Inoperability Propagation in Networks with Coupled Logistic Dynamics

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Abstract. Networked infrastructures are essential for modern society but are vulnerable to various threats that can lead to significant disruptions. Traditional models often fail to capture the complex dynamics of these systems. To address this issue, we propose a novel framework based on coupled logistic dynamics, where each unit is represented as a node in a coupled logistic system. In this model, damage growth within individual nodes follows logistic patterns and propagates through interdependent units via coupling mechanisms. Our analysis demonstrates that degradation spreads radially across the network, with delays in damage that scale logarithmically with respect to the coupling strength and the distance from the initial failure. We conclude with numerical results that support our theoretical findings, offering insights into the resilience and vulnerability of networked infrastructures.

Keywords: Logistic growth · Cascading failures · Interdependent networks · System resilience · Damage propagation

1 Introduction

In modern society, networked infrastructures play a critical role in sustaining essential services, facilitating economic activities, and ensuring the well-being of communities. However, these infrastructures are vulnerable to various threats, including natural disasters, cyberattacks, equipment failures, and operational errors, which can lead to disruptions, downtime, and cascading failures with far-reaching consequences [1,4,11]. Understanding the dynamics of failures and inoperabilities in network infrastructures is essential for enhancing resilience, mitigating risks, and maintaining the reliability of critical systems.

Traditional approaches to modelling failures and inoperabilities in network infrastructures often rely on simplistic assumptions or linear frameworks that fail to capture the complex interdependencies and nonlinear dynamics inherent in these systems [5,10,12]. To address this limitation, we propose a novel modelling framework based on coupled logistic dynamics [6,9], which offers a more realistic

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representation of the propagation and growth of damage within interconnected infrastructures.

Examples of logistic growth of damage within individual units or components of larger infrastructure systems include:

- **Transformer in a Power Grid:** A fault in a transformer can initially cause gradual increases in temperature or vibration. As the fault progresses, the degradation rate accelerates due to thermal runaway or insulation breakdown, leading to rapid damage growth. Eventually, the rate stabilizes as the transformer fails catastrophically or protective mechanisms, such as overcurrent relays or automatic shutdown systems, activate.
- **Critical Component in a Transportation System:** Components like bridges or tunnels can suffer damage from corrosion, fatigue, or structural deficiencies. Corrosion in a bridge, for example, may start slowly but accelerate as the structure's integrity deteriorates. The damage growth eventually stabilizes when the bridge approaches collapse or remediation measures, such as repairs or load restrictions, are implemented.
- **Network Router in a Communication Network:** Routers can experience faults from hardware malfunctions, software bugs, or cyberattacks. Initial faults may cause minor network disruptions, but as the faults propagate, the impact on performance escalates rapidly, leading to congestion and service outages. The damage growth stabilizes as failover mechanisms or redundancy features activate to restore normal operation or isolate the affected component.
- **Pump in a Water Distribution System:** Pumps can experience wear, mechanical failures, or cavitation issues. A fault may initially cause a gradual decrease in efficiency, but as it worsens, it leads to accelerated deterioration due to increased vibrations or overheating. The damage growth stabilizes as protective measures, such as automatic shutdown systems or maintenance protocols, activate.

In these examples, damage growth within a single unit or component typically follows a logistic pattern: slow initial progression, accelerating deterioration, and eventual stabilization as critical thresholds are reached or intervention measures are implemented.

Notice that, although to the best of our knowledge this work is the first modelling failure/inoperability propagation via coupled logistic dynamical networks, in recent literature logistic functions have been adopted as a convenient tool in the context of the study of cascading effects: in [2] a simulation-based model for the spread of failures is provided where the breakdown of the elements is modelled as a logistic function; in [13] the influence of the stress-dependent wear out on the failure propagation dynamics is modelled as a logistic function; in [7] a model for cascading failures in a network is developed where the probability that an overloaded node fails as a logistic function.

In our proposed framework, each unit or component within the network is modelled as a node in a coupled logistic system, where the growth of damage within a unit follows logistic patterns once it exceeds a certain threshold.

Moreover, the damage can propagate to interdependent units through coupling mechanisms, triggering logistic growth of damage in those units as well. This coupling captures the feedback loops, dependencies, and interactions present in networked infrastructures, where failures in one component can propagate and amplify throughout the system.

