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Non-monotonic Lunar Plasma Sheaths

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We argue that the electrostatic field profile observed by ARTEMIS in the neighbourhood of the lunar surface is sustained by singular electron and ion velocity distribution functions. These are singular solutions of the steady state, two species Vlasov-Poisson equations. The energy distributions of the hot, finite mass, mobile ions is assumed to be log singular at the position of the electric potential's minimum. We show that the electron energy distributions on opposite sides of this minimum are not equal. This leads to a jump discontinuity of the electron distribution across its separatrix. A simple relation exists between the difference of these two electron distributions and that of the ions. The distributions of both species are given in terms of elementary functions and they meet smooth boundary conditions at one plasma end. Simple, finite amplitude profiles of the electric potential result from Poisson equation, which are smoothly, but non monotonically and non symmetrically distributed in space. Three such solutions are investigated in detail as appropriate for non monotonic double layers and for a plasma of semi-infinite extent bounded by a surface.

Our starting observational data is the electric potential space distribution as observed by the ARTEMIS spacecraft [1]. This distribution is characterized by an asymmetric shape about its minimum and by by an asymptotic decay towards two different values as the space coordinate x approaches $\pm\infty$ and it is modelled in Fig. 1.

To afford a concise treatment for both the electron (subscript e) and ion (i) populations, we assume that quantities in the plasma depend on position Lx and particle velocity $v_0 v$ and denote by Φ_0, n_0, $L = \surd(4\pi e n_0/\Phi_0)$, $v_0 = \surd(e\Phi_0/m_e)$, $m_{e,i}$, $-Z_e e, Z_i e$, $n_0 f_{e,i}(x,v)/(Z_{e,i} v_0)$ the scales of voltage, density, length and velocity, the electron and ion masses and charges (with $Z_e = 1$), and their velocity distributions. We write the potential as

$$\Phi(Lx) = \min\Phi + \Phi_0\phi(x), \quad (1)$$

$$\text{where } \Phi_0 = \max\Phi - \min\Phi, 0 \le \phi \le 1 \quad (2)$$

and we set

$$f_{e,i}(x,v) + f_{e,i}(x,-v) = \surd(2\mu_{e,i}) F_{e,i}(u_{e,i})/Z_{e,i}, \quad (3)$$

where $\mu_{e,i} = m_{e,i}/(m_e Z_{e,i})$ and

$$u_{e,i} = \mu_{e,i} v^2/2 - V_{e,i}, V_e = \phi, V_i = 1 - \phi \quad (4)$$

are the particle total and potential energies. Finally, we assume that the plasma be in steady state and collisionless, so that $F_{e,i}$ solve the time independent Vlasov equation, thus being constant along particle trajectories.

In each domain $1, 2$, where $\phi' \ne 0$ (Fig. 1), we set $\phi(x) = \eta, \phi''(x) = \phi_{xx}(\eta)$ and we write Poisson equation as

$$\phi_{xx}(\phi) = n_e - n_i,\ n_{e,i} = \int_{-V_{e,i}}^{\infty} dt \frac{F_{e,i}(t)}{\surd(t + V_{e,i})}. \quad (5)$$

This equation is to be solved under the condition that, on the high potential boundary, ϕ has its absolute maximum ($\phi = 1, \phi' = 0$, Eq. (2) and Fig. 1), and

$$F_e(w) = F_{e0}(w), F_i(u) = F_{i0}(u), \text{for } u \ge 0. \quad (6)$$

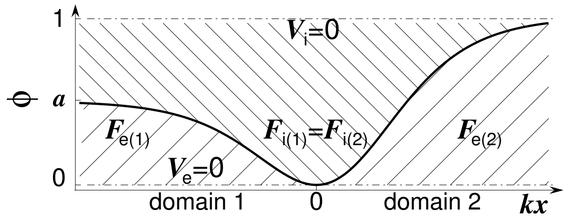

FIG. 1. An asymmetric potential ϕ vs. coordinate x. Note the zero values of the electron and ion potential energies $V_{e,i}$ (dot-dashed lines) and the loci of trapped electrons [ions] (wide [narrow] hatched area). In domains $1, 2$, $\phi' \ne 0$ and the electron [ion] distributions $F_{e(1,2)}$ [$F_{i(1,2)}$] differ [coincide].

