THE EFFECT OF THE KINEMATIC HARDENING MODULUS ON THE STRESS DELAY IN A COMPLEX TENSION-TORSION TEST

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1. INTRODUCTION

Although the constitutive equations of elastic-plastic materials have been studied for many years, the possibility today of obtaining the numerical solution of difficult boundary-value problems has increased even further the need for accurate knowledge of the mechanical response of the materials.

One of the best ways of establishing the adequacy of an elastic-plastic constitutive model is to apply it to the study of complex tension-torsion tests, and compare the results with the experimental data obtained by submitting thin-walled tubular specimens to combined tension-torsion processes (see [1], [2], [3] and [4]). Indeed, if a tension process is followed by a torsion process, the direction of the stress follows the orthogonal change in direction of the strain trajectory after a certain angular delay; the ability to account for this phenomenon is a good test for a constitutive law.

In [5] a general kinematic hardening model is presented; it is shown that a suitable choice of parameters allows for results in excellent agreement with the available experimental data. The method is, however, somewhat complicated and the constitutive response to tension-torsion processes must be calculated numerically.

In the present paper we confine ourselves to an examination of the classical kinematic hardening rule proposed by MELAN [6], according to which the rate of the elastic range centre is parallel to the rate of plastic deformation. In this way a constitutive equation is obtained which, in the case of tension-torsion processes, can be integrated analytically. This makes it possible to carry out a qualitative analysis of the effect of the kinematic hardening modulus on the delay with which the stress follows the strain trajectory. It is shown that, in spite of the simplicity of the hardening rule used, the model gives a fairly good account of essential phenomenological aspects.

2. CONSTITUTIVE HYPOTHESES

In this chapter, for the reader's convenience, we shall briefly present, in axiomatic form, certain elements of the theory of infinitesimal plasticity which, as shown in [9], can be deduced from the general theory of materials with elastic range formulated in [7] and [8] on the assumption, accepted in the present paper, that the displacement gradient from a fixed reference configuration is small. We shall begin with a number of indispensable definitions.

A *deformation process* or, more briefly, a *history* is a continuous and continuously piecewise differentiable mapping, defined on the closed real interval [0,1] with values in Sym, the space of all the second-order symmetric tensors,

$$\hat{E} : [0,1] \rightarrow \mathrm{Sym} , \qquad \tau \longmapsto \hat{E}(\tau) . \tag{2.1}$$

The value $\hat{E}(\tau)$ at instant τ of a history $\hat{E}$ is interpreted as the infinitesimal deformation, i.e., the symmetrical part of the displacement gradient starting from a fixed reference configuration, in a fixed material point. At each instant τ in which $\hat{E}$ is differentiable, $\dot{E}$ represents the value of the derivate of $\hat{E}$ at instant τ; for each τ for which $\dot{E}$ is discontinuous we shall indicate the right-hand derivative as $\dot{E}$.

Let us define the section and continuation operation on the set $\mathfrak{F}$ of all the histories. For $\hat{E} \in \mathfrak{F}$ and $\tau \in [0,1]$, the *τ-section* of $\hat{E}$ is the history $\hat{E}_\tau$ such that

$$\tau' \longmapsto \hat{E}_\tau(\tau') := \hat{E}(\tau\tau') , \quad \tau' \in [0,1] ; \tag{2.2}$$

a *continuation* $\hat{G}$ of $\hat{E}$ is any history $\hat{G}$ such that $\hat{G}_\tau = \hat{E}$. For $A \in \mathrm{Sym}$, a continuation of $\hat{E}$ *up to* A is a continuation of $\hat{E}$ of which the final value $\hat{E}(1)$ coincides with A.

The materials being considered here are elastic-plastic isotropic solid whose mechanical response to deformation processes is described by a frame-indifferent and rate-independent constitutive functional. All deformation processes are thought to originate from such a state. Physically, this means carrying out experiments only on identically prepared specimens. To be more precise, taking

$$\mathfrak{D} := \{ \hat{E} \in \mathfrak{F} \ / \ \hat{E}(0) = 0 \} , \tag{2.3}$$

to be the set of all histories whose initial value is the origin of Sym, the material response is described by a constitutive functional

$$\tilde{T} : \mathfrak{D} \rightarrow \mathrm{Sym} , \qquad T = \tilde{T}(\hat{E}) , \tag{2.4}$$

whose value $\tilde{T}(\hat{E})$ gives the Cauchy stress at the end of history $\hat{E}$.

Since the constitutive response is not affected by the strain rate, we can interpret $\tilde{T}(\hat{E}_\tau)$, the Cauchy stress associated with the τ-section of $\hat{E}$, as the stress attained at instant τ during history $\hat{E}$. To make this clearer, below we shall write $\hat{T}_E(\tau)$ instead of $\tilde{T}(\hat{E}_\tau)$.

The kind of constitutive response is further specified by the notion of elastic range and plastic history (see [7], Section 3 and 5, respectively). Elastic range $E(\hat{E}_\tau)$ at time τ corresponding to history $\hat{E} \in \mathfrak{D}$ is the closure of an arcwise connected open subset of Sym, whose boundary is attainable from interior points only; it contains $\hat{E}(\tau)$ and its points are interpreted as the infinitesimal deformations from the reference configuration to configurations which are elastically accessible from the current configuration.

A history $\hat{G}$ with $\hat{E}_\tau = \hat{G}_{\tau_1}$ is called an *elastic continuation* of $\hat{E}$ if $\hat{G}$ remains in $E(\hat{E}_\tau)$ *i.e.*, if $\hat{G}(\tau) \in E(\hat{E}_\tau)$ for each $\tau' \in [\tau_1,1]$. Functional $\tilde{T}$, restricted to the set of elastic continuations of $\hat{E}_\tau$, is path-independent; in other words, if $\hat{G}$ and $\hat{H}$ are two elastic continuations of $\hat{E}_\tau$ such that $\hat{G}(1) = \hat{H}(1)$ we have $\tilde{T}(\hat{G}) = \tilde{T}(\hat{H})$. Moreover, the elastic range is invariant for elastic continuations, in that, for each elastic continuation $\hat{G}$ of $\hat{E}_\tau$, we have

$$E(\hat{E}_\tau) = E(\hat{G}) \,. \tag{2.5}$$

Given a history $\hat{E} \in \mathfrak{D}$, a history $\hat{E}^p \in \mathfrak{T}$ is called a *plastic history* corresponding to $\hat{E}$ if, for each $\tau \in [0,1]$, $\hat{E}^p(\tau) \in E(\hat{E}_\tau)$ and $\tilde{T}(\hat{G}) = 0$, for each elastic continuation $\hat{G}$ of $\hat{E}_\tau$ up to $\hat{E}^p(\tau)$. Thus, at each instant τ, $\hat{E}^p(\tau)$ defines an unstressed configuration which is elastically accessible from the current configuration.

