# On a Cahn–Hilliard system with convection and dynamic boundary conditions

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**Abstract**

This paper deals with an initial and boundary value problem for a system coupling equation and boundary condition both of Cahn–Hilliard type; an additional convective term with a forced velocity field, which could act as a control on the system, is also present in the equation. Either regular or singular potentials are admitted in the bulk and on the boundary. Both the viscous and pure Cahn–Hilliard cases are investigated, and a number of results is proven about existence of solutions, uniqueness, regularity, continuous dependence, uniform boundedness of solutions, strict separation property. A complete approximation of the problem, based on the regularization of maximal monotone graphs and the use of a Faedo–Galerkin scheme, is introduced and rigorously discussed.

**Key words:** Cahn–Hilliard system, convection, dynamic boundary condition, initial-boundary value problem, well-posedness, regularity of solutions.

**AMS (MOS) Subject Classification:** 35K61, 35K25, 76R05, 80A22

# 1 Introduction

This paper is concerned with the following Cahn-Hilliard system with convection:

$$\partial_t \rho + \nabla\rho \cdot u - \Delta\mu = 0 \quad \text{and} \quad \tau_\Omega \partial_t \rho - \Delta\rho + f'(\rho) = \mu \quad \text{in } Q := \Omega \times (0,T), \tag{1.1}$$

where $\Omega$ denotes a bounded three-dimensional domain and $T > 0$ is a fixed final time. The unknowns are $\rho$, the order parameter, and $\mu$, the chemical potential; $f'$ stands for the derivative of a double-well potential $f$, $u$ is a given velocity field and $\tau_\Omega$ is a nonnegative constant. According to whether $\tau_\Omega$ is positive or zero, we speak of viscous Cahn–Hilliard or pure Cahn–Hilliard system, respectively.

The equations in (1.1) provide a description of the evolution phenomena related to solid-solid phase separations with convection leaded by the term $\nabla\rho \cdot u$, for some fixed velocity vector $u$. Let us refer to [1, 5, 21, 22, 35] for some pioneering contributions on the class of Cahn–Hilliard problems. In general, an evolution process goes on with diffusion; however, for the process of phase separation there is a structural difference since each phase concentrates and the so-called spinodal decomposition occurs. A discussion on the modeling approach for phase separation, spinodal decomposition and mobility of atoms between cells can be found in [8, 16, 23, 29, 36]).

Typical and important examples of $f$ are the so–called *classical regular potential* and the *logarithmic double-well potential*. They are given by

$$f_{reg}(r) := \frac{1}{4}\,(r^2 - 1)^2\,, \quad r \in \mathbb{R}, \tag{1.2}$$

$$f_{log}(r) := ((1+r)\ln(1+r) + (1-r)\ln(1-r)) - cr^2\,, \quad r \in (-1,1), \tag{1.3}$$

where $c > 1$ is such that $f_{log}$ is nonconvex. Another example is the following *double obstacle potential*:

$$f_{2obs}(r) := -cr^2 \quad \text{if } |r| \le 1 \quad \text{and} \quad f_{2obs}(r) := +\infty \quad \text{if } |r| > 1, \tag{1.4}$$

where $c > 0$. In cases like (1.4), one has to split $f$ into a non-differentiable convex part (the indicator function of $[-1,1]$ in the present example) and a smooth perturbation. Accordingly, one has to replace the derivative of the convex part by the subdifferential and interpret the second identity in (1.1) as a differential inclusion. In order to incorporate cases like (1.4) in our analysis, we allow $f'$ to be expressed by the sum $\beta + \pi$, where $\beta$ is the subdifferential of a convex and lower semicontinuous function $\widehat{\beta} : \mathbb{R} \to [0,+\infty]$ such that $\widehat{\beta}(0) = 0$, and $\pi$ is the Lipschitz continuous derivative of the concave perturbation $\widehat{\pi} : \mathbb{R} \to \mathbb{R}$. Thus, we have that $f = \widehat{\beta} + \widehat{\pi}$ represents a possibly non-smooth double-well potential.

In order to set an initial-boundary value problem for (1.1), we have to specify initial and boundary conditions. As far as the latter are concerned, the classical ones are the homogeneous Neumann boundary conditions, namely

$$\partial_\nu \mu = 0, \quad \partial_\nu \rho = 0 \quad \text{on } \Sigma := \Gamma \times (0,T), \tag{1.5}$$

where $\Gamma$ stands for the smooth boundary of $\Omega$ and $\partial_\nu$ denotes the outward normal derivative. In the present work, on the contrary we tackle two dynamic boundary conditions for $\mu$ and $\rho$ so to obtain a system of Cahn–Hilliard type also on the boundary. Namely, we complement (1.1) with

$$\partial_t \rho_\Gamma + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{and} \quad \tau_\Gamma \partial_t \rho_\Gamma + \partial_\nu \rho - \Delta_\Gamma \rho_\Gamma + f'_\Gamma(\rho_\Gamma) = \mu_\Gamma \quad \text{on } \Sigma, \tag{1.6}$$

where $\mu_\Gamma$ and $\rho_\Gamma$ are the traces of $\mu$ and $\rho$, respectively, $\Delta_\Gamma$ is the Laplace-Beltrami operator on the boundary, $\tau_\Gamma$ is a nonnegative constant, and $f'_\Gamma = \beta_\Gamma + \pi_\Gamma$ comes out from another potential $f_\Gamma = \widehat{\beta}_\Gamma + \widehat{\pi}_\Gamma$ with the same behavior as $f$, the two potentials being not completely independent but related by a suitable growth condition. Then, it turns out that initial conditions should be prescribed both in the bulk and on the boundary.

Therefore, by considering everything, the resulting initial and boundary value problem reads

$$\partial_t \rho + \nabla \rho \cdot u - \Delta \mu = 0 \quad \text{in } Q, \tag{1.7}$$
$$\tau_\Omega \partial_t \rho - \Delta \rho + \beta(\rho) + \pi(\rho) \ni \mu \quad \text{in } Q, \tag{1.8}$$
$$\rho_\Gamma = \rho_{|\Sigma}, \quad \mu_\Gamma = \mu_{|\Sigma} \quad \text{and} \quad \partial_t \rho_\Gamma + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma, \tag{1.9}$$
$$\tau_\Gamma \partial_t \rho_\Gamma + \partial_\nu \rho - \Delta_\Gamma \rho_\Gamma + \beta_\Gamma(\rho_\Gamma) + \pi_\Gamma(\rho_\Gamma) \ni \mu_\Gamma \quad \text{on } \Sigma, \tag{1.10}$$
$$\rho(0) = \rho_0 \quad \text{in } \Omega \quad \text{and} \quad \rho_\Gamma(0) = \rho_{0|\Gamma} \quad \text{on } \Gamma. \tag{1.11}$$

Up to our knowledge, in the case of a pure Cahn–Hilliard system, that is, with $\tau_\Omega = \tau_\Gamma = 0$, and without convective term ($u = 0$), the problem (1.7)–(1.11) has been firstly formulated by Goldstein, Miranville and Schimperna [27] and analyzed from various viewpoints in other contributions (see [7–9, 28]); moreover, in the case of general potentials, the problem has been deeply investigated in [15] from the point of view of existence, uniqueness and regularity of the weak solution (see also [24] for an optimal control problem) by using an abstract approach. Here, instead, we face with *the full system* (1.7)–(1.11) by a complete approximation procedure, which involves not only a regularization of graphs but the setting of a precise Faedo–Galerkin scheme. Moreover, in the viscous case with both $\tau_\Omega$ and $\tau_\Gamma$ positive, we can prove the uniform boundedness of both the chemical potential and the order parameter, up to the boundary, and we are even able to show the strict separation property in the case of logarithmic potentials like $f_{log}$ in (1.3). In addition to this, we did our best to try to keep minimal assumptions on the velocity field $u$, concerning summability and time derivation (see the later (2.21) and (2.47)). So, we think that our contribution could be a useful tool for studying other problems, which possibly involve other equations with coupled terms, and in particular for investigating optimal control problems.

Let us now review some related literature. It turns out that some class of Cahn–Hilliard system, possibly including dynamic boundary conditions, has collected a noteworthy interest in recent years: we can quote [10, 32, 34, 37, 38, 43] among other contributions. In case of no convective term in (1.7), and assuming the homogeneous boundary condition $\partial_\nu \mu = 0$ (i.e., the first condition in (1.5)) and the condition (1.10) with $\tau_\Gamma > 0$ and $\mu_\Gamma$ as a given datum, the problem has been first addressed in [25]: the well-posedness and the large time behavior of solutions have been studied for regular potentials $f$ and $f_\Gamma$, as well as for various singular potentials like the ones in (1.3) and (1.4). One can see [25, 26]: in these two papers the authors were able to overcome the difficulties due to singularities using a set of assumptions for $\beta$, $\pi$ and $\beta_\Gamma$, $\pi_\Gamma$ that gives the role of the dominating potential to $f$ and entails some technical difficulties. The subsequent papers [17–19] follow a different approach (firstly considered in [6] to investigate the Allen–Cahn equation with dynamic boundary conditions), which consists in letting $f_\Gamma$ be the leading potential with respect to $f$: by this the analysis turns out to be simpler. The paper [17] contains many results about existence, uniqueness and regularity of solutions for general potentials that include (1.2)–(1.3), and are valid for both the viscous and pure cases, i.e., by assuming just $\tau_\Omega \geq 0$. Moreover, the optimal boundary control problems for the viscous and pure Cahn–Hilliard equation are discussed in [19] and [18], respectively, in analogy with the corresponding contributions [13, 20] for the Allen–Cahn equation. The paper [14] deals with the well-posedness of the same system, but in which also an additional mass constraint on the boundary is imposed. In addition, we aim to emphasize that Cahn–Hilliard systems have been rather investigated from the viewpoint of optimal control. In this connection, we point out the contributions [44, 45] dealing with the convective Cahn–Hilliard equation; the case with a nonlocal potential is studied in [39]. We also refer to [11, 30, 42, 46] and quote the paper [12] investigating the second-order optimality conditions for the state system considered in [19]. There also exist articles addressing some discretized versions of general Cahn–Hilliard systems, cf. [31, 41].

The present paper is organized as follows. In the next two sections, we list our assumptions and notations, state our results and give the relations between weak solutions and the above boundary value problem. Sections 4 is devoted to continuous dependence and uniqueness, while the existence of a solution is shown in Section 6 by taking the limit of suitable approximating problems studied in Section 5. Finally, Section 7 is devoted to our regularity results.

## 2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. First of all, the set $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, connected and smooth. As in the Introduction, $\nu$ is the outward unit normal vector field on $\Gamma := \partial\Omega$, and $\partial_\nu$ and $\Delta_\Gamma$ stand for the corresponding normal derivative and the Laplace-Beltrami operator, respectively. Furthermore, we denote by $\nabla_\Gamma$ the surface gradient and write $|\Omega|$ and $|\Gamma|$ for the volume of $\Omega$ and the area of $\Gamma$, respectively.

If $X$ is a Banach space, $\|\cdot\|_X$ denotes both its norm and the norm of $X^3$. Moreover, $X^*$ is the dual space of $X$, and $\langle\,\cdot\,,\,\cdot\,\rangle_X$ is the dual pairing between $X^*$ and $X$. The only exception from the convention for the norms is given by the spaces $L^p$ constructed on $\Omega$, $\Gamma$, $Q$, and $\Sigma$, for $1 \leq p \leq \infty$, whose norms are denoted by $\|\cdot\|_p$. Furthermore, we put

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega) \quad \text{and} \quad W := H^2(\Omega), \tag{2.1}$$

$$H_\Gamma := L^2(\Gamma)\,, \quad V_\Gamma := H^1(\Gamma) \quad \text{and} \quad W_\Gamma := H^2(\Gamma), \tag{2.2}$$

$$\mathcal{H} := H \times H_\Gamma\,, \quad \mathcal{V} := \{(v, v_\Gamma) \in V \times V_\Gamma : \ v_\Gamma = v_{|\Gamma}\} \quad \text{and} \quad \mathcal{W} := \bigl(W \times W_\Gamma\bigr) \cap \mathcal{V}\,. \tag{2.3}$$

In the sequel, we work in the framework of the Hilbert triplet $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$. Thus, we have $\langle (g, g_\Gamma), (v, v_\Gamma)\rangle_{\mathcal{V}} = \int_\Omega gv + \int_\Gamma g_\Gamma v_\Gamma$ for every $(g, g_\Gamma) \in \mathcal{H}$ and $(v, v_\Gamma) \in \mathcal{V}$. Next, we introduce the generalized mean value, the related spaces and the operator $\mathcal{N}$ we widely use throughout the paper. The former is defined by

$$\operatorname{mean} g^* := \frac{\langle g^*, (1,1)\rangle_{\mathcal{V}}}{|\Omega| + |\Gamma|} \quad \text{for } g^* \in \mathcal{V}^* \tag{2.4}$$

and reduces to

$$\operatorname{mean} g^* = \frac{\int_\Omega v + \int_\Gamma v_\Gamma}{|\Omega| + |\Gamma|} \quad \text{if } g^* = (v, v_\Gamma) \in \mathcal{H}\,. \tag{2.5}$$

Of course, the components of the pair $(1,1)$ in (2.4) are the constant functions 1 on $\Omega$ and $\Gamma$, respectively. We stress that the function

$$\mathcal{V} \ni (v, v_\Gamma) \mapsto \|\nabla v\|_H^2 + \|\nabla_\Gamma v_\Gamma\|_{H_\Gamma}^2 + |\operatorname{mean}(v, v_\Gamma)|^2$$

yields the square of a Hilbert norm on $\mathcal{V}$ that is equivalent to the natural one. In particular, we have, for every $(v, v_\Gamma) \in \mathcal{V}$,

$$\|(v, v_\Gamma)\|_{\mathcal{V}} \le C_\Omega \bigl( \|\nabla v\|_H + \|\nabla_\Gamma v_\Gamma\|_{H_\Gamma} + |\operatorname{mean}(v, v_\Gamma)| \bigr), \tag{2.6}$$

where $C_\Omega$ depends only on $\Omega$. Now, we set

$$\mathcal{V}_{*0} := \{ g^* \in \mathcal{V}^* : \ \operatorname{mean} g^* = 0 \}, \quad \mathcal{H}_0 := \mathcal{H} \cap \mathcal{V}_{*0} \quad \text{and} \quad \mathcal{V}_0 := \mathcal{V} \cap \mathcal{V}_{*0}. \tag{2.7}$$

Notice the difference between $\mathcal{V}_{*0}$ and the dual space $\mathcal{V}_0^* = (\mathcal{V}_0)^*$. At this point, it is clear that the function

$$\mathcal{V}_0 \ni (v, v_\Gamma) \mapsto \|(v, v_\Gamma)\|_{\mathcal{V}_0} := \bigl( \|\nabla v\|_H^2 + \|\nabla_\Gamma v_\Gamma\|_{H_\Gamma}^2 \bigr)^{1/2} \tag{2.8}$$

is a Hilbert norm on $\mathcal{V}_0$ which is equivalent to the usual one. This has the following consequence: for every $g^* \in \mathcal{V}_{*0}$, there exists a unique pair $(\xi, \xi_\Gamma) \in \mathcal{V}_0$ such that

$$\int_\Omega \nabla \xi \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma = \langle g^*, (v, v_\Gamma)\rangle_{\mathcal{V}} \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{2.9}$$

Indeed, the right-hand side of (2.9), restricted to the pairs $(v, v_\Gamma) \in \mathcal{V}_0$, defines a continuous linear functional on $\mathcal{V}_0$ with respect to its natural norm ($\mathcal{V}_0$ is a subspace of $\mathcal{V} \subset V \times V_\Gamma$), and thus also with respect to the norm (2.8). Therefore, by the Riesz representation theorem, there exists a unique pair $(\xi, \xi_\Gamma) \in \mathcal{V}_0$ such that

$$\int_\Omega \nabla \xi \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma = \langle g^*, (v, v_\Gamma)\rangle_{\mathcal{V}} \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}_0.$$

On the other hand, the same relation holds true for $(v, v_\Gamma) = (1,1)$, since $\operatorname{mean} g^* = 0$. As $\mathcal{V} = \mathcal{V}_0 \oplus \operatorname{span}\{(1,1)\}$, we obtain (2.9). This allows us to define $\mathcal{N} : \mathcal{V}_{*0} \to \mathcal{V}_0$ by setting:

$$\text{for } g^* \in \mathcal{V}_{*0}, \ \ \mathcal{N} g^* \text{ is the unique pair } (\xi, \xi_\Gamma) \in \mathcal{V}_0 \text{ satisfying (2.9).} \tag{2.10}$$

We notice that $\mathcal{N}$ is linear, symmetric, and bijective. Therefore, if we set

$$\|g^*\|_* := \|\mathcal{N} g^*\|_{\mathcal{V}_0}, \quad \text{for } g^* \in \mathcal{V}_{*0}, \tag{2.11}$$

then we obtain a Hilbert norm on $\mathcal{V}_{*0}$, which turns out to be equivalent to the norm induced by the norm of $\mathcal{V}^*$. For a future use, we collect some properties of $\mathcal{N}$. By just applying the definition, we have that

$$\langle g^*, \mathcal{N}g^* \rangle_{\mathcal{V}} = \|g^*\|_*^2 \qquad \text{if } g^* \in \mathcal{V}_{*0}, \tag{2.12}$$

$$\int_\Omega \nabla w \cdot \nabla \xi + \int_\Gamma \nabla_\Gamma w_\Gamma \cdot \nabla_\Gamma \xi_\Gamma = \|(w, w_\Gamma)\|_{\mathcal{H}}^2$$
$$\text{if } (w, w_\Gamma) \in \mathcal{V}_0 \text{ and } (\xi, \xi_\Gamma) = \mathcal{N}(w, w_\Gamma)\,. \tag{2.13}$$

By accounting for the symmetry of $\mathcal{N}$, we also have (where, here and in the sequel, $\mathcal{N}$ is applied to $\mathcal{V}_{*0}$-valued functions as well)

$$\langle \partial_t g^*, \mathcal{N}g^* \rangle_{\mathcal{V}} = \frac{1}{2}\,\frac{d}{dt}\,\|g^*\|_*^2 \quad \text{if } g^* \in H^1(0,T;\mathcal{V}_{*0}), \tag{2.14}$$

$$\int_\Omega \nabla w \cdot \nabla \xi + \int_\Gamma \nabla_\Gamma w_\Gamma \cdot \nabla_\Gamma \xi_\Gamma = \frac{1}{2}\,\frac{d}{dt}\,\|(w, w_\Gamma)\|_{\mathcal{H}}^2$$
$$\text{if } (w, w_\Gamma) \in L^2(0,T;\mathcal{V}),\ \ \partial_t(w, w_\Gamma) \in L^2(0,T;\mathcal{V}_{*0}),\ \ (\xi, \xi_\Gamma) = \mathcal{N}(\partial_t(w, w_\Gamma))\,. \tag{2.15}$$

Now, we list our assumptions. For the structure of our system, we postulate:

$$\tau_\Omega \text{ and } \tau_\Gamma \ \text{ are nonnegative real numbers}\,; \tag{2.16}$$

$$\widehat{\beta},\, \widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty] \quad \text{are convex, proper and l.s.c. with} \quad \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0\,; \tag{2.17}$$

$$\widehat{\pi},\, \widehat{\pi}_\Gamma : \mathbb{R} \to \mathbb{R} \quad \text{are of class } C^2 \text{ with Lipschitz continuous first derivatives.} \tag{2.18}$$

We set, for convenience,

$$\beta := \partial\widehat{\beta}\,, \quad \beta_\Gamma := \partial\widehat{\beta}_\Gamma\,, \quad \pi := \widehat{\pi}' \quad \text{and} \quad \pi_\Gamma := \widehat{\pi}'_\Gamma, \tag{2.19}$$

and assume that, with some positive constants $C$ and $\eta$,

$$D(\beta_\Gamma) \subseteq D(\beta) \quad \text{and} \quad |\beta^\circ(r)| \le \eta|\beta^\circ_\Gamma(r)| + C \quad \text{for every } r \in D(\beta_\Gamma). \tag{2.20}$$

In (2.20), the symbols $D(\beta)$ and $D(\beta_\Gamma)$ denote the domains of $\beta$ and $\beta_\Gamma$, respectively. More generally, we use the notation $D(\mathcal{G})$ for every maximal monotone graph $\mathcal{G}$ in $\mathbb{R} \times \mathbb{R}$, as well as for the maximal monotone operators induced on $L^2$ spaces. Moreover, for $r \in D(\mathcal{G})$, $\mathcal{G}^\circ(r)$ stands for the element of $\mathcal{G}(r)$ having minimum modulus.

