On anomalous diffusion and fluctuation-dissipation relations

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In his seminal paper on the Brownian Motion, Einstein found the first example of fluctuation-dissipation relation. In the absence of external forcing one has

$$\langle x(t)\rangle = 0 \quad , \quad \langle x^2(t)\rangle \simeq 2Dt \; , \tag{1}$$

where x is the position of the Brownian particle and D the diffusion coefficient. Once a small constant external force F is applied one has a linear drift $\overline{\delta x}(t) \simeq \mu F t$ which is proportional to $\langle x^2(t)\rangle$ and the Einstein relation holds: $\mu = \beta D$.

However, it is now well established that in many systems the mean square displacement of a tagged particle does not grow linearly with time and anomalous diffusion can be observed, i.e.

$$\langle x^2(t)\rangle \sim t^{2\nu} \;\text{ with }\; \nu \neq 1/2, \tag{2}$$

see for instance [1].

Here we consider a couple of one-dimensional models, showing subdiffusive behaviour, and focus on the study of the relation between the response to an external driving force and unperturbed correlations. In particular, we study the dynamics of a single particle making a random walk on a "comb" lattice and the single-file model, i.e. a gas of hard rods coupled to an external thermal bath. In both systems we observe that the equilibrium fluctuation-dissipation theorem in the form of the Einstein relation is satisfied also in the subdiffusive regime, i.e. the response to an external force and spontaneous fluctuations are proportional. On the other hand, introducing "non equilibrium" conditions through a stationary current, in the form of unbalanced transition probabilities for the comb or with dissipative interactions for the single-file, we find in both cases strong violations of the Einstein relation (see figure 1 and reference [2]).

For the comb model, where the transition rates $W[(x,y) \to (x',y')]$ of the process are explicitly known, we are able to write down an out of equilibrium fluctuation-dissipation relation [3–5]

$$\overline{\delta x}(t) = \frac{1}{2}\left[\langle x(t)x(t)\rangle - \langle x(t)x(0)\rangle - \langle x(t)A(t,0)\rangle\right], \tag{3}$$

where $A(t,0) = \sum_{t'=0}^{t} B(t')$ and

$$B[(x,y)] = \sum_{(x',y')} (x'-x)W[(x,y)\to(x',y')], \tag{4}$$

which allows us to recover an exact relation between response function and unperturbed correlation functions (see figure 1). The observable B gives an effective measure of the propensity of the system to leave a certain state (x,y). Relation (3) is quite general and holds for all kinds of non-equilibrium states, stationary or not. It has been used to build very efficient field-free algorithms for the measure of the response function in the framework of aging systems [6].

In conclusion, we find that the Einstein formula holds also for models showing anomalous diffusion, provided that no currents are present in the system. On the contrary, when non equilibrium conditions are considered, strong violations occur, and a generalized non-equilibrium relation has to be taken into account.

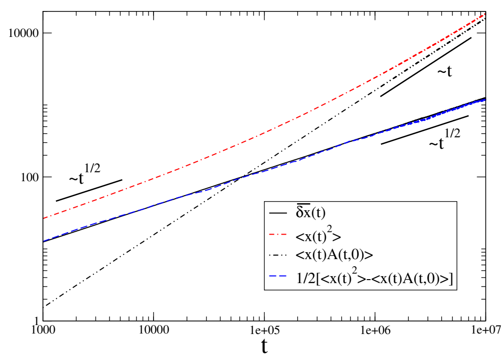

FIG. 1. Response function (black line) and m.s.d. (red dotted line) measured in the comb model. The correlation function $\langle x(t)A(t,0)\rangle$ (black dotted line) yields the right correction to recover the full response function (blue dotted line), in agreement with the FDR (3).

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