Here we assume that, for each history $\hat{E} \in \mathfrak{D}$ there exist, precisely, a plastic history $\hat{E}^p$ corresponding to $\hat{E}$. We also assume there is no plastic change of volume *i.e.*,

$$\mathrm{tr}\hat{E}^p(\tau) = 0 \,, \qquad \text{for each } \tau \in [0,1] \,, \tag{2.6}$$

where tr is the trace functional.

Since the material is isotropic, it can be proved that $\hat{E}^p(0) = 0$ (see [7], subsection 7.1) and therefore

$$\hat{E}^p \in \mathfrak{D} \,, \quad \text{for each } \hat{E} \in \mathfrak{D} \,. \tag{2.7}$$

Besides, if $\hat{E} \in \mathfrak{D}$ and $\hat{G}$ is an elastic continuation of $\hat{E}$ such that $\hat{G}_\tau = \hat{E}$, it is easy to verify that

$$\hat{G}^p(\tau') = \hat{E}^p(1) \,, \qquad \text{for each } \tau' \in [\tau,1] \tag{2.8}$$

(see [7], prop. 5.2) and therefore

$$\dot{G}^p(\tau') = 0 \,, \qquad \text{for each } \tau' \in [\tau,1] \,. \tag{2.9}$$

Let $\hat{E}$ and $\hat{E}^p \in \mathfrak{D}$, be a history and its plastic history, respectively. In this paper, we accept the classic hypothesis that the material response to elastic deformations with respect to the current unstressed configuration is not affected by the past deformation process. We also deal esclusively with isotropic materials subject to infinitesimal deformations. Consequently, we assume that there exist two constants λ and μ, the so-called Lame' moduli, such that, in view also of (2.6), we have

$$\hat{T}_E(\tau) = T^*(\hat{E}(\tau) - \hat{E}^p(\tau)) = 2\mu(\hat{E}(\tau) - \hat{E}^p(\tau)) + \lambda(\mathrm{tr}\hat{E}(\tau))\, I \,, \tag{2.10}$$

(here I is the identity tensor). Application T^*, which depends on the material but not on the history, is called the *structural mapping* [10].

The image of the elastic range under) T^*, *i.e.*, the set of all the stresses which are elastically accessible starting from that current, and its boundary, are called *stress range* and *yield surface*, respectively.

For each tensor $A \in$ Sym the *deviator* of A will be indicated by

$$A_0 := A - \tfrac{1}{3}(\mathrm{tr}A)\, I \,. \tag{2.11}$$

The following relations can be immediately deduced from (2.6), (2.10) and (2.11):

$$\hat{T}_E(\tau)_0 = 2\mu(\hat{E}(\tau)_0 - \hat{E}^p(\tau)) \,, \tag*{(2.12)$_1$}$$

$$\mathrm{tr}\hat{T}_E(\tau) = 3\chi\mathrm{tr}\hat{E}(\tau) \,, \tag*{(2.12)$_2$}$$

where

$$\chi := \tfrac{1}{3}(2\mu + 3\lambda) \tag{2.13}$$

is the *bulk modulus*.

Here we accept Drucker's postulate in one of its equivalent versions (see [11]). As is well known, important properties of the elastic range and of the plastic strain rate follow from this postulate (see [8], prop. 7.6). We shall now proceed to list such properties. For this purpose let Sym_0 be the subset of Sym made up of all the traceless tensors and let

S be the closure of an open set of Sym_0. It should be remembered that, for each point V belonging to the boundary ∂S of S, the normal cone of S in V is the (possibly empty) subset $C_S(V)$ of Sym_0 defined by

$$C_S(S) := \{ A \in Sym_0\backslash\{0\} \ / \ (V - T) \cdot A \geq 0 \text{ for each } T \in S \}, \qquad (2.14)$$

where for A and B belonging to Sym, $A \cdot B := tr(AB)$.

For each history $\hat{E} \in \mathfrak{D}$ and for each $\tau \in [0,1]$,

i) *(convexity rule)* there exists a convex subset $E_0(\hat{E}_\tau)$ of Sym_0 such that the elastic range $E(\hat{E}_\tau)$ corresponding to $\hat{E}_\tau$ is the cylinder

$$E(\hat{E}_\tau) = \{ E \in Sym \ / \ E_0 \in E_0(\hat{E}_\tau) \} ; \qquad (2.15)$$

ii) *(flow rule)* the plastic strain rate $\dot{E}^p(\tau)$ is either null or belongs to the normal cone of $\partial E_0(\hat{E}_\tau)$ at $(\hat{E}_\tau)_0$.

In view of the applications we have in mind, the hypothesis may be advanced that the material satisfies the v. Mises criterion. To be more precise, let us suppose that, for each history $\hat{E} \in \mathfrak{D}$ there exists a history $\hat{C}_E$ with values in Sym_0 such that $\hat{C}_E(0) = 0$ and, for each $\tau \in [0,1]$, the set $E_0(\hat{E}_\tau)$ is the ball

$$E_0(\hat{E}_\tau) = \{ E \in Sym_0 \ / \ \|E - \hat{C}_E(\tau)\| \leq \rho \}, \qquad (2.16)$$

where, for $A \in Sym$, $\|A\| := A \cdot A = trA^2$.

In the present paper we do not intend to take isotropic hardening into account; therefore, let us suppose ρ is an independent positive constant of the history (see [5], Sez. 3).