For the data, we make the following assumptions:

$$u \in L^2(0,T;L^3(\Omega))^3, \quad \operatorname{div} u = 0 \quad \text{in } Q \quad \text{and} \quad u \cdot \nu = 0 \quad \text{on } \Sigma\,; \tag{2.21}$$

$$(\rho_0\,,\, \rho_{0|\Gamma}) \in \mathcal{V}\,, \quad \widehat{\beta}(\rho_0) \in L^1(\Omega) \quad \text{and} \quad \widehat{\beta}_\Gamma(\rho_{0|\Gamma}) \in L^1(\Gamma)\,; \tag{2.22}$$

$$m_0 := \operatorname{mean}(\rho_0, \rho_{0|\Gamma}) \in \operatorname{int} D(\beta_\Gamma). \tag{2.23}$$

Let us come to our notion of solution. It is a triple of pairs, $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\zeta, \zeta_\Gamma))$, that satisfies a rather low level of regularity, in principle. Indeed, we just require that

$$(\mu, \mu_\Gamma) \in L^2(0,T;\mathcal{V}), \tag{2.24}$$

$$(\rho, \rho_\Gamma) \in H^1(0,T;\mathcal{V}^*) \cap L^\infty(0,T;\mathcal{V}), \tag{2.25}$$

$$(\zeta, \zeta_\Gamma) \in L^2(0,T;\mathcal{H}), \tag{2.26}$$

$$\tau_\Omega \partial_t \rho \in L^2(0,T;H) \quad \text{and} \quad \tau_\Gamma \partial_t \rho_\Gamma \in L^2(0,T;H_\Gamma)\,. \tag{2.27}$$

We have written, e.g., $\tau_\Omega \partial_t \rho$ in (2.27) instead of $\partial_t(\tau_\Omega \rho)$. We similarly behave throughout the paper, in particular in the forthcoming (2.29), in order to avoid a heavy notation. The problem to be solved is stated in a weak form, owing to the assumptions (2.21) on $u$. Namely, we require that

$$\langle \partial_t(\rho,\rho_\Gamma),(v,v_\Gamma)\rangle_{\mathcal{V}} - \int_\Omega \rho u\cdot\nabla v + \int_\Omega \nabla\mu\cdot\nabla v + \int_\Gamma \nabla_\Gamma\mu_\Gamma\cdot\nabla_\Gamma v_\Gamma = 0$$
$$\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V}, \tag{2.28}$$

$$\tau_\Omega \int_\Omega \partial_t\rho\, v + \tau_\Gamma \int_\Gamma \partial_t\rho_\Gamma\, v_\Gamma + \int_\Omega \nabla\rho\cdot\nabla v + \int_\Gamma \nabla_\Gamma\rho_\Gamma\cdot\nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega \big(\zeta+\pi(\rho)\big)v + \int_\Gamma \big(\zeta_\Gamma+\pi_\Gamma(\rho_\Gamma)\big)v_\Gamma = \int_\Omega \mu v + \int_\Gamma \mu_\Gamma v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V}, \tag{2.29}$$

$$\zeta\in\beta(\rho) \quad \text{a.e. in } Q \quad \text{and} \quad \zeta_\Gamma\in\beta_\Gamma(\rho_\Gamma) \quad \text{a.e. on } \Sigma, \tag{2.30}$$

$$\rho(0)=\rho_0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \rho_\Gamma(0)=\rho_{0|\Gamma} \quad \text{a.e. on } \Gamma. \tag{2.31}$$

We observe that any weak solution to problem (2.28)–(2.31) satisfies

$$\partial_t \operatorname{mean}(\rho,\rho_\Gamma)=0, \quad \text{whence} \quad \operatorname{mean}(\rho,\rho_\Gamma)(t)=m_0 \quad \text{for every } t\in[0,T]. \tag{2.32}$$

Indeed, it suffices to take $(v,v_\Gamma)=(|\Omega|+|\Gamma|)^{-1}(1,1)$ in (2.28).

However, one can wonder whether the solution enjoys the better regularity

$$\partial_t(\rho,\rho_\Gamma)=(\partial_t\rho,\partial_t\rho_\Gamma)\in L^2(0,T;\mathcal{H}) \quad \text{and} \quad (\mu,\mu_\Gamma)\in L^2(0,T;\mathcal{W}), \tag{2.33}$$

$$(\rho,\rho_\Gamma)\in L^2(0,T;\mathcal{W}), \tag{2.34}$$

and actually satisfies the boundary value problems presented in the Introduction, i.e.,

$$\partial_t\rho + \nabla\rho\cdot u - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{2.35}$$

$$\partial_t\rho_\Gamma + \partial_\nu\mu - \Delta_\Gamma\mu_\Gamma = 0 \quad \text{a.e. on } \Sigma, \tag{2.36}$$

$$\tau_\Omega\partial_t\rho - \Delta\rho + \zeta + \pi(\rho) = \mu \quad \text{a.e. in } Q, \tag{2.37}$$

$$\tau_\Gamma\partial_t\rho_\Gamma + \partial_\nu\rho - \Delta_\Gamma\rho_\Gamma + \zeta_\Gamma + \pi_\Gamma(\rho_\Gamma) = \mu_\Gamma \quad \text{a.e. on } \Sigma. \tag{2.38}$$

This is not obvious. For instance, it is not clear whether the derivative $\partial_t(\rho,\rho_\Gamma)$ can be replaced by $(\partial_t\rho,\partial_t\rho_\Gamma)$, since the components of the test functions $(v,v_\Gamma)\in\mathcal{V}$ used in (2.28) are not independent. In the first result we present, we answer the above questions. However, for future use, it is convenient to prepare a more general tool.

**Theorem 2.1.** *Assume* (2.16)–(2.20) *for the structure,* (2.21) *for the velocity field and*

$$((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))\in L^2(0,T;\mathcal{V}\times\mathcal{V}\times\mathcal{H}) \quad \textit{with} \quad (\tau_\Omega\partial_t\rho,\tau_\Gamma\partial_t\rho_\Gamma)\in L^2(0,T;\mathcal{H})\,.$$

*Then, we have the following statements:*
*i) if* $\rho\in L^2(0,T;W)$, $\partial_t(\rho,\rho_\Gamma)\in L^2(0,T;\mathcal{H})$ *and* (2.28) *is fulfilled, then*

$$(\mu,\mu_\Gamma)\in L^1(0,T;\mathcal{W}) \quad \textit{with}$$
$$\|(\mu,\mu_\Gamma)\|_{L^1(0,T;\mathcal{W})} \le C_1\big(\|(\mu,\mu_\Gamma)\|_{L^2(0,T;\mathcal{V})} + \|\partial_t(\rho,\rho_\Gamma)\|_{L^2(0,T;\mathcal{H})}$$
$$+ \|\rho\|_{L^2(0,T;W)}\|u\|_{L^2(0,T;H)}\big), \tag{2.39}$$

*where $C_1$ depends only on $\Omega$, and* (2.35)–(2.36) *hold true as well;*
*ii) if* (2.29) *is satisfied, then*

$$\begin{aligned}&(\rho,\rho_\Gamma)\in L^2(0,T;\mathcal{W}) \quad \textit{with}\\ &\|(\rho,\rho_\Gamma)\|_{L^2(0,T;\mathcal{W})}\le C_2\bigl(\|(\rho,\rho_\Gamma)\|_{L^2(0,T;\mathcal{V})}\\ &\qquad\qquad + \|((\mu,\mu_\Gamma),(\zeta,\zeta_\Gamma),(\tau_\Omega\partial_t\rho,\tau_\Gamma\partial_t\rho_\Gamma))\|_{L^2(0,T;\mathcal{H}\times\mathcal{H}\times\mathcal{H})}\bigr), \end{aligned} \tag{2.40}$$

*where $C_2$ depends only on $\Omega$, and* (2.37)–(2.38) *hold as well;*
*iii) if $\gamma:\mathbb{R}\to\mathbb{R}$ is monotone and Lipschitz continuous, and if* (2.29) *holds true with $\zeta_\Gamma\in\gamma(\rho_\Gamma)$ a.e. on $\Sigma$, then*

$$\begin{aligned}\|\zeta_\Gamma\|_{L^2(0,T;H_\Gamma)}&\le C_3\bigl(\|(\rho,\rho_\Gamma)\|_{L^2(0,T;\mathcal{V})}\\ &\qquad + \|((\mu,\mu_\Gamma),\zeta,(\tau_\Omega\partial_t\rho,\tau_\Gamma\partial_t\rho_\Gamma))\|_{L^2(0,T;\mathcal{H}\times H\times\mathcal{H})}\bigr), \end{aligned} \tag{2.41}$$

*where $C_3$ depends only on $\Omega$.*

*Assume, in addition, that $u$ belongs to $L^\infty(0,T;L^3(\Omega))$ and that*

$$((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))\in L^\infty(0,T;\mathcal{V}\times\mathcal{V}\times\mathcal{H}) \quad \textit{and} \quad (\tau_\Omega\partial_t\rho,\tau_\Gamma\partial_t\rho_\Gamma)\in L^\infty(0,T;\mathcal{H})\,.$$

*Then, we have the following statements:*
*iv) if $\rho\in L^\infty(0,T;W)$, $\partial_t(\rho,\rho_\Gamma)\in L^\infty(0,T;\mathcal{H})$ and* (2.28) *is fulfilled, then*

$$\begin{aligned}&(\mu,\mu_\Gamma)\in L^\infty(0,T;\mathcal{W}) \quad \textit{with}\\ &\|(\mu,\mu_\Gamma)\|_{L^\infty(0,T;\mathcal{W})}\le C_4\bigl(\|(\mu,\mu_\Gamma)\|_{L^\infty(0,T;\mathcal{V})}+\|\partial_t(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{H})}\\ &\qquad\qquad + \|\rho\|_{L^\infty(0,T;W)}\|u\|_{L^\infty(0,T;H)}\bigr), \end{aligned} \tag{2.42}$$

*where $C_4$ depends only on $\Omega$;*
*v) if* (2.29) *is satisfied, then*

$$\begin{aligned}&(\rho,\rho_\Gamma)\in L^\infty(0,T;\mathcal{W}) \quad \textit{with}\\ &\|(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{W})}\le C_5\bigl(\|(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{V})}\\ &\qquad\qquad + \|((\mu,\mu_\Gamma),(\zeta,\zeta_\Gamma),(\tau_\Omega\partial_t\rho,\tau_\Gamma\partial_t\rho_\Gamma))\|_{L^\infty(0,T;\mathcal{H}\times\mathcal{H}\times\mathcal{H})}\bigr), \end{aligned} \tag{2.43}$$

*where $C_5$ depends only on $\Omega$;*
*vi) if $\gamma:\mathbb{R}\to\mathbb{R}$ is monotone and Lipschitz continuous, and if* (2.29) *holds true with $\zeta_\Gamma\in\gamma(\rho_\Gamma)$ a.e. on $\Sigma$, then*

$$\begin{aligned}\|\zeta_\Gamma\|_{L^\infty(0,T;H_\Gamma)}&\le C_6\bigl(\|(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{V})}\\ &\qquad + \|((\mu,\mu_\Gamma),\zeta,(\tau_\Omega\partial_t\rho,\tau_\Gamma\partial_t\rho_\Gamma))\|_{L^\infty(0,T;\mathcal{H}\times H\times\mathcal{H})}\bigr), \end{aligned} \tag{2.44}$$

*where $C_6$ depends only on $\Omega$.*
*As a particular case of i) and ii), every solution to problem* (2.28)–(2.31) *satisfying* (2.24)–(2.27) *also fulfills* (2.34) *and* (2.37)–(2.38)*, and, if $\tau_\Omega$ and $\tau_\Gamma$ are strictly positive,* (2.33) *and* (2.35)–(2.36) *hold true as well.*

**Remark 2.2.** We stress that all of the constants appearing in the estimates (2.39)–(2.44) depend only on $\Omega$. In particular, the constants $C_3$ and $C_6$ do not depend on $\gamma$.

Our next results regard the well-posedness and the continuous dependence of the solution on the velocity field. They are as follows:

**Theorem 2.3.** *Assume* (2.16)–(2.20) *for the structure and* (2.21)–(2.23) *for the data. Then, problem* (2.28)–(2.31) *has a at least one solution* $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ *satisfying the regularity properties* (2.24)–(2.27), (2.34) *and the inequality*

$$\begin{aligned}&\|(\mu,\mu_\Gamma)\|_{L^2(0,T;\mathcal{V})} + \|(\rho,\rho_\Gamma)\|_{H^1(0,T;\mathcal{V}^*)\cap L^\infty(0,T;\mathcal{V})\cap L^2(0,T;\mathcal{W})}\\ &\quad + \|(\zeta,\zeta_\Gamma)\|_{L^2(0,T;\mathcal{H})} + \tau_\Omega^{1/2}\|\partial_t\rho\|_{L^2(0,T;H)} + \tau_\Gamma^{1/2}\|\partial_t\rho_\Gamma\|_{L^2(0,T;H_\Gamma)} \le K_1, \qquad (2.45)\end{aligned}$$

*for some constant* $K_1$ *that depends only on the structure of the system,* $\Omega$*,* $T$*, the initial data, and the norm of* $u$ *in* $L^2(0,T;L^3(\Omega))^3$*. Furthermore, the components* $\rho$ *and* $\rho_\Gamma$ *of any solution are uniquely determined, and the whole solution is unique if at least one of the operators* $\beta$ *and* $\beta_\Gamma$ *is single-valued.*

**Remark 2.4.** By combining the statements of Theorems 2.1 and 2.3, it is clear that estimates also hold for the norms of $(\mu,\mu_\Gamma)$ and $(\rho,\rho_\Gamma)$ in $L^2(0,T;\mathcal{W})$ with a constant $K_1'$ similar to $K_1$.

