SECOND ORDER SOLUTION OF CIRCULAR DISC IMPACT PROBLEM

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SUMMARY

The second order solution of the disc impact problem is derived for the initial stage of the disc motion. The disc is of circular shape and is originally floating on the free surface, then it suddenly starts to penetrate the water with a constant entry velocity. The boundary value problem for the second order velocity potential is of mixed type with a Dirichlet condition being imposed on the free surface and a Neumann boundary condition on the disc surface. Both conditions are given in terms of the first order solution derived in a previous paper. The second order problem is complicated by singular behavior of the velocity potential on the free surface. For this reason the solution is presented as the sum of three different contributions. The first two terms are introduced to match the singular behavior of the velocity potential on the free surface whereas the third one, which is regular at the disc edge, provides the complete satisfaction of the boundary conditions both over the free surface and on the body.

1. INTRODUCTION

The sudden vertical motion of a circular disc initially floating on a still liquid surface is considered. It is assumed that no air is entrained at the plate-liquid interface, and that the liquid flow is axisymmetric and potential. Both surface tension and the gravity are not taken into account.

In [1] the hydrodynamic loads generated during the early stage after the sudden start of the disc have been derived through a small time expansion procedure. Therein the first order velocity potential was obtained and used to derive the boundary value problem for the second order velocity potential. Differently from the two-dimensional case [2], in the corresponding axisymmetric problem the second order solution cannot be analytically derived. The solution of the corresponding boundary value problem is made further complicated by the singularity of the second order velocity potential about the disc edge. Owing to this singularity, in [1] only the most divergent part of the second order velocity potential was used to derive an asymptotic estimate of the hydrodynamic loads acting on the disc. However, the comparison with the fully nonlinear numerical results established in [3] revealed that the incompleteness of the second order outer solution in the axisymmetric problem makes the asymptotic estimate valid for a much shorter time interval with respect to that found in the two-dimensional case.

On the basis of the above considerations, complete second order velocity potential is derived here. Due to the singularity of the velocity potential, the solution is presented as the sum of three contributions. The first two terms are introduced to match the singular behavior of the velocity potential at the edge of the disc and the third one, which is regular, is aimed at satisfying the boundary conditions all over both the free surface and the body surface.

2. BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER VELOCITY POTENTIAL

In [1] the early stage after the sudden vertical start of a circular disc was investigated through a small time expansion procedure. The velocity potential φ and the free surface elevation η were presented in the forms

$$\varphi(r,z,t) = \dot{h}(t)\varphi_0(r,z) + \tilde{\varphi}_1(r,z,t) \ , \tag{1}$$

$$\eta(r,t) = h(t)\eta_0(r) + \tilde{\eta}_1(r,t) \ , \tag{2}$$

where r, z are the non-dimensional radial and vertical coordinates and $h(t)$ is the penetration depth of the disc. As $t \to +0$, the leading order terms in (1) and (2) satisfy the boundary value problem with mixed boundary conditions

$$\begin{array}{rcll} \Delta\varphi_0 & = & 0 \ , & (z<0) \\ \dfrac{\partial\varphi_0}{\partial z} & = & -1 \ , & (z=0, r<1) \\ \varphi_0 & = & 0 \ , & (z=0, r>1) \\ \eta_0 & = & \dfrac{\partial\varphi_0}{\partial z} \ , & (z=0, r>1) \\ \varphi_0 & \to & 0 \ , & (r^2+z^2\to\infty) \end{array} \tag{3}$$

which is well known as the pressure-impulse problem [4]. The solution to the BVP (3) can be derived by using the oblate spheroidal variables which are defined as

$$r = (1-\mu^2)^{1/2}(1+\xi^2)^{1/2} \ , \ z = \mu\xi, \tag{4}$$

where μ varies from -1 to $+1$ with its extreme values representing the negative and positive z-axis, respectively. The variable μ is zero over the portion of the $z=0$ plane with $r>1$ (i.e. the undisturbed free surface), whereas the variable ξ is zero over the disc surface. Within the oblate spheroidal variables the solution of the boundary value problem (3) is

$$\varphi_0(\mu,\xi) = \frac{2}{\pi}\mu\left(1-\xi\arctan\left(\xi^{-1}\right)\right) \ . \tag{5}$$