Specifically, under the assumption of logistic growth and considering a small coupling among the subsystems, an initial failure or damage affecting one unit leads to degradations in other systems that follow a similar logistic pattern. These degradations are essentially delayed versions of the original damage, with the same growth shape but shifted in time. Moreover, such degradation propagates radially in the network and the delay experienced by the units scales with the product of the logarithm of the coupling strength and the distance from the initiator in terms of the number of hops in the graph. Numerical results confirm the theoretical findings.

2 Materials and Methods

In this section, we present a comprehensive modeling framework to understand the dynamics of failures and inoperabilities in networked infrastructures characterized by logistic-like failure dynamics, which provide a realistic representation of damage propagation within interdependent systems.

This section is structured as follows:

1. After introducing some necessary preliminary definitions, we examine the dynamics of two interdependent systems governed by coupled logistic equations. This initial analysis allows us to understand the basic interactions and propagation mechanisms between a pair of systems. We identify the fixed points, analyze the stability, and derive the conditions under which the growth of damage follows logistic patterns.
2. Next, we extend the model to n interdependent systems. We generalize the coupled logistic equations to accommodate multiple systems, each influenced by its neighbours through a defined adjacency matrix. By applying perturbation theory, we derive the equations governing the evolution of damage in this multi-system framework. We explore the propagation of initial perturbations across the network and establish the scaling relationships for the delay in the spread of damage.
3. Finally, we introduce an algorithm for calculating the *path degeneracy*, z_{ii_0}, which measures the number of shortest paths from the initially perturbed node to other nodes in the network. This metric is crucial for understanding the distribution of dependencies and the potential for cascading failures. We provide a step-by-step procedure to compute z_{ii_0} using matrix operations and illustrate its application within the context of our coupled logistic framework.

2.1 Notation and Graph Theory

We denote vectors with boldface lowercase letters and matrices with uppercase letters.

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a graph with N nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and e edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where $(v_j, v_i) \in \mathcal{E}$ captures the existence of a link from node v_i to node v_j. A graph is said to be undirected if $(v_j, v_i) \in E$ whenever $(v_i, v_j) \in E$, while is said to be directed otherwise. In the following, we consider an undirected graph. An undirected graph is *connected* if each node can be reached by every other node via the edges. Let the neighbourhood $\mathcal{D}_i$ of a node $v_i \in \mathcal{V}$ be the set of nodes $v_j \in \mathcal{V}$ such that $(v_i, v_j) \in \mathcal{E}$; similarly, the p-neighborhood $\mathcal{D}_i^p$ of a node $v_i \in \mathcal{V}$ is the set of nodes $v_j \in \mathcal{V}$ such that the length of the minimum path from v_i to v_j is p. The *degree* d_i of a node v_i is the number of its incoming edges, i.e., $d_i = |\mathcal{D}_i|$. The *adjacency matrix* associated to a graph $\mathcal{G}$ with n nodes is the $n \times n$ matrix A such that its (i, j)-th entry a_{ij} is $a_{ij} = 1$ if $(v_j, v_i) \in E$ and $A_{ij} = 0$, otherwise.

2.2 Two Coupled Systems

Let us analyze the dynamics of two interdependent systems (referred to as system X and system Y) governed by coupled logistic equations:

$$\begin{cases} \dfrac{d\,y(t)}{dt} &= y(t)(1 - y(t)) + \epsilon\, x(t)(1 - y(t)) \\[2ex] \dfrac{d\,x(t)}{dt} &= x(t)(1 - x(t)) + \epsilon\, y(t)(1 - x(t)) \end{cases} \tag{1}$$

Here, $0 \le \epsilon \ll 1$ quantifies the interaction between systems X and Y. The system exhibits two fixed points: $(x, y) = (0, 0)$ (unstable) and $(x, y) = (1, 1)$ (stable).

Notice that when $x \gg \epsilon$ and $1 - x \gg \epsilon$, we can approximate its dynamics as $\frac{d\,x(t)}{dt} \approx x(t)(1 - x(t))$ (a similar result holds for for $\frac{d\,y(t)}{dt}$). This means that in a range $\Delta < x, y < 1 - \Delta$ with $\epsilon \ll \Delta \ll 1$, the growth of x has the same shape as the growth of y since $\epsilon \ll 1$; in particular, as clarified later in the text, they both exhibit logistic growth and will therefore appear as time-shifted versions of each other.