The set

$$E := \{ E \in Sym \ / \ \|E_0\| \leq \rho \} \qquad (2.17)$$

is called the *initial elastic range* of the material. Let $\hat{E} \in \mathfrak{D}$ be a history; since $\hat{E}$ is a continuous function and $\hat{E}(0) = 0$, for τ small enough $\hat{E}(\tau) \in E$. Let us suppose that the image of [0,1] under $\hat{E}$ is not entirely contained in E, and let $\tau^* \in]0,1[$ be the greatest value of τ for which the image $[0,\tau^*]$ under $\hat{E}$ is contained in E; τ^* is the *instant of first yielding* of the material.

For V, the three-dimensional real inner product space, let e_1, e_2, e_3 be an othonormal basis of V, fixed once and for all. Let us consider a pure tension process, *i.e.*, a history $\hat{E} \in \mathfrak{D}$ such that

$$\hat{T}_E(\tau) = \sigma_{11}(\tau)\, e_1 \otimes e_1 \,, \quad \tau \in [0,1] \,, \tag{2.18$_1$}$$

so that

$$\hat{T}_E(\tau)_0 = \tfrac{1}{3}\sigma_{11}(\tau)(2e_1 \otimes e_1 - (e_2 \otimes e_2 + e_3 \otimes e_3)) \,, \quad \tau \in [0,\, 1] \,, \tag{2.18$_2$}$$

and let τ^* be the corresponding instant of first yielding. The quantity

$$\sigma^* := \sigma_{11}(\tau^*) \tag{2.19}$$

is the *first yielding tension* of the material. In order to ascertain the relation between σ^* and ρ let us note that, in view of (2.9) and $(2.12)_1$, it turns out that, for $\tau \in [0,\tau^*]$,

$$\hat{E}(\tau)_0 = \tfrac{1}{6}(\sigma_{11}(\tau)/\mu)(2e_1 \otimes e_1 - (e_2 \otimes e_2 + e_3 \otimes e_3)) \,. \tag{2.20}$$

Therefore

$$\hat{E}(\tau^*)_0 = \tfrac{1}{6}(\sigma^*/\mu)(2e_1 \otimes e_1 - (e_2 \otimes e_2 + e_3 \otimes e_3)) \tag{2.21}$$

and it can be deduced from (2.16) and (2.19) that

$$\rho = \sqrt{2/3}\; \sigma^*/(2\mu) \,. \tag{2.22}$$

From the flow rule and from (2.16) we have

$$\dot{E}^P(\tau) = \|\dot{E}^P(\tau)\|\; \hat{N}_E(\tau) \,, \tag{2.23}$$

where

$$\hat{N}_E(\tau) := (1/\rho)((\hat{E}_\tau)_0 - \hat{C}_E(\tau)) \tag{2.24}$$

is the outward unit normal vector to $\partial E_0(\hat{E}_\tau)$ at $(\hat{E}_\tau)_0$.

In order to take the Bauschinger effect into account, let us accept the classic kinematic hardening rule proposed by MELAN [6]. Let us suppose, that is, that there exists a non-negative constant η, called the kinematic hardening modulus, such that for each history $\hat{E} \in \mathfrak{D}$ and for each $\tau \in [0,1]$ we have

$$\hat{C}_E(\tau) = (1 + \eta)\hat{E}^p(\tau) \,. \tag{2.25}$$

In particular, a material for which we have $\eta = 0$ is called *ideally plastic.*

Within the framework of this theory, the use of the hardening rule (2.25) requires some caution. Indeed, if for some history $\hat{E} \in \mathfrak{D}$ and for some $\tau \in [0,1]$ we do not have

$$\eta \le \rho(\|\hat{E}^p(\tau)\|)^{-1} \,, \tag{2.26}$$

$\hat{E}^p(\tau)$ does not belong to $E_0(\hat{E}_\tau)$, in contrast with the definition of plastic history given above. More general hardening rules like that to be found in [5] may ensure, thanks to a suitable choice of parameters, that the inequality (2.26) is satisfied during every history $\hat{E} \in \mathfrak{D}$. Rule (2.25) may therefore be considered as an approximation of these more general rules, which is applicable when the value of $\|\hat{E}^p(\tau)\|$ during the history is sufficiently small.

As proved in [5] and [9], the evolution of the corresponding plastic deformation is governed, for each history $\hat{E} \in \mathfrak{D}$, not only by the flow rule (2.23) but also by the following relation

$$\|\dot{E}^p(\tau)\| = \begin{cases} 0 \text{ if } \|\hat{E}(\tau)_0 - \hat{C}_E(\tau)\| < \rho \\ 0 \text{ if } \|\hat{E}(\tau)_0 - \hat{C}_E(\tau)\| = \rho \text{ and } \hat{N}_E(\tau)\cdot\dot{E}(\tau)_0 \le 0 \\ (1/(1+\eta))\, \hat{N}_E(\tau)\cdot\dot{E}(\tau)_0 \\ \quad \text{if } \|\hat{E}(\tau)_0 - \hat{C}_E(\tau)\| = \rho \text{ and } \hat{N}_E(\tau)\cdot\dot{E}(\tau)_0 > 0 \,. \end{cases} \tag{2.27}$$

When one of the first two cases contemplated in the right-hand side of (2.27) occurs, the material behaves elastically; the third case is known as the *plastic loading condition* .

Relation $(2.27)_3$ is typical of the present constitutive theory; indeed, it prescribes $\|\dot{E}^p(\tau)\|$ as a function of the deviatoric strain rate $\dot{E}(\tau)_0$, rather than of the deviatoric stress rate $\dot{T}(\tau)_0$, as happens in the classic theory (see [12], p. 33). Although the present model gives the same result as the classic one, it simplifies the search for the solution of the constitutive equation, as will become clear in the following chapter.

