} are the symplectic eigenvalues that can be computed by

$$d_{\pm}^2 = \frac{\Delta(\sigma_{AB}) \pm \sqrt{\Delta(\sigma_{AB})^2 - 4 \det \sigma_{AB}}}{2}, \quad (96)$$

with $\Delta(\sigma_{AB}) = \det \sigma_A + \det \sigma_B + 2 \det \epsilon$. Here $\det \sigma_{AB}$ and $\Delta(\sigma_{AB})$ are global symplectic invariants, while $\det \sigma_A$, $\det \sigma_B$, and $\det \epsilon$ are local symplectic invariants.

The matrix ϵ encodes the information about the correlations, which can also reveal the presence of entanglement. In the case of a two-mode Gaussian state, the positive partial transpose (PPT) criterion [48], named in this case Simon's criterion, provides a necessary and sufficient condition for entanglement [49, 50]. Indeed, given a two-mode Gaussian state ρ_{AB} with covariance matrix σ_{AB} , the state is entangled iff σ_{AB} is positive definite, $d_- \geq \frac{1}{2}$ and $\tilde{d}_- < \frac{1}{2}$, where \tilde{d}_- is the symplectic eigenvalue of the partial transposed covariance matrix $\tilde{\sigma}_{AB}$, which is given by

$$\tilde{d}_- = \sqrt{\frac{\tilde{\Delta} - \sqrt{\tilde{\Delta}^2 - 4 \det \sigma_{AB}}}{2}}, \quad (97)$$

with $\tilde{\Delta} = \det \sigma_A + \det \sigma_B - 2 \det \epsilon$. Furthermore, a necessary, but not sufficient, condition for entanglement is $\det \epsilon < 0$.

Generally, when one is only interested in the correlations properties of a bipartite system, a different way of writing the covariance matrix σ_{AB} is helpful. In fact, any two-mode covariance matrix can be brought into a normal form via local symplectic transformations, namely for any σ_{AB} there exists a symplectic matrix $\mathbf{S}_N = \mathbf{S}_A \oplus \mathbf{S}_B$, with \mathbf{S}_A and \mathbf{S}_B acting on the first and second mode respectively, such that the transformed matrix $\sigma_{AB}^N = \mathbf{S}_N^T \sigma_{AB} \mathbf{S}_N$ can be expressed as

$$\sigma_{AB}^N = \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}, \quad (98)$$

where $\det \sigma_A = a^2$, $\det \sigma_B = b^2$, $\det \epsilon = c_+ c_-$, and $\det \sigma_{AB} = (ab - c_+^2)(ab - c_-^2)$. Such four real parameters, which are uniquely determined, up to a common sign flip between c_- and c_+ , allow to study the correlations between the parties in an easy and correct way, since local transformations do not change the amount of correlations. From a thermodynamical study, however, this approach is not justified, since local squeezing and their relative direction play an important role in energy exchanges, as we saw in the previous section.

On the other hand, the treatment of two-mode Gaussian states in the most general form involves too many parameters, thus a full study of the thermodynamics in such case is beyond the scope of this paper. Our aim is to show that in the presence of correlations new phenomena arise and to set up the framework for future work. Then, in the following we will limit our study only to local thermal states, correlated in two possible ways, characterized by the choices: $c_+ = c_- = c$ (Type-I) and $c_+ = -c_- = c$ (Type-II), respectively. Type-I class contains only separable states, while Type-II can also describe entangled states. This approach will give us general hints about the role of correlations in the studied thermodynamical processes.

B. Type-I correlated states

The covariance matrix for Type-I correlated states is given by

$$\sigma_{AB}^I = \begin{pmatrix} \frac{2N_A+1}{2} & 0 & c & 0 \\ 0 & \frac{2N_A+1}{2} & 0 & c \\ c & 0 & \frac{2N_B+1}{2} & 0 \\ 0 & c & 0 & \frac{2N_B+1}{2} \end{pmatrix}. \quad (99)$$

The constraints of Eq. (95) impose that c is bounded as

$$|c| \leq \sqrt{N_A N_B}. \quad (100)$$

The coherent contribution to the modes is described by the complex parameters α and δ for system and bath, respectively. All states belonging to this class are not entangled.

1. Frequency converter/beam splitter

The variation of the internal energy of the thermodynamical system for this class of states can be computed according to Eq. (55). Since $\epsilon = c\mathbb{I}_2$, one has

$$\Delta E_A = \omega_A \sin^2 \theta (N_B - N_A + |\delta|^2 - |\alpha|^2) + \omega_A \sin 2\theta [c \cos \varphi + \text{Re}(\alpha \delta^* e^{-i\varphi})]. \quad (101)$$

From Eq. (56) one also has

$$\sigma'_B = \left(\frac{2N_A+1}{2} \sin^2 \theta + \frac{2N_B+1}{2} \cos^2 \theta - c \sin 2\theta \cos \varphi \right) \mathbb{I}_2, \quad (102)$$

and hence the heat is given by

$$\Delta Q = \omega_B [(N_A - N_B) \sin^2 \theta - c \sin 2\theta \cos \varphi]. \quad (103)$$

Notice that inequality (69) for uncorrelated input states can now be violated.