**Theorem 2.5.** *Under the assumptions* (2.16)–(2.20) *on the structure and* (2.21)–(2.23) *on the data, let* $u_i$*,* $i=1,2$*, be two choices of* $u$*, and let* $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ *be the difference of two corresponding solutions. Then the inequality*

$$\begin{aligned}&\|(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{V}_0^*)\cap L^2(0,T;\mathcal{V})} + \tau_\Omega^{1/2}\|\partial_t\rho\|_{L^\infty(0,T;H)} + \tau_\Gamma^{1/2}\|\partial_t\rho_\Gamma\|_{L^\infty(0,T;H_\Gamma)}\\ &\le K_2\|u_1-u_2\|_{L^2(0,T;L^3(\Omega))} \qquad (2.46)\end{aligned}$$

*holds true for some constant* $K_2$ *that depends only on the structure of the system,* $\Omega$*,* $T$*, the initial data, and the norms of* $u_i$*,* $i=1,2$*, in* $L^2(0,T;L^3(\Omega))^3$*.*

Under additional assumptions on the initial data and on the velocity $u$, we can ensure further regularity for the solution. Namely, we have the following result:

**Theorem 2.6.** *In addition to the assumptions* (2.16)–(2.20) *for the structure and* (2.21)–(2.23) *for the data, suppose that* $\tau_\Omega$ *and* $\tau_\Gamma$ *are strictly positive and that*

$$u \in H^1(0,T;L^{3/2}(\Omega)) \cap L^\infty(0,T;L^3(\Omega)), \qquad (2.47)$$

$$\rho_0 \in H^2(\Omega)\,, \quad \rho_{0|\Gamma} \in H^2(\Gamma)\,, \quad \beta^\circ(\rho_0) \in L^2(\Omega) \quad \text{and} \quad \beta_\Gamma^\circ(\rho_{0|\Gamma}) \in L^2(\Gamma)\,. \qquad (2.48)$$

*Then, problem* (2.28)–(2.31) *has a at least one solution* $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ *that also satisfies*

$$\begin{aligned}&(\mu,\mu_\Gamma) \in L^\infty(0,T;\mathcal{W})\,, \quad (\rho,\rho_\Gamma) \in W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W})\\ &\quad \text{and} \quad (\zeta,\zeta_\Gamma) \in L^\infty(0,T;\mathcal{H}), \qquad (2.49)\\ &\|(\mu,\mu_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} + \|(\rho,\rho_\Gamma)\|_{W^{1,\infty}(0,T;\mathcal{H})\cap H^1(0,T;\mathcal{V})\cap L^\infty(0,T;\mathcal{W})}\\ &\quad + \|(\zeta,\zeta_\Gamma)\|_{L^\infty(0,T;\mathcal{H})} \le K_3, \qquad (2.50)\end{aligned}$$

*with a constant* $K_3$ *that depends only on the structure of the system,* $\Omega$*,* $T$*, the initial data, and the norm of* $u$ *in* $H^1(0,T;L^{3/2}(\Omega)) \cap L^\infty(0,T;L^3(\Omega))$*. In particular, the components* $(\mu,\mu_\Gamma)$ *and* $(\rho,\rho_\Gamma)$ *are bounded.*

**Remark 2.7.** As $\Omega \subset \mathbb{R}^3$ and $\mathcal{W} \subset C^0(\overline{\Omega}) \times C^0(\overline{\Gamma})$ due to the Sobolev inequalities, from standard embedding results (cf., e.g., [40, Sect. 8, Cor. 4]) and (2.49) it follows that even $\rho \in C^0(\overline{Q})$ and $\rho_\Gamma \in C^0(\overline{\Sigma})$. Moreover, a part of the result of Theorem 2.6 still holds true without assuming that $\tau_\Omega$ and $\tau_\Gamma$ are strictly positive, provided that the initial data satisfy the additional condition

$$\begin{aligned}&\bigl(-\Delta\rho_0 + (\beta_\varepsilon + \pi)(\rho_0), -\Delta_\Gamma \rho_{0|\Gamma} + \partial_\nu \rho_0 + (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(\rho_{0|\Gamma})\bigr) \\ &\quad \text{belongs to a bounded subset of } \mathcal{V} \text{ for every } \varepsilon \in (0,1). \end{aligned} \tag{2.51}$$

With respect to the previous statement, we miss the conditions $\partial_t(\rho, \rho_\Gamma) \in L^\infty(0,T;\mathcal{H})$ and $(\mu, \mu_\Gamma) \in L^\infty(0,T;\mathcal{W})$ (see the forthcoming Remark 7.1 for details). If the double-well potentials in the bulk and on the boundary are the same potential of logarithmic type as in the next (2.52)–(2.53), then it is easy to find sufficient conditions on $\rho_0$ for (2.51) to hold. Indeed, one can assume that $\|\rho_0\|_\infty < 1$ and $(\Delta\rho_0, \Delta_\Gamma\rho_{0|\Gamma} - \partial_\nu\rho_0) \in \mathcal{V}$.

Our last result requires potentials of logarithmic type (see (1.3)) with the same domain. Namely, we require that

$$\beta,\, \beta_\Gamma : (-1,1) \to \mathbb{R} \quad \text{are } C^2 \text{ functions with} \tag{2.52}$$

$$\lim_{r\searrow -1} \beta(r) = \lim_{r\searrow -1} \beta_\Gamma(r) = -\infty \quad \text{and} \quad \lim_{r\nearrow 1} \beta(r) = \lim_{r\nearrow 1} \beta_\Gamma(r) = +\infty \,. \tag{2.53}$$

**Theorem 2.8.** *In addition to the assumptions* (2.16)–(2.20) *on the structure, assume that* $\tau_\Omega$ *and* $\tau_\Gamma$ *are strictly positive and that* $\beta$ *and* $\beta_\Gamma$ *satisfy* (2.52)–(2.53)*. Moreover, assume that* $u$ *and* $\rho_0$ *satisfy* (2.21), (2.47) *and*

$$\rho_0 \in W, \qquad \rho_{0|\Gamma} \in W_\Gamma, \qquad \inf \rho_0 > -1 \quad \textit{and} \quad \sup \rho_0 < 1 \,. \tag{2.54}$$

*Then the unique solution* $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ *satisfies*

$$\rho_* \le \rho(x,t) \le \rho^* \quad \textit{for all } (x,t) \in \overline{Q}, \tag{2.55}$$

*for some constants* $\rho_*, \rho^* \in (-1,1)$ *that depend only on the structure of the system,* $\Omega$*,* $T$*, the initial data, and the norm of* $u$ *in* $H^1(0,T;L^{3/2}(\Omega)) \cap L^\infty(0,T;L^3(\Omega))$*.*

**Theorem 2.9.** *In addition to* (2.16)–(2.20)*, assume that* $\tau_\Omega$ *and* $\tau_\Gamma$ *are strictly positive, that* $\beta$ *and* $\beta_\Gamma$ *satisfy* (2.52)–(2.53)*, and that* $\beta$*,* $\pi$*,* $\beta_\Gamma$ *and* $\pi_\Gamma$ *are of class* $C^2$*. Moreover, assume that* $\rho_0$ *satisfies* (2.54)*. Finally, let* $u_i \in H^1(0,T;L^3(\Omega))$*,* $i=1,2$*, be two choices of* $u$ *satisfying* (2.21)*, and let* $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ *be the difference of the corresponding solutions. Then the inequality*

$$\begin{aligned}&\|(\mu,\mu_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} + \|(\rho,\rho_\Gamma)\|_{W^{1,\infty}(0,T;\mathcal{H})\cap H^1(0,T;\mathcal{V})\cap L^\infty(0,T;\mathcal{W})} \\ &\le K_4 \|u_1 - u_2\|_{H^1(0,T;L^3(\Omega))}\end{aligned} \tag{2.56}$$

*holds true for some constant* $K_4$ *that depends only on the structure of the system,* $\Omega$*,* $T$*, the initial data, and the norms of* $u_i$*,* $i=1,2$*, in* $H^1(0,T;L^3(\Omega))$*.*

Throughout the paper, we will repeatedly use Young's inequality

$$a\,b \le \delta\, a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for all } a,b \in \mathbb{R} \text{ and } \delta > 0, \tag{2.57}$$

Hölder's inequality, and the Sobolev inequality related to the continuous embedding $V \subset L^p(\Omega)$ with $p \in [1,6]$ (since $\Omega$ is three-dimensional, bounded and smooth). Besides, this embedding is compact for $p < 6$, and the same holds for the analogous spaces on the boundary. It follows that the embeddings $\mathcal{V} \subset \mathcal{H}$ and $\mathcal{H} \subset \mathcal{V}^*$ are compact as well. In particular, we have the compactness inequality

$$\begin{aligned} &\|(v, v_\Gamma)\|_{\mathcal{H}} \leq \delta\big(\|\nabla v\|_H + \|\nabla_\Gamma v_\Gamma\|_{H_\Gamma}\big) + C_\delta\, \|(v, v_\Gamma)\|_{\mathcal{V}^*} \\ &\quad \text{for every } (v, v_\Gamma) \in \mathcal{V} \text{ and } \delta > 0, \end{aligned} \tag{2.58}$$

where $C_\delta$ depends only on $\Omega$ and $\delta$. Finally, we set, for brevity,

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for } 0 < t \leq T, \tag{2.59}$$

and simply write $Q$ and $\Sigma$ if $t = T$.

We conclude this section by stating a general rule concerning the constants that appear in the estimates to be performed in the sequel. The small-case symbol $c$ stands for a generic constant whose values might change from line to line (and even within the same line) and depend only on $\Omega$, on the shape of the nonlinearities, and on the constants and the norms of the functions involved in the assumptions of our statements. In particular, the values of $c$ do not depend on $\varepsilon$ if this parameter is considered. A small-case symbol with a subscript like $c_\delta$ (in particular, with $\delta = \varepsilon$) indicates that the constant might depend on the parameter $\delta$, in addition. On the contrary, we mark precise constants that we can refer to by using different symbols, like in (2.20) and in (2.45).

# 3 Strong solutions

This section is devoted to the proof of Theorem 2.1. Our argument relies on a result on an elliptic problem. Thus, we prove the following lemma:

**Lemma 3.1.** *Let $\gamma : \mathbb{R} \to \mathbb{R}$ be monotone and Lipschitz continuous, and assume that $(w, w_\Gamma) \in \mathcal{V}$ and $(g, g_\Gamma) \in \mathcal{H}$ satisfy*

$$\int_\Omega \nabla w \cdot \nabla v + \int_\Gamma \nabla_\Gamma w_\Gamma \cdot \nabla_\Gamma v_\Gamma + \int_\Gamma \gamma(w_\Gamma) v_\Gamma = \int_\Omega g v + \int_\Gamma g_\Gamma v_\Gamma \quad \textit{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{3.1}$$

*Then we have that*

$$(w, w_\Gamma) \in \mathcal{W} \quad \textit{and} \quad \|(w, w_\Gamma)\|_{\mathcal{W}} + \|\gamma(w_\Gamma)\|_{H_\Gamma} \leq C_\Omega\big(\|(w, w_\Gamma)\|_{\mathcal{V}} + \|(g, g_\Gamma)\|_{\mathcal{H}}\big), \tag{3.2}$$

*where $C_\Omega$ depends only on $\Omega$. Moreover, $(w, w_\Gamma)$ solves the boundary value problem*

$$-\Delta w = g \quad \textit{a.e. in } \Omega, \quad \textit{and} \quad \partial_\nu w - \Delta_\Gamma w_\Gamma + \gamma(w_\Gamma) = g_\Gamma \quad \textit{a.e. on } \Gamma\,. \tag{3.3}$$

*Proof.* We use well-known estimates from the theory of traces and elliptic equations. The values of $c$ will depend only on $\Omega$. We set, for brevity, $M := \|(w, w_\Gamma)\|_{\mathcal{V}} + \|(g, g_\Gamma)\|_{\mathcal{H}}$. By taking any $v \in H^1_0(\Omega)$ and testing (3.1) by $(v, 0)$, we obtain the first identity in (3.3) in the sense of distributions. In particular, we have $\Delta w = -g \in H$. By combining this with $w_{|\Gamma} = w_\Gamma \in V_\Gamma$, we deduce that

$$w \in H^{3/2}(\Omega) \quad \text{and} \quad \|w\|_{H^{3/2}(\Omega)} \le c \left( \|\Delta w\|_H + \|w_\Gamma\|_{V_\Gamma} \right) \le c\, M.$$

It follows that

$$\partial_\nu w \in H_\Gamma \quad \text{and} \quad \|\partial_\nu w\|_{H_\Gamma} \le c \left( \|w\|_{H^{3/2}(\Omega)} + \|\Delta w\|_H \right) \le c\, M,$$

as well as the validity of the formula

$$\int_\Omega \nabla w \cdot \nabla v = - \int_\Omega \Delta w\, v + \int_\Gamma \partial_\nu w\, v_{|\Gamma} \quad \text{for every } v \in V.$$

By replacing $-\Delta w$ by $g$, comparing with (3.1), and noticing that for every $v_\Gamma \in V_\Gamma$ there exists some $v \in V$ such that $(v, v_\Gamma) \in \mathcal{V}$, we deduce that

$$\int_\Gamma \nabla_\Gamma w_\Gamma \cdot \nabla_\Gamma v_\Gamma + \int_\Gamma \gamma(w_\Gamma) v_\Gamma = \int_\Gamma (g_\Gamma - \partial_\nu w) v_\Gamma \quad \text{for every } v_\Gamma \in V_\Gamma\,. \tag{3.4}$$

In particular, by choosing $v_\Gamma = \gamma(w_\Gamma)$, we obtain that

$$\int_\Gamma \gamma'(w_\Gamma) |\nabla_\Gamma w_\Gamma|^2 + \int_\Gamma |\gamma(w_\Gamma)|^2 = \int_\Gamma (g_\Gamma - \partial_\nu w) \gamma(w_\Gamma),$$

whence immediately

$$\|\gamma(w_\Gamma)\|_{H_\Gamma} \le \|g_\Gamma - \partial_\nu w\|_{H_\Gamma} \le c\, M,$$

which is a part of (3.2). Then, we can rewrite (3.4) in the form

$$\int_\Gamma \nabla_\Gamma w_\Gamma \cdot \nabla_\Gamma v_\Gamma = \int_\Gamma (g_\Gamma - \partial_\nu w - \gamma(w_\Gamma)) v_\Gamma \quad \text{for every } v_\Gamma \in V_\Gamma\,.$$

This implies the second identity in (3.3) (at least in a generalized sense), as well as

$$\Delta_\Gamma w_\Gamma \in H_\Gamma \quad \text{and} \quad \|\Delta_\Gamma w_\Gamma\|_{H_\Gamma} \le \|g_\Gamma - \partial_\nu w - \gamma(w_\Gamma)\|_{H_\Gamma} \le c\, M\,.$$

Therefore, we also have that

$$w_\Gamma \in W_\Gamma \quad \text{and} \quad \|w_\Gamma\|_{W_\Gamma} \le c\big( \|w_\Gamma\|_{V_\Gamma} + \|\Delta_\Gamma w_\Gamma\|_{H_\Gamma} \big) \le c\, M\,.$$

We conclude that

$$w \in W \quad \text{and} \quad \|w\|_W \le c\big( \|\Delta w\|_H + \|w_\Gamma\|_{W_\Gamma} \big) \le c\, M\,.$$

Therefore, both the regularity and the estimate of (3.2) are completely proved, and the equations (3.3) hold almost everywhere. □

**Proof of Theorem 2.1.** In order to prove *i*) and *iv*), we account for (2.21), which implies that $-\int_\Omega \rho\, u \cdot \nabla v = \int_\Omega \nabla \rho \cdot u\, v$ a.e. in $(0,T)$ for every $v \in V$, and rewrite (2.28) a.e. in $(0,T)$ with this substitution. Then, for a.a. $t \in (0,T)$, we apply Lemma 3.1 with

$$\gamma = 0, \quad (w, w_\Gamma) = (\mu, \mu_\Gamma)(t), \quad g = -(\partial_t \rho + \nabla \rho \cdot u)(t) \quad \text{and} \quad g_\Gamma = -\partial_t \rho_\Gamma(t),$$

by observing that $\|\nabla\rho(t)\cdot u(t)\|_2 \le \|\nabla\rho(t)\|_6\|u(t)\|_3 \le c\|\rho(t)\|_W\|u(t)\|_3$, where $c$ depends only on $\Omega$. Then, we take the norms of both sides of (3.2) in $L^1(0,T)$ or in $L^\infty(0,T)$ to deduce (2.39) and (2.42), respectively, and notice that (3.3) coincides with (2.35)–(2.36). To prove *ii*) and *v*), we apply Lemma 3.1 for a.a. $t\in(0,T)$ with

$$\gamma = 0, \quad (w,w_\Gamma) = (\rho,\rho_\Gamma)(t), \quad g = \bigl(\mu - \tau_\Omega\partial_t\rho - \zeta - \pi(\rho)\bigr)(t) \\ \text{and} \quad g_\Gamma = \bigl(\mu_\Gamma - \tau_\Gamma\partial_t\rho_\Gamma - \zeta_\Gamma - \pi_\Gamma(\rho_\Gamma)\bigr)(t),$$

and argue as before. Finally, to prove *iii*) and *vi*), we apply Lemma 3.1 for a.a. $t\in(0,T)$ with $\gamma$ as in the statement, $(w,w_\Gamma)$ and $g$ as in the previous step, and

$$g_\Gamma = \bigl(\mu_\Gamma - \tau_\Gamma\partial_t\rho_\Gamma - \pi_\Gamma(\rho_\Gamma)\bigr)(t).$$

Then, we write the estimate for $\zeta_\Gamma$ of (3.1) and take the norms of both sides in $L^2(0,T)$ or in $L^\infty(0,T)$. □

# 4 Continuous dependence and uniqueness

In this section, we give the proof of Theorem 2.5 concerning continuous dependence on the velocity field $u$ and derive the uniqueness part of Theorem 2.3.