It is known that in a small time expansion procedure the first order velocity potential provides the boundary conditions for the second order potential. For the circular disc impact problem, it was shown in [1] that the dynamic boundary condition on the free surface reads

$$\frac{\partial\tilde{\varphi}_1}{\partial t} = -\frac{1}{2}\dot{h}^2\left[\varphi_{0,z}\right]^2 + o(1) \quad (r>1, z=0) \ , \tag{6}$$

whereas the body boundary condition takes the form

$$\frac{\partial\tilde{\varphi}_1}{\partial z} = -h\dot{h}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_0}{\partial r}\right) + o(h) \quad (r<1, z=0) \ . \tag{7}$$

Equations (6) and (7) indicate that the velocity potential $\tilde{\varphi_1}$ can be decomposed as

$$\tilde{\varphi_1} = \dot{h}h\varphi_1(r,z) + \mathcal{D}(t)\varphi_e(r,z) + \tilde{\varphi_2}(r,z,t), \tag{8}$$

where $\varphi_1(r,z)$ satisfies equations

$$\begin{aligned} \Delta\varphi_1 &= 0 \quad (z<0) \\ \varphi_1 &= -\frac{1}{2}\left[\frac{\partial\varphi_0}{\partial z}\right]^2 \quad (z=0, r>1) \\ \frac{\partial\varphi_1}{\partial z} &= -\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\varphi_0}{\partial r}\right] \quad (z=0, r<1) \end{aligned} \tag{9}$$

and $\varphi_e(r,z)$ is the eigensolution of the boundary value problem (9) with the least singularity at the disc edge. In equation (8), the function $\mathcal{D}(t)$ should be obtained by matching the outer and inner solutions.

By substituting the velocity potential φ_0 given by equation (5) into the free surface and body boundary conditions of the boundary value problem (9) we obtain

$$\varphi_1(r,0) = -\frac{2}{\pi^2}\left[\frac{1}{\sqrt{r^2-1}} - \arcsin\left(\frac{1}{r}\right)\right]^2 \quad (r>1) \tag{10}$$

and

$$\frac{\partial\varphi_1}{\partial z}(r,0) = -\frac{2}{\pi}\left[\frac{2}{\sqrt{1-r^2}} + \frac{r^2}{(1-r^2)^{\frac{3}{2}}}\right] \quad (r<1). \tag{11}$$

The singularity of these conditions about the edge of the disc makes the solution of the boundary value problem (9) rather complicated. A way to overcome the difficulty is to present the solution as the sum of three different contributions. The first two are aimed at representing the singular part of the free surface boundary condition and the third one is designed to satisfy the boundary conditions all over the free surface and the body surface.

3. DIVERGENT CONTRIBUTIONS TO THE SECOND ORDER VELOCITY POTENTIAL

From equation (10) it can be seen that approaching the edge of the disc, $r \to 1+0$, the velocity potential behaves as

$$\varphi_1(r,0) \sim -\frac{2}{\pi^2}\frac{1}{r^2-1} + \frac{4}{\pi^2}\frac{\arcsin(1/r)}{\sqrt{r^2-1}}. \tag{12}$$

Such a behavior can be represented by a 3D dipole of intensity A distributed along the circle $r=1$. This dipole distribution induces a velocity potential given by

$$\begin{aligned} \phi_D(r,z) &= -\frac{A}{\pi}\frac{1}{(\delta^2+4r)^{3/2}}\left[rn_r\left(\frac{E(\gamma)}{1-\gamma^2}\left(\frac{2}{\gamma^2}-1\right)\right.\right. \\ &\left.\left. -\frac{2}{\gamma^2}F(\gamma)\right) + \frac{E(\gamma)}{1-\gamma^2}(-n_r + zn_z)\right], \end{aligned}$$

where

$$\delta^2 = (r-1)^2 + z^2 \qquad , \gamma^2 = \frac{4r}{\delta^2+4r} \ ,$$

n_r and n_z specify the dipole orientation,

$$F(\gamma) = \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1-\gamma^2\cos^2\alpha}},$$