The work performed on the system writes

$$\Delta W_A = (\omega_A - \omega_B) [(N_B - N_A) \sin^2 \theta + c \sin 2\theta \cos \varphi] + \omega_A \left[\sin^2 \theta (|\delta|^2 - |\alpha|^2) + \sin 2\theta \text{Re}(\alpha \delta^* e^{-i\varphi}) \right]. \quad (104)$$

Note first that for $\omega = \omega_A = \omega_B$, i.e. for a beam splitter, the work performed on the system is independent of the correlations and one recovers Eq. (73). Indeed, the increase (decrease) in the internal energy due to correlations is exactly balanced by the heat released (absorbed) by the bath. The work in this case depends only on the coherence terms described by the displacement operators.

Let us assume for simplicity that the coherent signal is set to zero and $\omega_A = \omega_B$, which implies that $\Delta W_A = 0$. Consider now the case when $N_A > N_B$, namely system is hotter than the bath, and focus on the heat flow. As long as condition (100) is also satisfied, the heat is negative if

$$c > \frac{1}{2} (N_A - N_B) \frac{\tan \theta}{\cos \varphi}, \quad (105)$$

when $\cos \varphi > 0$ (note that $\tan \theta > 0$, since $\theta \in [0, \frac{\pi}{2}]$). If $\cos \varphi < 0$, we have a negative heat flow for

$$c < \frac{1}{2} (N_A - N_B) \frac{\tan \theta}{\cos \varphi}. \quad (106)$$

This can lead to an apparent violation of the second law: after the process the bath is colder, even if initially $T_A > T_B$ and no work is performed on the system. Finally, notice that for $\varphi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ the presence of initial correlations does not affect any thermodynamical quantity.

2. Parametric amplifier

For initial state with covariance matrix of the form (99), the variation of the internal energy from Eq. (59) is given by

$$\Delta E_A = \omega_A \sinh^2 r \left(N_A + N_B + 1 + |\delta|^2 + |\alpha|^2 \right) + \omega_A \sinh 2r \operatorname{Re} \left(\alpha \delta e^{-i\psi} \right), \quad (107)$$

which does not depend on the correlations, since $\epsilon = c \mathbb{I}_2$ and so $\operatorname{Tr}[\tilde{R}_\psi \epsilon] = 0$.

The final covariance matrix of the bath reads

$$\sigma'_B = \left(\frac{2N_A + 1}{2} \sinh^2 r + \frac{2N_B + 1}{2} \cosh^2 r \right) \mathbb{I}_2 + c \sinh 2r \tilde{R}_\psi, \quad (108)$$

and hence

$$\det \sigma'_B = \left(\frac{2N_A + 1}{2} \sinh^2 r + \frac{2N_B + 1}{2} \cosh^2 r \right)^2 - c^2 \sinh^2 2r. \quad (109)$$

The heat exchanged can then be expressed as

$$\Delta Q = \omega_B \sqrt{\left(\frac{2N_A + 1}{2} \sinh^2 r + \frac{2N_B + 1}{2} \cosh^2 r \right)^2 - c^2 \sinh^2 2r} - \omega_B \frac{2N_B + 1}{2}. \quad (110)$$

For fixed N_A and N_B the heat has its maximum value for $c = 0$, for which $\Delta Q = \omega_B \sinh^2 r (N_A + N_B + 1) \geq 0$. Notice that such value also saturates inequality (90), which holds only for factorized initial states. For increasing values of the correlation $|c|$ the heat decreases, and may even become negative, differently from the case of Sec. IV B for uncorrelated input states. In fact, for

$$|c| > \frac{\sqrt{\left(\frac{2N_A + 1}{2} \sinh^2 r + \frac{2N_B + 1}{2} \cosh^2 r \right)^2 - \left(\frac{2N_B + 1}{2} \right)^2}}{\sinh 2r}, \quad (111)$$

one has $\Delta Q < 0$. Note, however, that $|c|$ cannot be arbitrarily large since necessarily $|c| \leq \sqrt{N_A N_B} \equiv c_M$ in order to guarantee a physical state. For instance, if $r = 1$, $N_A = 20$ and $N_B = 10$, then we have $\Delta Q < 0$ if $13.90 < |c| \leq c_M = 14.14$. Since the right-hand side in (111) increases with r while c is bounded, notice also that there exists a maximum value of r for which the condition (111) can be satisfied while keeping N_A and N_B fixed. Anyway, the minimum of ΔQ versus the correlations is reached for $|c| = c_M$.

Finally, the work performed on the system is

$$\begin{aligned} \Delta W_A = & \omega_A \sinh 2r \operatorname{Re} \left(\alpha \delta e^{-i\psi} \right) + \omega_A \sinh^2 r \left(N_A + N_B + 1 + |\delta|^2 + |\alpha|^2 \right) \\ & + \omega_B \sqrt{\left(\frac{2N_A + 1}{2} \sinh^2 r + \frac{2N_B + 1}{2} \cosh^2 r \right)^2 - (c \sinh 2r)^2} - \omega_B \frac{2N_B + 1}{2}. \end{aligned} \quad (112)$$

Note that the dependency of the work on the correlations cannot be eliminated even for $\omega_A = \omega_B$, differently from the case of frequency conversion. Moreover, for increasing values of $|c|$, ΔW_A decreases. Therefore, we have shown that the presence of correlations, although pertaining to separable states, allows new phenomena for this process, such as negative heat flows, which are impossible in the absence of correlations. This may happen even when the bath is colder than the system.