Suppose that at $t = 0$, $x(t)$ starts at 0 while $y(t)$ starts at a very small $y_0 \to 0$. When $y(t)$ is small, the term $y(t)(1 - y(t)) \approx y(t)$ dominates, leading to an initial exponential growth of $y(t)$. As $y(t)$ grows, the term $\epsilon y(t)$ becomes increasingly significant in the equation for $\frac{d\,x(t)}{dt}$; thus, $x(t)$ will also start growing, but since $\epsilon \ll 1$, $x(t)$ will remain relatively small compared to exponentially growing $y(t)$. Eventually, $x(t)$ will grow "almost logistically" in the region $[\Delta, 1 - \Delta]$, mirroring the previous growth of $y(t)$ in the same region.

To calculate the time shift, we need to understand how $x(t), y(t)$ reach values $\overline{x}, \overline{y} \approx \Delta$ with $\epsilon \ll \Delta \ll 1$ so that the approximation $\frac{d\,x(t)}{dt} \approx x(t)(1 - x(t))$, $\frac{d\,y(t)}{dt} \approx y(t)(1 - y(t))$ holds. To consider the growth away from the unstable fixed point, we analyze the linearized system:

$$\begin{cases} \frac{d\,y(t)}{dt} &= y(t) + \epsilon\, x(t) \\ \\ \frac{d\,x(t)}{dt} &= x(t) + \epsilon\, y(t) \end{cases} \tag{2}$$

One approach to understanding this system's behaviour is to resort to perturbation theory: by expanding our functions in powers of ϵ,

$$\begin{cases} y(t, \epsilon) = y^{(0)}(t) + \epsilon y^{(1)}(t) + \dots \\ x(t, \epsilon) = x^{(0)}(t) + \epsilon x^{(1)}(t) + \dots \end{cases}$$

Equation (2) becomes

$$\begin{cases} \frac{d}{dt}\left(y^{(0)}(t) + \epsilon y^{(1)}(t) + \dots\right) = \left(y^{(0)}(t) + \epsilon y^{(1)}(t) + \dots\right) + \epsilon\left(x^{(0)}(t) + \epsilon x^{(1)}(t) + \dots\right) \\ \\ \frac{d}{dt}\left(x^{(0)}(t) + \epsilon x^{(1)}(t) + \dots\right) = \left(x^{(0)}(t) + \epsilon x^{(1)}(t) + \dots\right) + \epsilon\left(y^{(0)}(t) + \epsilon y^{(1)}(t) + \dots\right) \end{cases}$$

This system is satisfied for every ϵ if all the terms corresponding to the same power in ϵ are equal, leading to:

$$\begin{cases} \dfrac{dy^{(0)}}{dt} &= y^{(0)} \\ \dfrac{dx^{(0)}}{dt} &= x^{(0)} \\ \dfrac{dy^{(1)}}{dt} &= y^{(1)} + x^{(0)} \\ \dfrac{dx^{(1)}}{dt} &= x^{(1)} + y^{(0)} \\ & \dots \end{cases} \tag{3}$$

Notice that considering terms only up to the first order, the perturbation expansions of Eq. (1) and Eq. (2) yield the same equations for the evolution of $x^{(0)}$, $y^{(0)}$, $x^{(1)}$, and $y^{(1)}$; however, working with the linearised system facilitates keeping track of combinatorial factors. To satisfy the initial conditions $x(t = 0, \epsilon) = x_0$, $y(t = 0, \epsilon) = y_0$ for all ϵ, we impose the initial conditions $x^{(0)}(t = 0) = x_0$, $y^{(0)}(t = 0) = y_0$, and $x^{(k)}(t = 0) = y^{(k)}(t = 0) = 0$ for all $k > 0$.

