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THERMODYNAMICS OF NO-TENSION MATERIALS

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Abstract. After a brief review of the thermodynamics of isotropic elastic materials, this paper presents a constitutive equation for no-tension materials in the presence of thermal expansion that accounts for the temperature-dependence of their material's constants. Specifically, under the hypothesis of infinitesimal strains, an explicit expression is given for free energy from which the internal energy, entropy and stress are obtained. Then, the basic equations of the thermo-mechanical equilibrium of a no-tension solid are presented, and we observe that, under the further hypothesis of an infinitesimal strain rate, thermo-mechanical uncoupling occurs. Finally, a study is performed of a circular ring made of a no-tension material subjected to a plane stress under the action of both two uniform radial pressures exerted on the inner and outer boundary and a temperature distribution varying linearly with the radius; all material constants are assumed temperature-independent, except for Young's modulus, which depends linearly on the temperature. The stress field, displacement field, fractures, free energy and entropy are explicitly calculated and compared with these same quantities for a circular ring made of a linear elastic material.

1. INTRODUCTION

With the aim of modelling the behaviour of masonry structures, the constitutive equation of materials not withstanding tension has been studied by many authors under isothermal conditions (Del Piero, 1989, Panzeca *et al.*, 1988). The infinitesimal strain is assumed to be the sum of a positive semi-definite inelastic part and an elastic part on which the stress, negative semi-definite, depends linearly. Moreover, the stress and the inelastic strain, which can be interpreted as fracture strain, are orthogonal. Thus, one obtains a non-linear hyperelastic material, called *masonry-like* or *no-tension* material.

The existence and the uniqueness of the solution to this equation have been proven and the solution itself calculated explicitly in the isotropic case. More recently, suitable numerical techniques have been studied which allow application of the constitutive equation to solution of the equilibrium problem of masonry solids through the finite element method (Lucchesi *et al.* , 1994, 1995, 1996a). These techniques have yielded sound results mainly in the study of arches and vaults.

However, there are many engineering problems in which the presence of thermal dilatation must be accounted for; consider for example the influence of thermal variations on stress fields in masonry bridges (Guidi, 1906), or the thermo-mechanical behaviour of the refractory materials used in the iron and steel industry (Kienow *et al.*, 1966), and finally, geological problems connected with the presence of volcanic calderas such as that of Pozzuoli (Como *et al.*, 1989). In many such cases the thermal variation during the thermo-mechanical process under examination is so high that the dependence of the material constants on temperature cannot be ignored.

In what follows we set forth a constitutive equation for isotropic no-tension materials under non-isothermal conditions. The theory presented here has allowed study of numerical techniques for solution to the equilibrium problem of masonry-like solids in the presence of thermal loads via the finite element method (Lucchesi *et al.*, 1996b).

Section 2 is dedicated to a brief review of the thermodynamic theory of isotropic elastic materials (Truesdell *et al.*, 1965, Carlson, 1972). In Section 3 we present the assumptions underlying the procedure, *i.e.*, that the thermal expansion is a spherical tensor of type $\beta(\theta)\mathbf{I}$, where $\beta(\theta)$ is a material function of the temperature θ and $\mathbf{I}$ the identity tensor, and then explicitly define the free energy as a function of both θ and the measure of strain $\mathbf{A} = \mathbf{V} - \mathbf{I}$, with $\mathbf{V}$ the left stretch tensor. In view of the target applications, we assume that the displacement gradient $\mathbf{H}$ is small. Moreover, although no limitations are placed on the range of temperature variation, we do assume that $\beta(\theta)$, $\beta'(\theta)$ and $\beta''(\theta)$ are also small. These hypotheses allow us to express free energy as function of the infinitesimal strain $\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$. Once the free energy has been thus approximated, we can then deduce the internal energy, entropy and stress. We thereby obtain a non-linear elastic material that in the absence of thermal variation, conforms to the definition of masonry-like materials presented in Del Piero (1989) and Panzeca *et al.* (1988). We then define the specific heat at constant strain and prove that (by virtue of our assumtions regarding $\mathbf{E}$, function β and its first and second derivatives) its value coincides with the specific heat at constant stress, within a second-order error with respect to $\|\mathbf{H}\|$. By assuming the classical Fourier hypothesis for heat flux, the material presented in this paper is completely characterized by five functions of the temperature: Young's modulus, the Poisson ratio, coefficient of thermal expansion $\beta(\theta)$, conductivity and specific heat. In fact, when these material functions are known, the thermodynamic potentials (and consequently the thermo-mechanical behaviour) of the material is determined. At this point, once the energy equation has been obtained, we are in a position to write the basic equations of the thermoelastic theory for no-tension materials. Just as in the linear elastic case, these equations are: the strain-displacement relation, the equation of equilibrium, the constitutive equations for the stress and the heat flux and the equilibrium energy equation. The system we obtain is coupled because the coefficient of the temperature in the energy equation depends on strain and strain rate. In particular, if we assume that the strain rate is small, then the thermoelastic equilibrium equations are uncoupled and can be integrated separately.

Treatment of the theory is fully three-dimensional; as the example presented in Section 5 deals with plane stress, the Appendix provides relative expressions of the free energy, entropy and stress.

In Section 4, by limiting ourselves to the case of thermo-mechanical uncoupling, we consider the equilibrium problem of no-tension solids in the presence of thermal variations and under suitable hypotheses of regularity, we prove that the solution to the boundary-value problem is unique in terms of stress, a well-established result for no-tension materials under isothermal conditions (Lucchesi *et al.*, 1996a).

Finally, in Section 5, still under the hypothesis of thermo-mechanical uncoupling, we consider a circular ring made of a no-tension material subjected to a plane stress consequent to the action of both two uniform radial pressures acting on the inner and outer boundary and a temperature distribution varying linearly with the radius. We assume that the coefficient of thermal expansion $\beta(\theta)$ and Young's modulus are linear functions of the temperature and the Poisson ratio is constant. Once the stress field and corresponding fractures in the circular ring have been explicitly determined, free energy and entropy are calculated as functions of the radius. The quantities obtained are compared with those relative to a circular ring made of a linear elastic material.

2. BACKGROUND THERMODYNAMICS

This section outlines some concepts of thermodynamics and thermoelasticity (Truesdell *et al.*, 1965, Carlson, 1972) necessary for treatment of the theory developed in the next section.

Let $\mathcal{V}$ be a three-dimension linear space and Lin be the space of second order tensors, equipped with the inner product $\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}\,(\mathbf{A}^T\mathbf{B})$, $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$, with $\mathbf{A}^T$ the transpose of $\mathbf{A}$. Let us indicate as Sym, Sym^+ and Sym^- the subsets of Lin constituted by symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively.

Let $\mathcal{B}_0$ be the reference configuration of a body and $\mathcal{B}_t$ the configuration at time t, during a motion. Furthermore, let $\mathbf{T}$ be the Cauchy stress tensor and $\mathbf{b}$ the body force. We assume that the kinetic energy is nil and then the following balance laws,

$$\int_{\partial\mathcal{P}} \mathbf{Tn}\, dA + \int_{\mathcal{P}} \mathbf{b}\; dV = 0\,, \quad \int_{\partial\mathcal{P}} (\mathbf{x} - \mathbf{o}) \times \mathbf{Tn}\, dA + \int_{\mathcal{P}} (\mathbf{x} - \mathbf{o}) \times \mathbf{b}\, dV = 0 \qquad (2.1)$$

hold for every time t and every part $\mathcal{P}$ of $\mathcal{B}_t$, with $\mathbf{o}$ a point of the Euclidean space and $\mathbf{n}$ the unit outward normal to the boundary $\partial\mathcal{P}$ of $\mathcal{P}$. Under suitable hypotheses on $\mathbf{b}$ and $\mathbf{T}$, from relations (2.1) we deduce

$$\mathrm{div}\,\mathbf{T} + \mathbf{b} = \mathbf{0}\,, \qquad \mathbf{T} = \mathbf{T}^T\,, \qquad (2.2)$$

where div $\mathbf{T}$ is the divergence of $\mathbf{T}$. Now let

$$w(\mathcal{P}) = \int_{\partial\mathcal{P}} (\mathbf{Tn})\cdot\mathbf{v}\, dA + \int_{\mathcal{P}} \mathbf{b}\cdot\mathbf{v}\, dV \qquad (2.3)$$

be the power of external loads acting on $\mathcal{P}$, where $\mathbf{v}$ is the velocity. Since we have supposed that the kinetic energy is nil, from the theorem of power expended we obtain

$$w(\mathcal{P}) = \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\, dV \ , \tag{2.4}$$

with $\mathbf{D}$ the symmetric part of the velocity gradient and $\mathbf{T}\cdot\mathbf{D}$ the *stress power.*

The first law of thermodynamics postulates the existence of a real function ε, defined on $\mathcal{B}_t$, called *internal energy* per unit mass, such that for each part $\mathcal{P}$ of $\mathcal{B}_t$ we have

$$\frac{d}{dt}\int_{\mathcal{P}} \varepsilon\, dm = w(\mathcal{P}) - \int_{\partial\mathcal{P}} \mathbf{q}\cdot\mathbf{n}\, dA + \int_{\mathcal{P}} s\, dm \,, \tag{2.5}$$

where $\mathbf{q}$ is the *heat flux vector* per unit surface area, s the *heat supply* per unit mass, $\int_{\mathcal{P}} \varepsilon\, dm$ the internal energy of $\mathcal{P}$ and $-\int_{\partial\mathcal{P}} \mathbf{q}\cdot\mathbf{n}\, dA + \int_{\mathcal{P}} s\, dm$ the *heat flux* into $\mathcal{P}$.

Let ρ be the density in the current configuration, from (2.5), in view of the divergence theorem we deduce

$$\int_{\mathcal{P}} \dot{\varepsilon}\, \rho\, dV = w(\mathcal{P}) - \int_{\mathcal{P}} \mathrm{div}\mathbf{q}\, dV + \int_{\mathcal{P}} s\, \rho\, dV \,, \tag{2.6}$$

to which corresponds the local form of the first law of thermodynamics

$$\dot{\varepsilon}\, \rho = \mathbf{T}\cdot\mathbf{D} - \mathrm{div}\mathbf{q} + \rho\, s \,. \tag{2.7}$$

The second law of thermodynamics postulates the existence of a real function η, defined on $\mathcal{B}_t$, called the *entropy* per unit mass, such that for each part $\mathcal{P}$ of $\mathcal{B}_t$ we have

$$\frac{d}{dt}\int_{\mathcal{P}} \eta\, dm \geq -\int_{\partial\mathcal{P}} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\, dA + \int_{\mathcal{P}} \frac{s}{\theta}\, dm \,, \tag{2.8}$$

where $\theta \in [\theta_1, \theta_2]$, with $\theta_1 > 0$, is the *absolute temperature,* $\int_{\mathcal{P}} \eta\, dm$ is the *entropy* of $\mathcal{P}$ and $-\int_{\partial\mathcal{P}} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\, dA + \int_{\mathcal{P}} \frac{s}{\theta}\, dm$ is the *entropic flux* of $\mathcal{P}$.