Relations (2.23), (2.24) and (2.27) can be conveniently made into a single equation. Indeed, given a history $\hat{E} \in \mathfrak{D}$ and after obtaining from it the corresponding *deviatoric history* $\hat{E}_0 : [0,1] \to Sym_0$,

$$\hat{E}_0(\tau) := \hat{E}(\tau)_0 = \hat{E}(\tau) - (\mathrm{tr}\hat{E}(\tau))\ I\ , \quad \tau \in [0,1]\ , \tag{2.28}$$

let us put

$$\hat{X}_E(\tau) := (\hat{E}_0(\tau) - \hat{C}_E(\tau))\ ; \tag{2.29}$$

from (2.23), (2.27) and (2.29) we then obtain

$$\dot{X}_E(\tau) = \begin{cases} \dot{E}_0(\tau) \text{ if } \|\hat{X}(\tau)\| < \rho \\ \dot{E}_0(\tau) \text{ if } \|\hat{X}(\tau)\| = \rho \text{ and } \hat{X}_E(\tau)\cdot\dot{E}_0(\tau) \le 0 \\ \dot{E}_0(\tau) - (1/\rho^2)(\hat{X}_E(\tau)\cdot\dot{E}_0(\tau))\hat{X}(\tau) \\ \qquad \text{if } \|\hat{X}(\tau)\| = \rho \text{ and } \hat{X}_E(\tau)\cdot\dot{E}_0(\tau) > 0\ , \end{cases} \tag{2.30}$$

with the initial condition

$$\hat{X}_E(0) = 0\ , \tag{2.31}$$

a direct consequence of (2.3), (2.7), (2.25) and (2.29).

In [13] it is proved that, for every $\hat{E} \in \mathfrak{D}$, the derivative of which is a function of bounded variation on [0,1], the equation (2.30) with the initial condition (2.31) provides one and only one solution $\hat{X}$, which is lipschitzian on [0,1] and derivable to the right in every point.

For each history $\hat{E}$ assigned, the integration of equation (2.30) with initial condition (2.31) makes it possible to determine $\hat{E}^p$, with the help of (2.23), (2.25) and (2.29). The stress can then be calculated from relation (2.12).

3. A COMPLEX LOADING PROCESS

This section deals with a deviatoric deformation process which can be obtained, using the experimental methodologies described in detail in [2] and [3], in a thin-walled tubular specimen submitted to combined tension-torsion loads; the analytical solution of the relevant equation (2.30) is found. Since our purpose here is purely demonstrative, we have confined ourselves to considering a deformation process made up of two normal segments (see Figures 1 and 2) but the extension of this method of solution to processes made up of two segments at different angles raises no particular problems. After this, the results, calculated for different values of the kinematic hardening modulus, are compared with the experimental data contained in [1].

Let α and β be two real positive numbers. Let us consider the deviatoric deformation process $\hat{E}_0 : [0,1] \to Sym_0$, defined as follows

$$\hat{E}_0(\tau) = \epsilon_{11}(\tau)\,(e_1 \otimes e_1 - e_2 \otimes e_2) + \epsilon_{12}(e_1 \otimes e_2 + e_2 \otimes e_1), \tag{3.1}$$

where, for τ^*, the instant of first yielding, we have

$$\epsilon_{11}(\tau) = \begin{cases} \alpha\tau & \text{for } 0 \le \tau \le \tau_1 \\ \alpha\tau_1 & \text{for } \tau_1 \le \tau \le 1\,, \end{cases} \tag{3.2}$$

$$\epsilon_{12}(\tau) = \begin{cases} 0 & \text{for } 0 \le \tau \le \tau_1 \\ \beta(\tau-\tau_1) & \text{for } \tau_1 \le \tau \le 1, \end{cases} \tag{3.3}$$

for some fixed $\tau_1 \in]\tau^*, 1[$.

In view of (2.29), (2.30) and (3.1), for each $\tau \in [0,1]$, we can write

$$\hat{X}_E(\tau) = \xi_{11}(\tau)(e_1 \otimes e_1 - e_2 \otimes e_2) + \xi_{12}(\tau)(e_1 \otimes e_2 + e_2 \otimes e_1). \tag{3.4}$$

For $\tau \in [0, \tau^*]$ the material behaves elastically and we find $\dot{X}_E(\tau) = \dot{E}_0(\tau)$, while for $\tau \in [\tau^*, \tau_1]$ we have plastic loading and from (2.30), (3.1) and (3.2) we can immediately deduce that $\dot{X}_E(\tau) = 0$. Therefore

$$\hat{X}_E(\tau) = \begin{cases} \hat{E}_0(\tau)\,, & \text{for } \tau \in [0, \tau^*] \\ \hat{E}_0(\tau^*)\,, & \text{for } \tau \in [\tau^*, \tau_1]\,. \end{cases} \tag{3.5}$$

For $\tau \in [\tau_1,1]$ the plastic loading situation continues and we therefore have $\|\hat{X}_E(\tau)\| = \rho$ which, in view of (3.4), implies that

$$2(\xi_{11}{}^2 + \xi_{12}{}^2) = \rho^2\,. \tag{3.6}$$

This last relation suggests we should put (v. Fig. 1)

$$\xi_{11} = (\tfrac{1}{2}\sqrt{2}\,\rho)\sin\gamma \quad ; \quad \xi_{12} = (\tfrac{1}{2}\sqrt{2}\,\rho)\cos\gamma\,. \tag{3.7}$$