Now, suppose the perturbation starts in system Y, i.e., $x(t = 0) = x_0 = 0$ and $y(t = 0) = y_0 \ll 1$. With the given initial conditions, we have that

$$y^{(0)}(t) = e^t y_0$$

and thus

$$y(t) = e^t y_0 + \mathcal{O}(\epsilon).$$

With the given initial conditions, $x^{(0)}(t) = 0$; thus, the first non-vanishing term is $x^{(1)}$ with initial condition $x^{(1)}(t = 0) = 0$:

$$\frac{dx^{(1)}(t)}{dt} = x^{(1)}(t) + y^{(0)}(t) = x^{(1)}(t) + e^t y_0$$

Thus, by solving the above differential equation, we have that

$$x^{(1)}(t) = \int_0^t e^{t-\tau} e^{\tau} y_0 \, d\tau = te^t y_0$$

and hence

$$x(t) = \epsilon\, t\, e^t y_0 + O(\epsilon^2) \approx \epsilon\, t\, y(t) + O(\epsilon^2).$$

Since both $x(t)$ and $y(t)$ are growing, system X will reach the same state as system Y after a delay τ such that $x(t + \tau) = y(t)$.

Let us now characterize such delay τ. Considering only the smallest non-vanishing terms, we solve the following system of equations:

$$\begin{cases} x(t+\tau) = y(t) \\ x(t) \approx \epsilon y_0 t e^t = \epsilon y_0 f(t) \\ y(t) \approx y_0 e^t \end{cases} \tag{4}$$

where $f(w) = we^w$ and its inverse function is typically referred to as the *Lambert function* W [8]. In particular, for z real with $z \geq 3$, a reasonably accurate estimation[1] W_0 for the Lambert function is given by [3]:

$$W_0(z) = \ln z - \ln\ln z + \underbrace{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{km} (\ln\ln z)^{m+1} (\ln z)^{-k-m-1}}_{\omega_{k,m}(z)}$$

For large real values of z, the term $\omega_{k,m}(z)$ in the above expression becomes negligible and we have that

$$W_0(z) \approx \ln(z) - \ln(\ln(z)) + o(1).$$

We are now in a position to characterize the delay τ. In particular, the equation $x(t + \tau) = y(t)$ yields

$$\epsilon(t+\tau)e^{t+\tau} = e^t,$$

i.e.,

$$(t+\tau)e^{t+\tau} = \frac{e^t}{\epsilon}.$$

At this point applying the Lambert function $W(\cdot)$ on both sides, and by considering the approximation $W_0(\cdot)$ we obtain

$$\begin{aligned} t + \tau &= W\left(\frac{e^t}{\epsilon}\right) \approx W_0\left(\frac{e^t}{\epsilon}\right) \approx \ln\left(\frac{e^t}{\epsilon}\right) - \ln\ln\left(\frac{e^t}{\epsilon}\right) + o(1) \\ &= t - \ln(\epsilon) - \ln(t - \ln(\epsilon)) + o(1) \end{aligned}$$

[1] This term is the principal branch, see [3] for details.

and thus

$$\tau \approx -\ln(\epsilon) - \ln(t - \ln(\epsilon)) + o(1).$$

The above expression, apart from a logarithmic correction term $\ln(t - \ln(\epsilon))$, which is negligible for small ϵ, follows the scaling

$$\tau \sim -\ln(\epsilon). \tag{5}$$

2.3 Extension to n Subsystems

This reasoning can be extended to the dynamics of $i = 1, \ldots, n$ interdependent systems governed by coupled logistic equations:

$$\frac{d\,y_i(t)}{dt} = y_i(t)(1 - y_i(t)) + \left(\epsilon \sum_j a_{ij}\, y_j(t)\right)(1 - y_i(t))\;. \tag{6}$$

Here, the $n \times n$ adjacency matrix A with entries $a_{ij} \in \{0, 1\}$ represents dependencies among nodes ($a_{ij} = 1$ if system i depends on system j; $a_{ij} = 0$ otherwise) while ϵ measures the strength of the dependencies. The unperturbed system has two fixed points, one featuring $y_i = 0$ (unstable) and one with $y_i = 1$ (stable). Also in the perturbed case, the fixed points for the coupled systems are $y_i = 0\ \forall i$ (unstable) and $y_i = 1\ \forall i$ (stable).