C. Type-II correlated states

The covariance matrix for Type-II correlated states is given by

$$\sigma_{AB}^{II} = \begin{pmatrix} \frac{2N_A + 1}{2} & 0 & c & 0 \\ 0 & \frac{2N_A + 1}{2} & 0 & -c \\ c & 0 & \frac{2N_B + 1}{2} & 0 \\ 0 & -c & 0 & \frac{2N_B + 1}{2} \end{pmatrix}, \quad (113)$$

These states are locally thermal and can be separable or entangled, depending on the range of c . The constraints in Eq. (95) impose the conditions

$$|c| \leq \sqrt{N_A(1+N_B)}, \quad (114)$$

if $N_A \leq N_B$, or

$$|c| \leq \sqrt{N_B(1+N_A)} \quad (115)$$

if $N_A > N_B$. By applying Simon's criterion, we know that system and bath are entangled iff

$$|c| > \sqrt{N_A N_B}. \quad (116)$$

1. Frequency converter/beam splitter

For this class of states ΔE_A , given by Eq. (55), reads

$$\Delta E_A = \omega_A \sin^2 \theta \left(N_B - N_A + |\delta|^2 - |\alpha|^2 \right) + \omega_A \sin 2\theta \operatorname{Re} \left(\alpha \delta^* e^{-i\varphi} \right), \quad (117)$$

which is independent of the correlations, differently from the Type-I states. In fact, since $\epsilon = c\sigma_Z$, then $\operatorname{Tr} [R_\varphi \epsilon] = 0$ for any φ . The final covariance matrix of the bath is

$$\sigma'_B = \left(\frac{2N_A+1}{2} \sin^2 \theta + \frac{2N_B+1}{2} \cos^2 \theta \right) \mathbb{I}_2 - c \sin 2\theta \bar{R}_\varphi, \quad (118)$$

where

$$\bar{R}_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ -\sin \varphi & -\cos \varphi \end{pmatrix}, \quad (119)$$

and its determinant is given by

$$\det \sigma'_B = \left(\frac{2N_A+1}{2} \sin^2 \theta + \frac{2N_B+1}{2} \cos^2 \theta \right)^2 - c^2 \sin^2 2\theta. \quad (120)$$

Hence, the stronger are the correlations, the lower is the final temperature of the bath. Since $\det \sigma_B = \left(\frac{2N_B+1}{2} \right)^2$, the heat exchanged can be expressed as

$$\Delta Q = \omega_B \sqrt{\left(N_A \sin^2 \theta + N_B \cos^2 \theta + \frac{1}{2} \right)^2 - (c \sin 2\theta)^2} - \omega_B \frac{2N_B+1}{2}. \quad (121)$$

The heat ΔQ has a maximum for $c = 0$, for which inequality (90) is saturated, and decreases for increasing values of $|c|$. Since for fixed N_A and N_B the state is entangled only if $|c| > \sqrt{N_A N_B}$, the decrease of ΔQ is emphasized the more the initial state is entangled. For

$$|c| > \frac{\sqrt{\left(N_A \sin^2 \theta + N_B \cos^2 \theta + \frac{1}{2} \right)^2 - \left(\frac{2N_B+1}{2} \right)^2}}{\sin 2\theta}, \quad (122)$$

along with condition (114) or (115), the heat flow becomes negative. Anomalous heat flows can be found also in this case, with general enhancement for increasing correlations/entanglement.

2. Parametric amplifier

The variation of the internal energy (59) for initial states with covariance matrix (113) reads

$$\Delta E_A = \omega_A \sinh^2 r \left(N_A + N_B + 1 + |\delta|^2 + |\alpha|^2 \right) + \omega_A \sinh 2r \left[c \cos \psi + \operatorname{Re} \left(\alpha \delta e^{-i\psi} \right) \right]. \quad (123)$$

Since the final covariance matrix of the bath is $\sigma'_B = \left(\frac{2N_A+1}{2} \sinh^2 r + \frac{2N_B+1}{2} \cosh^2 r + c \sinh 2r \cos \psi\right) \mathbb{I}_2$, the corresponding heat is given by

$$\Delta Q = \omega_B [(N_A + N_B + 1) \sinh^2 r + c \sinh 2r \cos \psi]. \quad (124)$$

Hence, the work performed on the system writes

$$\begin{aligned} \Delta W_A = & \omega_A \left[\sinh^2 r \left(|\delta|^2 + |\alpha|^2 \right) + \sinh 2r \operatorname{Re} (\alpha \delta e^{-i\psi}) \right] \\ & + (\omega_A + \omega_B) [(N_B + N_A + 1) \sinh^2 r + c \sinh 2r \cos \psi]. \end{aligned} \quad (125)$$

Both ΔE_A and ΔQ depend in the same way on the correlations, and their effect adds up in the work. The strength of correlations rule the sign of all these thermodynamic quantities. For instance, we have $\Delta Q < 0$ iff

$$c < - \left(\frac{N_A + N_B + 1}{2} \right) \frac{\tanh r}{\cos \psi}, \quad (126)$$

for $\cos \psi > 0$, or

$$c > - \left(\frac{N_A + N_B + 1}{2} \right) \frac{\tanh r}{\cos \psi}, \quad (127)$$

for $\cos \psi < 0$, with the additional constraint (114) or (115). Stronger correlations, in the sense of greater values of $|c|$, do not result automatically in lower values for ΔQ and ΔW_A , due to a not trivial dependence on the phase ψ . Finally, notice that for $\psi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ the presence of initial correlations does not affect any thermodynamical quantity.