For $\mathbf{g} = \text{grad}\theta$ the spatial gradient of temperature, in view of (2.7), from (2.8), by applying the localization theorem we obtain

$$\rho\, \dot{\eta}\, \theta \geq \frac{1}{\theta}\, \mathbf{q}\cdot\mathbf{g} + \rho\, \dot{\varepsilon} - \mathbf{T}\cdot\mathbf{D}\,. \tag{2.9}$$

By introducing the *free energy* per unit mass $\psi = \varepsilon - \eta\, \theta$, from (2.9) we deduce the *dissipation inequality*

$$\rho\, \dot{\psi} + \rho\, \eta\, \dot{\theta} - \mathbf{T}\cdot\mathbf{D} + \frac{1}{\theta}\, \mathbf{q}\cdot\mathbf{g} \leq 0\,. \tag{2.10}$$

An *elastic material* is defined by constitutive equations providing functions ψ, $\mathbf{T}$, η and $\mathbf{q}$ whenever the deformation gradient $\mathbf{F} \in \text{Lin}^+$, temperature θ and the temperature gradient $\mathbf{g}$ are known. A well-known consequence (Truesdell *et al.*, 1965) of the second law of thermodynamics is that ψ, $\mathbf{T}$ and η do not depend on $\mathbf{g}$. Thus, for elastic materials, the following constitutive relations hold:

$$\begin{aligned} \psi &= \hat{\psi}(\mathbf{F}, \theta)\,, \\ \mathbf{T} &= \hat{\mathbf{T}}(\mathbf{F}, \theta)\,, \\ \eta &= \hat{\eta}(\mathbf{F}, \theta)\,, \\ \mathbf{q} &= \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g})\,. \end{aligned} \tag{2.11}$$

Moreover, from the second law it is possible to deduce the relations (Truesdell *et al.*, 1965)

$$\rho\, \partial_F \hat{\psi}\cdot\dot{F} - \mathbf{T}\cdot\mathbf{D} = 0\,, \tag{2.12}$$

$$\hat{\eta} = -\, \partial_\theta\, \hat{\psi}\,, \tag{2.13}$$

$$\mathbf{q}\cdot\mathbf{g} \leq 0\,. \tag{2.14}$$

From (2.12) we obtain

$$\hat{\mathbf{T}} = \rho\, (\partial_F\, \hat{\psi})\, \mathbf{F}^T = \rho\, \mathbf{F}\, (\partial_F\, \hat{\psi})^T\,. \tag{2.15}$$

Moreover, in view of (2.12), from (2.7) we arrive at the *energy equation*

$$\rho\,\dot{\eta}\,\theta = -\operatorname{div}\mathbf{q} + s\,\rho\,. \tag{2.16}$$

In view of (2.11), the internal energy ε has constitutive equation

$$\hat{\varepsilon}(\mathbf{F}, \theta) = \hat{\psi}(\mathbf{F}, \theta) + \theta\,\hat{\eta}(\mathbf{F}, \theta)\,. \tag{2.17}$$

The *specific heat at constant strain* per unit mass is defined by the relation

$$\hat{C}_E(\mathbf{F}, \theta) = \partial_\theta\,\hat{\varepsilon}(\mathbf{F}, \theta) \tag{2.18}$$

and therefore, by accounting for (2.17) and (2.13), we have

$$\hat{C}_E(\mathbf{F}, \theta) = \theta\,\partial_\theta\,\hat{\eta}(\mathbf{F}, \theta)\,. \tag{2.19}$$

If the elastic material is isotropic, then in place of $(2.11)_1$ we can write

$$\psi = \tilde{\psi}(\mathbf{B}, \theta)\,, \tag{2.20}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green tensor, and ψ depends on principal invariants of $\mathbf{B}$. Now we whish to prove that

$$\mathbf{T} = \tilde{\mathbf{T}}(\mathbf{B}, \theta) = 2\,\rho\,\partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\,\mathbf{B}\,. \tag{2.21}$$

To this end, we observe that for $\mathbf{H} \in$ Lin, we have

$$(\partial_F\,\hat{\psi}(\mathbf{F}, \theta))\cdot\mathbf{H} = \partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\cdot(\partial_F\,(\mathbf{F}\mathbf{F}^T)[\mathbf{H}]) = \partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\cdot(\mathbf{F}\mathbf{H}^T + \mathbf{H}\mathbf{F}^T) =$$

$$\operatorname{tr}(\partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\,\mathbf{H}\mathbf{F}^T + \partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\,\mathbf{F}\mathbf{H}^T) = \operatorname{tr}(\mathbf{F}^T\,\partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\,\mathbf{H} + \partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\,\mathbf{F}\mathbf{H}^T) =$$

$$(\partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)^T\,\mathbf{F} + \partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\,\mathbf{F})\cdot\mathbf{H}\,,$$

and then deduce

$$(\partial_F\,\hat{\psi}(\mathbf{F}, \theta)) = (\partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)^T + \partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta))\;\mathbf{F} = 2\,\partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)\,\mathbf{F}\,, \tag{2.22}$$

where the last step is justified by the fact that $\partial_{\mathbf{B}}\tilde{\psi}(\mathbf{B}, \theta)$ is symmetric, as the derivatives of the principal invariants of $\mathbf{B}$ are symmetric and commute. The desired result follows from $(2.15)_1$.

For $\mathbf{A} = \mathbf{V} - \mathbf{I}$, with $\mathbf{V} = (\mathbf{B})^{1/2}$ the left stretch tensor, from (2.21), using the equality $\partial_A\psi(A,$

$\theta)\cdot\mathbf{H} = 2\,\partial_B\tilde{\psi}(\mathbf{B}, \theta)(\mathbf{I} + \mathbf{A})\cdot\mathbf{H}$, for each $\mathbf{H} \in$ Lin, we get

$$\mathbf{T} = \mathbf{T}(\mathbf{A}, \theta) = \rho\, \partial_A\psi(\mathbf{A}, \theta)\, (\mathbf{I} + \mathbf{A})\,. \tag{2.23}$$

From the preceding relations it is easy to arrive at

$$\eta(\mathbf{A}, \theta) = -\, \partial_\theta\, \psi(\mathbf{A}, \theta)\,, \tag{2.24}$$

$$\varepsilon(\mathbf{A}, \theta) = \psi(\mathbf{A}, \theta) + \theta\, \eta(\mathbf{A}, \theta)\,, \tag{2.25}$$

$$C_E(\mathbf{A}, \theta) = \theta\, \partial_\theta\, \eta(\mathbf{A}, \theta)\,. \tag{2.26}$$

3. ELASTIC MATERIALS THAT DO NOT WITHSTAND TENSION

The aim of this section is to formulate a thermoelastic constitutive theory of isotropic no-tension materials. We assume that the thermal expansion is the spherical tensor $\beta(\theta)\mathbf{I}$, where $\beta(\theta)$ is a material function of the temperature θ called *coefficient of thermal expansion.* Firstly, we set forth the explicit expression of the free energy as a function of $\mathbf{A} \in$ Sym and temperature $\theta \in [\theta_1, \theta_2]$. Subsequently, under the hypothesis of infinitesimal strain, we deduce approximate expressions for the thermodynamic potentials and stress as functions of the symmetric part of the displacement gradient $\mathbf{E}$ and θ.

Let $E(\theta)$, $\nu(\theta)$ and $\beta(\theta)$ be temperature-dependent material functions, such that

$$E(\theta) > 0\,, \qquad 0 \le \nu(\theta) < \tfrac{1}{2}\,, \qquad \text{for each } \theta \in [\theta_1, \theta_2]\,, \qquad \beta(\theta_0) = 0\,, \tag{3.1}$$

with $\theta_0 \in [\theta_1, \theta_2]$ as the reference temperature, and let us set

$$\gamma(\theta) = \frac{2\,\nu(\theta)}{1 - 2\,\nu(\theta)}\,. \tag{3.2}$$

For a_1, a_2, a_3 with $a_1 \le a_2 \le a_3$ the eigenvalues of $\mathbf{A}$, let us consider the following subsets of Sym $\times\, [\theta_1, \theta_2]$

$$\mathcal{R}_1 = \{(\mathbf{A}, \theta) \mid 2(a_3 - \beta(\theta)) + \gamma(\theta)\, (\mathrm{tr}\mathbf{A} - 3\,\beta(\theta)) \le 0\}\,, \tag{3.3}$$

$$\mathcal{R}_2 = \{(\mathbf{A}, \theta) \mid a_1 - \beta(\theta) \ge 0\}\,, \tag{3.4}$$

$$\mathbb{R}_3 = \{(\mathbf{A}, \theta) \mid a_1 - \beta(\theta) \leq 0\,,\ \gamma(\theta)\,(a_1 - \beta(\theta)) \ + 2\,(1 + \gamma(\theta)\,)\ (a_2 - \beta(\theta)) \geq 0\}, \qquad (3.5)$$

$$\mathbb{R}_4 = \{(\mathbf{A}, \theta) \mid \gamma(\theta)\,(a_1 - \beta(\theta)) \ + 2\,(1 + \gamma(\theta)\,)\ (a_2 - \beta(\theta)) \leq 0\,,$$

$$2(a_3 - \beta(\theta)) + \gamma(\theta)\,(\mathrm{tr}\mathbf{A} - 3\,\beta(\theta)) \geq 0\}\,. \qquad (3.6)$$

Now we are in a position to define the free energy function $\psi(\mathbf{A}, \theta)$ that in the four regions $\mathbb{R}_i$ has the following expressions

$$\psi(\mathbf{A}, \theta) = \xi(\theta) + \frac{E(\theta)}{2(1 + \nu(\theta))\,\rho}\,\{\frac{\nu(\theta)}{1 - 2\,\nu(\theta)}\,(\mathrm{tr}\mathbf{A} - 3\,\beta(\theta))^2 +$$

$$(a_1 - \beta(\theta))^2 + (a_2 - \beta(\theta))^2 + (a_3 - \beta(\theta))^2\}\,, \qquad \text{for } (\mathbf{A}, \theta) \in \mathbb{R}_1\,, \qquad (3.7)$$

$$\psi(\mathbf{A}, \theta) = \xi(\theta)\,, \qquad \text{for } (\mathbf{A}, \theta) \in \mathbb{R}_2\,, \qquad (3.8)$$

$$\psi(\mathbf{A}, \theta) = \xi(\theta) + \frac{E(\theta)}{2\rho}\,(a_1 - \beta(\theta))^2\,, \qquad \text{for } (\mathbf{A}, \theta) \in \mathbb{R}_3\,, \qquad (3.9)$$

$$\psi(\mathbf{A}, \theta) = \xi(\theta) + \frac{E(\theta)}{2\,(1 - \nu^2(\theta))\rho}\,\{(a_1 - \beta(\theta))^2 + (a_2 - \beta(\theta))^2 +$$

$$2\nu\,(a_1 - \beta(\theta))\,(a_2 - \beta(\theta))\}\,, \qquad \text{for } (\mathbf{A}, \theta) \in \mathbb{R}_4\,, \qquad (3.10)$$

where $\xi(\theta)$ is a material function which will be specified in the following.

Since we are interested in considering infinitesimal strain, we suppose that there exists $\delta \in [0, 1)$ such that

$$\|\mathbf{H}\| \leq \delta\,, \quad |\beta(\theta)| \leq \delta, \quad |\beta'(\theta)| \leq \delta\,, \quad |\beta''(\theta)| \leq \delta\,, \ \text{for each } \theta \in [\theta_1, \theta_2]\,, \qquad (3.11)$$

where $\mathbf{H} = \mathbf{F} - \mathbf{I}$ is the displacement gradient and $\|\ \|$ is the norm induced by the scalar product $\cdot$ in Lin, $\|\mathbf{H}\| = (\mathbf{H}\cdot\mathbf{H})^{1/2}$. By designating

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T) \qquad (3.12)$$

the infinitesimal strain, from the relation

$$(\mathbf{I} + \mathbf{A})^2 = \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}\mathbf{H}^T\,, \qquad (3.13)$$

from $(3.11)_1$ and (3.12), we immediately deduce that

$$\mathbf{E} = O(\delta)^{(1)}, \quad \mathbf{A} = O(\delta), \quad \mathbf{A} - \mathbf{E} = O(\delta^2). \tag{3.14}$$

Thus, for e_1, e_2, e_3 the eigenvalues of $\mathbf{E}$ with $e_1 \le e_2 \le e_3$, within an error of order $o(\delta^2)^{(2)}$, we have

$$\psi(\mathbf{E}, \theta) = \xi(\theta) + \frac{E(\theta)}{2(1 + \nu(\theta))\,\rho} \{\frac{\nu(\theta)}{1 - 2\,\nu(\theta)} (\mathrm{tr}\mathbf{E} - 3\,\beta(\theta))^2 +$$

$$(e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + (e_3 - \beta(\theta))^2\}, \qquad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_1, \tag{3.15}$$

$$\psi(\mathbf{E}, \theta) = \xi(\theta), \qquad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_2, \tag{3.16}$$

$$\psi(\mathbf{E}, \theta) = \xi(\theta) + \frac{E(\theta)}{2\,\rho} (e_1 - \beta(\theta))^2, \qquad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_3, \tag{3.17}$$

$$\psi(\mathbf{E}, \theta) = \xi(\theta) + \frac{E(\theta)}{2\,(1 - \nu^2(\theta))\rho} \{(e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 +$$

$$2\nu\,(e_1 - \beta(\theta))\,(e_2 - \beta(\theta))\}, \qquad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_4, \tag{3.18}$$

where, within an error of order $O(\delta^2)$, regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ can be considered defined in terms of the eigenvalues of $\mathbf{E}$, instead of those of $\mathbf{A}$.