**Proof of Theorem 2.5.** We take two choices $u_i$, $i=1,2$, of $u$ and consider two corresponding solutions $((\mu_i,\mu_{i\Gamma}),(\rho_i,\rho_{i\Gamma}),(\zeta_i,\zeta_{i\Gamma}))$. We set $\rho := \rho_1-\rho_2$ and similarly define the other differences, according to the notation of the statement. We observe that $\operatorname{mean}(\rho,\rho_\Gamma)=0$ by the conservation property (2.32), applied to $(\rho_i,\rho_{i\Gamma})$ for $i=1,2$, whence $(\xi,\xi_\Gamma)(s) := \mathcal{N}((\rho,\rho_\Gamma)(s))$ is well defined for every $s\in[0,T]$. Thus, we write equation (2.28) at the time $s$ for both solutions, test the difference by $(\xi,\xi_\Gamma)(s)$ and integrate with respect to $s$ over $(0,t)$, where $t\in(0,T)$. Owing to (2.14), we obtain the identity

$$\frac12\|(\rho,\rho_\Gamma)(t)\|_*^2 + \int_{Q_t}\nabla\mu\cdot\nabla\xi + \int_{\Sigma_t}\nabla_\Gamma\mu_\Gamma\cdot\nabla_\Gamma\xi_\Gamma = \int_{Q_t}(\rho_1u_1-\rho_2u_2)\cdot\nabla\xi\,. \tag{4.1}$$

At the same time, we write equation (2.29) at the time $s$ for both solutions, test the difference by $(\rho,\rho_\Gamma)(s)$, integrate over $(0,t)$, and add the same term $\int_0^t\|(\rho,\rho_\Gamma)(s)\|_{\mathcal{H}}^2\,ds$ to both sides, for convenience. We obtain that

$$\begin{aligned}&\frac{\tau_\Omega}{2}\int_\Omega|\rho(t)|^2 + \frac{\tau_\Gamma}{2}\int_\Gamma|\rho_\Gamma(t)|^2 + \int_0^t\|\rho(s)\|_V^2\,ds + \int_0^t\|\rho_\Gamma(s)\|_{V_\Gamma}^2\,ds + \int_{Q_t}\zeta\rho + \int_{\Sigma_t}\zeta_\Gamma\rho_\Gamma \\ &= \int_{Q_t}\bigl\{\rho^2 - \bigl(\pi(\rho_1)-\pi(\rho_2)\bigr)\rho\bigr\} + \int_{\Sigma_t}\bigl\{\rho_\Gamma^2 - \bigl(\pi_\Gamma(\rho_{1\Gamma})-\pi_\Gamma(\rho_{2\Gamma})\bigr)\rho_\Gamma\bigr\} \\ &\quad + \int_{Q_t}\mu\rho + \int_{\Sigma_t}\mu_\Gamma\rho_\Gamma\,. \end{aligned}\tag{4.2}$$

At this point, we add these equalities to each other. By the definition of $\mathcal{N}$, the last two integrals of (4.2) and the ones on the left-hand side of (4.1) cancel each other. Moreover, the terms involving $\zeta$ and $\zeta_\Gamma$ are nonnegative by monotonicity. Thus, by owing to the Lipschitz continuity of $\pi$ and $\pi_\Gamma$, we deduce that

$$\frac{1}{2}\|(\rho,\rho_\Gamma)(t)\|_*^2 + \frac{\tau_\Omega}{2}\int_\Omega |\rho(t)|^2 + \frac{\tau_\Gamma}{2}\int_\Gamma |\rho_\Gamma(t)|^2 + \int_0^t \|(\rho,\rho_\Gamma)(s)\|_{\mathcal{V}}^2\,ds$$
$$\le \int_{Q_t} |\rho u_1 + \rho_2 u|\,|\nabla\xi| + c\int_0^t \|(\rho,\rho_\Gamma)(s)\|_{\mathcal{H}}^2\,ds =: I_1 + I_2,$$

and we now treat the contributions $I_1$ and $I_2$ on the right-hand side separately. We account for the Hölder, Sobolev and Young inequalities, and use the definitions (2.8) and (2.11). We have that

$$I_1 \le \int_0^t \bigl(\|\rho(s)\|_6\,\|u_1(s)\|_3 + \|\rho_2(s)\|_6\,\|u(s)\|_3\bigr)\|\nabla\xi(s)\|_2\,ds$$
$$\le \frac{1}{4}\int_0^t \|(\rho,\rho_\Gamma)(s)\|_{\mathcal{V}}^2\,ds + c\int_0^t \|u_1(s)\|_3^2\,\|(\rho,\rho_\Gamma)(s)\|_*^2\,ds$$
$$+ c\,\|\rho_2\|_{L^\infty(0,T;V)}^2 \int_0^t \|u(s)\|_3^2\,ds + \int_0^t \|(\rho,\rho_\Gamma)(s)\|_*^2\,ds\,.$$

We deal with $I_2$ as follows, invoking the compactness inequality (2.58):

$$I_2 \le \frac{1}{4}\int_0^t \|(\rho,\rho_\Gamma)(s)\|_{\mathcal{V}}^2\,ds + c\int_0^t \|(\rho,\rho_\Gamma)(s)\|_*^2\,ds\,.$$

At this point, we collect all of these inequalities, observe that the function $s \mapsto \|u_1(s)\|_3^2$ belongs to $L^1(0,T)$ by (2.21), and apply the Gronwall lemma. We immediately deduce (2.46) with a constant whose dependence agrees with that asserted in the statement of Theorem 2.5. With this, the proof is complete.

**Partial uniqueness and uniqueness.** Next, we derive the uniqueness part of Theorem 2.3. Uniqueness for $(\rho,\rho_\Gamma)$ clearly follows by taking $u_1 = u_2$ in (2.46). Assume now that $\beta$ is single-valued. This implies that $\zeta = \beta(\rho)$ is uniquely determined as well. Next, by Theorem 2.1, (2.37)–(2.38) hold true. From (2.37), we deduce uniqueness for the component $\mu$ of the solution. This also implies uniqueness for $\mu_\Gamma = \mu_{|\Sigma}$, and (2.38) yields uniqueness for $\zeta_\Gamma$. Assume now that $\beta_\Gamma$ is single-valued. In this case, we first derive uniqueness for $\zeta_\Gamma = \beta_\Gamma(\rho_\Gamma)$, then for $\mu_\Gamma$ by owing to (2.38). On the other hand, the first equation (2.28) with $(\rho,\rho_\Gamma)$ completely known implies that the difference of the components $(\mu,\mu_\Gamma)$ of two solutions is space independent, whence it has the form $t \mapsto \varphi(t)(1,1)$ for some $\varphi \in L^2(0,T)$, since the second component is the trace of the first one. But $\varphi$ must vanish since $\mu_\Gamma$ is unique. This implies that $\mu$ is unique as well. Finally, (2.37) yields uniqueness for $\zeta$. □

# 5 Approximation

In this section, we construct and solve an approximating problem depending on the small parameter $\varepsilon \in (0,1)$, which is understood to be fixed throughout the whole section. This problem is simply obtained by modifying (2.28)–(2.31) as follows: instead of $\tau_\Omega$ and $\tau_\Gamma$, we take the strictly positive constants

$$\tau_\Omega^\varepsilon := \max\{\tau_\Omega, \varepsilon\} \quad \text{and} \quad \tau_\Gamma^\varepsilon := \max\{\tau_\Gamma, \varepsilon\}, \tag{5.1}$$

and replace the functionals $\widehat{\beta}$ and $\widehat{\beta}_\Gamma$ and the operators $\beta$ and $\beta_\Gamma$ by the following Moreau and Yosida regularizations $\widehat{\beta}_\varepsilon$, $\widehat{\beta}_{\Gamma,\varepsilon}$, $\beta_\varepsilon$, $\beta_{\Gamma,\varepsilon}$ (see, e.g., [4, pp. 28 and 39]):

$$\begin{aligned}
\widehat{\beta}_\varepsilon(r) &:= \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon}|r-s|^2 + \widehat{\beta}(s) \right\} = \int_0^r \beta_\varepsilon(s)ds, \\
\widehat{\beta}_{\Gamma,\varepsilon}(r) &:= \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon\eta}|r-s|^2 + \widehat{\beta}_\Gamma(s) \right\} = \int_0^r \beta_{\Gamma,\varepsilon}(s)ds, \\
\beta_\varepsilon(r) &:= \frac{1}{\varepsilon}\bigl(r - (I+\varepsilon\beta)^{-1}(r)\bigr), \\
\beta_{\Gamma,\varepsilon}(r) &:= \frac{1}{\varepsilon\eta}\bigl(r - (I+\varepsilon\eta\beta_\Gamma)^{-1}(r)\bigr)
\end{aligned}$$

for all $r \in \mathbb{R}$, where $\eta > 0$ is the same constant as in the assumption (2.20). We point out that (2.17) and (2.19) hold also for the approximations. Moreover, we have that

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r), \quad 0 \le \widehat{\beta}_{\Gamma,\varepsilon}(r) \le \widehat{\beta}_\Gamma(r) \quad \text{for every } r \in \mathbb{R}, \tag{5.2}$$

$$|\beta_\varepsilon(r)| \le |\beta^\circ(r)|, \quad \bigl|\beta_{\Gamma,\varepsilon}(r)\bigr| \le \bigl|\beta_\Gamma^\circ(r)\bigr| \quad \text{for every } r \in D(\beta). \tag{5.3}$$

Furthermore, (2.20) also holds true for $\beta_\varepsilon$ and $\beta_{\Gamma,\varepsilon}$ with the same constants (see [6, Lemma 4.4]). We thus write

$$|\beta_\varepsilon(r)| \le \eta|\beta_{\Gamma,\varepsilon}(r)| + C \quad \text{for every } r \in \mathbb{R}. \tag{5.4}$$

Since $\beta_\varepsilon$ and $\beta_{\Gamma,\varepsilon}$ have the same sign, we see that (5.4) and the Young inequality yield

$$\beta_{\Gamma,\varepsilon}(r)\beta_\varepsilon(r) \ge \frac{1}{2\eta}\,|\beta_\varepsilon(r)|^2 - C_\eta \quad \text{for every } r \in \mathbb{R}, \tag{5.5}$$

with a similar constant $C_\eta$. We also notice that the inclusion $D(\beta_\Gamma) \subseteq D(\beta)$ (see (2.20)) and (2.23) imply that

$$\beta_\varepsilon(r)(r-m_0) \ge \delta_0|\beta_\varepsilon(r)| - C_0 \quad \text{and} \quad \beta_{\Gamma,\varepsilon}(r)(r-m_0) \ge \delta_0|\beta_{\Gamma,\varepsilon}(r)| - C_0 \tag{5.6}$$

for every $r \in \mathbb{R}$ and every $\varepsilon \in (0,1)$, where $\delta_0$ and $C_0$ are some positive constants that depend only on $\beta$, $\beta_\Gamma$ and on the position of $m_0$ in the interior of $D(\beta_\Gamma)$ and of $D(\beta)$ (see, e.g. [25, p. 908]).

The sought solution is a quadruple $(\mu^\varepsilon, \mu_\Gamma^\varepsilon, \rho^\varepsilon, \rho_\Gamma^\varepsilon)$ having the regularity properties

$$(\mu^\varepsilon, \mu_\Gamma^\varepsilon) \in L^2(0,T;\mathcal{V}) \cap L^1(0,T;\mathcal{W}), \tag{5.7}$$

$$(\rho^\varepsilon, \rho_\Gamma^\varepsilon) \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}) \cap L^2(0,T;\mathcal{W}) \tag{5.8}$$

and such that the 6-tuple $(\mu^\varepsilon, \mu_\Gamma^\varepsilon, \rho^\varepsilon, \rho_\Gamma^\varepsilon, \zeta^\varepsilon, \zeta_\Gamma^\varepsilon)$ obtained by setting

$$\zeta^\varepsilon := \beta_\varepsilon(\rho^\varepsilon) \quad \text{and} \quad \zeta_\Gamma^\varepsilon := \beta_{\Gamma,\varepsilon}(\rho_\Gamma^\varepsilon) \tag{5.9}$$

solves the following problem:

$$\int_\Omega \partial_t\rho^\varepsilon\, v + \int_\Gamma \partial_t\rho_\Gamma^\varepsilon\, v_\Gamma - \int_\Omega \rho^\varepsilon u\cdot\nabla v + \int_\Omega \nabla\mu^\varepsilon\cdot\nabla v + \int_\Gamma \nabla\mu_\Gamma^\varepsilon\cdot\nabla v_\Gamma = 0$$
$$\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V}, \tag{5.10}$$
$$\tau_\Omega^\varepsilon \int_\Omega \partial_t\rho^\varepsilon\, v + \tau_\Gamma^\varepsilon \int_\Gamma \partial_t\rho_\Gamma^\varepsilon\, v_\Gamma + \int_\Omega \nabla\rho^\varepsilon\cdot\nabla v + \int_\Gamma \nabla_\Gamma\rho_\Gamma^\varepsilon\cdot\nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega \bigl(\zeta^\varepsilon + \pi(\rho^\varepsilon)\bigr)v + \int_\Gamma \bigl(\zeta_\Gamma^\varepsilon + \pi_\Gamma(\rho_\Gamma^\varepsilon)\bigr)v_\Gamma = \int_\Omega \mu^\varepsilon v + \int_\Gamma \mu_\Gamma^\varepsilon v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V}, \tag{5.11}$$
$$\rho^\varepsilon(0) = \rho_0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \rho_\Gamma^\varepsilon(0) = \rho_{0|\Gamma} \quad \text{a.e. on } \Gamma. \tag{5.12}$$

We have written the sum of two integrals instead of a duality in (5.10), in accordance with the requirement (5.8) on $(\rho,\rho_\Gamma)$.

The aim of this section is to solve the approximating problem (5.9)–(5.12). In this respect, we have the following result.

**Theorem 5.1.** *Assume* (2.16)–(2.20) *and* (5.1) *for the structure and* (2.21)–(2.23) *for the data. Then the problem* (5.9)–(5.12) *has a unique solution* $(\mu^\varepsilon,\mu_\Gamma^\varepsilon,\rho^\varepsilon,\rho_\Gamma^\varepsilon)$ *with the regularity properties* (5.7)–(5.8).

The rest of the section is devoted to the proof of Theorem 5.1. Since the approximating problem (5.9)–(5.12) is a particular case of problem (2.28)–(2.31) and the operators $\beta_\varepsilon$ and $\beta_{\Gamma,\varepsilon}$ are single-valued, uniqueness has been already established in the previous section. As for existence, we use a slightly modified Faedo–Galerkin scheme with a proper choice of the Hilbert basis. We introduce the operator $\mathcal{A}\in\mathcal{L}(\mathcal{V};\mathcal{V}^*)$ by setting

$$\langle \mathcal{A}(w,w_\Gamma),(v,v_\Gamma)\rangle_{\mathcal{V}} := \int_\Omega \nabla w\cdot\nabla v + \int_\Gamma \nabla_\Gamma w_\Gamma\cdot\nabla_\Gamma v_\Gamma \quad \text{for } (w,w_\Gamma),(v,v_\Gamma)\in\mathcal{V}, \tag{5.13}$$

and notice that $\mathcal{A}$ is nonnegative and weakly coercive. Indeed, we have that

$$\langle \mathcal{A}(v,v_\Gamma),(v,v_\Gamma)\rangle_{\mathcal{V}} + \|(v,v_\Gamma)\|_{\mathcal{H}}^2 = \|(v,v_\Gamma)\|_{\mathcal{V}}^2 \quad \text{for every } (v,v_\Gamma)\in\mathcal{V}. \tag{5.14}$$

Moreover, as the embedding $\mathcal{V}\subset\mathcal{H}$ is compact, the resolvent of $\mathcal{A}$ is compact as well, and the spectrum of $\mathcal{A}$ reduces to a discrete set of eigenvalues, the eigenvalue problem being

$$(e,e_\Gamma)\in\mathcal{V}\setminus\{(0,0)\} \quad \text{and} \quad \mathcal{A}(e,e_\Gamma) = \lambda(e,e_\Gamma)\,. \tag{5.15}$$

More precisely, we can rearrange the eigenvalues and choose the eigenvectors in order that

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \dots \quad \text{and} \quad \lim_{j\to\infty}\lambda_j = +\infty, \tag{5.16}$$
$$\mathcal{A}(e^j,e_\Gamma^j) = \lambda_j(e^j,e_\Gamma^j) \quad \text{and} \quad \int_\Omega e^i e^j + \int_\Gamma e_\Gamma^i e_\Gamma^j = \delta_{ij} \quad \text{for } i,j=1,2,\dots, \tag{5.17}$$

and $\{(e^j,e_\Gamma^j)\}$ generates a dense subspace of both $\mathcal{V}$ and $\mathcal{H}$. We notice that

$$\int_\Omega \nabla e^i\cdot\nabla e^j + \int_\Gamma \nabla_\Gamma e_\Gamma^i\cdot\nabla_\Gamma e_\Gamma^j = \lambda_i\Bigl(\int_\Omega e^i e^j + \int_\Gamma e_\Gamma^i e_\Gamma^j\Bigr) = \lambda_i\delta_{ij} \quad \text{for } i,j=1,2,\dots.$$

We also observe that every element $(w, w_\Gamma) \in \mathcal{H}$ can be written as

$$(w, w_\Gamma) = \sum_{j=1}^{\infty} w_j (e^j, e^j_\Gamma) \quad \text{with} \quad \sum_{j=1}^{\infty} |w_j|^2 = \|(w, w_\Gamma)\|^2_{\mathcal{H}} < +\infty,$$

and that (on account of (5.14))

$$(w, w_\Gamma) \in \mathcal{V} \quad \text{if and only if} \quad \sum_{j=1}^{\infty} (1 + \lambda_j) |w_j|^2 < +\infty\,.$$

Namely, the last sum yields the square of a norm on $\mathcal{V}$ that is equivalent to $\|\cdot\|_{\mathcal{V}}$. In particular, we have the following property (the finite sum is the $\mathcal{H}$-projection on the subspace $\mathcal{V}_n$ defined below):

$$\|(w^n, w^n_\Gamma)\|_{\mathcal{V}} \le C_\Omega \|(w, w_\Gamma)\|_{\mathcal{V}} \quad \text{if} \quad (w^n, w^n_\Gamma) = \sum_{j=1}^{n} w_j (e^j, e^j_\Gamma), \tag{5.18}$$

where $C_\Omega$ depends only on $\Omega$. At this point, we set

$$\mathcal{V}_n := \operatorname{span}\{(e^j, e^j_\Gamma) : \ 1 \le j \le n\} \quad \text{and} \quad \mathcal{V}_\infty := \bigcup_{j=1}^{\infty} \mathcal{V}_n = \operatorname{span}\{(e^j, e^j_\Gamma) : \ j \ge 1\}, \tag{5.19}$$

and, for every $n \ge 1$, we look for a quadruple $(\mu^n, \mu^n_\Gamma, \rho^n, \rho^n_\Gamma)$ satisfying

$$(\mu^n, \mu^n_\Gamma) \in L^2(0, T; \mathcal{V}_n) \quad \text{and} \quad (\rho^n, \rho^n_\Gamma) \in H^1(0, T; \mathcal{V}_n), \tag{5.20}$$

$$\begin{aligned} &\int_\Omega \partial_t \rho^n \, v + \int_\Gamma \partial_t \rho^n_\Gamma \, v_\Gamma - \int_\Omega \rho^n u \cdot \nabla v + \int_\Omega \nabla \mu^n \cdot \nabla v + \int_\Gamma \nabla_\Gamma \mu^n_\Gamma \cdot \nabla_\Gamma v_\Gamma \\ &\quad + \frac{1}{n} \int_\Omega \mu^n v + \frac{1}{n} \int_\Gamma \mu^n_\Gamma v_\Gamma = 0 \\ &\quad \text{a.e. in } (0, T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}_n, \end{aligned} \tag{5.21}$$

$$\begin{aligned} &\tau^\varepsilon_\Omega \int_\Omega \partial_t \rho^n \, v + \tau^\varepsilon_\Gamma \int_\Gamma \partial_t \rho^n_\Gamma \, v_\Gamma + \int_\Omega \nabla \rho^n \cdot \nabla v + \int_\Gamma \nabla_\Gamma \rho^n_\Gamma \cdot \nabla_\Gamma v_\Gamma \\ &\quad + \int_\Omega \bigl(\beta_\varepsilon(\rho^n) + \pi(\rho^n)\bigr) v + \int_\Gamma \bigl(\beta_{\Gamma,\varepsilon}(\rho^n_\Gamma) + \pi_\Gamma(\rho^n_\Gamma)\bigr) v_\Gamma = \int_\Omega \mu^n v + \int_\Gamma \mu^n_\Gamma v_\Gamma \\ &\quad \text{a.e. in } (0, T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}_n, \end{aligned} \tag{5.22}$$

$$\rho^n(0) = \rho^n_0 \quad \text{a.e. in } \Omega, \tag{5.23}$$

where $\rho^n_0$ is defined by the conditions $(\rho^n_0, \rho^n_{0|\Gamma}) \in \mathcal{V}_n$ and

$$\int_\Omega \rho^n_0 v + \int_\Gamma \rho^n_{0|\Gamma} v_\Gamma = \int_\Omega \rho_0 v + \int_\Gamma \rho_{0|\Gamma} v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}_n. \tag{5.24}$$

Thus, $\rho^n_0$ is the first component of the orthogonal projection of $(\rho_0, \rho_{0|\Gamma})$ on $\mathcal{V}_n$. We have

$$\|\rho^n_0\|_H \le \|(\rho^n_0, \rho^n_{0|\Gamma})\|_{\mathcal{H}} \le \|(\rho_0, \rho_{0|\Gamma})\|_{\mathcal{H}} \quad \text{and} \quad \|(\rho^n_0, \rho^n_{0|\Gamma})\|_{\mathcal{V}} \le C_\Omega \|(\rho_0, \rho_{0|\Gamma})\|_{\mathcal{V}}, \tag{5.25}$$

the second one on account of (5.18).