D. The role of correlations on heat exchanges: an information-theoretic approach

In the previous subsection we have focused on the role of initial correlations between system and bath, considering the heat flow and the work performed on the system. Here, following an information-theoretic approach, we consider a different perspective by studying how the variation of correlations between the initial and final state can determine the heat flows for a general entropy-preserving transformation.

For generic interaction between two bosonic systems the heat is evaluated by Eq. (28). As a consequence, the sign of the heat is determined by the sign of the variation of the bath von Neumann entropy, namely we have $\Delta Q > 0$ iff $\Delta S_B = S(\rho'_B) - S(\rho_B) > 0$. Since for entropy-preserving transformations we have $\Delta I(A : B) = \Delta S_A + \Delta S_B$, the heat exchange is positive iff $\Delta I(A : B) > \Delta S_A$. When both $\Delta S_A > 0$ and $\Delta S_B > 0$, both the intrinsic temperatures of system and bath increase, thus leading to an increase in the correlations. Conversely, if the process lowers both temperatures, it must also decrease the total amount of correlations.

For bipartite pure states the von Neumann entropy of the marginal state(s) is a well defined measure of entanglement. As a consequence, for a pure initial state one has $\Delta Q > 0$ iff the amount of entanglement increases, while a decreasing entanglement implies that the temperature of the bath decreases, independently of the temperature of the system. Notice that this may also happen for $W = 0$, i.e. when no external work is performed. A paradigmatic example is the case of two pure squeezed states in orthogonal direction under a balanced beam splitter, producing a twin-beam state, and the reversed transformation.

Specifically for Gaussian states, a quantifier of the information encoded in a state ρ is provided by the Rényi-2 entropy [51–53]

$$S_2(\rho) = \frac{1}{2} \ln(\det \sigma), \quad (128)$$

where σ is the covariance matrix associated to ρ . By comparing with Eq. (47), we notice that the Rényi-2 entropy just replaces the concave function $g(x)$ of Eq. (27) appearing in the von Neumann entropy by a different concave function, namely $\ln(x + \frac{1}{2})$. Recalling Eq. (45), one also has $S_2(\rho) = -\ln(2\mu_\rho)$.

The Rényi-2 entropy satisfies the strong subadditivity inequality for all Gaussian states and coincides up to a constant with the Wigner entropy. Typically, this link allows for a fundamental simplification of the problem of characterizing entropy production, as one can map an open system dynamics into a Fokker-Planck equation for the Wigner function, and hence employ tools of classical stochastic processes to obtain simple expressions for entropy production rate and entropy flux rate from the system to the environment [53–57]. The Rényi-2 entropy can also be

used to define a Gaussian entanglement measure, the Gaussian Renyi-2 (GR2) entanglement, which for pure states ψ_{AB} is given by

$$\mathcal{E}_2(\psi_{AB}) = \frac{1}{2} \ln(\det \sigma_B) = S_2(\rho_B). \quad (129)$$

The bound energy can be reformulated in terms of the Renyi-2 entropy, namely $B(\chi_B) = \omega_B \exp(S_2(\chi_B))$, and hence the heat rewrites

$$\Delta Q = \omega_B \{ \exp[S_2(\rho'_B)] - \exp[S_2(\rho_B)] \}. \quad (130)$$

Then, for pure Gaussian states the heat has a clear interpretation in terms of the variation of the GR2 entanglement, i.e.

$$\Delta Q = \omega_B \{ \exp[\mathcal{E}_2(\rho'_{AB})] - \exp[\mathcal{E}_2(\rho_{AB})] \}. \quad (131)$$

Here again we see that if the state loses entanglement in the thermodynamical transformation then $\Delta Q < 0$, namely the bath becomes colder. Conversely, if the transformation increases the entanglement, the bound energy of the bath increases.

In the case of mixed states we expect that also correlations of separable states are involved in the heat exchange, as we have already seen in many previous examples. The Renyi-2 entropy leads to a well-defined measure of correlations for Gaussian states, namely the Renyi-2 mutual information

$$I_2(A : B) = \frac{1}{2} \ln \left(\frac{\det \sigma_A \det \sigma_B}{\det \sigma_{AB}} \right). \quad (132)$$

Let us consider its variation under Gaussian transformations

$$\Delta I_2(A : B) = \frac{1}{2} \ln \left(\frac{\det \sigma'_A \det \sigma'_B}{\det \sigma_A \det \sigma_B} \right), \quad (133)$$

where we used the fact that $\det \sigma'_{AB} = \det \sigma_{AB}$. Equivalently, one has

$$\Delta I_2(A : B) = \Delta S_2(\rho_A) + \ln \left(\frac{B(\rho'_B)}{B(\rho_B)} \right), \quad (134)$$

which implies

$$B(\rho'_B) = \left[\exp \left(\Delta I_2(A : B) - \Delta S_2(\rho_A) \right) \right] B(\rho_B). \quad (135)$$

The relevant point of equality (135) is the fact that it relates initial and final temperature of the bath by a multiplicative factor directly related to information quantities. In this context then the bath is heated iff $\Delta I_2(A : B) > \Delta S_2(\rho_A)$. Finally, we recall that the Renyi-2 entropy and mutual information have already been explicitly related to the entropy production in irreversible processes [53–56], also assessed experimentally [57]. Hence, further developments may consider irreversible entropy production mechanisms in the present framework.