From relation (3.4), within an error of order $o(\delta)$, we can write

$$\mathbf{T} = \mathbf{T}(\mathbf{E}, \theta) = \rho\, \partial_{\mathbf{E}} \psi(\mathbf{E}, \theta); \tag{3.19}$$

then, in view of expressions (3.15) - (3.18), by indicating the normalized eigenvectors of $\mathbf{E}$ as $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, accounting for the following relations

$$D_E\, e_1 = \mathbf{q}_1 \otimes \mathbf{q}_1, \qquad D_E\, e_2 = \mathbf{q}_2 \otimes \mathbf{q}_2, \qquad D_E\, e_3 = \mathbf{q}_3 \otimes \mathbf{q}_3, \tag{3.20}$$

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[1] Given a mapping B from a neighborhood of 0 in $\mathbb{R}$ into a vector space with norm $\|\cdot\|$, we write $B(\delta) = O(\delta)$ if there exist $k > 0$ and $k' > 0$ such that $\|B(\delta)\| < k\,|\delta|$ whenever $|\delta| < k'$.

[2] Given a mapping B from a neighborhood of 0 in $\mathbb{R}$ into a vector space with norm $\|\cdot\|$, we write $B(\delta) = o(\delta)$ if for each $k > 0$ there is $k' > 0$ such that $\|B(\delta)\| < k\,|\delta|$ whenever $|\delta| < k'$.

(Lucchesi *et al.*, 1996a) and ignoring terms of order $o(\delta)$, we get

$$T(\mathbf{E}, \theta) = \frac{E(\theta)}{1 + \nu(\theta)} \{\mathbf{E} - \beta(\theta)\, \mathbf{I} + \frac{\nu(\theta)}{1 - 2\,\nu(\theta)} \operatorname{tr}(\mathbf{E} - \beta(\theta)\, \mathbf{I})\, \mathbf{I}\}, \qquad \text{for } (\mathbf{E}, \theta) \in \mathbb{R}_1 , \tag{3.21}$$

$$T(\mathbf{E}, \theta) = \mathbf{0} , \qquad \text{for } (\mathbf{E}, \theta) \in \mathbb{R}_2 , \tag{3.22}$$

$$T(\mathbf{E}, \theta) = E(\theta)\, (e_1 - \beta(\theta))\, \mathbf{q}_1 \otimes \mathbf{q}_1 , \qquad \text{for } (\mathbf{E}, \theta) \in \mathbb{R}_3 , \tag{3.23}$$

$$T(\mathbf{E}, \theta) = \frac{E(\theta)}{1 - \nu^2(\theta)} \{[e_1 - \beta(\theta) + \nu(\theta)\, (e_2 - \beta(\theta))]\mathbf{q}_1 \otimes \mathbf{q}_1 + [e_2 - \beta(\theta) + \nu(\theta)\, (e_1 - \beta(\theta))]\mathbf{q}_2 \otimes \mathbf{q}_2 \} , \qquad \text{for } (\mathbf{E}, \theta) \in \mathbb{R}_4 , \tag{3.24}$$

where $\mathbf{0}$ is the null tensor.

It immediately follows from relations (3.21)-(3.24) that $\mathbf{T}$ and $\mathbf{E}$ are coaxial; moreover, from the definition of regions $\mathbb{R}_i$ we deduce that $\mathbf{T}$ is negative semi-definite. For each $\theta \in [\theta_1, \theta_2]$, let $\mathbb{D}(\theta)$ be the definite positive fourth order tensor

$$\mathbb{D}(\theta) = \frac{1 + \nu(\theta)}{E(\theta)} \mathbb{I} - \frac{\nu(\theta)}{E(\theta)} \mathbf{I} \otimes \mathbf{I} , \tag{3.25}$$

where $\mathbb{I}$ is the fourth order identity tensor and $\mathbf{I} \otimes \mathbf{I}$ is the fourth order tensor defined by $\mathbf{I} \otimes \mathbf{I}\, [\mathbf{B}] = (\operatorname{tr} \mathbf{B})\, \mathbf{I}$, $\mathbf{B} \in$ Lin; let us now put

$$\mathbf{E}^e(\mathbf{E}, \theta) = \mathbb{D}(\theta)\, [\mathbf{T}(\mathbf{E}, \theta)] , \tag{3.26}$$

$$\mathbf{E}^a(\mathbf{E}, \theta) = \mathbf{E} - \beta(\theta)\, \mathbf{I} - \mathbf{E}^e(\mathbf{E}, \theta) . \tag{3.27}$$

We purpose to prove that $\mathbf{E}^a(\mathbf{E}, \theta)$ is positive semi-definite and orthogonal to $\mathbf{T}(\mathbf{E}, \theta)$,

$$\mathbf{E}^a(\mathbf{E}, \theta) \in \mathrm{Sym}^+ , \qquad \mathbf{E}^a(\mathbf{E}, \theta) \cdot \mathbf{T}(\mathbf{E}, \theta) = 0 . \tag{3.28}$$

To this end, we consider the four regions $\mathbb{R}_i$ separately. If $(\mathbf{E}, \theta) \in \mathbb{R}_1$, we have $\mathbb{D}(\theta)[\mathbf{T}] = \mathbf{E} - \beta(\theta)\, \mathbf{I}$, then

$$\mathbf{E}^a(\mathbf{E}, \theta) = \mathbf{0} \tag{3.29}$$

and it is trivial to verify (3.28). For $(\mathbf{E}, \theta) \in \mathcal{R}_2$, it holds that

$$\mathbf{E}^a(\mathbf{E}, \theta) = \mathbf{E} - \beta(\theta)\,\mathbf{I} \tag{3.30}$$

which is positive semi-definite by virtue of (3.4). If $(\mathbf{E}, \theta) \in \mathcal{R}_3$, we have

$$\mathbf{E}^a(\mathbf{E}, \theta) = [e_2 - \beta(\theta) + \nu(\theta)\,(e_1 - \beta(\theta))]\mathbf{q}_2 \otimes \mathbf{q}_2 +$$

$$[e_3 - \beta(\theta) + \nu(\theta)\,(e_1 - \beta(\theta))]\mathbf{q}_3 \otimes \mathbf{q}_3 \tag{3.31}$$

which results positive semi-definite in view of (3.2) and (3.5). Also, by comparing (3.31) and (3.23) we can state that $\mathbf{T}$ and $\mathbf{E}^a$ are orthogonal. Finally, if $(\mathbf{E}, \theta) \in \mathcal{R}_4$ we have

$$\mathbf{E}^a(\mathbf{E}, \theta) = \frac{1}{1 - \nu(\theta)}\,[e_3 - \beta(\theta) + \nu(\theta)\,(e_1 + e_2 - e_3 - \beta(\theta))]\mathbf{q}_3 \otimes \mathbf{q}_3 \tag{3.32}$$

and from (3.2), (3.6) and (3.24), conditions (3.28) must follow.

It is easy to verify that in absence of temperature variations, the materials characterized by the free energy given in (3.15)-(3.18) conforms to the isothermal no-tension material studied by Del Piero (1989) and Panzeca *et al.* (1988).

Let

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2\,(1 + \nu)} \quad \text{and} \quad 3\,\chi = \frac{E}{1 - 2\nu}, \tag{3.33}$$

be the Lamé moduli and the coefficient of volumetric expansion of the material, respectively. From (3.15)-(3.18), by taking into account of (3.7)-(3.10), we can deduce the expression of the entropy η and the internal energy ε within an error of order $o(\delta^2)$.

For $(\mathbf{E}, \theta) \in \mathcal{R}_1$,

$$\eta(\mathbf{E}, \theta) = -\,\xi'(\theta) - \frac{\lambda'(\theta)}{2\rho}\,(\mathrm{tr}\mathbf{E} - 3\,\beta(\theta))^2 +$$

$$-\,\frac{\mu'(\theta)}{\rho}\,\{(e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + (e_3 - \beta(\theta))^2\} +$$

$$\frac{3\,\chi(\theta)\,\beta'(\theta)}{\rho}\,(\mathrm{tr}\mathbf{E} - 3\,\beta(\theta))\,, \tag{3.34}$$

$$\varepsilon(\mathbf{E},\theta) = \xi(\theta) - \theta\xi'(\theta) + \frac{\lambda(\theta) - \theta\lambda'(\theta)}{2\rho}\,(\mathrm{tr}\mathbf{E} - 3\,\beta(\theta))^2 +$$

$$+\frac{\mu(\theta) - \theta\mu'(\theta)}{\rho}\,\left\{(e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + (e_3 - \beta(\theta))^2\right\} +$$

$$\frac{3\theta\,\chi(\theta)\,\beta'(\theta)}{\rho}\,(\mathrm{tr}\mathbf{E} - 3\,\beta(\theta))\,, \tag{3.35}$$

for $(\mathbf{E},\theta) \in \mathcal{R}_2$,

$$\eta(\mathbf{E},\theta) = -\,\xi'(\theta)\,, \tag{3.36}$$

$$\varepsilon(\mathbf{E},\theta) = \xi(\theta) - \theta\xi'(\theta)\,, \tag{3.37}$$

for $(\mathbf{E},\theta) \in \mathcal{R}_3$,

$$\eta(\mathbf{E},\theta) = -\,\xi'(\theta) - \frac{E'(\theta)}{2\,\rho}\,(e_1 - \beta(\theta))^2 +$$

$$\frac{E(\theta)\,\beta'(\theta)}{\rho}\,(e_1 - \beta(\theta))\,, \tag{3.38}$$

$$\varepsilon(\mathbf{E},\theta) = \xi(\theta) - \theta\xi'(\theta) + \frac{E(\theta) - \theta E'(\theta)}{2\,\rho}\,(e_1 - \beta(\theta))^2 +$$

$$\frac{\theta E(\theta)\,\beta'(\theta)}{\rho}\,(e_1 - \beta(\theta))\,, \tag{3.39}$$

for $(\mathbf{E},\theta) \in \mathcal{R}_4$,

$$\eta(\mathbf{E},\theta) = -\,\xi'(\theta) - \frac{E'(\theta)(1 - \nu(\theta)^2) + 2E(\theta)\nu(\theta)\nu'(\theta)}{2\,\rho\,(1 - \nu(\theta)^2)^2}\,\{(e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2\} +$$

$$-\,\frac{E'(\theta)\nu(\theta)(1 - \nu(\theta)^2) + E(\theta)\nu'(\theta)(1 + \nu(\theta)^2)}{\rho\,(1 - \nu(\theta)^2)^2}\,(e_1 - \beta(\theta))\,(e_2 - \beta(\theta)) +$$

$$\frac{E(\theta)\beta'(\theta)}{\rho\,(1 - \nu(\theta))}\,(e_1 + e_2 - 2\,\beta(\theta))\,, \tag{3.40}$$

$$\varepsilon(\mathbf{E}, \theta) = \xi(\theta) - \theta\xi'(\theta) + \frac{(E(\theta) - \theta E'(\theta))(1 - \nu(\theta)^2) - 2\theta E(\theta)\nu(\theta)\nu'(\theta)}{2\,\rho\,(1 - \nu(\theta)^2)^2}\,\{(e_1-\beta(\theta))^2 +$$

$$(e_2-\beta(\theta))^2\} + \frac{\theta E(\theta)\beta'(\theta)}{\rho\,(1 - \nu(\theta))}\,(e_1 + e_2 - 2\,\beta(\theta)) +$$

$$\frac{(E(\theta) - \theta E'(\theta))\nu(\theta)(1 - \nu(\theta)^2) - \theta E(\theta)\nu'(\theta)(1 + \nu(\theta)^2)}{\rho\,(1 - \nu(\theta)^2)^2}\,(e_1 - \beta(\theta))\,(e_2 - \beta(\theta))\,, \tag{3.41}$$

where ' denotes the derivative with respect to θ.