To understand the growth away from the unstable fixed point, we linearize Eq. (6), obtaining

$$\frac{d\,y_i(t)}{dt} = y_i(t) + \epsilon \sum_j a_{ij} y_j(t). \tag{7}$$

Then, applying perturbation theory with $y_i(t) = \sum_{k=0}^{\infty} \epsilon^k y_i^{(k)}(t)$, we get:

$$\begin{cases} \dfrac{dy_i^{(0)}(t)}{dt} & = y_i^{(0)}(t) \\ \dfrac{dy_i^{(1)}(t)}{dt} & = y_i^{(1)}(t) + \sum_j a_{ij} y_j^{(0)}(t) \\ & \vdots \\ \dfrac{dy_i^{(k+1)}(t)}{dt} & = y_i^{(k+1)}(t) + \sum_j a_{ij} y_j^{(k)}(t) \\ & \vdots \end{cases} \tag{8}$$

Considering an initial scenario where only one node i_0 is perturbed (i.e., $y_k(0) = y_0$ if $k = i_0$, $y_k(0) = 0$ otherwise), we have that all the zero-th terms are null $y_k^{(0)}(t = 0) = 0$ for $k \neq i_0$ while only $y_{i_0}^{(0)}(t)$ is non-zero and equal to

$$y_{i_0}^{(0)}(t) = y_0 e^t.$$

Thus, the only nodes with a non-zero first term in the equation

$$\frac{dy_i^{(1)}}{dt} = y_i^{(1)} + \sum_j a_{ij} y_j^{(0)}$$

are the immediate neighbors of i_0, collected in the set $\mathcal{D}^1$ i.e.,

$$\begin{cases} \dfrac{d\,y_{i_1}^{(1)}(t)}{dt} & = y_{i_1}^{(1)} + y_0 e^t \\ y_{i_1}^{(1)}(t) & = y_0 t e^t \\ i_1 \in \mathcal{D}^1; \end{cases} \tag{9}$$

for all the other nodes, $y_i^{(1)}(t) = 0$. Similarly, nodes with a first non-vanishing term of second order in ϵ are in $\mathcal{D}^2$, with equations:

$$\begin{cases} \dfrac{d\,y_{i_2}^{(2)}(t)}{dt} & = y_{i_2}^{(2)} + \displaystyle\sum_{i_1 \in \mathcal{D}^1} y_{i_1}^{(1)} \\ y_{i_2}^{(2)}(t) & = y_0 \dfrac{t^2}{2} e^t \sum_{i_1 \in \mathcal{D}^1} a_{i_1 i_2} \\ i_2 \in \mathcal{D}^2 \end{cases} \tag{10}$$

Generalizing, nodes at distance k from the source have equations of the form:

$$\begin{cases} \dfrac{d\,y_{i_k}^{(k)}(t)}{dt} & = y_{i_k}^{(k)} + \displaystyle\sum_{i_{k-1} \in \mathcal{D}^{k-1}} a_{i_k i_{k-1}} y_{i_{k-1}}^{(k-1)} \\ y_{i_k}^{(k)}(t) & = y_0 \frac{t^k}{k!} e^t z_{i_k i_0} \\ z_{i_k i_0} & = \displaystyle\sum_{i_{k-1} \in \mathcal{D}^{k-1}} \cdots \sum_{i_1 \in \mathcal{D}^1} \sum_{i_0 \in \mathcal{D}^0} a_{i_k i_{k-1}} \cdots a_{i_2 i_1} a_{i_1 i_0} \\ i_k \in \mathcal{D}^k, \end{cases} \tag{11}$$

where $z_{i_k i_0}$ accounts for the number of shortest paths of the same length from i_0 to i_k.

Similarly to the case of two coupled subsystems, let us now identify the delay τ_{i_k} such that $y_{i_k}(t + \tau_{i_k}) = y_{i_1}(t)$. To this end, we observe that it holds

$$y_{i_k}(t) \approx \epsilon^k y_{i_k}^{(k)}(t) + \mathcal{O}(\epsilon^{k+1}) = \epsilon^k y_0 \frac{t^k}{k!} e^t z_{i_k i_0} + \mathcal{O}(\epsilon^{k+1}).$$

Therfore, the delay τ_{i_k} is such that

$$\epsilon^k y_0 \frac{(t + \tau_{i_k})^k}{k!} e^{t+\tau_{i_k}} z_{i_k i_0} = y_0 e^t.$$