$$E(\gamma) = \int_0^{\pi/2} \sqrt{1-\gamma^2\cos^2\alpha}\,\mathrm{d}\alpha$$

are the first and second complete elliptic integrals, respectively [5]. Intensity and orientation of the dipole can be properly chosen to reproduce the singular behavior of the second order velocity potential about the edge of the disc. If we assume the dipoles along the unit circle are oriented in the horizontal direction and normal to the circle, i.e. $n_r = 1, n_z = 0$, we obtain

$$\begin{aligned} \phi_D(r,z) &= -\frac{A}{\pi}\frac{1}{(\delta^2+4r)^{3/2}}\left[r\left(\frac{E(\gamma)}{1-\gamma^2}\left(\frac{2}{\gamma^2}-1\right)\right.\right. \\ &\left.\left. -\frac{2}{\gamma^2}F(\gamma)\right) - \frac{E(\gamma)}{1-\gamma^2}\right] \quad . \end{aligned} \tag{13}$$

If we take $z=0$ and let $r \to 1+0$, we find

$$\phi_D(r,0) \sim -\frac{A}{\pi}\frac{1}{4^{3/2}}\left[r\left(\frac{E(\gamma)}{1-\gamma^2}(2-1) - 2F(\gamma)\right) - \frac{E(\gamma)}{1-\gamma^2}\right]$$

where $\gamma \to 1$ and $E(\gamma) \to 1$, whereas $F(\gamma)$ has a logarithmic singularity. Therefore

$$\phi_D(r,0) \sim -\frac{A}{8\pi}\left[\frac{1}{1-\gamma^2}(r-1) - 2F(\gamma)\right] \qquad \text{as } r \to 1{+}0.$$

From the definitions, as $r \to 1{+}0$ we have that $1/(\gamma^2-1) \sim 4/(r-1)^2$ and thus

$$\phi_D(r,0) \sim -\frac{A}{8\pi}\left[\frac{4}{r-1} - 2F(\gamma)\right] \qquad \text{as } r \to 1+0 \ . \tag{14}$$

The comparison of the leading order term in (14) with equation (12) indicates that these two equations match each other in the leading order as $r \to 1+0$ if $A = 2/\pi$.

Although the dipole distribution allowed us to represent the most divergent part of the second order velocity potential, it introduces another low-order singularity due to the logarithmic singularity of the function $F(\gamma)$ as $\gamma \to 1$. Such a singularity can be removed by adding the velocity potential induced by a circular source located at $(1,0)$, which is given by

$$\phi_S(r,z) = -\frac{B}{\pi}\frac{1}{(\delta^2+4r)^{1/2}}F(\gamma) \ . \tag{15}$$

If we take $B = A/2 = 1/\pi$, then such a velocity potential compensates the singular behavior of the second term in equation (14).

On the basis of the above considerations, the combination of the dipole (13) and source (15) velocity potentials

$$\begin{aligned} \phi^A(r,z) = &-\frac{2}{\pi^2}\frac{1}{(\delta^2+4r)^{3/2}}\left[r\left(\frac{E(\gamma)}{1-\gamma^2}\left(\frac{2}{\gamma^2}-1\right)\right.\right. \\ &\left.\left.-\frac{2}{\gamma^2}F(\gamma)\right) - \frac{E(\gamma)}{1-\gamma^2}\right] - \frac{1}{\pi^2}\frac{1}{(\delta^2+4r)^{1/2}}F(\gamma) \end{aligned} \tag{16}$$

correctly matches the most divergent part of the second order velocity potential given by equation (10).

The velocity potential ϕ^A derived above, only fits the most divergent contribution in equation (12). In order to derive a harmonic function that matches the square root singularity at the edge of the disc, we can use the fact that if $\varphi(r,z)$ is a harmonic function, then its z-derivative is also harmonic function. Hence, if we differentiate with respect to z the velocity potential $\varphi_0(r,z)$ given by equation (5), we obtain that

$$\varphi_{0,z}(\mu,\xi) = \frac{2}{\pi}\left[\left(1-\xi\arctan\frac{1}{\xi}\right)\xi\frac{(1-\mu^2)}{(\mu^2+\xi^2)} + \mu\left(-\arctan\frac{1}{\xi}+\frac{\xi}{(\xi^2+1)}\right)\mu\frac{(1+\xi^2)}{(\mu^2+\xi^2)}\right] \qquad (17)$$