VI. WORK EXTRACTION SCHEMES

Our final aim is to suggest possible implementations for work extraction. For a bipartite system, described by a global state ρ_{AB} , the extractable work is upper bounded by the global free energy $F(\rho_{AB})$. In most cases it is not possible to have full access to both the system and the bath, since one may perfectly control the former but not the latter. Hence, extracting the whole $F(\rho_{AB})$ might be an impossible task. On the other hand, if one considers only the system then it is possible to extract at most $F(\rho_A)$, which is generally far below $F(\rho_{AB})$. To enhance the amount of extractable work one can make the system interact with the bath via an entropy-preserving process, increasing the free energy to $F(\rho'_A) > F(\rho_A)$, and then extract work from the system. Here we do not formalize a working medium to explicitly implement the work extraction from the system, but we will instead focus on engineering the interaction via bilinear transformation of the modes pertaining to system and bath, in order to achieve the optimal increase of the free energy of the system.

In this context, for a fair treatment the relevant quantity is the variation $\tilde{W} \equiv \Delta F_A - W$, which represents the balance between the increase of the free energy of the system and the cost employed by the interaction $W =$

$\Delta E_A + \Delta E_B$ (which is zero for the passive beam-splitter). Recalling that the free energy bounds the extractable work, then a positive value of \tilde{W} is intended as a positive contribution for work extraction in the first-stage of a work engine, where just system and bath interact. We also notice the possibility of having both $\Delta F_A > 0$ and $W < 0$, which means that an increase of the system free energy can be contextually accompanied by work extraction [58]. Notice also the following equivalent expression for \tilde{W}

$$\tilde{W} = -(\Delta E_B + \Delta B_A), \quad (136)$$

which formally corresponds to the work extracted from the bath. For the study of \tilde{W} , we will consider bipartite states in Eq. (21) with no coherent signal on the system, i.e. $\alpha = 0$, since the corresponding contribution to the free energy of the states can be trivially extracted by the inverse unitary displacement with no entropy exchange. Since the bath is considered as not directly accessible, its coherent signal cannot be extracted with the same procedure, and it must be generally taken into account in the engineering of the transformation.

The net increase in the extractable work \tilde{W} is given by the two contributions ΔE_B and ΔB_A , which for a general two-mode bosonic state can be expressed as

$$\Delta E_B = \omega_B \left(\langle b^\dagger b \rangle_{\rho'_B} - \langle b^\dagger b \rangle_{\rho_B} \right), \quad (137)$$

$$\Delta B_A = \omega_A \left[g^{-1} (S(\rho'_A)) - g^{-1} (S(\rho_A)) \right], \quad (138)$$

respectively. Therefore, to get a positive \tilde{W} a competition appears between an appropriate reduction of the internal energy of the bath and a sufficient decrease in the bound energy—and hence entropy—of the system.

For Gaussian states undergoing a symplectic transformation Γ as in Eq. (38), the variation of the internal energy of the bath is given by

$$\Delta E_B = \frac{\omega_B}{2} \text{Tr} [B\sigma_B B^T + C\sigma_A C^T + C\epsilon B^T + B\epsilon^T C^T] + \frac{\omega_B}{2} \left(\| \langle BR_B \rangle \|^2 - \| \langle R_B \rangle \|^2 - \text{Tr} [\sigma_B] \right), \quad (139)$$

where we considered that the system does not have coherent signal.

The variation of the system bound energy is given by $\Delta B_A = \omega_A (\sqrt{\det \sigma'_A} - \sqrt{\det \sigma_A})$, where the covariance matrix of the system after the process reads as

$$\sigma'_A = A\sigma_A A^T + D\sigma_B D^T + A\epsilon D^T + D\epsilon^T A^T. \quad (140)$$

The net increase in the free energy of the system \tilde{W} depends on the symplectic transformation Γ that makes the system interact with the bath. Typically, Γ is characterized by some parameters that reflect possible configurations of an experimental setup. Since \tilde{W} is a function of the transformation, i.e. $\tilde{W} = \tilde{W}(\Gamma)$, we can look for the appropriate configuration of Γ such that $\tilde{W}(\Gamma) > 0$ and is maximum.

A. Frequency converter/beam splitter

For this process the variation of the internal energy of the bath is given by

$$\Delta E_B = \frac{\omega_B}{2} \sin^2 \theta (\text{Tr} [\sigma_A] - \text{Tr} [\sigma_B] - 2|\delta|^2) - \frac{\omega_B}{2} \sin 2\theta \text{Tr} [R_\varphi^T \epsilon], \quad (141)$$

whereas the final covariance matrix of the system reads

$$\sigma'_A = \cos^2 \theta \sigma_A + \sin^2 \theta R_\varphi \sigma_B R_\varphi^T + \frac{1}{2} \sin 2\theta (\epsilon R_\varphi^T + R_\varphi \epsilon^T). \quad (142)$$

Since for this process $\Delta E_B = -\frac{\omega_B}{\omega_A} \Delta E_A$ [58], one has $\tilde{W} = \frac{\omega_B}{\omega_A} \Delta E_A - \Delta B_A$. Hence, the employed transformation must increase the internal energy of the system while decreasing its intrinsic temperature.