**The discrete problem.** By (5.20), we have to look for $(\mu^n,\mu^n_\Gamma)$ and $(\rho^n,\rho^n_\Gamma)$ given by

$$(\mu^n,\mu^n_\Gamma)(t)=\sum_{j=1}^n \mu_j(t)(e^j,e^j_\Gamma) \quad \text{and} \quad (\rho^n,\rho^n_\Gamma)(t)=\sum_{j=1}^n \rho_j(t)(e^j,e^j_\Gamma)$$

for some $\mu_j\in L^2(0,T)$ and $\rho_j\in H^1(0,T)$. Let us introduce the $n$-vectors $\overline{\mu}:=(\mu_j)$ and $\overline{\rho}:=(\rho_j)$. Then, by rewriting the system (5.21)–(5.22) just with $(v,v_\Gamma)=(e^i,e^i_\Gamma)$ for $i=1,\dots,n$, we see that it takes the form

$$\overline{\rho}\,'(t)-U(t)\,\overline{\rho}(t)+D_n\,\overline{\mu}(t)=0 \quad \text{and} \quad B\,\overline{\rho}\,'(t)+D\,\overline{\rho}(t)+F(\overline{\rho}(t))=\overline{\mu}(t), \tag{5.26}$$

where $D_n:=\operatorname{diag}(\lambda_1+\frac{1}{n},\dots,\lambda_n+\frac{1}{n})$, $D:=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$, $F:\mathbb{R}^n\to\mathbb{R}^n$ is Lipschitz continuous, and the matrices $U=(u_{ij})\in L^2(0,T;\mathbb{R}^{n\times n})$ and $B=(b_{ij})\in\mathbb{R}^{n\times n}$ are given by

$$u_{ij}(t):=\int_\Omega e^j u(t)\cdot\nabla e^i \quad \text{for a.a. } t\in(0,T) \quad \text{and} \quad b_{ij}:=\tau_\Omega^\varepsilon\int_\Omega e^je^i+\tau_\Gamma^\varepsilon\int_\Gamma e^j_\Gamma e^i_\Gamma,$$

for $i,j=1,\dots,n$. By adding the second identity in (5.26) to the first one multiplied by $D_n^{-1}$, we obtain the equivalent system

$$(D_n^{-1}+B)\,\overline{\rho}\,'(t)+V(t)\,\overline{\rho}(t)+F(\overline{\rho}(t))=0 \quad \text{and} \quad \overline{\mu}(t)=B\,\overline{\rho}\,'(t)+D\,\overline{\rho}(t)+F(\overline{\rho}(t)),$$

where $V:=D-D_n^{-1}U$ belongs to $L^2(0,T;\mathbb{R}^{n\times n})$ and $D_n^{-1}+B$ is invertible, as we verify. To this end, we show that $B$ is positive definite. Indeed, for any vector $y=(y_1,\dots,y_n)\in\mathbb{R}^n$, by setting $(v,v_\Gamma):=\sum_{j=1}^n y_j(e^j,e^j_\Gamma)$, we have that

$$\begin{aligned}(By)\cdot y&=\sum_{i,j=1}^n b_{ij}y_jy_i=\tau_\Omega^\varepsilon\int_\Omega\sum_{i=1}^n y_ie^i\sum_{j=1}^n y_je^j+\tau_\Gamma^\varepsilon\int_\Omega\sum_{i=1}^n y_ie^i_\Gamma\sum_{j=1}^n y_je^j_\Gamma\\&=\tau_\Omega^\varepsilon\int_\Omega|v|^2+\tau_\Gamma^\varepsilon\int_\Gamma|v_\Gamma|^2\geq\varepsilon\|(v,v_\Gamma)\|^2_{\mathcal{H}}=\varepsilon\|y\|^2_{\mathbb{R}^n}\,.\end{aligned}$$

Hence, $D_n^{-1}+B$ is positive definite as well, thus invertible. On the other hand, (5.23) is equivalent to an initial condition for $\overline{\rho}$. Therefore, the discrete problem (5.20)–(5.23) has a unique solution.

At this point, our aim is to show that the solutions to the discete problem converge to a solution to the approximating problem (5.9)–(5.12) as $n$ tends to infinity, at least for a subsequence. To this end, we start estimating and find bounds that do not depend on $n$. On the contrary, they can depend on $\varepsilon$.

**An a priori estimate.** We test (5.21), written at the time $s$, by $(\mu^n,\mu^n_\Gamma)(s)$ and integrate over $(0,t)$ with respect to $s$ to find that

$$\begin{aligned}&\int_{Q_t}\partial_t\rho^n\,\mu^n+\int_{\Sigma_t}\partial_t\rho^n_\Gamma\,\mu^n_\Gamma+\int_{Q_t}|\nabla\mu^n|^2+\int_{\Sigma_t}|\nabla_\Gamma\mu^n_\Gamma|^2\\&\quad+\frac{1}{n}\int_{Q_t}|\mu^n|^2+\frac{1}{n}\int_{\Sigma_t}|\mu^n_\Gamma|^2=\int_{Q_t}\rho^nu\cdot\nabla\mu^n\,.\end{aligned}$$

Next, we test (5.22) by $\partial_t(\rho^n,\rho_\Gamma^n)(s)$, integrate over $(0,t)$ with respect to $s$, and add the same terms $\int_{Q_t}\rho^n\partial_t\rho^n$ and $\int_{\Sigma_t}\rho_\Gamma^n\partial_t\rho_\Gamma^n$ to both sides for convenience. We obtain that

$$\begin{aligned}
&\tau_\Omega^\varepsilon\int_{Q_t}|\partial_t\rho^n|^2+\tau_\Gamma^\varepsilon\int_{\Sigma_t}|\partial_t\rho_\Gamma^n|^2+\frac12\,\|(\rho^n,\rho_\Gamma^n)(t)\|_{\mathcal{V}}^2+\int_\Omega\widehat{\beta}_\varepsilon(\rho^n(t))+\int_\Gamma\widehat{\beta}_{\Gamma,\varepsilon}(\rho_\Gamma^n(t))\\
&=\frac12\,\|(\rho^n,\rho_\Gamma^n)(0)\|_{\mathcal{V}}^2+\int_\Omega\widehat{\beta}_\varepsilon(\rho^n(0))+\int_\Gamma\widehat{\beta}_{\Gamma,\varepsilon}(\rho_\Gamma^n(0))+\int_{Q_t}\mu^n\partial_t\rho^n+\int_{\Sigma_t}\mu_\Gamma^n\partial_t\rho_\Gamma^n\\
&\quad+\int_{Q_t}\bigl(\rho^n-\pi(\rho^n)\bigr)\partial_t\rho^n+\int_{\Sigma_t}\bigl(\rho_\Gamma^n-\pi_\Gamma(\rho_\Gamma^n)\bigr)\partial_t\rho_\Gamma^n\,.
\end{aligned}$$

At this point, we add these equalities and notice that four terms cancel. Then, the remaining terms on the left-hand side are nonnegative, so that we can forget about four of them. Moreover, we use (5.1) and start estimating the right-hand side (also accounting for (5.18), (5.2) and (2.22)). We then arrive at the estimate

$$\begin{aligned}
&\int_{Q_t}|\nabla\mu^n|^2+\int_{\Sigma_t}|\nabla_\Gamma\mu_\Gamma^n|^2+\varepsilon\int_{Q_t}|\partial_t\rho^n|^2+\varepsilon\int_{\Sigma_t}|\partial_t\rho_\Gamma^n|^2+\frac12\,\|(\rho^n,\rho_\Gamma^n)(t)\|_{\mathcal{V}}^2\\
&\le\int_{Q_t}|\rho^n|\,|u|\,|\nabla\mu^n|+c+\frac{\varepsilon}{2}\int_{Q_t}|\partial_t\rho^n|^2+\frac{\varepsilon}{2}\int_{\Sigma_t}|\partial_t\rho_\Gamma^n|^2+c_\varepsilon\int_{Q_t}|\rho^n|^2+c_\varepsilon\int_{\Sigma_t}|\rho_\Gamma^n|^2+c_\varepsilon\,.
\end{aligned}$$

On the other hand, the Hölder, Sobolev and Young inequalities yield that

$$\begin{aligned}
&\int_{Q_t}|\rho^n|\,|u|\,|\nabla\mu^n|\le\int_0^t\|\rho^n(s)\|_6\,\|u(s)\|_3\,\|\nabla\mu^n(s)\|_2\,ds\\
&\le\frac12\int_{Q_t}|\nabla\mu^n|^2+c\int_0^t\|u(s)\|_3^2\,\|\rho^n(s)\|_V^2\,ds\,,
\end{aligned}$$

and we notice that the function $s\mapsto\|u(s)\|_3^2$ belongs to $L^1(0,T)$, by (2.21). Therefore, by rearranging and applying the Gronwall lemma, we can infer that

$$\|\nabla\mu^n\|_{L^2(0,T;H)}+\|\nabla_\Gamma\mu_\Gamma^n\|_{L^2(0,T;H_\Gamma)}+\|(\rho^n,\rho_\Gamma^n)\|_{H^1(0,T;\mathcal{H})\cap L^\infty(0,T;\mathcal{V})}\le c_\varepsilon\,. \tag{5.27}$$

**Consequence.** Just by Lipschitz continuity, we also have that

$$\|(\beta_\varepsilon+\pi)(\rho^n)\|_{L^\infty(0,T;H)}+\|(\beta_{\Gamma,\varepsilon}+\pi_\Gamma)(\rho_\Gamma^n)\|_{L^\infty(0,T;H_\Gamma)}\le c_\varepsilon\,.$$

On the other hand, if we test (5.22) by $(|\Omega|+|\Gamma|)^{-1}(1,1)$, then we obtain, for a.a. $t\in(0,T)$,

$$\begin{aligned}
&|\operatorname{mean}(\mu^n,\mu_\Gamma^n)(t)|\\
&\le c\,\bigl\{\|\partial_t\rho^n(t)\|_H+\|\partial_t\rho_\Gamma^n(t)\|_{H_\Gamma}+\|(\beta_\varepsilon+\pi)(\rho^n(t))\|_H+\|(\beta_{\Gamma,\varepsilon}+\pi_\Gamma)(\rho_\Gamma^n(t))\|_{H_\Gamma}\bigr\}\,.
\end{aligned}$$

Therefore, we have shown that $\operatorname{mean}(\mu^n,\mu_\Gamma^n)$ is bounded in $L^2(0,T)$, so that (5.27) and (2.6) allow us to conclude that

$$\|(\mu^n,\mu_\Gamma^n)\|_{L^2(0,T;\mathcal{V})}\le c_\varepsilon\,. \tag{5.28}$$

**Conclusion.** We account for (5.27)–(5.28) and use standard weak and weak star compactness results, as well as the Aubin-Lions lemma (see, e.g., [33, Thm. 5.1, p. 58]). It follows that

$$(\mu^n, \mu^n_\Gamma) \to (\mu^\varepsilon, \mu^\varepsilon_\Gamma) \qquad \text{weakly in } L^2(0,T;\mathcal{V}), \tag{5.29}$$

$$\begin{aligned}(\rho^n, \rho^n_\Gamma) \to (\rho^\varepsilon, \rho^\varepsilon_\Gamma) \qquad & \text{weakly star in } H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}) \\ & \text{and strongly in } L^2(0,T;\mathcal{H}),\end{aligned} \tag{5.30}$$

as $n$ tends to infinity, at least for a subsequence. By Lipschitz continuity, we also deduce that $(\beta_\varepsilon + \pi)(\rho^n)$ and $(\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(\rho^n_\Gamma)$ converge to $(\beta_\varepsilon + \pi)(\rho^\varepsilon)$ and $(\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(\rho^\varepsilon_\Gamma)$ strongly in $L^2(0,T;H)$ and in $L^2(0,T;H_\Gamma)$, respectively. Moreover, $\rho^n u$ converges to $\rho^\varepsilon u$ weakly in $L^2(0,T;L^2(\Omega))$, since $u \in L^2(0,T;L^3(\Omega))$ and $\rho^n$ is bounded in $L^\infty(0,T;L^6(\Omega))$, by the Sobolev inequality. Finally, $(\rho^n, \rho^n_\Gamma)(0)$ converges to $(\rho^\varepsilon, \rho^\varepsilon_\Gamma)(0)$ at least weakly in $\mathcal{H}$, so that (5.12) is satisfied.

Now, we recall (5.19) for the definition of $\mathcal{V}_\infty$, and take an arbitrary $\mathcal{V}_\infty$-valued step function $(v, v_\Gamma)$. Since the range of $(v, v_\Gamma)$ is finite-dimensional, there exists some $m$ such that $(v, v_\Gamma)(t) \in \mathcal{V}_m$ for a.a. $t \in (0,T)$. It follows that $(v, v_\Gamma)(t) \in \mathcal{V}_n$ for a.a. $t \in (0,T)$ and every $n \geq m$, so that we can test (5.21) and (5.22), written at the time $t$, by $(v, v_\Gamma)(t)$ and integrate over $(0,T)$. At this point, it is straightforward to deduce that $(\mu^\varepsilon, \mu^\varepsilon_\Gamma)$, $(\rho^\varepsilon, \rho^\varepsilon_\Gamma)$ and the functions $\zeta^\varepsilon$ and $\zeta^\varepsilon_\Gamma$ given by (5.9) satisfy the integrated version of (5.10)–(5.11) for every such step functions, namely, we have that

$$\int_Q \partial_t \rho^\varepsilon \, v + \int_\Sigma \partial_t \rho^\varepsilon_\Gamma \, v_\Gamma - \int_Q \rho^\varepsilon u \cdot \nabla v + \int_Q \nabla \mu^\varepsilon \cdot \nabla v + \int_\Sigma \nabla \mu^\varepsilon_\Gamma \cdot \nabla v_\Gamma = 0 \,,$$

$$\begin{aligned}&\tau^\varepsilon_\Omega \int_Q \partial_t \rho^\varepsilon \, v + \tau^\varepsilon_\Gamma \int_\Sigma \partial_t \rho^\varepsilon_\Gamma \, v_\Gamma + \int_Q \nabla \rho^\varepsilon \cdot \nabla v + \int_\Sigma \nabla_\Gamma \rho^\varepsilon_\Gamma \cdot \nabla_\Gamma v_\Gamma \\ &\quad + \int_Q \bigl(\zeta^\varepsilon + \pi(\rho^\varepsilon)\bigr) v + \int_\Sigma \bigl(\zeta^\varepsilon_\Gamma + \pi_\Gamma(\rho^\varepsilon_\Gamma)\bigr) v_\Gamma = \int_Q \mu^\varepsilon v + \int_\Sigma \mu^\varepsilon_\Gamma v_\Gamma \,.\end{aligned}$$

By density, the same equations hold true for every $(v, v_\Gamma) \in L^2(0,T;\mathcal{V})$. This implies that (5.10)–(5.11) hold a.e. in $(0,T)$ and for every $(v, v_\Gamma) \in \mathcal{V}$, as desired. We notice that (5.10) and (5.11) are formally equal to (2.28) and (2.29), respectively. Moreover, by accounting for (2.21), we can replace the term $-\int_\Omega \rho^\varepsilon u \cdot \nabla v$ by the expression $\int_\Omega \nabla \rho^\varepsilon \cdot u \, v$ in (5.10) and notice that $\nabla \rho^\varepsilon \cdot u$ belongs to $L^2(0,T;H)$, since $\rho^\varepsilon \in L^\infty(0,T;L^6(\Omega))$ and $u \in L^2(0,T;L^3(\Omega))$. This, and what we already know for the other terms, allow us to apply *i)* and *ii)* of Theorem 2.1. We then deduce the full regularity (5.7)–(5.8), by starting from the lower regularity already established. □

# 6 Existence

This section is devoted to the conclusion of the proof of Theorem 2.3. Namely, we show that the solutions to the approximating problems converge to a solution to problem (2.28)–(2.31) satisfying (2.45). We recall that the constant mean value property (2.32) is also satisfied by the solutions to the $\varepsilon$-approximating problems. In performing our estimates, we avoid the superscript $\varepsilon$ in the notation of the solution, for simplicity, writing it only at the end of each step.