where the relations

$$\xi_z = \mu\frac{(1+\xi^2)}{(\mu^2+\xi^2)} \quad \xi_r = \xi\frac{(1-\mu^2)^{1/2}(1+\xi^2)^{1/2}}{(\mu^2+\xi^2)}$$

$$\mu_z = \xi\frac{(1-\mu^2)}{(\mu^2+\xi^2)} \quad \mu_r = -\mu\frac{(1-\mu^2)^{1/2}(1+\xi^2)^{1/2}}{(\mu^2+\xi^2)}$$

were used. Equation (17) can be further simplified thus obtaining

$$\phi^B(\mu,\xi) = \frac{2}{\pi}\left(-\arctan\frac{1}{\xi}+\frac{\xi}{\xi^2+\mu^2}\right) \quad , \qquad (18)$$

which along the free surface, $z=0$, $\mu=0$, behaves as

$$\phi^B = \frac{2}{\pi}\left(\frac{1}{\xi}-\arctan\frac{1}{\xi}\right) = \frac{2}{\pi}\left(\frac{1}{\sqrt{r^2-1}}-\arcsin\frac{1}{r}\right) \quad .$$

4. REGULAR CONTRIBUTION

The velocity potentials ϕ^A and ϕ^B, although matching the singular behavior of φ_1 about the edge of the disc, do not satisfy the free surface boundary condition (10). Moreover, contributions of these potential to the body boundary condition (11) is not clear sofar. However, we can assume from now on that the second order velocity potential φ_1 can be represented in the form:

$$\varphi_1 = \phi^A + \phi^B + \phi^C \quad .$$

If we substitute this decomposition into the free surface condition (10) and the body boundary condition (11), we obtain the corresponding boundary conditions for the velocity potential ϕ^C. After some mathematics we obtain that on the free surface, $z=0, r>1$, the velocity potential ϕ^C has to satisfy the condition

$$\begin{aligned}\phi^C(r,0) &= -\frac{2}{\pi^2}\left[\frac{1}{\xi(r)}-\arctan\frac{1}{\xi(r)}\right]^2 \\ &- \frac{2}{\pi}\left[\frac{1}{\xi(r)}-\arctan\frac{1}{\xi(r)}\right]+\frac{1}{\pi^2}\frac{E(\gamma)}{r-1} \qquad (19)\end{aligned}$$

which, as $r\to 1+0$, approaches the limit value

$$\phi^C(1,0) = \frac{1}{2}-\frac{4}{\pi^2}+\frac{1}{2\pi^2} = 0.14537586 \quad .$$

The graph of the function $\phi^C(r,0)$ for $r>1$ is drawn in Fig.1.

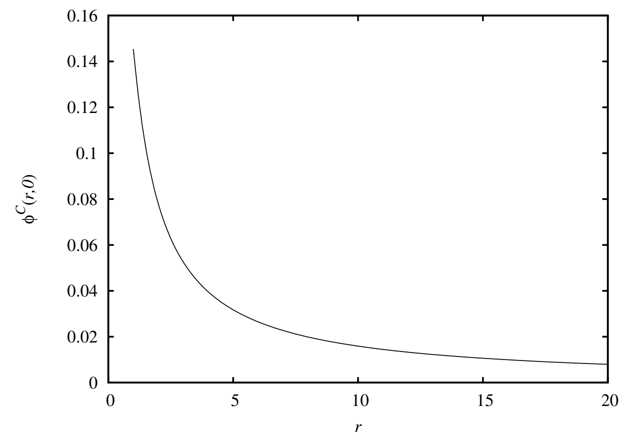

Figure 1: Velocity potential on the free surface, $r>1$.