Since the example considered in Section 5 deals with the equilibrium problem of a circular ring made of a no-tension material subjected to a plane stress, the corresponding expressions for free energy, stress and entropy are reported in the Appendix.

From (3.21)-(3.24) and (3.34), (3.36), (3.38), (3.40) we obtain the *Maxwell relation*

$$\partial_\theta\,\mathbf{T}(\mathbf{E}, \theta) = -\,\rho\,\partial_E\,\eta(\mathbf{E}, \theta)\,. \tag{3.42}$$

Moreover, from (2.26), disregarding terms of order $O(\delta^2)$, we get the specific heat C_E at constant strain

$$C_E(\mathbf{E}, \theta) = \theta\,\partial_\theta\eta(\mathbf{E}, \theta) = -\,\theta\,\xi''(\theta)\,. \tag{3.43}$$

Since the thermodynamic potentials are defined within an arbitrary constant, we assume that they vanish for $\mathbf{E} = \mathbf{0}$ and $\theta = \theta_0$; in other words, we suppose that the equalities

$$\xi(\theta_0) = \xi'(\theta_0) = 0 \tag{3.44}$$

hold. From (3.43), in view of (3.44), we deduce the following relation

$$\xi(\theta) = \int_{\theta_0}^{\theta} C_E(0, \theta')\,d\theta' \; - \theta \int_{\theta_0}^{\theta} \frac{1}{\theta'}\,C_E(0, \theta')\,d\theta' \;\;, \tag{3.45}$$

which allows to determine the function $\xi(\theta)$, once the specific heat is known.

Let

$$\mathcal{h}(\mathbf{T}, \theta) = \varepsilon(\mathbf{E}(\mathbf{T}, \theta), \theta) - \frac{1}{\rho} \mathbf{T}\cdot\mathbf{E}(\mathbf{T}, \theta) \tag{3.46}$$

be the *enthalpy* per unit mass. We shall denote by

$$C_T(\mathbf{T}, \theta) = \partial_\theta \mathcal{h}(\mathbf{T}, \theta) \tag{3.47}$$

the *specific heat at constant stress* per unit mass. We intend to verify that C_E and C_T coincide, within an error of order $O(\delta^2)$. Setting

$$\bar{\eta}(\mathbf{T}, \theta) = \eta(\mathbf{E}(\mathbf{T}, \theta), \theta)\,, \tag{3.48}$$

by analogy to (2.26), let us start by supposing that

$$C_T(\mathbf{T}, \theta) = \theta\, \partial_\theta\, \bar{\eta}(\mathbf{T}, \theta) \tag{3.49}$$

holds true. In fact, in virtue of (2.13) and (3.19), we get

$$C_T(\mathbf{T}, \theta) = \partial_\theta\, \varepsilon(\mathbf{E}, \theta) + \partial_E\, \varepsilon(\mathbf{E}, \theta)\cdot\partial_\theta\, \mathbf{E}(\mathbf{T}, \theta) - \frac{1}{\rho}\mathbf{T}\cdot\partial_\theta\, \mathbf{E}(\mathbf{T}, \theta) =$$

$$= \partial_\theta\, \psi(\mathbf{E}, \theta) + \eta(\mathbf{E}, \theta) + \theta\, \partial_\theta\, \eta(\mathbf{E}, \theta) + (\partial_E\, \psi(\mathbf{E}, \theta) + \theta\, \partial_E\, \eta(\mathbf{E}, \theta))\cdot\partial_\theta\, \mathbf{E}(\mathbf{T}, \theta) +$$

$$- \frac{1}{\rho}\mathbf{T}\cdot\partial_\theta\, \mathbf{E}(\mathbf{T}, \theta) =$$

$$\theta\, \partial_\theta\, \eta(\mathbf{E}, \theta) + \theta\, \partial_E\, \eta(\mathbf{E}, \theta)\cdot\partial_\theta\, \mathbf{E}(\mathbf{T}, \theta) = \theta\, \partial_E\, \bar{\eta}(\mathbf{T}, \theta)\,. \tag{3.50}$$

By using the preceding relation we shall prove that, for $(\mathbf{E}, \theta)$ belonging to $\mathcal{R}_1$ and for $\mathbf{T}$ given in (3.21) we have

$$C_T(\mathbf{T}, \theta) - C_E(\mathbf{E}, \theta) = \frac{\theta}{2\rho\, \mu(\theta)\, \chi(\theta)} \{2\mu(\theta)\, m'(\theta)^2 + 4\, \mu(\theta)\, m'(\theta)\, \chi'(\theta)\, \mathrm{tr}\mathbf{E} +$$

$$- \frac{2}{3} [3\, \lambda(\theta)(2\mu'(\theta))^2 + 3\lambda'(\theta)^2 + 4\mu'(\theta)\, \lambda'(\theta)) +$$

$$- \mu(\theta)\, \lambda'(\theta)\big(3\lambda'(\theta) + 4\mu'(\theta)\big)\big]\, (\mathrm{tr}\mathbf{E})^2 + 4\, \chi(\theta)\, \mu'(\theta)^2\, \|\mathbf{E}\|^2 \Big\}, \quad (3.51)$$

where $m(\theta) = - 3\, \chi(\theta)\beta(\theta)$. To this end we observe that, in view of (3.50), (3.43) and (3.42) we can write,

$$C_T(\mathbf{T}, \theta) - C_E(\mathbf{E}, \theta) = \theta\{\partial_\theta \eta(\mathbf{E}(\mathbf{T}, \theta), \theta) - \partial_\theta \eta(\mathbf{E}, \theta)\} = \theta\, \partial_E \eta(\mathbf{E}, \theta)\cdot\partial_\theta \mathbf{E}(\mathbf{T}, \theta) =$$

$$- \frac{\theta}{\rho}\, \partial_\theta \mathrm{T}(\mathbf{E}, \theta)\cdot\partial_\theta \mathbf{E}(\mathbf{T}, \theta) . \quad (3.52)$$

On the other hand, using the chain rule and the relation $\mathbf{E} = \frac{1}{2\mu(\theta)}\, \mathbf{T} - \frac{\lambda(\theta)}{6\mu(\theta)\chi(\theta)}\, (\mathrm{tr}\, \mathbf{T})\, \mathbf{I} + \beta(\theta)\mathbf{I}$ obtained from (3.21), we get

$$\partial_\theta \mathbf{E}(\mathbf{T}, \theta) = - \partial_T\mathbf{E}(\mathbf{T}, \theta)\, [\partial_\theta \mathbf{T}(\mathbf{E}, \theta)] =$$

$$- \Big(\frac{1}{2\, \mu(\theta)}\, \mathbb{I} - \frac{\lambda(\theta)}{6\, \mu(\theta)\, \chi(\theta)}\, \mathbf{I}\otimes\mathbf{I}\Big)\, [\partial_\theta \mathbf{T}(\mathbf{E}, \theta)] =$$

$$- \frac{1}{2\, \mu(\theta)}\, \partial_\theta \mathbf{T}(\mathbf{E}, \theta) + \frac{\lambda(\theta)}{6\, \mu(\theta)\, \chi(\theta)}\, \mathrm{tr}(\partial_\theta \mathbf{T}(\mathbf{E}, \theta))\, \mathbf{I} . \quad (3.53)$$

Since, in view of (3.21) we have

$$\partial_\theta \mathbf{T}(\mathbf{E}, \theta) = m'(\theta)\, \mathbf{I} + 2\, \mu'(\theta)\, \mathbf{E} + \lambda'(\theta)\, \mathrm{tr}\mathbf{E}\, \mathbf{I} , \quad (3.54)$$

from (3.52) and (3.53) we obtain (3.51).

From relations (3.52), (3.11) and the fact that $m(\theta) = O(\delta)$ and $m'(\theta) = O(\delta)$, we arrive at the desired result

$$C_T(\mathbf{T}, \theta) = C_E(\mathbf{E}, \theta), \quad (3.55)$$

within an error of order $O(\delta^2)$. It can be proved that equality (3.55) also holds for $(\mathbf{E}, \theta)$ belonging to the other regions $\mathbb{R}_2$, $\mathbb{R}_3$ and $\mathbb{R}_4$.

In order to complete the system of constitutive equations, we assume the usual relation for heat flux:

$$\mathbf{q} = - \kappa(\theta)\, \mathbf{g} , \quad (3.56)$$

where $\kappa(\theta) \geq 0$ is the *conductivity coefficient.*

The material having constitutive relations (3.15)-(3.18), (3.34)-(3.41), (3.44), (3.21)-(3.24), (3.43) and (3.56) is wholly characterized by five temperature-dependent functions $E(\theta)$, $\nu(\theta)$, $\beta(\theta)$, $\kappa(\theta)$ and $C_E(\theta)$. Moreover, from relations (3.42), $(3.43)_1$ and (3.19), we can deduce within an error of order $o(\delta)$ that

$$\dot{\eta}(\mathbf{E}, \theta) = \partial_E \eta(\mathbf{E}, \theta)\cdot\dot{\mathbf{E}} + \partial_\theta \eta(\mathbf{E}, \theta)\dot{\theta} = -\frac{1}{\rho}\,\partial_\theta \mathbf{T}(\mathbf{E}, \theta)\cdot\dot{\mathbf{E}} + \frac{1}{\theta}\,C_E(\mathbf{E}, \theta)\,\dot{\theta}\,. \tag{3.57}$$