**First a priori estimate.** We test (5.10) and (5.11), written at the time $s$, by $(\mu, \mu_\Gamma)(s)$ and $\partial_t(\rho, \rho_\Gamma)(s)$, respectively. Then, we integrate over $(0,t)$ and sum up. Moreover, we add the same terms $\int_{Q_t} \rho\partial_t\rho$ and $\int_{\Sigma_t} \rho_\Gamma\partial_t\rho_\Gamma$ to both sides. Since some terms cancel each other, we obtain the identity

$$\begin{aligned}
&\int_{Q_t} |\nabla\mu|^2 + \int_{\Sigma_t} |\nabla_\Gamma\mu_\Gamma|^2 + \tau_\Omega^\varepsilon \int_{Q_t} |\partial_t\rho|^2 + \tau_\Gamma^\varepsilon \int_{\Sigma_t} |\partial_t\rho_\Gamma|^2 \\
&\quad + \frac{1}{2}\,\|(\rho,\rho_\Gamma)(t)\|_{\mathcal{V}}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\rho(t)) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\rho_\Gamma(t)) \\
&= \frac{1}{2}\,\|(\rho_0,\rho_{0|\Gamma})\|_{\mathcal{V}}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\rho_0) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\rho_{0|\Gamma}) \\
&\quad + \int_{Q_t} \bigl(\rho-\pi(\rho)\bigr)\partial_t\rho + \int_{\Sigma_t} \bigl(\rho_\Gamma-\pi_\Gamma(\rho_\Gamma)\bigr)\partial_t\rho_\Gamma + \int_{Q_t} \rho u\cdot\nabla\mu\,.
\end{aligned}$$

Now, we observe that

$$\int_{Q_t} \rho u\cdot\nabla\mu \le \int_0^t \|\rho(s)\|_6\,\|u(s)\|_3\,\|\nabla\mu(s)\|_2\,ds \le \frac{1}{2}\int_{Q_t} |\nabla\mu|^2 + c\int_0^t \|u(s)\|_3^2\,\|\rho(s)\|_V^2\,ds\,,$$

and that the function $s\mapsto \|u(s)\|_3^2$ belongs to $L^1(0,T)$, by (2.21). Therefore, also on account of (5.2) and (2.22), we easily conclude from Gronwall's lemma that

$$\begin{aligned}
&\|\nabla\mu^\varepsilon\|_{L^2(0,T;H)} + \|\nabla_\Gamma\mu_\Gamma^\varepsilon\|_{L^2(0,T;H_\Gamma)} + \|(\rho^\varepsilon,\rho_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{V})} \\
&\quad + \|\widehat{\beta}_\varepsilon(\rho^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|\widehat{\beta}_{\Gamma,\varepsilon}(\rho_\Gamma^\varepsilon)\|_{L^\infty(0,T;L^1(\Gamma))} \\
&\quad + (\tau_\Omega^\varepsilon)^{1/2}\|\partial_t\rho^\varepsilon\|_{L^2(0,T;H)} + (\tau_\Gamma^\varepsilon)^{1/2}\|\partial_t\rho_\Gamma^\varepsilon\|_{L^2(0,T;H_\Gamma)} \le c\,.
\end{aligned} \tag{6.1}$$

**Consequence.** By testing (5.10) with an arbitrary $(v,v_\Gamma)\in L^2(0,T;\mathcal{V})$, and owing to the assumptions (2.21) on $u$, we have that

$$\begin{aligned}
&\langle\partial_t(\rho,\rho_\Gamma),(v,v_\Gamma)\rangle_{\mathcal{V}} \\
&\le \|\nabla\mu\|_{L^2(0,T;H)}\|v\|_{L^2(0,T;V)} + \|\nabla_\Gamma\mu_\Gamma\|_{L^2(0,T;H_\Gamma)}\|v_\Gamma\|_{L^2(0,T;V_\Gamma)} \\
&\quad + \|\rho\|_{L^\infty(0,T;L^6(\Omega))}\,\|u\|_{L^2(0,T;L^3(\Omega))}\,\|\nabla v\|_{L^2(0,T;L^2(\Omega))}\,.
\end{aligned}$$

Then, the continuous embedding $V\subset L^6(\Omega)$ and (6.1) imply that

$$\|\partial_t(\rho^\varepsilon,\rho_\Gamma^\varepsilon)\|_{L^2(0,T;\mathcal{V}^*)} \le c\,. \tag{6.2}$$

**Second a priori estimate.** We account for (2.23) and test (5.11) by the $\mathcal{V}_0$-valued function $(\rho-m_0,\rho_\Gamma-m_0)$ a.e. in $(0,T)$ without integrating with respect to time. Setting $\alpha:=\mathrm{mean}(\mu,\mu_\Gamma)$ a.e. in $(0,T)$ for a while, we obtain

$$\begin{aligned}
&\int_\Omega \beta_\varepsilon(\rho)(\rho-m_0) + \int_\Gamma \beta_{\Gamma,\varepsilon}(\rho_\Gamma)(\rho_\Gamma-m_0) \\
&= -\,\tau_\Omega^\varepsilon\int_\Omega \partial_t\rho(\rho-m_0) - \tau_\Gamma^\varepsilon\int_\Gamma \partial_t\rho_\Gamma(\rho_\Gamma-m_0) - \int_\Omega |\nabla\rho|^2 - \int_\Gamma |\nabla_\Gamma\rho_\Gamma|^2 \\
&\quad - \int_\Omega \pi(\rho)(\rho-m_0) - \int_\Gamma \pi_\Gamma(\rho_\Gamma)(\rho_\Gamma-m_0) \\
&\quad + \int_\Omega (\mu-\alpha)(\rho-m_0) + \int_\Gamma (\mu_\Gamma-\alpha)(\rho_\Gamma-m_0)
\end{aligned} \tag{6.3}$$

a.e. in $(0,T)$. Observe that, in the right-hand side of (6.3) the integrals involving the gradients are bounded in $L^\infty(0,T)$, due to (6.1). Then, by using the inner product in $\mathcal{H}$, the corresponding Schwarz inequality, and the Lipschitz continuity of $\pi$ and $\pi_\Gamma$, we deduce that

$$\begin{aligned}
&\int_\Omega \beta_\varepsilon(\rho)(\rho-m_0) + \int_\Gamma \beta_{\Gamma,\varepsilon}(\rho_\Gamma)(\rho_\Gamma-m_0) \\
&\leq \bigl|\bigl((\tau_\Omega^\varepsilon\partial_t\rho, \tau_\Gamma^\varepsilon\partial_t\rho_\Gamma),(\rho-m_0,\rho_\Gamma-m_0)\bigr)_{\mathcal{H}}\bigr| + c \\
&\quad + \bigl|\bigl((\pi(\rho),\pi_\Gamma(\rho_\Gamma)),(\rho-m_0,\rho_\Gamma-m_0)\bigr)_{\mathcal{H}}\bigr| \\
&\quad + \bigl|\bigl((\mu-\alpha,\mu_\Gamma-\alpha),(\rho-m_0,\rho_\Gamma-m_0)_{\mathcal{H}}\bigr)\bigr| \\
&\leq \bigl\{\|(\tau_\Omega^\varepsilon\partial_t\rho, \tau_\Gamma^\varepsilon\partial_t\rho_\Gamma)\|_{\mathcal{H}} + c\,\|(\rho,\rho_\Gamma)\|_{\mathcal{H}} + c + \|(\mu-\alpha,\mu_\Gamma-\alpha)\|_{\mathcal{H}}\bigr\} \times \\
&\quad \times \|(\rho-m_0,\rho_\Gamma-m_0)\|_{\mathcal{H}} + c\,.
\end{aligned}$$

Hence, in view of (6.1) and (5.6), we deduce that

$$\int_\Omega |\beta_\varepsilon(\rho)| + \int_\Gamma |\beta_{\Gamma,\varepsilon}(\rho_\Gamma)| \leq c\|(\mu-\alpha,\mu_\Gamma-\alpha)\|_{\mathcal{H}} + \psi_\varepsilon \tag{6.4}$$

where $\psi_\varepsilon$ is bounded in $L^2(0,T)$ uniformly with respect to $\varepsilon$. On the other hand, owing to the definition (2.8) and recalling that $\|\cdot\|_{\mathcal{V}_0}$ is a norm on $\mathcal{V}_0$ that is equivalent to the standard one, we have that

$$\|(\mu-\alpha,\mu_\Gamma-\alpha)\|_{\mathcal{H}} \leq c\,\|(\mu-\alpha,\mu_\Gamma-\alpha)\|_{\mathcal{V}_0} = c\,\|(\nabla\mu,\nabla_\Gamma\mu_\Gamma)\|_{\mathcal{H}}\,.$$

Since the last term is bounded in $L^2(0,T)$ by (6.1), the inequality (6.4) implies that

$$\|\beta_\varepsilon(\rho)\|_{L^2(0,T;L^1(\Omega))} + \|\beta_{\Gamma,\varepsilon}(\rho_\Gamma)\|_{L^2(0,T;L^1(\Gamma))} \leq c\,.$$

At this point, we can test (5.11) by $(1,1)$ and find a bound for $\operatorname{mean}(\mu,\mu_\Gamma)$ in $L^2(0,T)$. Combining it with (6.1), we conclude that

$$\|(\mu^\varepsilon,\mu_\Gamma^\varepsilon)\|_{L^2(0,T;\mathcal{V})} \leq c\,. \tag{6.5}$$

**Third a priori estimate.** We test (5.11), written at the time $s$, with $(\beta_\varepsilon(\rho),\beta_\varepsilon(\rho_\Gamma))(s)$ and integrate over $(0,t)$ with respect to $s$, obtaining the identity

$$\begin{aligned}
&\tau_\Omega^\varepsilon \int_\Omega \widehat{\beta}_\varepsilon(\rho(t)) + \tau_\Gamma^\varepsilon \int_\Gamma \widehat{\beta}_\varepsilon(\rho_\Gamma(t)) + \int_{Q_t} \beta_\varepsilon'(\rho)|\nabla\rho|^2 + \int_\Sigma \beta_{\Gamma,\varepsilon}'(\rho_\Gamma)|\nabla_\Gamma\rho_\Gamma|^2 \\
&\quad + \int_{Q_t} |\beta_\varepsilon(\rho)|^2 + \int_{\Sigma_t} \beta_{\Gamma,\varepsilon}(\rho_\Gamma)\,\beta_\varepsilon(\rho_\Gamma) \\
&= \tau_\Omega^\varepsilon \int_\Omega \widehat{\beta}_\varepsilon(\rho_0) + \tau_\Gamma^\varepsilon \int_\Gamma \widehat{\beta}_\varepsilon(\rho_{0|\Gamma}) + \int_{Q_t} \bigl(\mu-\pi(\rho)\bigr)\beta_\varepsilon(\rho) + \int_{\Sigma_t} \bigl(\mu_\Gamma-\pi_\Gamma(\rho_\Gamma)\bigr)\beta_\varepsilon(\rho_\Gamma)\,.
\end{aligned}$$

All of the terms on the left-hand side are nonnegative but the last one, for which we have, thanks to (5.5),

$$\int_{\Sigma_t} \beta_{\Gamma,\varepsilon}(\rho_\Gamma)\,\beta_\varepsilon(\rho_\Gamma) \geq \frac{1}{2\eta}\int_{\Sigma_t} |\beta_\varepsilon(\rho_\Gamma)|^2 - c\,.$$

Since the right-hand side can be easily handled by using the Young inequality, (5.2), (2.22), and the estimates (6.1) and (6.5), we conclude that

$$\|\zeta^\varepsilon\|_{L^2(0,T;H)} + \|\beta_\varepsilon(\rho_\Gamma^\varepsilon)\|_{L^2(0,T;H_\Gamma)} \le c\,. \tag{6.6}$$

**Fourth a priori estimate.** We apply the part *iii*) of Theorem 2.1 to the solution to the approximating problem with the choice $\gamma = \beta_{\Gamma,\varepsilon}$. As the constant $C_3$ does not depend on $\varepsilon$, inequality (2.41) yields a bound for $\zeta_\Gamma$ in terms of quantities that have already been estimated. Hence, we conclude that

$$\|\zeta_\Gamma^\varepsilon\|_{L^2(0,T;H_\Gamma)} \le c\,. \tag{6.7}$$

At this point, we can apply the part *ii*) of Theorem 2.1. We thus have

$$\|(\rho^\varepsilon, \rho_\Gamma^\varepsilon)\|_{L^2(0,T;\mathcal{W})} \le c\,. \tag{6.8}$$

**Conclusion.** We account for (6.1)–(6.8) and use standard weak and weak star compactness results as well as the Aubin-Lions lemma (see, e.g., [33, Thm. 5.1, p. 58]). We have

$$\begin{aligned}
(\mu^\varepsilon, \mu_\Gamma^\varepsilon) \to (\mu, \mu_\Gamma) \quad & \text{weakly in } L^2(0,T;\mathcal{V}),\\
(\rho^\varepsilon, \rho_\Gamma^\varepsilon) \to (\rho, \rho_\Gamma) \quad & \text{weakly star in } H^1(0,T;\mathcal{V}^*) \cap L^\infty(0,T;\mathcal{V}) \cap L^2(0,T;\mathcal{W})\\
& \text{and strongly in } L^2(0,T;\mathcal{H}),\\
\tau_\Omega^\varepsilon \partial_t \rho^\varepsilon \to \tau_\Omega \partial_t \rho \quad & \text{weakly in } L^2(0,T;H),\\
\tau_\Gamma^\varepsilon \partial_t \rho_\Gamma^\varepsilon \to \tau_\Gamma \partial_t \rho_\Gamma \quad & \text{weakly in } L^2(0,T;H_\Gamma),\\
(\zeta^\varepsilon, \zeta_\Gamma^\varepsilon) \to (\zeta, \zeta_\Gamma) \quad & \text{weakly in } L^2(0,T;\mathcal{H}),
\end{aligned}$$

as $\varepsilon$ tends to zero, at least for a subsequence. Moreover, $\rho^\varepsilon u$ converges to $\rho u$ weakly in $L^2(0,T;L^2(\Omega))$, since $u \in L^2(0,T;L^3(\Omega))$ and $\rho^\varepsilon$ converges to $\rho$ at least weakly star in $L^\infty(0,T;L^6(\Omega))$. At this point, it is straightforward to deduce that $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ satisfies the integrated version of (2.28)–(2.29) with time-dependent test function $(v, v_\Gamma) \in L^2(0,T;\mathcal{V})$, and this is equivalent to our formulation. Furthermore, thanks to the strong convergence of $(\rho^\varepsilon, \rho_\Gamma^\varepsilon)$ to $(\rho, \rho_\Gamma)$ and to well-known results on maximal monotone operators (see, e.g. [2, Proposition 2.2, p. 38]), we derive (2.30), i.e., $\zeta \in \beta(\rho)$ and $\zeta_\Gamma \in \beta_\Gamma(\rho_\Gamma)$. Besides, $(\rho^\varepsilon, \rho_\Gamma^\varepsilon)(0)$ converges to $(\rho, \rho_\Gamma)(0)$ at least weakly in $\mathcal{V}^*$, so that (2.31) holds true as well. Finally, the estimate (2.45) follows from lower semicontinuity. $\square$

# 7 Complements

This section is devoted to the proof of Theorems 2.6, 2.8 and 2.9. Our proofs rely on further a priori estimates on the solutions to the $\varepsilon$-approximating problems. However, in performing them, we proceed formally, for brevity. Also in this section, we write the superscript $\varepsilon$ in the notation for the solution only at the end of each step. From now on, we assume that $\tau_\Omega > 0$, $\tau_\Gamma > 0$ and that (2.47)–(2.48) hold true. We can also take $\varepsilon \le \min\{\tau_\Omega, \tau_\Gamma\}$, so that $\tau_\Omega^\varepsilon = \tau_\Omega$ and $\tau_\Gamma^\varepsilon = \tau_\Gamma$ (see (5.1)).

**Fifth a priori estimate.** We differentiate both (5.10) and (5.11) with respect to time. By noting that $\mathrm{mean}(\partial_t(\rho,\rho_\Gamma))=0$ by (2.32), we test the obtained equations by $(\xi,\xi_\Gamma):=\mathcal{N}(\partial_t(\rho,\rho_\Gamma))$ and $\partial_t(\rho,\rho_\Gamma)$, respectively. We obtain the identities

$$\begin{aligned}
&\int_{Q_t}\partial_t^2\rho\,\xi+\int_{\Sigma_t}\partial_t^2\rho_\Gamma\,\xi_\Gamma+\int_{Q_t}\nabla\partial_t\mu\cdot\nabla\xi+\int_{\Sigma_t}\nabla_\Gamma\partial_t\mu_\Gamma\cdot\nabla_\Gamma\xi_\Gamma\\
&=\int_{Q_t}\partial_t\rho\,u\cdot\nabla\xi+\int_{Q_t}\rho\,\partial_t u\cdot\nabla\xi\,,\\
&\frac{\tau_\Omega}{2}\int_\Omega|\partial_t\rho(t)|^2+\frac{\tau_\Gamma}{2}\int_\Gamma|\partial_t\rho_\Gamma(t)|^2+\int_{Q_t}|\nabla\partial_t\rho|^2+\int_{\Sigma_t}|\nabla_\Gamma\partial_t\rho_\Gamma|^2\\
&\quad+\int_{Q_t}\beta_\varepsilon'(\rho)|\partial_t\rho|^2+\int_{\Sigma_t}\beta_{\Gamma,\varepsilon}'(\rho_\Gamma)|\partial_t\rho_\Gamma|^2\\
&=\frac{\tau_\Omega}{2}\int_\Omega|\partial_t\rho(0)|^2+\frac{\tau_\Gamma}{2}\int_\Gamma|\partial_t\rho_\Gamma(0)|^2\\
&\quad-\int_{Q_t}\pi'(\rho)|\partial_t\rho|^2-\int_{\Sigma_t}\pi_\Gamma'(\rho_\Gamma)|\partial_t\rho_\Gamma|^2+\int_{Q_t}\partial_t\mu\partial_t\rho+\int_{\Sigma_t}\partial_t\mu_\Gamma\partial_t\rho_\Gamma\,.
\end{aligned}$$