These boundary conditions uniquely fix the distribution of the electrons in domain 2 and that of the positive energy ions (the former reach that boundary from domain 2, and the latter do so from any domain, on trajectories along which their distributions are constant).

To find the distributions of the negative energy ions and of the domain 1 electrons, we denote by $-p$ and $-q$ the total charge densities respectively at the potental minimum and maximum and we write

$$\phi_{xx}(\phi) = p + \int_0^{\phi} dt \frac{G_e(t)}{\surd(\phi - t)} = q - \int_{\phi}^{1} dt \frac{G_i(1-t)}{\surd(t - \phi)}. \quad (7)$$

By direct substitution and using principal value integration according to need, we prove that the solution of Eq. (5) obeys the relation (here α = e [i], β = i [e], $u > 0$)

$$F_\alpha(-u) = G_\alpha(u) + \int_0^\infty dt \frac{\surd u}{\pi\surd t}\left[\frac{F_\alpha(t)}{t+u} + \frac{F_\beta(t-1)}{t-u}\right]. \quad (8)$$

Now, we showed [2] that $F_i(-u)$ is algebraically continuous on the ion separatrix (i.e. for $u = 0$), and log singular at the potential's minimum and for zero velocity ($u = 1$). To account for these properties, in domain 2 we make the non trivial Ansätz

$$G_i(u) = \frac{2c}{\pi}[3d(2b - 5 + 5u)\tanh^{-1}\surd u + (e - 16u)\surd u], \quad (9)$$

where $e = 12 - r/c + d(5 - 12b), r = p - q$ and b, c, d are constants. Eqs. (6), (8) and (9) complete the derivation of the ion distribution in domain 2.

We know that the ion distributions in domains 1 and 2 concide and so do those of the positive energy electrons (such particles visit both domains on trajectories along which their distributions are constant). To find the distribution of the negative energy electrons in domain 1, we denote by $\Delta f(u) = f(u)|_{\mathrm{domain}\,2} - f(u)|_{\mathrm{domain}\,1}$ the difference of the values that a quantity f attains in domains 1 and 2 for the same value of its argument u. Using Eq. (8) and solving Abel's Eq. (7) for G_{e}, we find, for $u > 0$,

$$\Delta F_{\mathrm{e}}(-u) = \Delta G_{\mathrm{e}}(u) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \mathrm{d}t \frac{\Delta\phi_{xx}(t)}{\surd(u-t)}. \quad (10)$$

Now, Eqs. (7) and (9) give (here $y = \surd\phi$)

$$\phi_{xx}(\phi) = q + (1-\phi)(r - 12c\phi) + cdy(1-y)(5y+3b) \quad (11)$$

in domain 2. This turns into $\phi_{xx}|_{\mathrm{domain}\,1}$ if $y \mapsto -y$ [3] and, taking b, c, d from Eq. (9), Eq. (10) gives

$$\Delta\phi_{xx} = 8cdy(3b - 5\phi), \Delta F_{\mathrm{e}}(-u) = 6cd(2b - 5u). \quad (12)$$

Given $u > 0$, this provides the distribution of the negative energy electrons in domain 1 once it is known in domain 2 and completes the derivation of the particle distributions.

The remarkable result of Eq. (12) gives $\Delta F_{\mathrm{e}}(u-1)$ at no extra cost, as π times the coefficient of the log singular term in the ion distribution (Eqs. (8) and (9)). Note that, since $G_{\mathrm{i}}(1)$ must be non negative, so must be ΔF_{e}. Also, in going from positive to negative values of energy across the domain 1 branch of the separatrix, the electron distribution jumps from $F_{\mathrm{e0}}(0)$ to $F_{\mathrm{e0}}(0) - \Delta F_{\mathrm{e}}(0)$, and since this latter value must be non negative, from Eqs. (6) and (12), the following constraints hold for the coefficients in Eq. (9):

$$0 \le 12bcd \le F_{\mathrm{e0}}(0). \quad (13)$$

As an application of our results, we consider the potential distribution within a non monotonic double layer, for which both plasma boundaries may be set at infinity. Eq. (11) gives

$$\phi(x) = a\{2/[(1-\surd a) - (1+\surd a)\coth(kx)]\}^2, \quad (14)$$

where $a = \phi(-\infty)$ and $k = \surd[2c(1+\surd a)]$.