By then accounting for (3.21)-(3.24), (3.43) and (2.16), we can write the energy equation in the four regions $\mathcal{R}_i$:
for $(\mathbf{E}, \theta) \in \mathcal{R}_1$,

$$-\operatorname{div}\mathbf{q} + \theta\,\{2\mu'(\theta)\,\mathbf{E}\cdot\dot{\mathbf{E}} + \lambda'(\theta)\,\mathrm{tr}\mathbf{E}\,\mathrm{tr}\dot{\mathbf{E}} - 3(\chi\beta)'\,\mathrm{tr}\dot{\mathbf{E}}\} + \rho\,s = -\,\rho\theta\,\xi''(\theta)\,\dot{\theta}\,, \tag{3.58}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_2$,

$$-\operatorname{div}\mathbf{q} + \rho\,s = -\,\rho\theta\,\xi''(\theta)\,\dot{\theta}\,, \tag{3.59}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_3$,

$$-\operatorname{div}\mathbf{q} + \theta\,\{E'(\theta)\,(e_1 - \beta(\theta))\,\mathbf{q}_1\otimes\mathbf{q}_1\cdot\dot{\mathbf{E}} - E(\theta)\beta'(\theta)\mathbf{q}_1\otimes\mathbf{q}_1\cdot\dot{\mathbf{E}}\} + \rho\,s = -\,\rho\theta\,\xi''(\theta)\,\dot{\theta}\,, \tag{3.60}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_4$,

$$-\operatorname{div}\mathbf{q} + \theta\,\Big\{\,\frac{E'(\theta)(1 - \nu(\theta)^2) + 2E(\theta)\nu(\theta)\nu'(\theta)}{(1 - \nu(\theta)^2)^2}\,[(e_1 + \nu(\theta)\,e_2 - (1 + \nu(\theta))\,\beta(\theta))\,\mathbf{q}_1\otimes\mathbf{q}_1\cdot\dot{\mathbf{E}} +$$

$$(e_2 + \nu(\theta)\,e_1 - (1 + \nu(\theta))\,\beta(\theta))\,\mathbf{q}_2\otimes\mathbf{q}_2\cdot\dot{\mathbf{E}}\,] +$$

$$\frac{E(\theta)}{1 - \nu(\theta)^2}\,[(\nu'(\theta)\,e_2 - \beta'(\theta) - \nu'(\theta)\beta(\theta) - \nu(\theta)\beta'(\theta))\,\mathbf{q}_1\otimes\mathbf{q}_1\cdot\dot{\mathbf{E}} +$$

$$(\nu'(\theta)\,e_1 - \beta'(\theta) - \nu'(\theta)\beta(\theta) - \nu(\theta)\beta'(\theta))\,\mathbf{q}_2\otimes\mathbf{q}_2\cdot\dot{\mathbf{E}}\,]\Big\} + \rho\,s = -\,\rho\theta\,\xi''(\theta)\,\dot{\theta}\,. \tag{3.61}$$

Thus, the basic equations of the thermoelastic theory are the strain-displacement relation (3.12), the equation of equilibrium $(2.2)_1$, constitutive relations (3.21)-(3.24) and (3.56) for stress and heat flux, respectively, and finally, the equilibrium energy equation (3.58)-(3.61). The system of equations obtained is coupled because the coefficient of temperature on the left-hand side of the energy equation (3.58)-(3.61) depends on the strain and strain rate. In particular, if we assume

$$\dot{\mathbf{E}} = O(\delta) , \tag{3.62}$$

the energy equation (3.58)-(3.61) can be simplified. In fact, disregarding terms of order $O(\delta^2)$, we obtain

$$- \operatorname{div} \mathbf{q} + \rho\, s = - \rho\theta\, \xi''(\theta)\, \dot{\theta} \tag{3.63}$$

for all regions and the thermoelastic equilibrium is therefore governed by the following equations

$$\begin{aligned} &\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T) , \\ &\operatorname{div} \mathbf{T} + \mathbf{b} = \mathbf{0} , \\ &\mathbf{T} = \mathbf{T}(\mathbf{E}, \theta), \\ &\mathbf{q} = - \kappa(\theta)\, \mathbf{g} , \\ &- \operatorname{div}\mathbf{q} + \rho\, s = - \rho\theta\, \xi''(\theta)\, \dot{\theta} , \end{aligned} \tag{3.64}$$

and thermo-mechanical uncoupling occurs.

Finally, if for temperature we have

$$\theta = \theta_0 + o(\delta), \tag{3.65}$$

with θ_0 as the reference temperature, it holds that

$$\beta(\theta) = \beta(\theta_0) + \beta'(\theta_0)\, (\theta - \theta_0) + o(\delta) , \tag{3.66}$$

and then accounting for (3.1), we have within an error of order $o(\delta)$

$$\beta(\theta) = \beta'(\theta_0)\, (\theta - \theta_0) \tag{3.67}$$

in a neighbourhood of $\theta = \theta_0$. The quantity

$$\alpha = \beta'(\theta_0) \tag{3.68}$$

is the *linear coefficient of thermal expansion.* Moreover, in this case functions $E(\theta)$, $\nu(\theta)$, $\beta(\theta)$, $\kappa(\theta)$ and $C_E(\theta)$ can be supposed temperature-independent and coincident with their value at θ_0.

4. THE BOUNDARY-VALUE PROBLEM

The equilibrium problem for no-tension solids has been studied in recent years and the existence of a solution has been proven solely for a rather restricted class of load conditions (Anzellotti, 1985, Giaquinta *et al.* , 1985). However, the uniqueness of the solution is guaranteed only in terms of stress, in the sense that different displacement and strain fields can correspond to the same stress field. Similar considerations can be made for a no-tension material having the constitutive equation $\mathbf{T} = \mathbf{T}(\mathbf{E}, \theta)$ given in Section 3, which associates the stress $\mathbf{T}$ to each $(\mathbf{E}, \theta) \in \mathrm{Sym} \times [\theta_1, \theta_2]$, such that

$$\mathbf{T} = \mathbb{D}(\theta)^{-1}[\mathbf{E} - \beta(\theta)\,\mathbf{I} - \mathbf{E}^a(\mathbf{E}, \theta)]\,, \qquad (4.1)_1$$

$$\mathbf{T} \in \mathrm{Sym}^-\,, \qquad (4.1)_2$$

$$\mathbf{E}^a \in \mathrm{Sym}^+\,, \qquad (4.1)_3$$

$$\mathbf{T} \cdot \mathbf{E}^a = 0\,, \qquad (4.1)_4$$

where the fourth order tensor $\mathbb{D}(\theta)$ is given in (3.25).

In this section we prove that the stress field satisfying the uncoupled equilibrium equations $(3.64)_1$, $(3.64)_2$ and $(3.64)_3$ for no-tension solids subjected to thermal load is unique. To this end, let $\mathcal{B}$ be a body made of a masonry-like material and let $\mathcal{S}_u$ and $\mathcal{S}_f$ be two subsets of the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, such that their union covers $\partial\mathcal{B}$ and their interiors are disjoint.

A load $(\mathbf{b}, \theta, \mathbf{f}_0)$ defined in $\mathcal{B} \times \mathcal{B} \times \mathcal{S}_f$ with values in $\mathcal{V} \times [\theta_1, \theta_2] \times \mathcal{V}$ is *admissible*, if the corresponding boundary-value problem has a solution, *i. e.* if there exists a triple $[\mathbf{u}, \mathbf{E}, \mathbf{T}]$, constituted by stress field $\mathbf{T}$, strain field $\mathbf{E}$ and displacement field $\mathbf{u}$ defined on the closure $\bar{\mathcal{B}}$ of $\mathcal{B}$, piece-wise C^2, such that the equations $(3.64)_1$, $(3.64)_2$ and $(3.64)_3$ are satisfied on $\mathcal{B}$ and the boundary conditions

$$\mathbf{u} = \mathbf{0} \ \text{ on } \mathcal{S}_u, \qquad (4.2)_1$$

$$\mathbf{T}\mathbf{n} = \mathbf{f}_0 \ \text{ on } \mathcal{S}_f, \qquad (4.2)_2$$

hold, where $\mathbf{n}$ is the unit outward normal to $\mathcal{S}_f$.

It is easy to prove that if $(\mathbf{b}, \theta, \mathbf{f}_0)$ is an admissible load and $[\mathbf{u}_1, \mathbf{E}_1, \mathbf{T}_1]$ and $[\mathbf{u}_2, \mathbf{E}_2, \mathbf{T}_2]$ are two solutions to $(3.64)_1$, $(3.64)_2$, $(3.64)_3$ and (4.2), then $\mathbf{T}_1(x) = \mathbf{T}_2(x)$ for every $x \in \mathcal{B}$. In fact, the triple $[\overline{\mathbf{u}}, \overline{\mathbf{E}}, \overline{\mathbf{T}}]$ with $\overline{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, $\overline{\mathbf{E}} = \mathbf{E}_1 - \mathbf{E}_2$ and $\overline{\mathbf{T}} = \mathbf{T}_1 - \mathbf{T}_2$ satisfies $(3.64)_1$ and $(4.2)_1$; in addition, it satisfies $(4.2)_2$ and $(3.64)_2$ with $\mathbf{f}_0 = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$. Thus, in conformity with the hypothesis on the smoothness of the solutions, a simple application of the principle of virtual work proves that

$$\int_{\mathcal{B}} \overline{\mathbf{T}} \cdot \overline{\mathbf{E}}\, dV = 0 . \tag{4.3}$$

On the other hand,

$$\overline{\mathbf{E}} = \mathbf{E}_1 - \beta(\theta)\mathbf{I} + \beta(\theta)\mathbf{I} - \mathbf{E}_2 = \mathbf{E}_1^e + \mathbf{E}_1^a - \mathbf{E}_2^e - \mathbf{E}_2^a , \tag{4.4}$$

where $\mathbf{E}_1^e$, $\mathbf{E}_1^a$ and $\mathbf{E}_2^e$, $\mathbf{E}_2^a$ are the elastic and the inelastic strain corresponding to $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively. From (4.3), by using (4.4) we obtain

$$\int_{\mathcal{B}} \overline{\mathbf{T}} \cdot (\mathbf{E}_1^e - \mathbf{E}_2^e)\, dV = \int_{\mathcal{B}} \overline{\mathbf{T}} \cdot (\mathbf{E}_1^a - \mathbf{E}_2^a)\, dV ; \tag{4.5}$$

the first member in (4.5) is equal to $\int_{\mathcal{B}} \overline{\mathbf{T}} \cdot \mathbb{D}(\theta)^{-1}[\overline{\mathbf{T}}]\, dV$ and is therefore non-negative because $\mathbb{D}(\theta)$ is positive definite. By virtue of $(4.1)_4$, the second member of (4.5) results equal to

$$\int_{\mathcal{B}} (\mathbf{T}_1 \cdot \mathbf{E}_1^a + \mathbf{T}_2 \cdot \mathbf{E}_2^a)\, dV ,$$

which is non-positive because of $(4.1)_2$ and $(4.1)_3$. Then, we have

$$\int_{\mathcal{B}} \overline{\mathbf{T}} \cdot \mathbb{D}(\theta)^{-1}[\overline{\mathbf{T}}]\, dV = 0 ,$$

from which we obtain $\overline{\mathbf{T}} \cdot \mathbb{D}(\theta)^{-1}[\overline{\mathbf{T}}] = 0$ everywhere in $\mathcal{B}$ and thereby $\overline{\mathbf{T}} = 0$, that is the sought after result.

5. AN EXAMPLE

Let us consider a circular ring with inner radius r_1 and outer radius $r_2 = 2r_1$ made of a no-tension material and subjected to a plane stress under the action of two uniform radial pressures p_1 and p_2 acting on the internal and external boundary, respectively. Moreover, the ring is subjected to a temperature distribution θ depending linearly on the radius r

$$\theta(r) = \frac{\theta_2 - \theta_1}{r_2 - r_1} r + \frac{r_2\theta_1 - r_1\theta_2}{r_2 - r_1}, \tag{5.1}$$

where θ_1 and θ_2 are the temperature of the internal boundary and external boundary, respectively.

We assume that $\beta(\theta) = \alpha(\theta - \theta_0)$, with θ_0 as the reference temperature, and that the Poisson ratio ν does not depend on θ. With regard to Young's modulus E, we assume it to be a linear function of temperature varying from E_2 for $\theta = \theta_2$ to $E_1 = \frac{1}{2} E_2$ for $\theta = \theta_1$,

$$E(\theta) = \frac{E_2}{2(\theta_2 - \theta_1)} (\theta + \theta_2 - 2\theta_1) \ . \tag{5.2}$$

From (5.1) and (5.2) we get

$$E(r) = \frac{E_2}{2(r_2 - r_1)} r \ . \tag{5.3}$$

The choice of quantitites r_1, r_2, E_1, E_2 such that $\frac{r_1}{r_2} = \frac{E_1}{E_2} = \frac{1}{2}$, is linked to the fact that, if condition $r_2 E_1 = r_1 E_2$ holds true, then the equilibrium equation of the ring is easily integrable; the procedure set forth here for calculating the solution is independent of the value of $\frac{E_1}{E_2}$.