In the following we consider specific examples for the frequency converter transformations.

1. Coherence from the bath

In this simple example we consider a state of the form

$$I \otimes D(\delta)(\nu_{N_A} \otimes \nu_{N_B}) I \otimes D^\dagger(\delta), \quad (143)$$

namely two local thermal modes, with coherent signal in the bath mode. Using a frequency converter one has

$$\Delta F_A = \omega_A \sin^2 \theta |\delta|^2, \quad (144)$$

which has the maximum value $\Delta F_{\max} = \omega_A |\delta|^2$ for $U_{FC}(\frac{\pi}{2})$. This corresponds to a swap of the system and the bath state, which is physically not trivial for $\omega_A \neq \omega_B$. The net increase in the extractable work \tilde{W} is given by

$$\tilde{W} = (\omega_B - \omega_A) \sin^2 \theta (N_B - N_A) + \omega_B \sin^2 \theta |\delta|^2, \quad (145)$$

which is positive as long as

$$|\delta|^2 > \frac{\omega_A - \omega_B}{\omega_B} (N_B - N_A), \quad (146)$$

and maximum for $\theta = \pi/2$. If condition (146) is violated the cost of the transformation exceeds the increase in the extractable work. We observe that for a balanced beam splitter (i.e. $\omega_A = \omega_B$ and $\theta = \pi/4$) the transformation would leave system and bath at the same temperature, but would be less efficient since it would create correlations and also leave coherence (and hence free energy) in the bath. Finally, notice that when Eq. (146) holds along with

$$W = (\omega_A - \omega_B)(|\delta|^2 + N_B - N_A) < 0 \quad (147)$$

the increase of the system free energy is accompanied by work extraction.

2. Exploiting type-I correlated states

In the following example the optimal transformation strongly depends on the state parameters. We consider correlated local thermal states with $\epsilon = c \mathbb{I}_2$, which are therefore described by the covariance matrix given in (99). We have seen that the net increase in the free energy of the system is $\tilde{W} = \frac{\omega_B}{\omega_A} \Delta E_A - \Delta B_A$. According to Eq. (101), the variation of the internal energy of the system for this class of states is given by

$$\Delta E_A = \omega_A \sin^2 \theta (N_B - N_A + |\delta|^2) + \omega_A \sin 2\theta c \cos \varphi, \quad (148)$$

whereas the corresponding variation of the bound energy reads

$$\Delta B_A = \omega_A \sin^2 \theta (N_B - N_A) + \omega_A \sin 2\theta c \cos \varphi. \quad (149)$$

Hence, one obtains

$$\tilde{W} = (\omega_B - \omega_A) [\sin^2 \theta (N_B - N_A) + \sin 2\theta c \cos \varphi] + \omega_B \sin^2 \theta |\delta|^2. \quad (150)$$

Let us consider the case $\delta = 0$ in order to study the pure effect of correlations. For this class of states if $\omega_A = \omega_B$ one has $\tilde{W} = 0$, namely the procedure does not provide any advantage.

Suppose now that $\omega_B > \omega_A$. From Eq. (150) it is clear that the optimal choice of φ depends on the sign of c , namely if $c < 0$ then $\varphi = \pi$, while for $c > 0$ we choose $\varphi = 0$. For $N_B > N_A$ we have $\tilde{W} > 0$ for any θ . If $N_B < N_A$, a positive \tilde{W} can be obtained if

$$\tan \theta < \frac{2|c|}{N_A - N_B}. \quad (151)$$

In both cases the optimal value of θ maximizing \tilde{W} is given by

$$\theta_{\max} = \frac{1}{2} \arctan \left(\frac{2|c|}{N_A - N_B} \right), \quad (152)$$

with the suitable choice $\varphi = 0$ or $\varphi = \pi$. Notice that the more correlated is the state, the larger is the net increase of the system free energy.

When $\omega_B < \omega_A$, again the sign of c fixes φ at 0 or π . For $N_B < N_A$, we have $\tilde{W} > 0$ for any θ . Otherwise, if $N_B > N_A$, the net increase \tilde{W} is positive for

$$\tan \theta < \frac{2|c|}{N_B - N_A}. \quad (153)$$

The optimal value of θ is provided by

$$\theta_{\max} = \frac{1}{2} \arctan \left(\frac{2|c|}{N_B - N_A} \right). \quad (154)$$

Notice that both solutions (152) and (154) satisfy the respective conditions (151) and (152). In summary, for all values of N_A, N_B , and c , we can provide a transformation that increases the net free energy of the system, i.e. $\tilde{W} > 0$, for suitable choice of φ and θ .

Finally, note that for $c = 0$ the procedure runs only for $(\omega_B - \omega_A)(N_B - N_A) > 0$, consistently with Eq. (146), with optimal $\theta_{\max} = \pi/2$. In Figs. 1 and 2 we report examples of $\tilde{W}(\theta)$ as a function of θ for different values of c , for fixed N_A, N_B , and ω_B/ω_A .