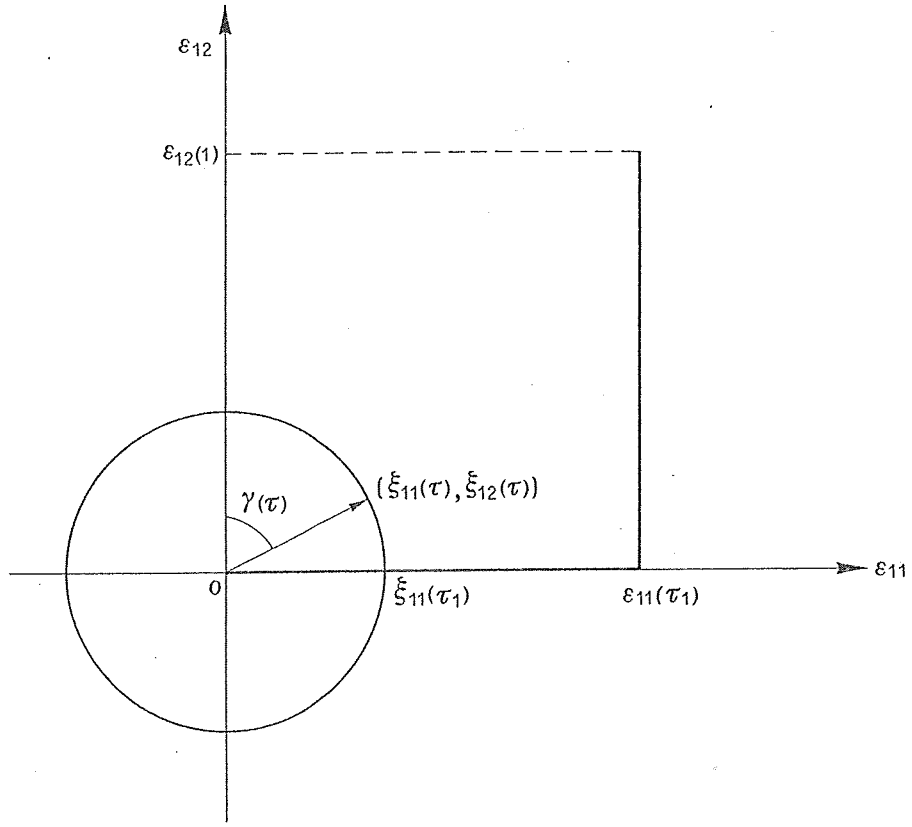

Fig. 1

From (3.1) - (3.4) and (3.7) we obtain the following equations, for $\tau \in [\tau_1, 1]$,

$$\hat{X}_E(\tau) = (\tfrac{1}{2}\sqrt{2}\,\rho)[\sin\gamma(\tau)(e_1 \otimes e_1 - e_2 \otimes e_2) +$$

$$+ \cos\gamma(\tau)(e_1 \otimes e_2 + e_2 \otimes e_1)] \; ; \tag{3.8}$$

$$\dot{E}_0(\tau) = \beta(e_1 \otimes e_2 + e_2 \otimes e_1) \quad . \tag{3.9}$$

We therefore have

$$\hat{X}_E \cdot \dot{E}_0 = \sqrt{2}\,\rho\beta\cos\gamma \tag{3.10}$$

and, in view of $(2.30)_3$, (3.8) - (3.10) we can write

$$\dot{X}_E = -\beta\sin\gamma[-\cos\gamma\,(e_1 \otimes e_1 - e_2 \otimes e_2) +$$

$$+ \sin\gamma(e_1 \otimes e_2 + e_2 \otimes e_1)] \ . \qquad (3.11)$$

On the other hand, deriving (3.8) with respect to τ, we obtain

$$\dot{X}_E = (\tfrac{1}{2}\sqrt{2}\,\rho\dot{\gamma})[\cos\gamma\,(e_1 \otimes e_1 - e_2 \otimes e_2) - \sin\gamma\,(e_1 \otimes e_2 + e_2 \otimes e_1)] \ . \qquad (3.12)$$

A comparison between (3.11) and (3.12) immediately provides the identity

$$\dot{\gamma} = -(\sqrt{2}\,\beta/\rho)\sin\gamma \ ; \qquad (3.13)$$

since $\gamma(\tau_1) = \pi/2$, as can be directly deduced from (3.1), (3.4), (3.5) and (3.7), we have

$$\int_{\tau_1}^{\tau} \frac{d\gamma}{\sin\gamma} = -\frac{\beta\sqrt{2}}{\rho}(\tau - \tau_1) \qquad (3.14)$$

which gives

$$\tan(\gamma/2) = \exp(-(\sqrt{2}\,\beta/\rho)(\tau - \tau_1)) \ . \qquad (3.15)$$

Equations (3.7) and (3.15) make it possible to calculate ξ_{11} and ξ_{12} for each $\tau \in [\tau_1, 1]$ and the value of the plastic deformation $\hat{E}^p(\tau)$ can be obtained from the relation

$$\hat{E}^p(\tau) = (1/(1+\eta))[(\epsilon_{11}(\tau) - \xi_{11}(\tau))\,(e_1 \otimes e_1 - e_2 \otimes e_2) +$$

$$+ (\epsilon_{12}(\tau) - \xi_{12}(\tau))(e_1 \otimes e_2 + e_2 \otimes e_1)] \ , \qquad (3.16)$$

which is a consequence of (2.25), (2.29) and (3.4). Finally, $(2.12)_1$ provides the following expression for the deviatoric part of the stress:

$$\hat{T}_E(\tau)_0 = \sigma_{11}(\tau)\,(e_1 \otimes e_1 - e_2 \otimes e_2) + \sigma_{12}(e_1 \otimes e_2 + e_2 \otimes e_1), \qquad (3.17)$$

where, as a result of (3.16),

$$\sigma_{11} = (2\mu/(1+\eta))(\eta\epsilon_{11} + \xi_{11}) \ ; \qquad (3.18)_1$$

$$\sigma_{12} = (2\mu/(1+\eta))(\eta\epsilon_{12} + \xi_{12}) \ . \qquad (3.18)_2$$

It is usual to represent the deviatoric deformation processes relevant to complex tension-torsion loads, such as that defined by (3.1), with the *deviatoric strain vector* [4]

$$e := \epsilon_{11}\, e_1 + {}^2\!/_3\sqrt{3}\,\epsilon_{12}\, e_2 \quad ; \tag{3.19}$$

the curve $\tau \mapsto e(\tau)$, $\tau \in [0,1]$, is called the *deviatoric strain trajectory* and

$$\lambda(\tau) := \int_0^{\tau} |\dot{e}(\tau')| d\tau' \tag{3.20}$$

is its length up to instant τ.

In particular, the deviatoric strain trajectory corresponding to the deviatoric process (3.1) is made up of two segments that are perpendicular to one another, as shown in Fig. 2. The first segment, whose length is

$$\lambda_1 := \lambda(\tau_1) = \alpha\tau_1 \; , \qquad (3.21)_1$$

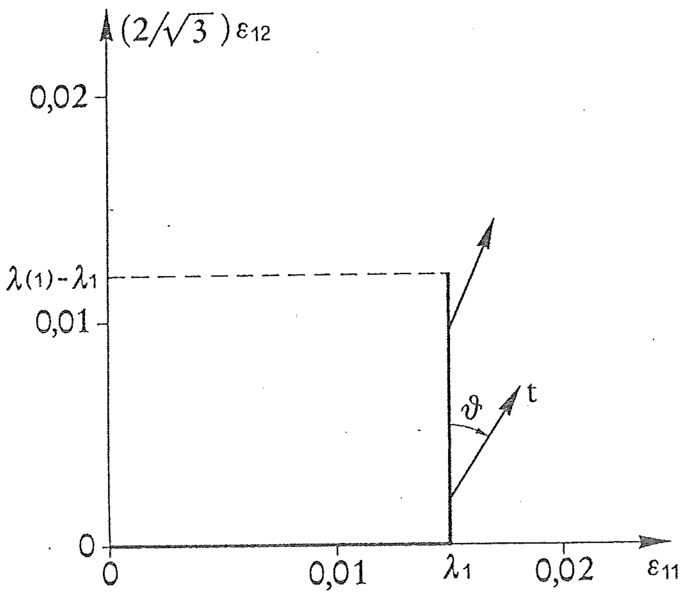

Fig. 2

corresponds to the interval $[0, \tau_1]$; the second segment, whose length is

$$\lambda(1) - \lambda_1 = {}^2\!/_3\sqrt{3}\, \beta(1 - \tau_1)\,, \qquad (3.21)_2$$