At this point, let us assume that τ_{i_k} is large enough to have that $k \ln \tau_{i_k} \ll \tau_{i_k}$; thus we can also assume that $k \ln(t + \tau_{i_k}) \ll t + \tau_{i_k}$ since $t/\ln t$ is an increasing function. Therefore, applying $\ln(\cdot)$ to both members and using

$$\ln\left((t + \tau_{i_k})^k e^{t+\tau_{i_k}}\right) \approx t + \tau_{i_k}$$

for $\tau_{i_k} \gg 1$, we obtain

$$\tau_{i_k} \sim -k \ln \epsilon + \ln k! - \ln z_{i_k i_0}.$$

Notice that $A(x) \sim B(x)$ is used here to mean that A scales like B for large x, i.e., $A(x) - B(x) = f(x)$ with $f(x) \ll A(x)$, $f(x) \ll B(x)$. Thus, if the dynamics start from an operating/non-failed system with a small failure $y_0 \ll 1$ in the system i_0, the delay of the i-th system scales as

$$\tau_{ii_0} \sim -d_{ii_0} \ln(\epsilon) - \ln(z_{ii_0}) + \ln(d_{ii_0}!) \tag{12}$$

where d_{ii_0} is the *chemical distance* (i.e., the length of the shortest path) from i_0 to i and z_{ii_0} counts the number of shortest paths from i to i_0 with distance d_{ii_0}.

2.4 Algorithm for Calculating the Path Degeneracy

To calculate z_{ii_0}, note that the first term is simply $z_{ii_0} = a_{ii_0}$ for all i that are neighbours of i_0. By defining

$$\left[B^{(d)}\right]_{ij} = \begin{cases} a_{ij} & \text{for } i \in \mathcal{D}^d \wedge j \in \mathcal{D}^{d-1} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

and $Z^{(d)} = \prod_{k=1}^{d} B^{(k)}$, we have that the matrix coefficients (even without the tree approximation) for elements at minimum distance $k+1$ are:

$$\sum_{j_k \in \mathcal{D}^k, \ldots, j_2 \in \mathcal{D}^2, j_1 \in \mathcal{D}^1} a_{ij_k} a_{j_k j_{k-1}} \ldots a_{j_2 j_1} a_{j_1 0} = \left[Z^{(k+1)}\right]_{ii_0} . \tag{14}$$

Thus, the algorithm for computing z_{ii_0} (using just for short the case of the dynamics starting from site 0) can be summarized as follows.

Algorithm 1. Compute z_{ii_0} for all nodes

```
Calculate the minimum distance matrix d_{ii_0}
Calculate the diameter D = max_i d_{ii_0}
D^0 ← {i_0}
z_{i_0 i_0} ← 1
for k = 1 ... D do
    D^k ← {i : d_{ii_0} = k}
    for i ∈ D^k do
        z_{ii_0} ← Σ_{j ∈ D^{k-1}} a_{ij} z_{ji_0}
    end for
end for
```

3 Results

In this section, we present the results of numerical simulations validating the theoretical predictions derived from our modeling framework. We focus on two scenarios: the dynamics of two interdependent systems governed by coupled logistic equations and the dynamics of a network of interdependent systems.

3.1 Interdependent Systems

First, we investigate the dynamics of two interdependent systems described by coupled logistic equations. Our theoretical analysis predicted that the delay in the failure of one system, given the failure of the other, scales logarithmically with the coupling strength. To verify this prediction, we performed numerical simulations varying the coupling strength ϵ over a range of values.

Figure 1 illustrates the relationship between the coupling strength ϵ and the corresponding delay τ in the failure of the second system, given the failure of the first. The numerical results align closely with the theoretical prediction, confirming that the delay indeed scales logarithmically with ϵ. Linear fits to the numerical data further support the validity of our theoretical analysis.