In Figs. 1 and 2, we show the asymmetric potential of Eq. (14) and the related particle distributions. For $w < 0$ these are

$$F_{\mathrm{e}}(w) = F_{\mathrm{e0}}(w) \quad \text{in domain 2,} \quad (15)$$

$$F_{\mathrm{e}}(w) = F_{\mathrm{e0}}(w) - \Delta F_{\mathrm{e}}(w) \quad \text{in domain 1,} \quad (16)$$

$$F_{\mathrm{i}}(w) = G_{\mathrm{i}}(-w) + \surd(\beta_{\mathrm{i}}/\pi)\mathrm{e}^{-\beta_{\mathrm{i}}w}\mathrm{erfc}(\surd[-\beta_{\mathrm{i}}w]) - \surd(\beta_{\mathrm{e}}/\pi)\mathrm{e}^{\beta_{\mathrm{e}}w}\mathrm{erfi}(\surd[-\beta_{\mathrm{e}}w]). \quad (17)$$

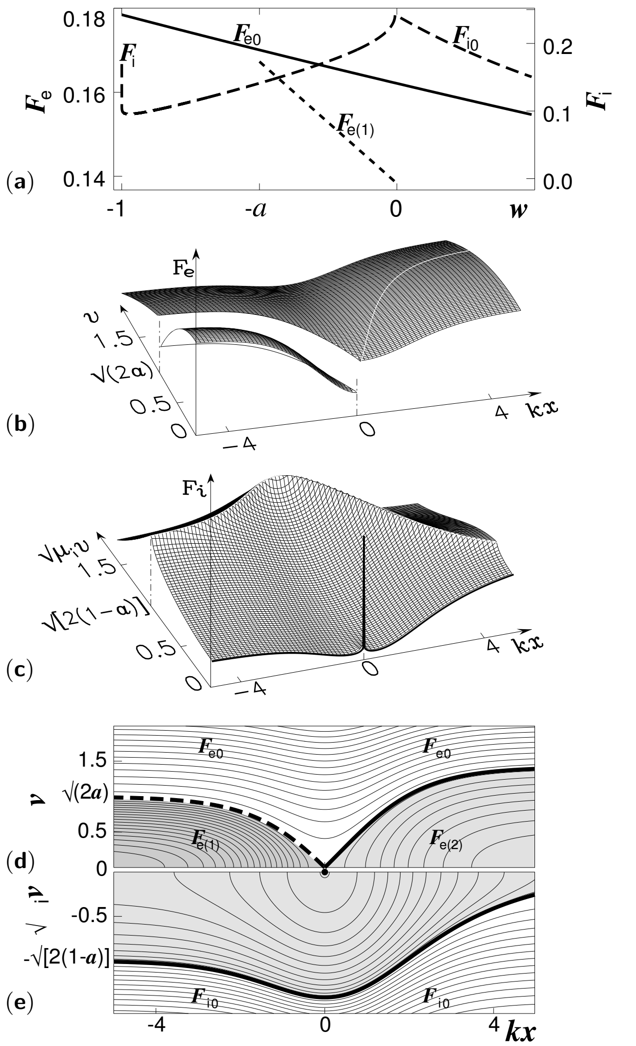

FIG. 2. (a) The electron [ion] distribution F_{e} [F_{i}] vs. energy w is discontinuous [log singular] at $w = 0$ [$w = -1$], i.e. on the left branch of the separatrix ((b) and dashed curve in (d)) [at $x = v = 0$, ((c) and bullet in (e))]. Shaded areas in (d) and (e) denote trapped particles. $F_{\mathrm{e}(1,2)}$ are the values of F_{e} in domains 1, 2 and $F_{\mathrm{e,i0}}$ are the boundary conditions of $F_{\mathrm{e,i}}$.