corresponds to the interval $[\tau_1, 1]$. At each instant $\tau \in [\tau_1, 1]$ we have

$$\lambda(\tau) = \lambda_1 + {}^2\!/_3\sqrt{3}\, \beta(\tau - \tau_1)\,. \qquad (3.22)$$

Similarly, it is usual to represent the mechanical response with the *deviatoric stress vector* [4]

$$\mathrm{t} := {}^3\!/_2\sigma_{11}\, \mathrm{e}_1 + \sqrt{3}\, \sigma_{12}\, \mathrm{e}_2\,. \qquad (3.23)$$

Tension-torsion experiments carried out on thin-walled tubes show that the direction of the deviatoric stress vector t and the tangent $\dot{\mathrm{e}}$ of the deviatoric strain trajectory only coincide when the curvature of the trajectory is very small, as is the case in the first segment of the deviatoric process (3.1). The angle $\theta(\tau)$ between vectors $\mathrm{t}(\tau)$ and $\dot{\mathrm{e}}(\tau)$,

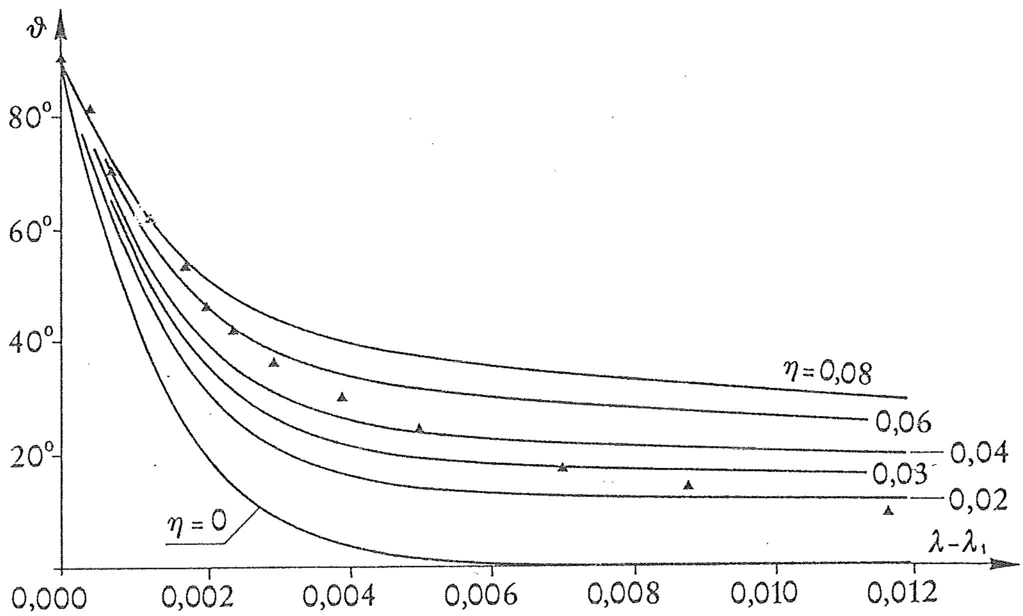

Fig. 3

$$\theta(\tau) := \cos^{-1}[\dot{e}(\tau)\cdot t(\tau)/(|\dot{e}(\tau)||t(\tau)|)]\,, \qquad (3.24)$$

is called the *delay angle* (see Fig 2).

For $\tau = \tau_1$, where the strain trajectory is right-angled, the delay angle coincides with $\pi/2$ and subsequently decreases until it disappears (see Fig. 3).

The change in direction of the strain vector also affects the length $|t|$ of the deviatoric stress vector, which rapidly decreases after the beginning of the second segment of the strain trajectory and then begins to increase again, as can be seen in Figure 4.

From (3.2), (3.3), (3.7), (3.18), $(3.21)_1$, (3.22) and (3.23) we can deduce that

$$t = (2\mu/(1+\eta))[(^3/_2\eta\lambda_1 + {}^3\!/_4\sqrt{2}\,\rho\sin\gamma)\, e_1 +$$

$$+ (^3/_2\eta(\lambda - \lambda_1) + \sqrt{3/2}\,\rho\cos\gamma)\, e_2]\,,\ \tau \in [\tau_1, 1]\,, \qquad (3.25)$$