We start by determining the solution for a linear elastic material. Let us indicate by σ_r, σ_φ, ε_r, ε_φ the radial and circumferential components of the stress and strain; by virtue of the constitutive equation for plane stress we have

$$\sigma_r(r) = \frac{E(r)}{1 - \nu^2} \{\varepsilon_\rho(r) + \nu\, \varepsilon_\varphi(r) - \alpha(1 + \nu)(\theta(r) - \theta_0)\}, \tag{5.4}$$

$$\sigma_\varphi(r) = \frac{E(r)}{1 - \nu^2} \{\varepsilon_\varphi(r) + \nu\, \varepsilon_\rho(r) - \alpha(1 + \nu)(\theta(r) - \theta_0)\}. \tag{5.5}$$

By imposing the equilibrium equation $\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0$, and accounting for (5.1) and (5.3) we get

$$r\left(\frac{d\varepsilon_r}{dr} + \nu\frac{d\varepsilon_\varphi}{dr}\right) + (2-\nu)\varepsilon_r - (1-2\nu)\varepsilon_\varphi = \alpha(1+\nu)\left(2\frac{\theta_2-\theta_1}{r_2-r_1}r + \frac{r_2\theta_1 - r_1\theta_2}{r_2-r_1} - \theta_0\right) \quad (5.6)$$

In turn, in view of relations linking radial displacement u to strains $\varepsilon_r = \frac{du}{dr}$ and $\varepsilon_\varphi = \frac{u}{r}$, (5.6) is equivalent to the linear ordinary differential equation

$$r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} - (1-\nu)\,u = \alpha(1+\nu)\left\{2\frac{\theta_2-\theta_1}{r_2-r_1}r^2 + \left(\frac{r_2\theta_1 - r_1\theta_2}{r_2-r_1} - \theta_0\right)r\right\}. \quad (5.7)$$

By performing the substitution $r = e^t$, we obtain the equation with constant coefficients

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - (1-\nu)\,u = \alpha(1+\nu)\left\{2\frac{\theta_2-\theta_1}{r_2-r_1}e^{2t} + \left(\frac{r_2\theta_1 - r_1\theta_2}{r_2-r_1} - \theta_0\right)e^t\right\} \quad (5.8)$$

whose general solution is

$$u(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \alpha(1+\nu)\left\{\frac{2}{5+\nu}\frac{\theta_2-\theta_1}{r_2-r_1}e^{2t} + \frac{1}{1+\nu}\left(\frac{r_2\theta_1 - r_1\theta_2}{r_2-r_1} - \theta_0\right)e^t\right\}, \quad (5.9)$$

where

$$\lambda_1 = -\frac{1+\sqrt{5-4\nu}}{2}, \qquad \lambda_2 = \frac{-1+\sqrt{5-4\nu}}{2} \quad (5.10)$$

and C_1 and C_2 are two integration constants. Now, starting with the radial displacement (5.9), and recalling that $r = e^t$, by using (5.4) and (5.5) we get the stress components

$$\sigma_r(r) = \frac{E_2}{2(r_2-r_1)(1-\nu^2)}\left\{C_1(\lambda_1+\nu)\,r^{\lambda_1} + C_2(\lambda_2+\nu)\,r^{\lambda_2} - \frac{\alpha(1-\nu^2)}{5+\nu}\frac{\theta_2-\theta_1}{r_2-r_1}r^2\right\}, \quad (5.11)$$

$$\sigma_\varphi(r) = \frac{E_2}{2(r_2-r_1)(1-\nu^2)}\left\{C_1(1+\nu\lambda_1)\,r^{\lambda_1} + C_2(1+\nu\lambda_2)\,r^{\lambda_2} - \frac{3\alpha(1-\nu^2)}{5+\nu}\frac{\theta_2-\theta_1}{r_2-r_1}r^2\right\}. \quad (5.12)$$

Quantities C_1 and C_2 are determined by imposing the boundary conditions $\sigma_r(r_1) = -p_1$ and $\sigma_r(r_2) = -p_2$ and take the form

$$C_1 = (\lambda_2 + \nu) \frac{r_2^{\lambda_2} \left(\frac{\alpha r_1^2}{5+\nu} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_1}{E_2} \right) - r_1^{\lambda_2} \left(\frac{\alpha r_2^2}{5+\nu} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_2}{E_2} \right)}{r_2^{\lambda_1} r_1^{\lambda_1} \left(r_1^{\sqrt{5-4\nu}} - r_2^{\sqrt{5-4\nu}} \right)}, \quad (5.13)$$

$$C_2 = (\lambda_1 + \nu) \frac{r_1^{\lambda_1} \left(\frac{\alpha r_2^2}{5+\nu} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_2}{E_2} \right) - r_2^{\lambda_1} \left(\frac{\alpha r_1^2}{5+\nu} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_1}{E_2} \right)}{r_2^{\lambda_1} r_1^{\lambda_1} \left(r_1^{\sqrt{5-4\nu}} - r_2^{\sqrt{5-4\nu}} \right)}. \quad (5.14)$$

For the sake of simplicity we shall limit ourselves to consider $\nu = 0$, in this case we have

$$\lambda_1 = - \frac{1+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{-1+\sqrt{5}}{2}, \quad (5.15)$$

$$C_1 = \lambda_2 \frac{r_2^{\lambda_2} \left(\frac{\alpha r_1^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_1}{E_2} \right) - r_1^{\lambda_2} \left(\frac{\alpha r_2^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_2}{E_2} \right)}{r_2^{\lambda_1} r_1^{\lambda_1} \left(r_1^{\sqrt{5}} - r_2^{\sqrt{5}} \right)}, \quad (5.16)$$

$$C_2 = \lambda_1 \frac{r_1^{\lambda_1} \left(\frac{\alpha r_2^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_2}{E_2} \right) - r_2^{\lambda_1} \left(\frac{\alpha r_1^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_1}{E_2} \right)}{r_2^{\lambda_1} r_1^{\lambda_1} \left(r_1^{\sqrt{5}} - r_2^{\sqrt{5}} \right)}, \quad (5.17)$$

and the elastic solution has the following components

$$\sigma_r(r) = \frac{E_2}{2(r_2 - r_1)} \left\{ C_1 \lambda_1 r^{\lambda_1} + C_2 \lambda_2 r^{\lambda_2} - \frac{\alpha}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} r^2 \right\}, \quad (5.18)$$

$$\sigma_\varphi(r) = \frac{E_2}{2(r_2 - r_1)} \left\{ C_1 r^{\lambda_1} + C_2 r^{\lambda_2} - \frac{3\alpha}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} r^2 \right\}, \quad (5.19)$$

$$\varepsilon_r(r) = C_1 \lambda_1 r^{\lambda_1 - 1} + C_2 \lambda_2 r^{\lambda_2 - 1} - \frac{\alpha}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} r + \alpha \left(\frac{\theta_2 - \theta_1}{r_2 - r_1} r + \frac{r_2\theta_1 - r_1\theta_2}{r_2 - r_1} - \theta_0 \right) r, \quad (5.20)$$

$$\varepsilon_\varphi(r) = C_1 r^{\lambda_1 - 1} + C_2 r^{\lambda_2 - 1} - \frac{3\alpha}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} r + \alpha \left(\frac{\theta_2 - \theta_1}{r_2 - r_1} r + \frac{r_2\theta_1 - r_1\theta_2}{r_2 - r_1} - \theta_0 \right) r, \quad (5.21)$$

$$u(r) = C_1 r^{\lambda_1} + C_2 r^{\lambda_2} + \alpha \left\{ \frac{2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} r^2 + \left(\frac{r_2\theta_1 - r_1\theta_2}{r_2 - r_1} - \theta_0 \right) r \right\}. \quad (5.22)$$

It is possible to prove that for some values of the constants, the stress field (5.18) - (5.19) is negative semi-definite, and that the elastic solution is then the solution corresponding to an elastic no-tension material. On the other hand, there exist values of the constants such that the radial stress is purely compressive, and circumferential stress is negative starting at $r = r_1$, vanishes at a point internal to the circular ring and becomes positive up to $r = r_2$. Thus, if the material does not withstand tension, the stress field (5.18) - (5.19) does not represent a solution to the equilibrium problem. With a procedure similar to that used in Padovani (1995) it is possible, by beginning with the elastic solution, to calculate a negative semi-definite stress field equilibrated with the loads which is therefore the solution to the equilibrium problem of the circular ring made of a material with constitutive equation (4.1). Such a stress field has components

$$\sigma_r(r) = \begin{cases} \dfrac{E_2}{2(r_2 - r_1)} \{\overline{C}_1 \lambda_1 r^{\lambda_1} + \overline{C}_2 \lambda_2 r^{\lambda_2} - \dfrac{\alpha}{5} \dfrac{\theta_2 - \theta_1}{r_2 - r_1} r^2\}, & r \in [r_1, r_0], \\ - \dfrac{r_2 p_2}{r}, & r \in [r_0, r_2], \end{cases} \tag{5.23}$$

$$\sigma_\varphi(r) = \begin{cases} \dfrac{E_2}{2(r_2 - r_1)} \{\overline{C}_1 r^{\lambda_1} + \overline{C}_2 r^{\lambda_2} - \dfrac{3\alpha}{5} \dfrac{\theta_2 - \theta_1}{r_2 - r_1} r^2\}, & r \in [r_1, r_0], \\ 0, & r \in [r_0, r_2], \end{cases} \tag{5.24}$$

where

$$\overline{C}_1 = \lambda_2 \frac{r_0^{\lambda_2} (\frac{\alpha r_1^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_1}{E_2}) - r_1^{\lambda_2} (\frac{\alpha r_0^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_0}{E_2})}{r_0^{\lambda_1} r_1^{\lambda_1} (r_1^{\sqrt{5}} - r_0^{\sqrt{5}})}, \tag{5.25}$$

$$\overline{C}_2 = \lambda_1 \frac{r_1^{\lambda_1} (\frac{\alpha r_0^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_0}{E_2}) - r_0^{\lambda_1} (\frac{\alpha r_1^2}{5} \frac{\theta_2 - \theta_1}{r_2 - r_1} - 2(r_2 - r_1) \frac{p_1}{E_2})}{r_0^{\lambda_1} r_1^{\lambda_1} (r_1^{\sqrt{5}} - r_0^{\sqrt{5}})}, \tag{5.26}$$

$$p_0 = \frac{r_2 p_2}{r_0}, \tag{5.27}$$

and r_0 is the unique root belonging to the interval $[r_1, r_2]$ of the equation

$$\overline{C}_1\, r_0^{\lambda_1} + \overline{C}_2\, r_0^{\lambda_2} - \frac{3\,\alpha}{5}\,\frac{\theta_2 - \theta_1}{r_2 - r_1}\, r_0^2 = 0\,. \tag{5.28}$$

The circular region Ω_1, with inner radius r_1 and outer radius r_0, is entirely compressed and does not contain fractures; in the remaining region Ω_2 with inner radius r_0 and outer radius r_2, the inelastic radial strain is null and the elastic radial strain is

$$\varepsilon_r^e(r) = -\,\frac{2(r_2 - r_1)\, r_2\, p_2}{E_2\, r^2}\,. \tag{5.29}$$