For $w > 0$, $F_{\mathrm{e,i}} = F_{\mathrm{e,i0}}$ (Eq. (6)), where

$$F_{\mathrm{e0}}(u) = \surd(\beta_{\mathrm{e}}/\pi)\mathrm{e}^{-\beta_{\mathrm{e}}(1+u)}, F_{\mathrm{i0}}(u) = \surd(\beta_{\mathrm{i}}/\pi)\mathrm{e}^{-\beta_{\mathrm{i}}u} \quad (18)$$

are the boundary conditions for $x \to \infty$. We assumed that electrons and ions are in charge neutrality there ($\phi'' = 0$ at infinity) and thermally distributed with density and temperature $n_0/Z_{\mathrm{e,i}}$ and $e\Phi_0 Z_{\mathrm{e,i}}/\beta_{\mathrm{e,i}}$ respectively.

The values of the parameters in Eqs. (9), (14), (17) and (18) used to draw Figs. 1 and 2 are $k = 0.2, c = k^2/[2(1+b)^2], b = \surd(r/c)/2 = \surd a = 1/\surd 2, d = 1-b, \beta_{\mathrm{e}} = 0.1, \beta_{\mathrm{i}} = 0.2$. These values give a maximum charge density and voltage drops across the double layer respectively of $p = 0.01$ times the electron charge density and

$\Phi_0 = 0.1$ times the electron temperature both taken as $x \to \infty$ and an ion to electron temperature ratio there of 0.5. The agreement of the potential profile of Fig. 1 with some waveforms of weak double layers [4] is worth noticing.

As a second application, we consider the potential distribution in the semi-infinite half space bounded by a plane surface where the potential reaches a maximum $\phi = \phi_0$. If this is also the absolute potential maximum, then, by construction (Eq. (2)), $\phi_0 = 1$, the surface occupies the high potential boundary at, say, $x = 0$ and it hosts the particle distributions' boundary conditions of Eq. (6): the other boundary shifts to infinity, where $\phi = \phi_\infty < \phi_0$. Quadrature of Eq. (11) gives

$$\phi(x) = \{\sqrt{\phi_0} + AB/[1 - (1+B)\coth^2(kx)]\}^2, \quad (19)$$

where $A = \sqrt{\phi_\infty} + \sqrt{\phi_0}$, $AB = 2(|\sqrt{\phi_0} - \sqrt{\phi_\infty}| - d)$, $k = \sqrt{[c(1+B)]}A$, and c, d are the constants appearing in Eq. (9).

A third application arises when $\phi_0 < 1$ is not the absolute potential maximum. Again by construction, the boundary hosting the particle distributions's conditions now shifts to infinity, where $\phi = \phi_\infty = 1$. The solution of Eq. (19) still holds provided $B \mapsto -B$. The plots of this solution and of the associated particle distributions are analogous to the ones in Figs. 1 and 2.

The fully analytical treatment we presented of steady state, non monotonic, asymmetric, electrostatic potential distributions and of the associated velocity distributions of collision-less electrons and ions offers some concomitant new results and advantages: (a) it involves both electrons and finite mass, mobile, hot ions; (b) it meets smooth boundary particle distributions at one plasma end (Eq. (6)); (c) it complies with the general result that the solutions of the steady state electron and ion Vlasov equations subject to smooth boundary conditions and supporting smooth, asymmetric potential profiles must necessarily be jump discontinuous and log singular [2, 5]; (d) it shows that, although the electron distribution on opposite sides of the potential minimum are not equal, they cannot be given arbitrarily and that, remarkably, they can be calculated directly from that of the ions at no extra cost (Eq. (12)); (e) it is not limited to small-amplitude potentials; (f) it affords an unified analysis of the potential distribution in different plasma settings (Eqs. (14) and (19)).

The simplicity of the singular solutions of the steady state, two species Vlasov-Poisson equations based on Eq. (9), their concise derivation, properties, versatile applications and agreement with observations should hopefully appear as convincing circumstances of their relevance. Their stability and their numerical reproduction will be dealt with elsewhere.

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