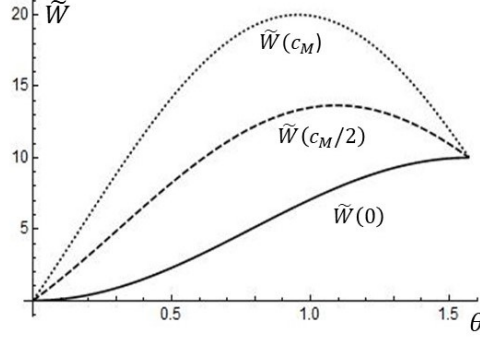


Figure 1. Plots of \tilde{W} for classically correlated thermal states with no coherent signal, $N_A = 5$, $N_B = 10$, and $\omega_B = 2\omega_A$, for different values of c . $\tilde{W}(c_M)$ represents \tilde{W} for the maximum value of the correlations $c_M = \sqrt{N_A N_B}$, while $\tilde{W}(c_M/2)$ for $c = c_M/2$. $\tilde{W}(0)$ is the extractable work for uncorrelated modes, which is maximized for $\theta = \pi/2$.

3. Exploiting type-II correlated states

We consider the type-II correlated states, whose covariance matrix is given in (113). We remind that the net increase in the free energy of the system reads as $\tilde{W} = \frac{\omega_B}{\omega_A} \Delta E_A - \Delta B_A$. For this class of states the variation of the internal energy is obtained by Eq. (117), namely

$$\Delta E_A = \omega_A \sin^2 \theta (N_B - N_A + |\delta|^2). \quad (155)$$

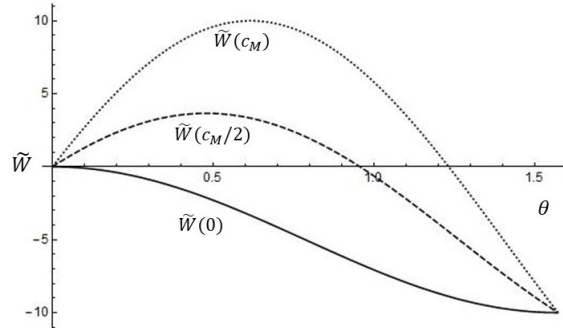


Figure 2. \tilde{W} as function of θ for classically correlated thermal states with no coherent signal, $N_A = 10$, $N_B = 5$, and $\omega_B = 2\omega_A$, for different values of c , as in Fig. 1. Since $N_A > N_B$, to get $\tilde{W} > 0$, θ must be smaller than a threshold that depends on N_A , N_B and c [see Eq. (151)]. When the modes are uncorrelated ($c = 0$) the work extraction scheme is useless.

The final covariance matrix of the system is given by

$$\sigma'_A = \left(\frac{2N_B + 1}{2} \sin^2 \theta + \frac{2N_A + 1}{2} \cos^2 \theta \right) \mathbb{I}_2 + c \sin 2\theta \bar{R}_\varphi, \quad (156)$$

and hence the corresponding variation of the bound energy writes

$$\Delta B_A = \omega_A \sqrt{\left(N_B \sin^2 \theta + N_A \cos^2 \theta + \frac{1}{2} \right)^2 - (c \sin 2\theta)^2} - \omega_A \frac{2N_A + 1}{2} \quad (157)$$

The net increase in the free energy of the system can be therefore expressed as

$$\tilde{W} = \omega_B \sin^2 \theta (N_B - N_A + |\delta|^2) + \omega_A \frac{2N_A + 1}{2} - \omega_A \sqrt{\left(N_B \sin^2 \theta + N_A \cos^2 \theta + \frac{1}{2} \right)^2 - (c \sin 2\theta)^2}. \quad (158)$$

For fixed values of state parameters N_A, N_B, c, δ the optimal value of θ is chosen to maximize \tilde{W} , possibly with $\tilde{W}_{\max} > 0$. For a better understanding of this case, let us explicitly consider a two-mode squeezed vacuum state, which represents a type-II pure entangled state between system and bath. The pertaining covariance matrix is obtained for $\sigma_A = \sigma_B = \frac{\cosh 2r}{2} \mathbb{I}_2$ and $c = \frac{\sinh 2r}{2}$, along with $\delta = 0$. Then Eq. (158) rewrites as

$$\tilde{W} = \omega_A \frac{\cosh 2r}{2} - \omega_A \sqrt{\frac{\cosh^2 2r}{4} - \frac{\sinh^2 2r}{4} \sin^2 2\theta}. \quad (159)$$

The maximal \tilde{W} is attained when $\theta = \frac{\pi}{4}$, for which

$$\tilde{W}_{\max} = \omega_A \sinh^2 r. \quad (160)$$

We notice that such an optimal transformation corresponds to a complete removal of correlations, which are used to increase the free energy of the system (and the bath). This effect can also be understood by means of the identity (85). In fact, since the two-mode squeezed state is generated by $U_{PA}(\xi)$ on the vacuum state $|0\rangle_A \otimes |0\rangle_B$ and $U_{FC}(\zeta)|0\rangle_A \otimes |0\rangle_B = 0$, one has

$$U_{FC}\left(\frac{\pi}{4}\right) U_{PA}(r)|0\rangle_A \otimes |0\rangle_B = (S_A(r)|0\rangle_A) \otimes (S_B(-r)|0\rangle_B). \quad (161)$$