The inelastic circumferential strain is a function $\varepsilon_\varphi^a(r)$, which needs to be determined and the inelastic radial strain is null. From the relation

$$\varepsilon_r(r) = \alpha\,\Big(\frac{\theta_2 - \theta_1}{r_2 - r_1}\, r + \frac{r_2\theta_1 - r_1\theta_2}{r_2 - r_1} - \theta_0\Big) - \frac{2(r_2 - r_1)\, r_2\, p_2}{E_2\, r^2}\,, \tag{5.30}$$

we get the radial displacement in Ω_2,

$$u(r) = \alpha\,\Big\{\frac{\theta_2 - \theta_1}{2(r_2 - r_1)}\, r^2 + \Big(\frac{r_2\theta_1 - r_1\theta_2}{r_2 - r_1} - \theta_0\Big)\, r\Big\} + \frac{2(r_2 - r_1)\, r_2\, p_2}{E_2}\Big(\frac{1}{r} - \frac{1}{r_0}\Big) +$$

$$\overline{C}_1\, r_0^{\lambda_1} + \overline{C}_2\, r_0^{\lambda_2} - \frac{\alpha(\theta_2 - \theta_1) r_0^2}{10(r_2 - r_1)}\,. \tag{5.31}$$

Finally, from

$$\varepsilon_\varphi(r) = \alpha\,\Big(\frac{\theta_2 - \theta_1}{r_2 - r_1}\, r + \frac{r_2\theta_1 - r_1\theta_2}{r_2 - r_1} - \theta_0\Big) + \varepsilon_\varphi^a(r) \tag{5.32}$$

and (5.31) we obtain the inelastic circumferential strain in Ω_2

$$\varepsilon_\varphi^a(r) = -\,\alpha\,\frac{\theta_2 - \theta_1}{2(r_2 - r_1)}\, r + \frac{2(r_2 - r_1)\, r_2\, p_2}{E_2}\Big(\frac{1}{r^2} - \frac{1}{r\, r_0}\Big) +$$

$$\Big(\overline{C}_1\, r_0^{\lambda_1} + \overline{C}_2\, r_0^{\lambda_2} - \frac{\alpha(\theta_2 - \theta_1) r_0^2}{10(r_2 - r_1)}\Big)\frac{1}{r}\,. \tag{5.33}$$

It is a simple matter to verify that $\varepsilon_\varphi^a(r_0) = 0$ and $\varepsilon_\varphi^a(r) > 0$ for $r \in (r_0, r_2]$.

The free energy and entropy for the circular ring can be calculated by using formulae (A.4), (A.5), (A.7), (A.8), (A.10) and (A.11) which have been included in Appendix for plane stress. In particular if we assume that the specific heat C_E is constant, then we have $\xi(\theta(r)) = C_E(\theta(r) - \theta_0) - C_E\,\theta(r)\ln[\theta(r)/\theta_0]$. The free energy has the expression

$$\psi(r) = \begin{cases} \xi(\theta(r)) + \dfrac{r_2 - r_1}{E_2\rho}\,\dfrac{(\sigma_r(r)^2 + \sigma_\varphi(r)^2)}{r}, & r \in [r_1, r_0], \\ \\ \xi(\theta(r)) + \dfrac{r_2 - r_1}{E_2\rho}\,\dfrac{\sigma_r(r)^2}{r}, & r \in [r_0, r_2] \end{cases} \qquad (5.34)$$

and the entropy is

$$\eta(r) = \begin{cases} -\,C_E\ln[\theta(r)/\theta_0] - \dfrac{(r_2 - r_1)^2}{E_2\rho(\theta_2 - \theta_1)}\,\dfrac{(\sigma_r(r)^2 + \sigma_\varphi(r)^2)}{r^2} + \dfrac{\alpha}{\rho}\,(\sigma_r(r) + \sigma_\varphi(r)), & r \in [r_1, r_0], \\ \\ -\,C_E\ln[\theta(r)/\theta_0] - \dfrac{(r_2 - r_1)^2}{E_2\rho(\theta_2 - \theta_1)}\,\dfrac{\sigma_r(r)^2}{r^2} + \dfrac{\alpha}{\rho}\,\sigma_r(r), & r \in [r_0, r_2]. \end{cases} \qquad (5.35)$$

Figures 1-4 show the behaviour of the radial and circumferential stresses, circumferential inelastic strain and radial displacement as r varies in the interval $[r_1, r_2]$. Figures 5 and 6 plot the difference between the free energy and function $\xi(\theta(r))$, and the sum of the entropy and the function $C_E\ln[\theta(r)/\theta_0]$ as functions of $r \in [r_1, r_2]$, respectively. The following parameters values have been used

$r_1 = 1$ m,
$r_2 = 2$ m,
$E_2 = 6.\ 10^9$ Pa,
$E_1 = 3.\ 10^9$ Pa,
$\nu = 0$,
$\theta_0 = 30°$ C,
$\theta_1 = 1200°$ C,
$\theta_2 = 20°$ C,
$p_1 = 1.\ 10^6$ Pa,
$p_2 = 1.\ 10^6$ Pa,

$\alpha = 1.\ 10^{-5}\ (^\circ C)^{-1}$,
$C_E = 1046$ J/Kg $^\circ$ C,
$\rho = 2000$ Kg/m^3.

The dotted line is the solution for a linear elastic material calculated according to (5.18), (5.19), (5.22), (A.4) and (A.10), while the continuous line represents the solution for a no-tension material. The value of the radius separating the compressed region and the cracked region, deduced from (5.28), is $r_0 \cong 1.226$ m.

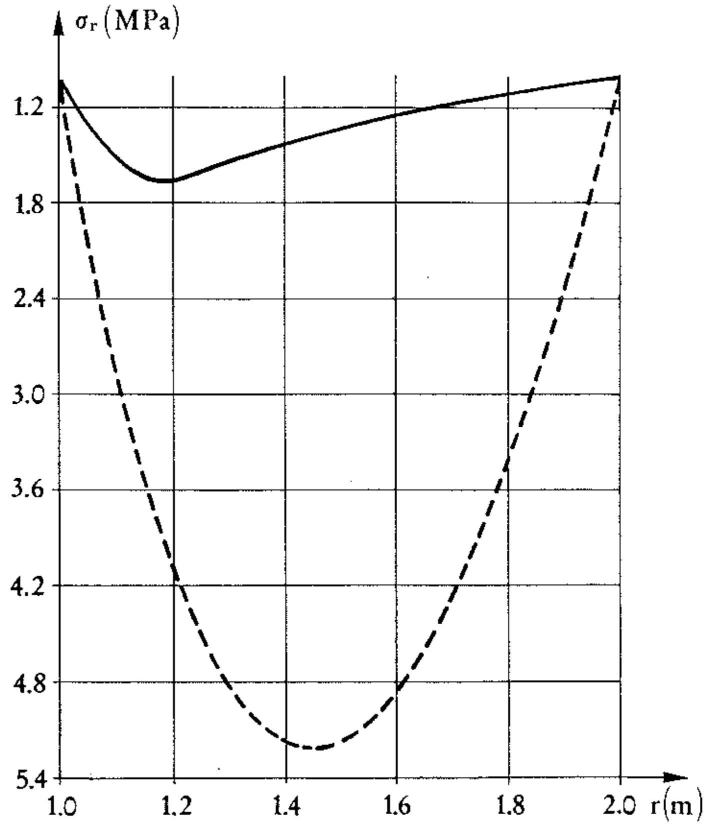

Figure 1. Radial stress σ_r vs. r.

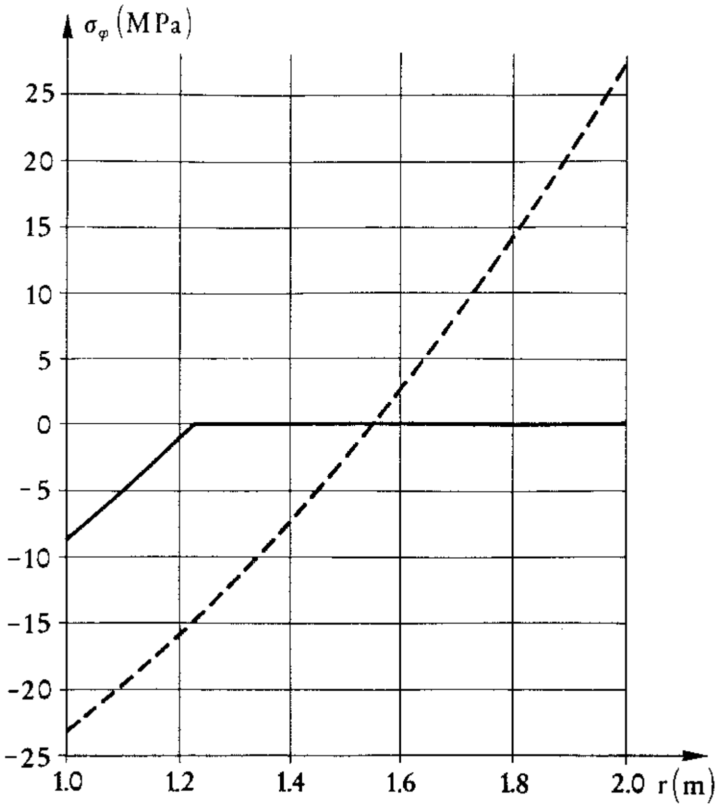

Figure 2. Circumferential stress σ_φ vs. r.

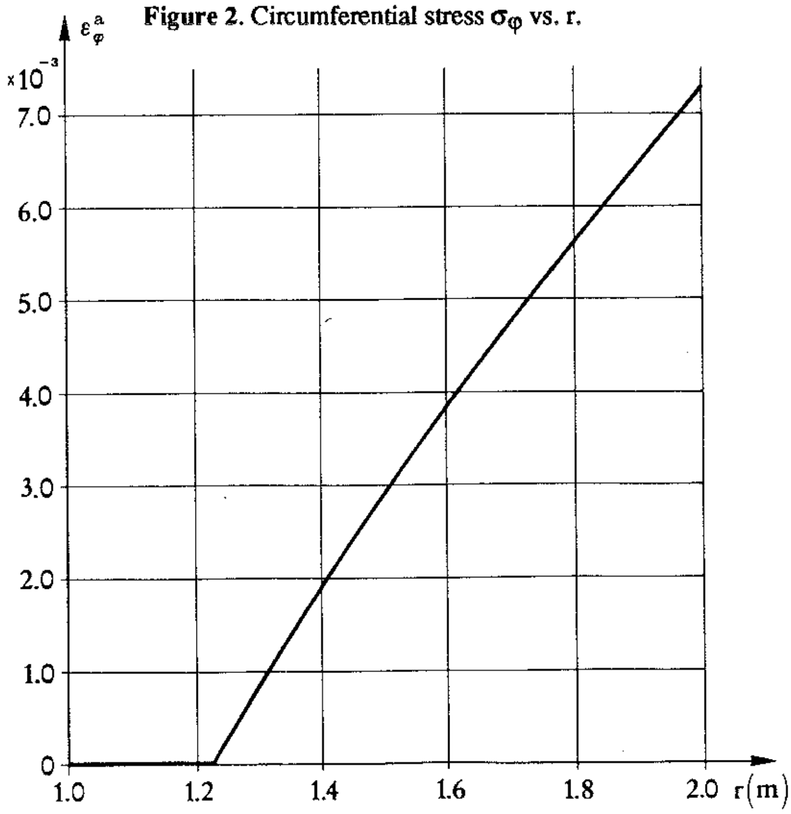

Figure 3. Circumferential inelastic strain ε_φ^a vs. r.