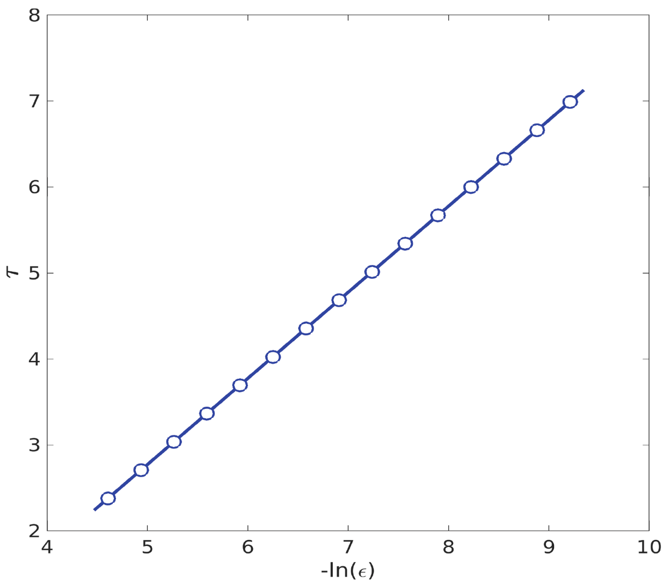

Fig. 1. By numerical solutions of the model, it is possible to estimate the delays τ as a function of the coupling ε. The numerical delays τ follows the theoretical prediction of Eq. 5 that assert that if a system Y fails, a system X dependent with intensity ε on X will fail with a delay τ that scales as $\tau \sim -\ln \epsilon$. We consider values of ε in $[10^{-4}, 10^{-2}]$ with initial conditions $y(t=0) = 10^{-6}$. Continuous lines are linear fit to the numerical data.

3.2 Network of Interdependent Systems

Next, we extend our analysis to a network of interdependent systems, where the failure of one system can propagate through the network, affecting interconnected systems. Our theoretical framework predicted that the delay in the failure of a system within the network scales logarithmically with the number of shortest paths connecting it to the initially failed system, as well as with the coupling strength.

Figure 2 presents the results of numerical simulations conducted on 100 random networks of interdependent systems. The plot demonstrates the logarithmic scaling of the delay τ with both the number of shortest paths connecting the failed system to the target system and the coupling strength ϵ. Linear fits to the numerical data confirm the consistency of our theoretical predictions with the simulated results.

Overall, our numerical simulations validate the theoretical framework proposed in this study, providing empirical evidence for the logarithmic scaling of delays in the failure of interdependent systems. These results underscore the importance of understanding system interdependencies and dynamics in enhancing the resilience and reliability of networked infrastructures.

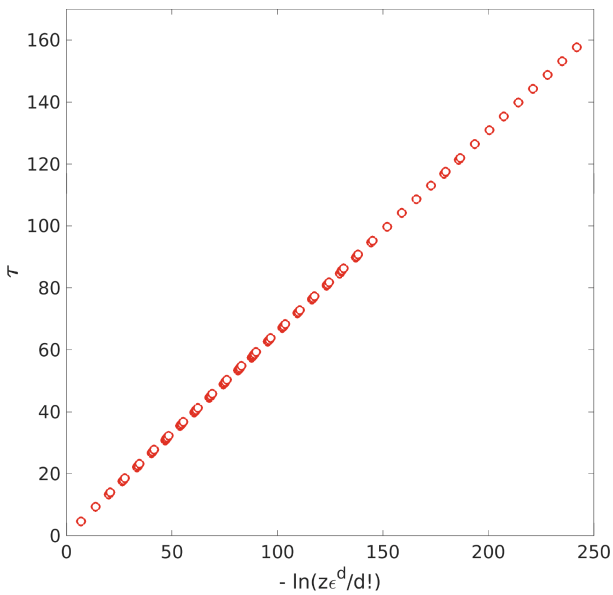

Fig. 2. By numerical solutions of the model, it is possible to estimate the delays τ as a function of the coupling ε. The numerical delays τ follows the theoretical prediction of Eq. (12) that assert in a network of n elements where the i_0-th system fails, a system i connected by z shortest paths of length d to i_0 fails with a delay τ that scales as $\tau \sim -\ln(z\epsilon^d/d!)$. We plot the results for 100 networks of $n = 50$ interdependent systems with average degree $k = 2.4$, coupling strength $\varepsilon = 10^{-3}$ and initial conditions $y_{i_0}(t = 0) = 10^{-6}$.

4 Conclusion

This study introduces a novel framework using coupled logistic dynamics to model the propagation and growth of damage in networked infrastructures. Traditional models often fail to capture the complexity of these systems, but our approach provides a more realistic representation by considering interdependencies and nonlinear dynamics.

Key findings include:

1. **Logarithmic Scaling:** The delay in failure propagation scales logarithmically with both the coupling strength and the number of shortest paths connecting the initial failure to the affected system.
2. **Radial Propagation:** Damage spreads radially, with delays increasing with the coupling strength and distance from the initial failure.

Numerical simulations support these theoretical results, emphasizing the importance of understanding interdependencies for enhancing system resilience. Future research should focus on incorporating real-world data and exploring advanced modeling techniques to improve predictive capabilities and risk management strategies.

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