On the body surface, $z=0, r<1$, the velocity potential ϕ^C has to satisfy the condition

$$\begin{aligned}\frac{\partial\phi^C}{\partial z}(r,0) &= -\frac{4}{\pi}\left(\frac{1}{(1-r^2)^{1/2}}+\frac{r^2}{2}\frac{1}{(1-r^2)^{3/2}}\right) \\ &- \frac{\partial\phi^A}{\partial z}(r,0)-\frac{\partial\phi^B}{\partial z}(r,0) \qquad (20)\end{aligned}$$

From the definition of ϕ^A it can be shown that

$$\frac{\partial\phi^A}{\partial z}(r,0)\equiv 0 \quad (r<1)$$

whereas from equation (18) we obtain that

$$\begin{aligned}\frac{\partial\phi^B}{\partial z}(r,0) &= \frac{2}{\pi}\left(\frac{1}{\mu}+\frac{1}{\mu^3}\right) \\ &= -\frac{2}{\pi}\left(\frac{1}{(1-r^2)^{1/2}}+\frac{1}{(1-r^2)^{3/2}}\right) \quad .\end{aligned}$$

where $\mu=-\sqrt{1-r^2}$ has been used for the lower side of the disc surface. With the help of these results, it can be shown that equation (20) provides

$$\frac{\partial\phi^C}{\partial z}(r,0) = 0 \quad (r<1) \quad . \qquad (21)$$

Hence, in order to completely determine the second order velocity potential, we need to find the velocity potential ϕ^C that satisfies the free surface boundary condition (19) and the body boundary condition (21). Such a boundary value problem can be solved numerically using the second Green's identity, as it is done in [3] for the fully nonlinear solution of the time domain problem. However, for the purpose of estimating the hydrodynamic loads acting on the disc during the early stage, the second order velocity potential is only needed on the disc surface. In this case the Sneddon's equation [6]

$$\phi^C(r,0) = \frac{2}{\pi}\int_0^\infty \phi^C[\sqrt{(1-r^2)\tau^2+1},0]\frac{d\tau}{\tau^2+1}$$

can be used. This equation provides the velocity potential on the disc as an integral of the velocity potential on the flat surface outside the disc in the case the normal derivative of the potential is zero over the disc.

In order to make a cross comparison of the results, in Fig.2 the velocity potential $\phi^C(r,0)$ for $r < 1$ obtained both by the BEM approach and by the Sneddon formula are shown. The two curves are essentially overlapped.

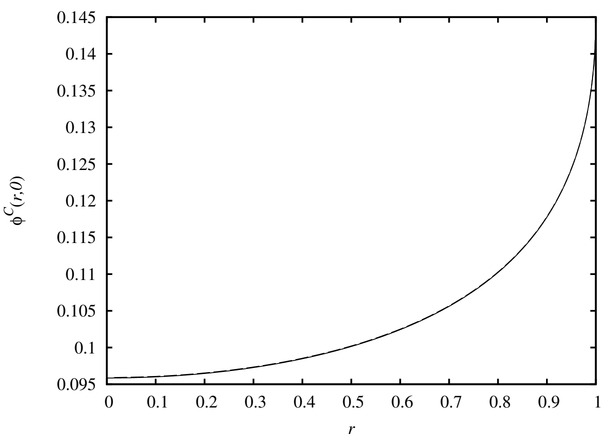

Figure 2: Velocity potential on the disc surface obtained by the Sneddon formula (*solid*) and by the BEM approach (*dash*), the two results being essentially overlapped.

5. CONCLUSIONS

The second order velocity potential of the sudden vertical motion of a disc initially floating on the free surface has been derived for the early stage of the impact. Owing to the singularities in the boundary conditions, the solution is presented as the sum of three different terms. The first one is a combination of circular dipole and source. This term fits the most divergent part of the free surface boundary condition. The second term is proportional to the z-derivative of the first order velocity potential and is introduced to represent the square root singularity in the free surface boundary condition. The third contribution, which is regular on the whole boundary and has a zero normal derivative on the disc surface, has been derived by using both a BEM approach and the Sneddon formula, the latter providing the velocity potential on the disc as an integral of the velocity potential on the free surface.

The second order velocity potential can now be used to derive the third order correction term in the asymptotics of the hydrodynamic loads acting on the disc during the early stage after the sudden start of the disc. Such asymptotic estimates and comparisons with corresponding fully nonlinear numerical calculations will be presented at the Workshop.

6. ACKNOWLEDGMENTS

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7. REFERENCES

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