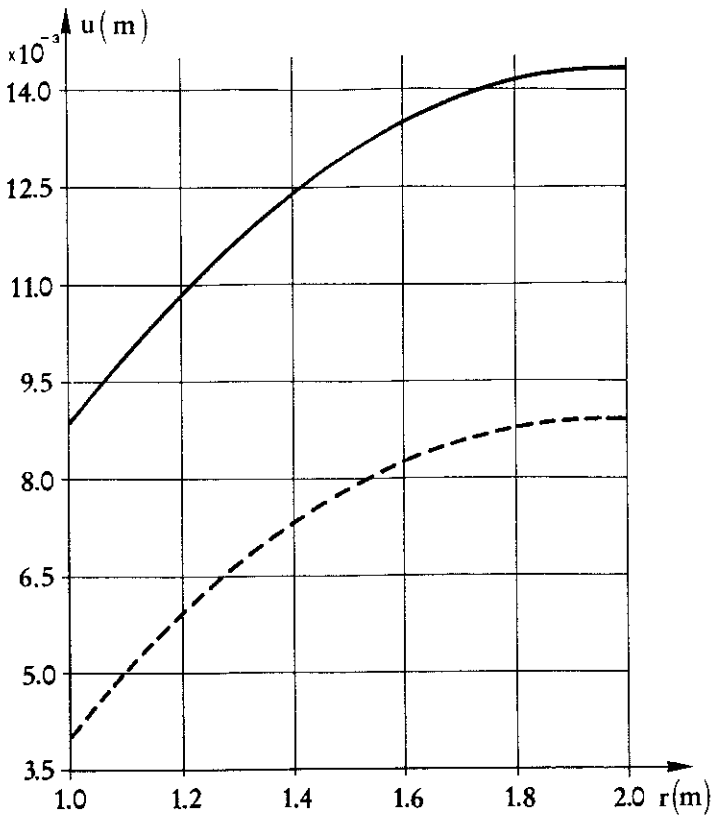

Figure 4. Radial displacement u vs. r.

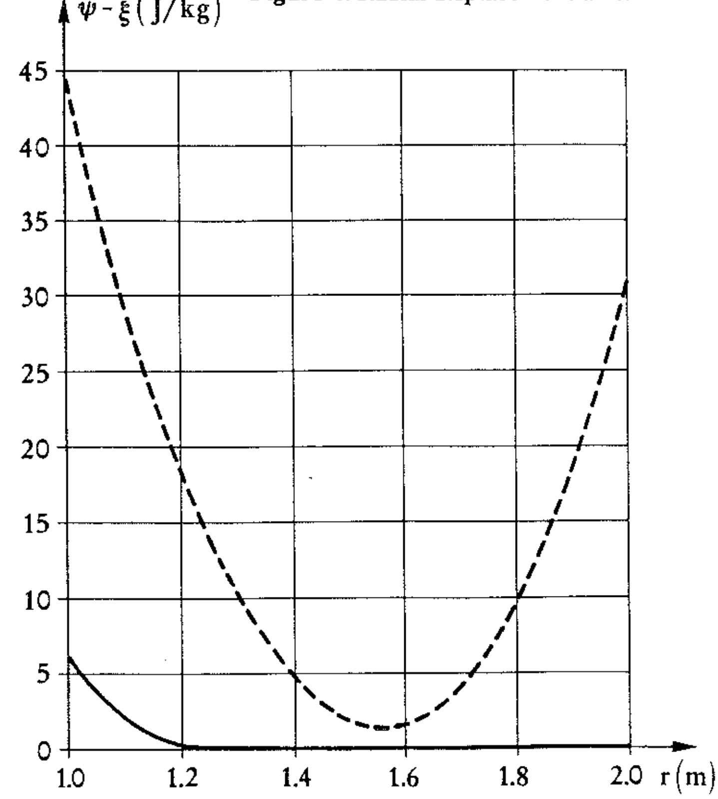

Figure 5. Difference between the free energy ψ and function $\xi(\theta(r))$ vs. r.

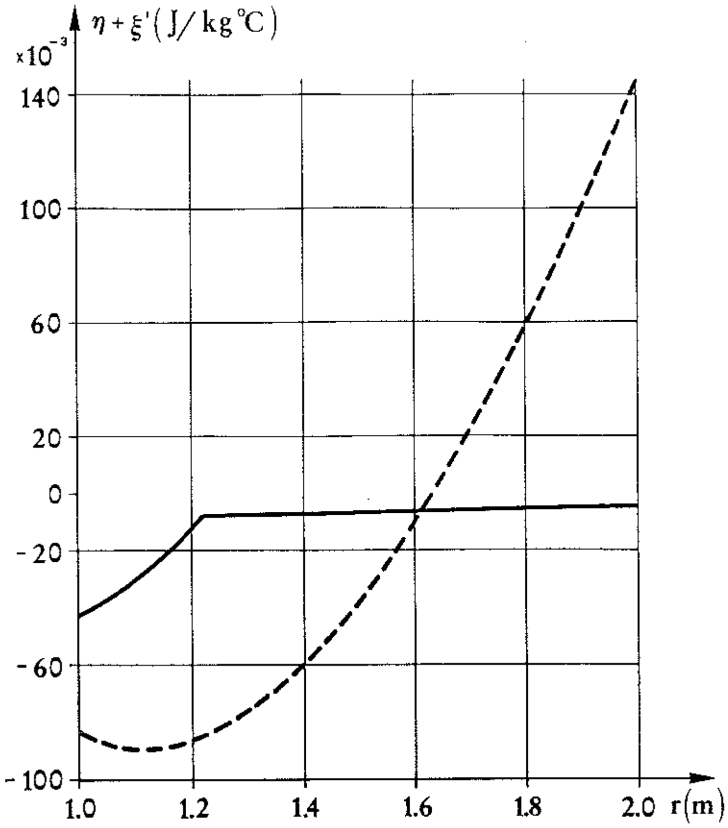

Figure 6. Sum of the entropy η and function $C_E \ln[\theta(r)/\theta_0]$ vs. r.

6. CONCLUSIONS

In this paper a constitutive equation for no-tension isotropic materials in the presence of thermal expansion is presented and the basic coupled equations of the thermoelastic equilibrium set forth while accounting for the temperature dependence of material constants. In the absence of thermal variations, this constitutive model coincides to the model for no-tension materials.

By limiting ourselves to consider thermo-mechanical uncoupling, we prove the uniqueness of the stress that solves the equilibrium problem of masonry-like solids subjected to thermal loads.

Since this kind of constitutive equation allows one to explicitly calculate not only the stress as function of temperature and strain, but also the derivative of the stress with respect to the strain, it is particularly suited for use in a finite element code for solving equilibrium problems of no-tension solids subjected to thermal loads via the Newton Raphson method, in a manner analogous to that performed in Lucchesi *et al.* (1994, 1995, 1996a).

APPENDIX

Let us consider a plane stress and suppose $t_3 = \mathbf{q}_3 \cdot \mathbf{T}\, \mathbf{q}_3 = 0$. Denoting by $\mathbf{E}$, the restriction of the strain tensor to the two-dimensional space orthogonal to $\mathbf{q}_3$ and by $e_1 \le e_2$, the eigenvalues of $\mathbf{E}$, we define the following subsets of $\mathrm{Sym} \times [\theta_1, \theta_2]$:

$$\mathcal{S}_1 = \{(\mathbf{E},\theta) \mid \gamma(\theta)(e_1 - \beta(\theta)) + 2(1+\gamma(\theta))(e_2 - \beta(\theta)) \le 0\} \quad , \tag{A.1}$$

$$\mathcal{S}_2 = \{(\mathbf{E},\ \theta) \mid \gamma(\theta)(e_1 - \beta(\theta)) + 2(1+\gamma(\theta))(e_2 - \beta(\theta)) > 0, \quad e_1 - \beta(\theta) \le 0\}\ , \tag{A.2}$$

$$\mathcal{S}_3 = \{(\mathbf{E},\ \theta) \mid e_1 - \beta(\theta) > 0\}\ . \tag{A.3}$$

The expression of the free energy in the three regions is

$$\psi(\mathbf{E},\ \theta) = \xi(\theta) + \frac{E(\theta)}{2\left(1 - \nu(\theta)^2\right)\rho}\ \{e_1^2 + e_2^2 + 2\nu(\theta)e_1e_2 +$$

$$2(1+\nu(\theta))\beta(\theta)(\beta(\theta) - e_1 - e_2)\ \}\ , \qquad \text{for } (\mathbf{E},\ \theta) \in \mathcal{S}_1\ , \tag{A.4}$$

$$\psi(\mathbf{E},\ \theta) = \xi(\theta) + \frac{E(\theta)}{2\,\rho}\ (e_1 - \beta(\theta))^2\ , \qquad \text{for } (\mathbf{E},\ \theta) \in \mathcal{S}_2\ , \tag{A.5}$$

$$\psi(\mathbf{E},\ \theta) = \xi(\theta)\ , \qquad \text{for } (\mathbf{E},\ \theta) \in \mathcal{S}_3\ . \tag{A.6}$$

Accounting for relations $D_E\, e_1 = \mathbf{q}_1 \otimes \mathbf{q}_1$, $D_E\, e_2 = \mathbf{q}_2 \otimes \mathbf{q}_2$, where $\mathbf{q}_1$ and $\mathbf{q}_2$ are the eigenvectors of $\mathbf{E}$ corresponding to eigenvalues e_1 and e_2 (Lucchesi *et al.*, 1996 a), for the stress we obtain
if $(\mathbf{E},\ \theta) \in \mathcal{S}_1$, then

$$\mathbf{T}(\mathbf{E},\ \theta) = \frac{E(\theta)}{1+\nu(\theta)}\ \{\mathbf{E} - \beta(\theta)\,\mathbf{I} + \frac{\nu(\theta)}{1-\nu(\theta)}\ \mathrm{tr}(\mathbf{E} - \beta(\theta)\,\mathbf{I})\,\mathbf{I}\}; \tag{A.7}$$

if $(\mathbf{E},\ \theta) \in \mathcal{S}_2$, then

$$\mathbf{T}(\mathbf{E},\ \theta) = E(\theta)\ (e_1 - \beta(\theta))\ \mathbf{q}_1 \otimes \mathbf{q}_1 \quad ; \tag{A.8}$$

if $(\mathbf{E},\ \theta) \in \mathcal{S}_3$, then

$$\mathbf{T}(\mathbf{E},\ \theta) = \mathbf{0}\ . \tag{A.9}$$

Finally, the entropy is

$$\eta(\mathbf{E}, \theta) = - \xi'(\theta) - \frac{E'(\theta)(1 - \nu^2(\theta)) + 2\nu(\theta)\nu'(\theta)E(\theta)}{2\rho(1 - \nu^2(\theta))^2} \{e_1^2 + e_2^2 + 2\nu(\theta)e_1e_2 +$$

$$2(1 + \nu(\theta))\beta(\theta)(\beta(\theta) - e_1 - e_2)\} - \frac{E(\theta)}{2\left(1 - \nu(\theta)^2\right)\rho} \{2\nu'(\theta)\, e_1\, e_2 +$$

$$2\nu'(\theta)\beta(\theta)(\beta(\theta) - e_1 - e_2) + 2\,(1 + \nu(\theta))\beta'(\theta)(\beta(\theta) - e_1 - e_2) +$$

$$2(1 + \nu(\theta))\beta(\theta)\beta'(\theta)\}\,, \qquad \text{for } (\mathbf{E}, \theta) \in \mathfrak{S}_1\,, \qquad (A.10)$$

$$\eta(\mathbf{E}, \theta) = - \xi'(\theta) - \frac{E'(\theta)}{2\,\rho}(e_1 - \beta(\theta))^2 + \frac{E(\theta)}{\rho}\,\beta'(\theta)(e_1 - \beta(\theta))\,, \quad \text{for } (\mathbf{E}, \theta) \in \mathfrak{S}_2\,, \qquad (A.11)$$

$$\eta(\mathbf{E}, \theta) = - \xi'(\theta)\,, \qquad \text{for } (\mathbf{E}, \theta) \in \mathfrak{S}_3\,. \qquad (A.12)$$

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