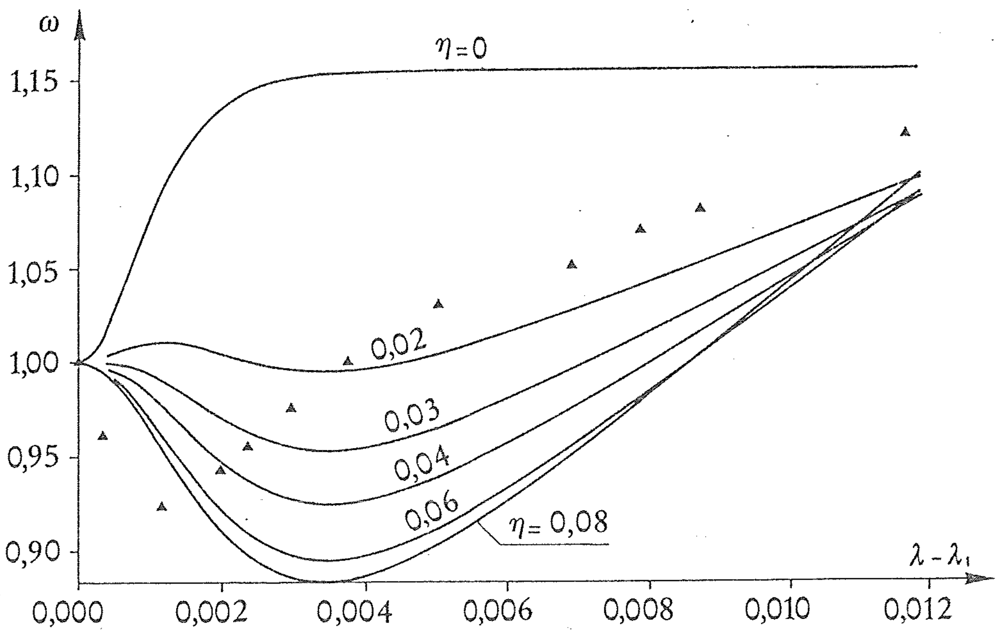

Fig. 4

from which, with the help of (3.24), we immediately obtain the delay angle

$$\theta(\tau) = \cos^{-1}[(2\mu/(1+\eta))(^3/_2\eta(\lambda-\lambda_1) + \sqrt{3/2}\,\rho\cos\gamma)\,/|t|] \,. \quad (3.26)$$

In particular, for ideally plastic materials η is zero and therefore (3.26) becomes

$$\theta(\tau) = \cos^{-1}(\cos\gamma\,/\,(\tfrac{3}{4}\sin^2\gamma + \cos^2\gamma)^{\frac{1}{2}})\,, \qquad (3.27)$$

so that the delay angle is independent of elastic modulus μ.

In order to study the affect of the kinematic hardening modulus, defined by equation (2.25), on the delay angle, let us determine the behaviour of $\cos\theta$ as a function of η, for fixed $\tau \in [\tau_1, 1]$. From (3.25) and (3.26) we obtain, after a number of computations,

$$\frac{d}{d\eta}\cos\theta =$$

$$^9/_8\sqrt{2}\,\rho\,(\lambda_1\eta + \tfrac{1}{2}\sqrt{2}\,\rho\sin\gamma)(^3/_2(\lambda-\lambda_1)\sin\gamma +$$

$$-\sqrt{3}\,\lambda_1\cos\gamma)|t|^{-3}\,, \qquad (3.28)$$

so that

$$\frac{d}{d\eta}\cos\theta \le 0 \quad \text{for } \tan\gamma \le \tfrac{2}{3}\sqrt{3}\,\lambda_1/(\lambda-\lambda_1) \qquad (3.29)$$

But

$$\tan\gamma = 2\tan(\gamma/2)/(1-\tan^2(\gamma/2)) \qquad (3.30)$$

and from (3.15), (3.22) and (3.29) we can deduce

$$\frac{d}{d\eta}\cos\theta \le 0 \qquad \text{for}$$

$$2(\lambda-\lambda_1)\exp(-(\tfrac{1}{2}\sqrt{6}/\rho)(\lambda-\lambda_1))/[1-\exp(-(\sqrt{6}/\rho)(\lambda-\lambda_1))] \le$$

$$\tfrac{2}{3}\sqrt{3}\,\lambda_1\,. \qquad (3.31)$$

Let $\varphi(\lambda - \lambda_1)$ be the left-hand side of the inequality (3.31). It is easy to ascertain that $\varphi(\lambda - \lambda_1)$ is a decreasing function, for $(\lambda - \lambda_1) > 0$, and that we get

$$\lim_{\lambda \to \lambda_1^+} \varphi(\lambda - \lambda_1) = \sqrt{2/3}\, \rho \,. \tag{3.32}$$

Therefore, if

$$\rho \leq \sqrt{2}\, \lambda_1 \,, \tag{3.33}$$

the delay angle θ is an increasing function of η for each $\tau \in [\tau_1, 1]$ or, equivalently, for each $\lambda \geq \lambda_1$.

In order to assess the effect of parameter η on the length $|t|$ of the deviatoric stress vector, when the deviatoric strain vector changes direction, let us put

$$\omega(\lambda - \lambda_1) := |t(\lambda - \lambda_1)| \,/\, |t(\lambda_1)| \,, \qquad \text{for } \lambda - \lambda_1 \geq 0 \,. \tag{3.34}$$

From (3.15) and (3.22) we can deduce the following equations:

$$\frac{d}{d\lambda} \sin\gamma = -\sqrt{\frac{3}{2}}\, \frac{1}{\rho} \sin\gamma \cos\gamma \,, \qquad (3.35)_1$$

$$\frac{d}{d\lambda} \cos\gamma = \sqrt{\frac{3}{2}}\, \frac{1}{\rho} (\sin\gamma)^2 \,, \qquad (3.35)_2$$

using which we find that

$$\frac{d\omega}{d\lambda} =$$

$$[{}^3/_2\eta^2(\lambda - \lambda_1) + {}^3/_2\eta(\lambda - \lambda_1)\sin^2\gamma +$$

$$+ \sqrt{3/2}\, \rho\eta\cos\gamma + {}^1/_4\sqrt{3/2}\, \rho\sin^2\gamma\cos\gamma +$$

$$- {}^3/_4\sqrt{3}\, \eta\lambda_1\sin\gamma\cos\gamma] \,/\, [(\eta\lambda_1 + {}^1/_2\sqrt{2}\, \rho)\, |t|] \,. \tag{3.36}$$