Now, we add these equalities to each other and treat the sum of the first two integrals by accounting for (2.14). Moreover, we can cancel four terms in the sum due to the definition of $\mathcal{N}$ (see (2.9)–(2.10)). Finally, we recall that $\beta_\varepsilon'$ and $\beta_{\Gamma,\varepsilon}'$ are nonnegative, and integrate by parts the integrals involving $u$ by using (2.21). We then obtain that

$$\begin{aligned}
&\frac12\,\|\partial_t(\rho,\rho_\Gamma)(t)\|_*^2+\frac{\tau_\Omega}{2}\int_\Omega|\partial_t\rho(t)|^2+\frac{\tau_\Gamma}{2}\int_\Gamma|\partial_t\rho_\Gamma(t)|^2+\int_{Q_t}|\nabla\partial_t\rho|^2+\int_{\Sigma_t}|\nabla_\Gamma\partial_t\rho_\Gamma|^2\\
&\le I_0-\int_{Q_t}\nabla\partial_t\rho\cdot u\,\xi-\int_{Q_t}\nabla\rho\cdot\partial_t u\,\xi-\int_{Q_t}\pi'(\rho)|\partial_t\rho|^2-\int_{\Sigma_t}\pi_\Gamma'(\rho_\Gamma)|\partial_t\rho_\Gamma|^2\,,
\end{aligned}\tag{7.1}$$

where

$$I_0:=\frac12\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2+\frac{\tau_\Omega}{2}\int_\Omega|\partial_t\rho(0)|^2+\frac{\tau_\Gamma}{2}\int_\Gamma|\partial_t\rho_\Gamma(0)|^2\,.\tag{7.2}$$

Now, we estimate the integrals involving $u$ by using the Hölder inequality, the continuous embedding $V\subset L^6(\Omega)$, the equivalence on $\mathcal{V}_0$ of the norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{V}_0}$, and the definition (2.11) of $\|\cdot\|_*$. We have

$$\begin{aligned}
&-\int_{Q_t}\nabla\partial_t\rho\cdot u\,\xi\le\int_0^t\|\nabla\partial_t\rho(s)\|_2\,\|u(s)\|_3\,\|\xi(s)\|_6\,ds\\
&\le\frac12\int_{Q_t}|\nabla\partial_t\rho|^2+c\int_0^t\|u(s)\|_3^2\,\|\xi(s)\|_V^2\,ds\\
&\le\frac12\int_{Q_t}|\nabla\partial_t\rho|^2+c\int_0^t\|u(s)\|_3^2\,\|(\xi,\xi_\Gamma)(s)\|_{\mathcal{V}_0}^2\,ds\\
&\le\frac12\int_{Q_t}|\nabla\partial_t\rho|^2+c\int_0^t\|u(s)\|_3^2\,\|\partial_t(\rho,\rho_\Gamma)(s)\|_*^2\,ds\,,
\end{aligned}$$

as well as

$$\begin{aligned}
&-\int_{Q_t}\nabla\rho\cdot\partial_t u\,\xi\le\int_0^t\|\nabla\rho(s)\|_6\,\|\partial_t u(s)\|_{3/2}\,\|\xi(s)\|_6\,ds\\
&\le c\int_0^t\|\nabla\rho(s)\|_V^2\,ds+c\int_0^t\|\partial_t u(s)\|_{3/2}^2\,\|\partial_t(\rho,\rho_\Gamma)(s)\|_*^2\,ds\,,
\end{aligned}$$

and we notice that the first term on the right-hand side is already bounded due to (6.8). In addition, the functions $s \mapsto \|u(s)\|_3^2$ and $s \mapsto \|\partial_t u(s)\|_{3/2}^2$ belong to $L^1(0,T)$, by (2.21) and (2.47). The last two terms on the right-hand side of (7.1) can easily be dealt with, by using the boundedness of $\pi'$ and $\pi'_\Gamma$ and the compactness inequality (2.58) in the following way:

$$
\begin{aligned}
&-\int_{Q_t} \pi'(\rho)|\partial_t\rho|^2 - \int_{\Sigma_t} \pi'_\Gamma(\rho_\Gamma)|\partial_t\rho_\Gamma|^2 \\
&\leq \frac{1}{2}\int_{Q_t} |\nabla\partial_t\rho|^2 + \frac{1}{2}\int_{\Sigma_t} |\nabla_\Gamma\partial_t\rho_\Gamma|^2 + c\int_0^t \|\partial_t(\rho,\rho_\Gamma)(s)\|_*^2\,ds\,.
\end{aligned}
$$

It remains to estimate the terms appearing in (7.2). To do that, we write (5.10)–(5.11) at time $t=0$ and account for the initial condition (5.12). We have

$$
\begin{aligned}
&\int_\Omega \partial_t\rho(0)v + \int_\Gamma \partial_t\rho_\Gamma(0)v_\Gamma + \int_\Omega \nabla\mu(0)\cdot\nabla v + \int_\Gamma \nabla_\Gamma\mu_\Gamma(0)\cdot\nabla_\Gamma v_\Gamma = \int_\Omega \rho_0\,u(0)\cdot\nabla v\,,\\
&\tau_\Omega\int_\Omega \partial_t\rho(0)\,v + \tau_\Gamma\int_\Gamma \partial_t\rho_\Gamma(0)\,v_\Gamma + \int_\Omega \nabla\rho_0\cdot\nabla v + \int_\Gamma \nabla_\Gamma\rho_{0|\Gamma}\cdot\nabla_\Gamma v_\Gamma\\
&\quad + \int_\Omega (\beta_\varepsilon+\pi)(\rho_0)v + \int_\Gamma (\beta_{\Gamma,\varepsilon}+\pi_\Gamma)(\rho_{0|\Gamma})v_\Gamma = \int_\Omega \mu(0)v + \int_\Gamma \mu_\Gamma(0)v_\Gamma\,,
\end{aligned}
$$

for every $(v,v_\Gamma)\in\mathcal{V}$. Now, we choose $(v,v_\Gamma)=(\xi,\xi_\Gamma):=\mathcal{N}(\partial_t(\rho,\rho_\Gamma)(0))$ in the first equality, $(v,v_\Gamma)=\partial_t(\rho,\rho_\Gamma)(0)$ in the second, and add. The terms involving $\mu(0)$ and $\mu_\Gamma(0)$ cancel out by the definition of $\mathcal{N}$ (see (2.9)–(2.10)). Moreover, invoking (2.12), we obtain that

$$
\begin{aligned}
&\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + \tau_\Omega\int_\Omega |\partial_t\rho(0)|^2 + \tau_\Gamma\int_\Gamma |\partial_t\rho_\Gamma(0)|^2\\
&= \int_\Omega \rho_0\,u(0)\cdot\nabla\xi - \int_\Omega \nabla\rho_0\cdot\nabla\partial_t\rho(0) - \int_\Gamma \nabla_\Gamma\rho_{0|\Gamma}\cdot\nabla_\Gamma\partial_t\rho_\Gamma(0)\\
&\quad - \int_\Omega (\beta_\varepsilon+\pi)(\rho_0)\,\partial_t\rho(0) - \int_\Gamma (\beta_{\Gamma,\varepsilon}+\pi_\Gamma)(\rho_{0|\Gamma})\,\partial_t\rho_\Gamma(0)\,,
\end{aligned}
$$

and we start estimating the right-hand side. For the first term, we account for the equivalence on $\mathcal{V}_0$ of the norms $\|\cdot\|_\mathcal{V}$ and $\|\cdot\|_{\mathcal{V}_0}$, and the definition (2.11) of $\|\cdot\|_*$ once more. Furthermore, we use the continuous embedding $W=H^2(\Omega)\subset C^0(\overline{\Omega})$ and the interpolation property, where $p,p_0,p_1\in[1,+\infty]$ and $\theta\in(0,1)$ satisfy $p_0\neq p_1$ and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ (see [3, p. 8 and Thm. 5.3.1 p. 113]),

$$
(L^{p_0}(\Omega),L^{p_1}(\Omega))_{\theta,p} = (L_{p_0p_0}(\Omega),L_{p_1p_1}(\Omega))_{\theta,p} = L_{pp}(\Omega) = L^p(\Omega)
$$

which gives in particular $(L^3(\Omega),L^{3/2}(\Omega))_{1/2,2}=L^2(\Omega)$ and thus the inequality

$$
\|u(0)\|_2 \leq c\,\|u\|_{H^1(0,T;L^{3/2}(\Omega))\cap L^2(0,T;L^3(\Omega))} \leq c\,.
$$

Hence, we can do the following computation:

$$
\begin{aligned}
&-\int_\Omega \rho_0\,u(0)\cdot\nabla\xi \leq \|\rho_0\|_\infty\,\|u(0)\|_2\,\|\nabla\xi\|_2\\
&\leq c\|\rho_0\|_W\,\|(\xi,\xi_\Gamma)\|_{\mathcal{V}_0} \leq c\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_* \leq \frac{1}{2}\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + c\,.
\end{aligned}
$$

We deal with the next two integrals by integrating by parts and using some of the assumptions (2.48):

$$
\begin{aligned}
&- \int_\Omega \nabla\rho_0 \cdot \nabla\partial_t\rho(0) - \int_\Gamma \nabla_\Gamma\rho_{0|\Gamma} \cdot \nabla_\Gamma\partial_t\rho_\Gamma(0) \\
&= \int_\Omega \Delta\rho_0\, \partial_t\rho(0) + \int_\Gamma (\Delta_\Gamma\rho_{0|\Gamma} - \partial_\nu\rho_0)\partial_t\rho_\Gamma(0) \le \delta \int_\Omega |\partial_t\rho(0)|^2 + \delta \int_\Gamma |\partial_t\rho_\Gamma(0)|^2 + c_\delta,
\end{aligned}
$$

where $\delta > 0$ is arbitrary. By invoking (5.3) for $\beta_\varepsilon$ and $\beta_{\Gamma,\varepsilon}$, and the assumptions (2.48), which also imply boundedness for $\rho_0$ and $\rho_{0|\Gamma}$, we find that

$$
\begin{aligned}
&- \int_\Omega (\beta_\varepsilon + \pi)(\rho_0)\, \partial_t\rho(0) - \int_\Gamma (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(\rho_{0|\Gamma})\, \partial_t\rho_\Gamma(0) \\
&\le \bigl(\|\beta^\circ(\rho_0)\|_2 + c\bigr)\|\partial_t\rho(0)\|_2 + \bigl(\|\beta^\circ_\Gamma(\rho_{0|\Gamma})\|_2 + c\bigr)\|\partial_t\rho_\Gamma(0)\|_2 \\
&\le \delta\|\partial_t\rho(0)\|_2^2 + \delta\|\partial_t\rho_\Gamma(0)\|_2^2 + c_\delta\,.
\end{aligned}
$$

Recalling all of the above estimates, and choosing $\delta > 0$ small enough, we see that $I_0 \le c$. At this point, we come back to (7.1) and apply the Gronwall lemma. We then conclude that

$$
\|\partial_t(\rho^\varepsilon, \rho^\varepsilon_\Gamma)\|_{L^\infty(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V})} \le c\,, \quad \text{whence} \quad \|(\rho^\varepsilon, \rho^\varepsilon_\Gamma)\|_{W^{1,\infty}(0,T;\mathcal{H})\cap H^1(0,T;\mathcal{V})} \le c\,. \tag{7.3}
$$

**Remark 7.1.** In connection with Remark 2.7, if $\tau_\Omega$ and $\tau_\Gamma$ are not supposed to be positive and (2.51) holds, one modifies the last estimates on the initial values as follows: we have

$$
\begin{aligned}
&- \int_\Omega \nabla\rho_0 \cdot \nabla\partial_t\rho(0) - \int_\Gamma \nabla_\Gamma\rho_{0|\Gamma} \cdot \nabla_\Gamma\partial_t\rho_\Gamma(0) \\
&\quad - \int_\Omega (\beta_\varepsilon + \pi)(\rho_0)\, \partial_t\rho(0) - \int_\Gamma (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(\rho_{0|\Gamma})\, \partial_t\rho_\Gamma(0) \\
&= - \int_\Omega \bigl(-\Delta\rho_0 + (\beta_\varepsilon + \pi)(\rho_0)\bigr)\partial_t\rho(0) - \int_\Gamma \bigl(\Delta_\Gamma\rho_{0|\Gamma} - \partial_\nu\rho_0 + (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(\rho_{0|\Gamma})\bigr)\partial_t\rho_\Gamma(0) \\
&\le \|-\Delta\rho_0 + (\beta_\varepsilon + \pi)(\rho_0), \Delta_\Gamma\rho_{0|\Gamma} - \partial_\nu\rho_0 + (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(\rho_{0|\Gamma})\|_{\mathcal{V}}\, \|\partial_t(\rho, \rho_\Gamma)(0)\|_{\mathcal{V}^*} \\
&\le \delta\|\partial_t(\rho, \rho_\Gamma)(0)\|^2_{\mathcal{V}^*} + c_\delta\,.
\end{aligned}
$$

This leads to an estimate that is somewhat weaker than (7.3) and yields a weaker result at the end of the procedure, as announced in the quoted remark.

**Sixth a priori estimate.** We set $\alpha := \mathrm{mean}(\mu, \mu_\Gamma)$ for a while and test (5.10) by the $\mathcal{V}_0$-valued function $(\mu, \mu_\Gamma) - \alpha(1,1)$. We obtain, for a.e. $t \in (0,T)$,

$$
\int_\Omega |\nabla\mu|^2 + \int_\Gamma |\nabla_\Gamma\mu_\Gamma|^2 = - \int_\Omega \partial_t\rho(\mu - \alpha) - \int_\Gamma \partial_t\rho_\Gamma(\mu_\Gamma - \alpha) + \int_\Omega \rho\, u\, \nabla\mu\,.
$$

Now, we recall that the norm (2.8) is equivalent on $\mathcal{V}_0$ to the natural norm. Thus, by also accounting for (2.47) and for (7.3), combined with the continuous embedding $V \subset L^6(\Omega)$, we may estimate the right-hand side a.e. in $(0,T)$ as follows:

$$
\begin{aligned}
&- \int_\Omega \partial_t\rho(\mu - \alpha) - \int_\Gamma \partial_t\rho_\Gamma(\mu_\Gamma - \alpha) + \int_\Omega \rho\, u\, \nabla\mu \\
&\le c\,\|\partial_t(\rho, \rho_\Gamma)\|_{\mathcal{V}^*}\|(\mu, \mu_\Gamma) - \alpha(1,1)\|_{\mathcal{V}_0} + \|\rho\|_6\, \|u\|_3\, \|\nabla\mu\|_2 \le c\bigl(\|\nabla\mu\|_2 + \|\nabla_\Gamma\mu_\Gamma\|_2\bigr).
\end{aligned}
$$

At this point, the Young inequality immediately yields that

$$\begin{aligned}&\|\nabla\mu^\varepsilon\|_{L^\infty(0,T;H)}+\|\nabla_\Gamma\mu_\Gamma^\varepsilon\|_{L^\infty(0,T;H_\Gamma)}\leq c\,,\quad\text{i.e.,}\\&\|(\mu^\varepsilon,\mu_\Gamma^\varepsilon)-\operatorname{mean}(\mu^\varepsilon,\mu_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{V})}\leq c\,.\end{aligned}\tag{7.4}$$

**Seventh a priori estimate.** We recall the estimate (6.4) already obtained, which holds a.e. in $(0,T)$ and also involves $\alpha := \operatorname{mean}(\mu,\mu_\Gamma)$. From (7.3) and (7.4), we infer that

$$\|\beta_\varepsilon(\rho)\|_{L^\infty(0,T;L^1(\Omega))}+\|\beta_{\Gamma,\varepsilon}(\rho_\Gamma)\|_{L^\infty(0,T;L^1(\Gamma))}\leq c.$$

We use this bound and (7.3) in the next estimate: we test (5.11) by $(1,1)/(|\Omega|+|\Gamma|)$ and obtain, for a.a. $t\in(0,T)$,

$$\begin{aligned}&|\operatorname{mean}(\mu,\mu_\Gamma)(t)|\leq c\,\|\partial_t(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{V}^*)}\\&\quad+c\|(\beta_\varepsilon+\pi)(\rho)\|_{L^\infty(0,T;L^1(\Omega))}+c\|(\beta_{\Gamma,\varepsilon}+\pi_\Gamma)(\rho_\Gamma)\|_{L^\infty(0,T;L^1(\Gamma))}\leq c\,.\end{aligned}$$

Combining this with (7.4), we conclude that

$$\|(\mu^\varepsilon,\mu_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{V})}\leq c,\quad\text{whence}\quad\|(\mu^\varepsilon,\mu_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{H})}\leq c\,.\tag{7.5}$$

**Eighth estimate.** At this point, we can test (5.11) by $(\beta_\varepsilon(\rho),\beta_\varepsilon(\rho_\Gamma))$ a.e. in $(0,T)$. By taking advantage of the above estimates and of (5.5), we immediately deduce that

$$\|\beta_\varepsilon(\rho^\varepsilon)\|_{L^\infty(0,T;H)}+\|\beta_\varepsilon(\rho_\Gamma^\varepsilon)\|_{L^\infty(0,T;H_\Gamma)}\leq c\,.\tag{7.6}$$

**Ninth a priori estimate.** We apply the part $vi)$ of Theorem 2.1 to the solution to the approximating problem with the choice $\gamma=\beta_{\Gamma,\varepsilon}$. As the constant $C_6$ does not depend on $\varepsilon$, inequality (2.44) yields a bound for $\zeta_\Gamma$ in terms of quantities that have already been estimated. Hence, we conclude that

$$\|\zeta_\Gamma^\varepsilon\|_{L^\infty(0,T;H_\Gamma)}\leq c\,.\tag{7.7}$$

At this point, we can apply the part $v)$ of Theorem 2.1. We thus have

$$\|(\rho^\varepsilon,\rho_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{W})}\leq c\,.\tag{7.8}$$

**Proof of Theorem 2.6.** We come back to the argument used for the existence part of proof of Theorem 2.3, recalling that the solution to the approximating problem converges to a solution to problem (2.28)–(2.31) in a proper topology, at least for a subsequence. In view of the estimates (7.3)–(7.8), the limiting solution also satisfies the further regularity specified by (2.49), and estimate (2.50) follows from semicontinuity. □

**Proof of Theorem 2.8.** We recall that $\mu$ and $\mu_\Gamma$ are bounded by Theorem 2.6. Thus, accounting for (2.53) and (2.54), we may choose $\rho_*,\rho^*\in(-1,1)$ with $\rho_*\leq\rho_0\leq\rho^*$ such that

$$\begin{aligned}&(\beta+\pi)(r)+\|\mu\|_\infty\leq 0\quad\text{and}\quad(\beta_\Gamma+\pi_\Gamma)(r)+\|\mu_\Gamma\|_\infty\leq 0\quad\text{for every }r\in(-1,\rho_*),\\&(\beta+\pi)(r)-\|\mu\|_\infty\geq 0\quad\text{and}\quad(\beta_\Gamma+\pi_\Gamma)(r)-\|\mu_\Gamma\|_\infty\geq 0\quad\text{for every }r\in(\rho^*,1).\end{aligned}$$

Then, we test (2.29) by $((\rho-\rho^*)^+,(\rho_\Gamma-\rho^*)^+)$, where $(\,\cdot\,)^+$ stands for the positive part, and integrate with respect to time. We obtain the identity

$$\begin{aligned}
&\tau_\Omega \int_\Omega |(\rho(t)-\rho^*)^+|^2 + \tau_\Gamma \int_\Gamma |(\rho_\Gamma(t)-\rho^*)^+|^2 \\
&\quad + \int_{Q_t} |\nabla(\rho-\rho^*)^+|^2 + \int_{\Sigma_t} |\nabla_\Gamma(\rho_\Gamma-\rho^*)^+|^2 \\
&= \int_{Q_t} \bigl(\mu-(\beta+\pi)(\rho)\bigr)(\rho-\rho^*)^+ \\
&\quad + \int_{\Sigma_t} \bigl(\mu_\Gamma-(\beta_\Gamma+\pi_\Gamma)(\rho_\Gamma)\bigr)(\rho_\Gamma-\rho^*)^+ \,.
\end{aligned}$$

Since the right-hand side is nonpositive, we conclude that $(\rho-\rho^*)^+=0$, i.e., $\rho\le\rho^*$. In the same way, one proves that $(\rho_*-\rho)^+=0$, i.e., $\rho\ge\rho_*$. □

Now, we start the proof of Theorem 2.9. Also in this case, we proceed formally. Moreover, in order to simplify the notation, we perform our estimates on the solutions to problem (2.28)–(2.31), directly, and avoid the approximating problem. For $i=1,2$, we denote, by $\mu_i$, $\mu_{i\Gamma}$, etc., the components of the solutions corresponding to $u_i$, while $\mu$, $\mu_\Gamma$, etc., are the differences, e.g., $\mu=\mu_1-\mu_2$, according to the notation of the statement. For brevity, we also set $u:=u_1-u_2$, as well as

$$f:=\widehat\beta+\widehat\pi\,, \quad f_\Gamma:=\widehat\beta_\Gamma+\widehat\pi_\Gamma\,, \quad \text{whence} \quad f'=\beta+\pi \quad \text{and} \quad f'_\Gamma=\beta_\Gamma+\pi_\Gamma\,.$$

Moreover, since the result given by Theorem 2.8 holds for both solutions, we can assume that $f'$, $f''$, $f'_\Gamma$ and $f''_\Gamma$ are bounded and Lipschitz continuous, the corresponding constants depending only on the previous assumptions on the structure, the norms of the velocity fields $u_i$ related to (2.47), and the assumptions (2.48) on the initial datum.