B. Parametric amplifier

The variation of the internal energy of the bath is given by

$$\Delta E_B = \frac{\omega_B}{2} \sinh^2 r (\text{Tr}[\sigma_A] + \text{Tr}[\sigma_B] + 2|\delta|^2) + \frac{\omega_B}{2} \sinh 2r \text{Tr}[\tilde{R}_\psi \epsilon], \quad (162)$$

whereas the final covariance matrix of the system reads

$$\sigma'_A = \cosh^2 r \sigma_A + \sinh^2 r \tilde{R}_\psi \sigma_B \tilde{R}_\psi + \frac{1}{2} \sinh 2r (\epsilon^T \tilde{R}_\psi + \tilde{R}_\psi \epsilon). \quad (163)$$

For this process we have $\Delta E_B = \frac{\omega_B}{\omega_A} \Delta E_A$ [58, 60], and hence $\tilde{W} = -\left(\frac{\omega_B}{\omega_A} \Delta E_A + \Delta B_A\right)$, which can also be expressed as $\tilde{W} = -\frac{\omega_B}{\omega_A} \Delta F_A - \left(1 + \frac{\omega_B}{\omega_A}\right) \Delta B_A$. In order to increase the extractable work ($\Delta F_A > 0$) we need $\Delta B_A < 0$ and $|\Delta B_A| > \frac{\omega_B}{\omega_A + \omega_B} \Delta F_A$.

We notice, however, that for Gaussian factorized input states one always has $\Delta B_A > 0$. In fact, for symmetry reasons $\Delta B_A = \Delta Q$ as in Eq. (90) with ω_A replacing ω_B , and hence the presence of initial correlations is needed to obtain $\tilde{W} > 0$. Let us consider then the last example.

1. *Exploiting type-II correlated states*

We consider the type-II correlated states, whose covariance matrix is given in (113). The variation of the internal energy is obtained by Eq. (123) with $\alpha = 0$, namely

$$\Delta E_A = \omega_A \sinh^2 r (N_A + N_B + 1 + |\delta|^2) + \omega_A \sinh 2r c \cos \psi, \quad (164)$$

whereas $\Delta B_A = \Delta Q$ as in Eq. (124) upon replacing ω_B with ω_A . Then one has

$$\tilde{W} = -\omega_B \sinh^2 r |\delta|^2 - (\omega_A + \omega_B) [(N_A + N_B + 1) \sinh^2 r + c \sinh 2r \cos \psi]. \quad (165)$$

For fixed values of state parameters N_A, N_B, c, δ , the optimal values of r and ψ are chosen to maximize \tilde{W} . For simplicity, let us consider the case $\omega = \omega_A = \omega_B$, $N_A = N_B = N$, and $c = \sqrt{N(N+1)}$, which corresponds to an input pure two-mode squeezed state displaced by a coherent signal δ in the bath mode. Then,

$$\tilde{W} = -\omega [(4N + 2 + |\delta|^2) \sinh^2 r + 2\sqrt{N(N+1)} \sinh 2r \cos \psi]. \quad (166)$$

Clearly, the optimal choice for ψ is $\psi = \pi$, and by solving $\partial \tilde{W} / \partial r = 0$ one obtains the optimal value of r as $r_{\max} = \operatorname{atanh} \frac{2\sqrt{N(N+1)}}{4N+2+|\delta|^2}$. The corresponding optimal net increase of the system free energy is given by

$$\tilde{W}_{\max} = \omega \frac{4N(N+1)(4N+2+|\delta|^2)}{(4N+2+|\delta|^2)^2 - 4N(N+1)}. \quad (167)$$

VII. CONCLUSIONS

In this work we have studied the thermodynamics of two bosonic systems that interact via entropy-preserving transformations in the mode operators. The first mode represents the thermodynamical system, while the second describes the bath which, differently from standard formulations of thermodynamics, is treated as a quantum system, namely it can possess coherence or squeezing, and can be correlated with the system. The main result of this work is the formulation of the first law of thermodynamics for any two-mode states, hence the balance between the heat (28) and the variation of the internal energy (36) of the system which gives the work performed onto the thermodynamical system in the entropy preserving transformation. We have systematically considered the two different bilinear transformations, namely the frequency converter/beam splitter and the parametric amplification. Although our results hold for any two-mode states, we have mainly focused on the case of Gaussian states, where the heat and the variation of the internal energy take simple expressions. In particular, we have derived the first law in the case of initial uncorrelated modes, providing the most general formulation for bilinear transformations. Moreover, we have considered the case of initial correlations between the thermodynamical system and the bath by analyzing two types of correlated states, one that considers only separable states and the other that enables also entanglement. The case of correlated states has also been considered from an information-theoretical point of view by means of the Renyi entropy of order two, thus showing how anomalous heat flows can occur by exploiting correlations. Finally, we have proposed work-extraction schemes, showing how one can engineer entropy-preserving transformations to increase the free energy of the system, namely the amount of extractable work, by letting the system interact with the bath, thus exploiting the presence of correlations, squeezing or coherence.

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