If we observe equation (3.36) it is immediately clear that, if the material is ideally plastic (*i.e.* if we have $\eta = 0$), ω is an increasing function for each $\lambda > \lambda_1$; if, on the other hand, η is positive, for ω to be an increasing function, $(\lambda - \lambda_1)$ must be sufficiently large. In the latter case, the effect of the sudden change in direction of the deviatoric strain trajectory on the behaviour of ω has realistically to be found in a small right-hand neighborhood of $\lambda = \lambda_1$. Given, then, that

$$\psi = \pi/2 - \gamma \,, \tag{3.37}$$

we shall confine ourselves to examining the behaviour of $\omega(\lambda)$ when $(\lambda - \lambda_1)$, and, therefore, ψ, too, are small enough quantities to allow the following approximations:

$$\cos\gamma = \sin\psi \simeq \psi \,, \qquad (3.38)_1$$

$$\sin\gamma = \cos\psi \simeq 1 \,, \qquad (3.38)_2$$

$$\lambda - \lambda_1 = -\sqrt{2/3}\,\rho\,\ln(\tan(\gamma/2)) \simeq -\sqrt{2/3}\,\rho\,\ln(1 - 2\psi^2) \simeq$$

$$\simeq 2\sqrt{2/3}\,\rho\psi^2 \,; \qquad (3.38)_3$$

the first identity in $(3.38)_3$ is a consequence of (3.15) and (3.22). With the help of approximations (3.38), we obtain from (3.36)

$$\frac{d\omega}{d\lambda} \simeq$$

$$\simeq \psi(\sqrt{3/2}\,\rho\eta + \tfrac{1}{4}\sqrt{3/2}\,\rho - \tfrac{3}{4}\sqrt{3}\,\eta\lambda_1)/((\eta\lambda_1 + \tfrac{1}{2}\sqrt{2}\,\rho)\,|t|) \,. \tag{3.39}$$

In applications such as those being considered in the present paper, in which we have

$$\rho/\lambda_1 < 1 \,, \tag{3.40}$$

it can reasonably be thought that, in order for a right-hand neighborhood of λ_1 where $\omega(\lambda)$ is decreasing, to exist, we have to have

$$\eta > \rho/(3\sqrt{2}\,\lambda_1 - 4\rho) \,. \tag{3.41}$$

We shall now give an estimation of the upper limit imposed on the value of η by condition (2.26), during the deviatoric process (3.1) (see the discussion preceding equation (2.26)).

For $\tau \in [0,\tau^*]$ we have $\hat{E}^p(\tau) = 0$. For $\tau \in [\tau^*,\tau_1]$, we can deduce from (2.25), (2.29), (3.1) - (3.3) and $(3.5)_2$ that we have

$$\|\hat{E}^p(\tau)\| = (1/(1 + \eta))\|\hat{E}_0(\tau) - \hat{E}_0(\tau^*)\| =$$

$$= \sqrt{2}\,\alpha\,(\tau - \tau^*)/(1 + \eta) \qquad (3.42)$$

and therefore, in view of (2.17) and $(3.21)_1$, that the greatest value of $\|\hat{E}^p(\tau)\|$, for $\tau \in [\tau^*,\tau_1]$ is

$$\sqrt{2}\,\alpha(\tau_1 - \tau^*)/(1 + \eta) = (\sqrt{2}\,\lambda_1 - \rho)/(1 + \eta)\ . \qquad (3.43)$$

For $\tau \in [\tau_1,1]$, (3.1) - (3.3), (3.7), (3.16) and (3.22) give

$$\|\hat{E}^p(\tau)\| = (\sqrt{2}/(1 + \eta))[(\lambda_1 - \tfrac{1}{2}\sqrt{2}\,\rho\sin\gamma)^2 +$$

$$+ (\tfrac{1}{2}\sqrt{3}\,(\lambda - \lambda_1) - \tfrac{1}{2}\sqrt{2}\,\rho\cos\gamma)^2]^{\frac{1}{2}}\ . \qquad (3.44)$$

Therefore, for $\tau \in [\tau_1, 1]$, we have

$$\|\hat{E}^p(\tau)\| < (\sqrt{2}/(1 + \eta))[\,\lambda_1{}^2 + \tfrac{3}{4}(\lambda - \lambda_1)^2 + \tfrac{1}{2}\rho^2]^{\frac{1}{2}} \qquad (3.45)$$

and so the greatest value attained by $\|\hat{E}^p(\tau)\|$ in the interval $[\tau_1, 1]$ is less than

$$(\sqrt{2}/(1 + \eta))[\lambda_1{}^2 + \tfrac{3}{4}(\lambda(1) - \lambda_1)^2 + \tfrac{1}{2}\rho^2]^{\frac{1}{2}}\ . \qquad (3.46)$$

The experimental data to be found in Figures 3 and 4, taken from [1], are relevant to complex tension-torsion tests carried out on a brass specimen with the following characteristic parameters:

$$\sigma^* = 140 \text{ MPa}\ , \qquad 2\mu = 70504 \text{ MPa}\ , \qquad (3.47)$$

and, therefore, in view of (2.22),

$$\rho = .00162\ . \qquad (3.48)$$

The lengths of the deviatoric strain trajectory are

$$\lambda_1 = .015\ , \qquad \lambda(1) - \lambda_1 = .012\ ; \qquad (3.49)$$

thus ρ/λ_1 = .108 and both the inequalities (3.33) and (3.40) are verified. Figure 3 shows the behaviour of the delay angle θ calculated for various values of η, with equation (3.26); this figure confirms that θ is an increasing function of η, for each $\lambda > \lambda_1$.

Figure 4, which shows the value of ω calculated for different values of η with equation (3.34), substantially confirms the estimation summarized in (3.39) and (3.41). Indeed, for ρ and λ_1 as in (3.48) and $(3.49)_1$, respectively, the right-hand side of the inequality (3.41) is about .0283.

Finally, it can be deduced from equations (3.43) and (3.46) that for ρ, λ_1 and $\lambda(1)$ as in (3.48) and (3.49), a sufficient condition for inequality (2.26) to be verified for each $\tau \in [0,1]$ proves to be $\eta \leq .067$.

4. CONCLUSIONS

Comparison between the experimental data and the calculated curves shows that, in the case in point, the kinematic hardening rule (2.25) can be used to give a fairly accurate approximate description of significant aspects of stress behaviour in a complex tension-torsion test. Moreover, examination of Figures 3 and 4 shows that the value of η for which the closest agreement with the experimental data is attained, still falls within a rather limited interval, even though such a value of η is a decreasing function of $(\lambda - \lambda_1)$. More complicated hardening rules should make it possible to obtain more accurate results; such an aim is achieved, for example, in [5], by allowing the centre of the elastic range to move in a direction which depends on the entire previous strain history. However, when the aim is to find explicit solutions to boundary-value problems, it is expedient to choose constitutive laws which, like the one proposed by MELAN, are very simple. Therefore, it seems worthwhile drawing attention to the fact that, unlike the ideally plastic rule, the MELAN hardening rule, in spite of its simplicity, accounts for phenomenological aspects generally considered important in testing elastic-plastic constitutive laws.

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