**First auxiliary estimate.** We write (2.28) for both solutions, take the difference and differentiate with respect to time. Then, we test the obtained equality by $(\xi,\xi_\Gamma):=\mathcal{N}(\partial_t(\rho,\rho_\Gamma))$ a.e. in $(0,T)$ and integrate over $(0,t)$. With the help of (2.14) and (2.11) we infer that

$$\frac12\,\|\partial_t(\rho,\rho_\Gamma)(t)\|_*^2 + \int_{Q_t}\nabla\partial_t\mu\cdot\nabla\xi + \int_{\Sigma_t}\nabla_\Gamma\partial_t\mu_\Gamma\cdot\nabla_\Gamma\xi_\Gamma = \int_{Q_t}\partial_t\bigl(\rho_1u_1-\rho_2u_2\bigr)\cdot\nabla\xi\,.$$

At the same time, we write (2.29) for both solutions, take the difference and differentiate it with respect to time; then, we test by $\partial_t(\rho,\rho_\Gamma)$ and integrate over $(0,t)$. Finally, we add the same integrals $\int_{Q_t}(\rho\,\partial_t\rho+\nabla\rho\cdot\nabla\partial_t\rho)$ and $\int_{\Sigma_t}(\rho_\Gamma\partial_t\rho_\Gamma+\nabla_\Gamma\rho_\Gamma\cdot\nabla_\Gamma\partial_t\rho_\Gamma)$ to both sides, for convenience. We obtain that

$$\begin{aligned}
&\frac{\tau_\Omega}{2}\int_\Omega|\partial_t\rho(t)|^2 + \frac{\tau_\Gamma}{2}\int_\Gamma|\partial_t\rho_\Gamma(t)|^2 + \int_{Q_t}|\nabla\partial_t\rho|^2 + \int_{\Sigma_t}|\nabla_\Gamma\partial_t\rho_\Gamma|^2 \\
&\quad + \frac12\,\|\rho(t)\|_V^2 + \frac12\,\|\rho_\Gamma(t)\|_{V_\Gamma}^2
\end{aligned}$$

$$
\begin{aligned}
&= - \int_{Q_t} \bigl( f''(\rho_1)\partial_t\rho_1 - f''(\rho_2)\partial_t\rho_2 \bigr) \partial_t\rho - \int_{\Sigma_t} \bigl( f''_\Gamma(\rho_{1\Gamma})\partial_t\rho_{1\Gamma} - f''_\Gamma(\rho_{2\Gamma})\partial_t\rho_{2\Gamma} \bigr) \partial_t\rho_\Gamma \\
&\quad + \int_{Q_t} \partial_t\mu \, \partial_t\rho + \int_{\Sigma_t} \partial_t\mu_\Gamma \, \partial_t\rho_\Gamma \\
&\quad + \int_{Q_t} \bigl( \rho \, \partial_t\rho + \nabla\rho \cdot \nabla\partial_t\rho \bigr) + \int_{\Sigma_t} \bigl( \rho_\Gamma \partial_t\rho_\Gamma + \nabla_\Gamma\rho_\Gamma \cdot \nabla_\Gamma\partial_t\rho_\Gamma \bigr) .
\end{aligned}
$$

At this point, we add these equalities to each other and employ the definition of $\mathcal{N}$ (see (2.9)–(2.10)) in order to cancel four terms in the sum. Moreover, we rearrange the right-hand side, account for (2.11) and the equivalence of (2.8) to the norm in $\mathcal{V}$ on the subspace $\mathcal{V}_0$, and use the boundedness and the Lipschitz continuity of both $f''$ and $f''_\Gamma$. We then obtain that

$$
\begin{aligned}
&\int_\Omega |\nabla\xi(t)|^2 + \int_\Gamma |\nabla_\Gamma\xi_\Gamma(t)|^2 + \int_\Omega |\partial_t\rho(t)|^2 + \int_\Gamma |\partial_t\rho_\Gamma(t)|^2 \\
&\quad + \int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{\Sigma_t} |\nabla_\Gamma\partial_t\rho_\Gamma|^2 + \|\rho(t)\|_V^2 + \|\rho_\Gamma(t)\|_{V_\Gamma}^2 \\
&\leq c \int_{Q_t} |\partial_t\rho| \, |u_1| \, |\nabla\xi| + c \int_{Q_t} |\partial_t\rho_2| \, |u| \, |\nabla\xi| + c \int_{Q_t} |\rho| \, |\partial_t u_1| \, |\nabla\xi| + c \int_{Q_t} |\rho_2| \, |\partial_t u| \, |\nabla\xi| \\
&\quad + c \int_{Q_t} |\rho| \, |\partial_t\rho_1| \, |\partial_t\rho| + c \int_{\Sigma_t} |\rho_\Gamma| \, |\partial_t\rho_{1\Gamma}| \, |\partial_t\rho_\Gamma| \\
&\quad + c \int_{Q_t} (|\rho_2| + 1) \, |\partial_t\rho|^2 + c \int_{\Sigma_t} (|\rho_{2\Gamma}| + 1) \, |\partial_t\rho_\Gamma|^2 \\
&\quad + c \int_0^t \|\rho(s)\|_V \, \|\partial_t\rho(s)\|_V \, ds + c \int_0^t \|\rho_\Gamma(s)\|_{V_\Gamma} \, \|\partial_t\rho_\Gamma(s)\|_{V_\Gamma} \, ds \leq c \sum_{j=1}^{10} I_j \, ,
\end{aligned}
$$

with obvious definitions of $I_1, \ldots, I_{10}$. We now estimate each of these integrals by using the Hölder, Sobolev and Young inequalities as follows. We have, for every $\delta > 0$,

$$
\begin{aligned}
I_1 &\leq \int_0^t \|\partial_t\rho(s)\|_6 \, \|u_1(s)\|_3 \, \|\nabla\xi(s)\|_2 \, ds \\
&\leq \delta \int_0^t \|\partial_t\rho(s)\|_V^2 \, ds + c_\delta \int_0^t \|u_1(s)\|_3^2 \, \|\nabla\xi(s)\|_2^2 \, ds \, , \\
I_2 &\leq \int_0^t \|\partial_t\rho_2(s)\|_6 \, \|u(s)\|_3 \, \|\nabla\xi(s)\|_2 \, ds \\
&\leq \delta \int_0^t \|u(s)\|_3^2 \, ds + c_\delta \int_0^t \|\partial_t\rho_2(s)\|_V^2 \, \|\nabla\xi(s)\|_2^2 \, ds \, , \\
I_3 &\leq \int_0^t \|\rho(s)\|_6 \, \|\partial_t u_1(s)\|_3 \, \|\nabla\xi(s)\|_2 \, ds \\
&\leq c \int_0^t \|\rho(s)\|_V^2 \, ds + \int_0^t \|\partial_t u_1(s)\|_3^2 \, \|\nabla\xi(s)\|_2^2 \, ds \, , \\
I_4 &\leq \int_{Q_t} |\nabla\xi(s)|^2 + c \, \|\rho_2\|_\infty^2 \int_{Q_t} |\partial_t u|^2 \, ,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &\le \int_0^t \|\rho(s)\|_3 \, \|\partial_t\rho_1(s)\|_3 \, \|\partial_t\rho(s)\|_3 \, ds \\
&\le \delta \int_0^t \|\partial_t\rho(s)\|_V^2 \, ds + c_\delta \int_0^t \|\partial_t\rho_1(s)\|_V^2 \, \|\rho(s)\|_V^2 \, ds \\
&= \delta \int_{Q_t} |\nabla\partial_t\rho|^2 + \delta \int_{Q_t} |\partial_t\rho|^2 + c_\delta \int_0^t \|\partial_t\rho_1(s)\|_V^2 \, \|\rho(s)\|_V^2 \, ds \, .
\end{aligned}
$$

Moreover, an analogous estimate holds for $I_6$. On the other hand, it is easy to see that

$$
I_7 \le c\,(1 + \|\rho_2\|_\infty) \int_{Q_t} |\partial_t\rho|^2 \, , \qquad I_8 \le c\,(1 + \|\rho_{2\Gamma}\|_\infty) \int_{\Sigma_t} |\partial_t\rho_\Gamma|^2 \, .
$$

Finally, $I_9$ and $I_{10}$ can be treated just with the Young inequality. Now, we observe that the functions

$$
s \mapsto \|u_1(s)\|_3^2 \, , \quad s \mapsto \|\partial_t\rho_i(s)\|_V^2 \, , \; i = 1,2 \, , \quad s \mapsto \|\partial_t u_1(s)\|_3^2 \, , \quad s \mapsto \|\partial_t\rho_{1\Gamma}(s)\|_{V_\Gamma}^2 \, ,
$$

all belong to $L^1(0,T)$. Hence, we collect all the inequalities we have obtained, choose $\delta$ small enough, and apply the Gronwall lemma. We conclude that

$$
\|(\nabla\xi, \nabla_\Gamma\xi_\Gamma)\|_{L^\infty(0,T;\mathcal{H})} + \|(\rho,\rho_\Gamma)\|_{W^{1,\infty}(0,T;\mathcal{H})\cap H^1(0,T;\mathcal{V})} \le c\,\|u\|_{H^1(0,T;L^3(\Omega))} \, , \tag{7.9}
$$

where we recall that $(\xi,\xi_\Gamma) := \mathcal{N}(\partial_t(\rho,\rho_\Gamma))$. Notice that (7.9) implies a part of (2.56).

**Second auxiliary estimate.** We write the equation (2.28) for both solutions and test the difference a.e. in $(0,T)$ by $(\mu,\mu_\Gamma)$. The same we do with (2.29), and test the difference by $-(\mu,\mu_\Gamma)$. Then, we sum up and have, a.e. in $(0,T)$,

$$
\begin{aligned}
&\|\mu\|_V^2 + \|\mu_\Gamma\|_{V_\Gamma}^2 \\
&= (\tau_\Omega - 1) \int_\Omega \partial_t\rho \, \mu + (\tau_\Gamma - 1) \int_\Gamma \partial_t\rho_\Gamma \, \mu_\Gamma + \int_\Omega \bigl(\rho_1 u_1 - \rho_2 u_2\bigr) \cdot \nabla\mu \\
&\quad + \int_\Omega \nabla\rho \cdot \nabla\mu + \int_\Gamma \nabla_\Gamma\rho_\Gamma \cdot \nabla_\Gamma\mu_\Gamma \\
&\quad + \int_\Omega \bigl(f'(\rho_1) - f'(\rho_2)\bigr)\mu + \int_\Gamma \bigl(f_\Gamma'(\rho_{1\Gamma}) - f_\Gamma'(\rho_{2\Gamma})\bigr)\mu_\Gamma \, .
\end{aligned}
$$

Now, we rearrange the right-hand side and use the boundedness and the Lipschitz continuity of $f'$ and $f'_\Gamma$, as well as the Hölder and Young inequalities. We obtain a.e. in $(0,T)$ that

$$
\begin{aligned}
&\|\mu\|_V^2 + \|\mu_\Gamma\|_{V_\Gamma}^2 \\
&\le \delta\,\|\mu\|_H^2 + c_\delta\|\partial_t\rho\|_H^2 + \delta\,\|\mu_\Gamma\|_{H_\Gamma}^2 + c_\delta\|\partial_t\rho_\Gamma\|_{H_\Gamma}^2 + \bigl(\|\rho\|_6\,\|u_1\|_3 + \|\rho_2\|_6\,\|u\|_3\bigr)\|\nabla\mu\|_2 \\
&\quad + \delta\,\|\nabla\mu\|_H^2 + c_\delta\|\nabla\rho\|_H^2 + \delta\,\|\nabla_\Gamma\mu_\Gamma\|_{\mathcal{H}}^2 + c_\delta\|\nabla_\Gamma\rho_\Gamma\|_{H_\Gamma}^2 \\
&\quad + \delta\,\|\mu\|_H^2 + c_\delta\|\rho\|_H^2 + \delta\,\|\mu_\Gamma\|_{H_\Gamma}^2 + c_\delta\|\rho_\Gamma\|_{H_\Gamma}^2 \, ,
\end{aligned}
$$

where $\delta > 0$ is arbitrary. By choosing $\delta$ small enough, using the Sobolev inequality, and recalling that $u_1 \in L^\infty(0,T;L^3(\Omega))$ and $\rho_2 \in L^\infty(0,T;V)$, we deduce that

$$
\|\mu\|_V^2 + \|\mu_\Gamma\|_{V_\Gamma}^2 \le c\bigl(\|\partial_t\rho\|_H^2 + \|\partial_t\rho_\Gamma\|_{H_\Gamma}^2 + \|\rho\|_V^2 + \|\rho_\Gamma\|_{V_\Gamma}^2 + \|u\|_3^2\bigr) \quad \text{a.e. in } (0,T).
$$

At this point, by accounting for (7.9), we conclude that

$$\|(\mu,\mu_\Gamma)\|_{L^\infty(0,T;\mathcal{V})} \le c\,\|u\|_{H^1(0,T;L^3(\Omega))}\,. \tag{7.10}$$

**Proof of Theorem 2.9.** We recall that (2.21) holds true for both $u_1$ and $u_2$ and rewrite the transport terms in (2.28) in the form $\int_\Omega \nabla\rho_i\cdot u_i\,v$. Then we take the difference of the equations, written for both solutions, and apply Lemma 3.1 for a.a. $t\in(0,T)$ with $\gamma=0$ and the following choice of $g$ and $g_\Gamma$:

$$g = \bigl(-\partial_t\rho - \nabla\rho_1\cdot u_1 + \nabla\rho_2\cdot u_2\bigr)(t) = \bigl(-\partial_t\rho - \nabla\rho_1\cdot u + \nabla\rho\cdot u_2\bigr)(t) \quad\text{and}\quad g_\Gamma = -\partial_t\rho_\Gamma(t)\,.$$

We then obtain that

$$\|(\mu,\mu_\Gamma)(t)\|_{\mathcal{W}} \le c\bigl(\|(\mu,\mu_\Gamma)(t)\|_{\mathcal{V}} + \|\partial_t\rho(t)\|_2 + \|u(t)\|_3 + \|\nabla\rho(t)\|_6 + \|\partial_t\rho_\Gamma(t)\|_2\,,$$

where $c$ depends only on $\Omega$ and the norms of $\nabla\rho_1$ and $u_2$ in the spaces $L^\infty(0,T;L^6(\Omega))$ and $L^\infty(0,T;L^3(\Omega))$, respectively. By combining this with (7.9)–(7.10), we deduce that

$$\|(\mu,\mu_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} \le c\,\|u\|_{H^1(0,T;L^3(\Omega))}\,,$$

which is a part of (2.56). In order to prove the remaining part of the estimate, we write (2.29) for both solutions, take the difference, and apply Lemma 3.1 for a.a. $t\in(0,T)$ with $\gamma=0$ and the choice

$$g = \bigl(-\tau_\Omega\partial_t\rho - f'(\rho_1) + f'(\rho_2) + \mu\bigr)(t) \quad\text{and}\quad g_\Gamma = \bigl(-\tau_\Gamma\partial_t\rho_\Gamma - f_\Gamma'(\rho_{1\Gamma}) + f_\Gamma'(\rho_{2\Gamma}) + \mu_\Gamma\bigr)(t).$$

We then obtain that

$$\|(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} \le c\bigl(\|(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{V})} + \|(g,g_\Gamma)\|_{L^\infty(0,T;\mathcal{H})}\bigr) \le c\,\|u\|_{H^1(0,T;L^3(\Omega))}\,,$$

where the last inequality follows from (7.9) and (7.10). With this, (2.56) is completely proved. □

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