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Orthomodular Structures of Decompositions

John Harding[*]

The key feature of a closed subspace of a Hilbert space is that it allows the vectors of the space to be represented as ordered pairs of vectors, one from the subspace and one from its orthogonal complement. So closed subspaces of a Hilbert space give direct product decompositions of the space, and conversely, every direct product decomposition of a Hilbert space gives a pair of complementary closed subspaces.

It turns out that the direct product decompositions of any set, vector space, group, ring, topological space, or uniform space naturally form an orthomodular poset, much as the closed subspaces of a Hilbert space form an orthomodular poset. Further, many common instances of orthomodular posets arise as such orthomodular posets of direct product decompositions.

In this talk we survey results of the past twenty years relating orthomodular posets of direct product decompositions to aspects of the quantum logic program. This includes the study of regularity, states, automorphisms, and their role in basic axiomatics. Also, as direct products have a standard categorical formulation, orthomodular posets of decompositions provide a means to link some of the recent categorical approaches to quantum mechanics to the quantum logic program.

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The Many Classical Faces of Quantum Structures

Chris Heunen[*]

Much about a quantum system is captured by the collection of its classical subsystems, such as algebraic, logical, and information-theoretic aspects. However, one can rigorously prove that the collection of classical subsystems cannot capture all information about a quantum system, for which more structure has to be added. This question is answered by the notion of an active lattice, which adds the information of how to switch between two classical viewpoints. After a survey of this area, I will discuss how to characterise active lattices, and how to reconstruct an operator algebra from its active lattice, parallelled with many illustrations from piecewise Boolean algebra.

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Is the Quantum State Real?

Matthew Leifer[*]

The status of the quantum state is perhaps the most controversial issue in the foundations of quantum theory. Is it an epistemic state (representing knowledge, information, or belief) or an ontic state (a direct reflection of reality)? In the ontological models framework, quantum states correspond to probability measures over more fundamental states of reality. The quantum state is then ontic if every pair of pure states corresponds to a pair of measures that do not overlap, and is otherwise epistemic. Recently, several authors have derived theorems that aim to show that the quantum state must be ontic in this framework. Each of these theorems involve auxiliary assumptions of varying degrees of plausibility. Without such assumptions, it has been shown that models exist in which the quantum state is epistemic. However, the definition of an epistemic quantum state used in these works is extremely permissive. Only two quantum states need correspond to overlapping measures and furthermore the amount of overlap may be arbitrarily small. In order to provide an explanation of quantum phenomena such as no-cloning and the indistinguishability of pure states, the amount of overlap should be comparable to the inner product of the quantum states. In this talk, I review the debate over the interpretation of quantum states and explain how the overlap of probability measures can be bounded using proofs of the Kochen-Specker theorem. In particular, I exhibit a family of states in d-dimensional Hilbert space for which the ratio of overlap to inner product must be $\leq de^{-cd}$ for some positive constant c, which severely constrains the epistemic interpretation of quantum states.

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Quantum Cellular Automata and Quantum Field Theory

*Paolo Perinotti**

In a recent paper [1], the mathematical structure of finite dimensional Hilbert spaces and the description of states, evolutions and measurements through density matrices, completely positive maps and POVMs has been derived from principles ruling information processing tasks. This result provides an information-theoretic basis for the abstract formalism of the theory of quantum systems.

In the present work we review subsequent efforts [2] aimed at an extension of the information-theoretical foundations to encompass also the strictly mechanical part of Quantum Theory. In particular, we will review the derivation of Weyl's, Dirac's and Maxwell's equations based on Quantum Cellular Automata, and from simple principles regarding their main computational features. These principles are unitarity, linearity, locality, homogeneity, isotropy.

Quantum systems constituting the Quantum Cellular automaton can be organised in a graph–where edges connect directly interacting systems–which is more precisely the Cayley graph of some finitely presented group. Under the further assumption that the group is (quasi-isometrically) embeddable in a Euclidean manifold, we will show that the only two possible unitary automata satisfying all our requirements give rise, in an appropriate limit, to Weyl's equation, and can be combined to obtain both Dirac [2] or Maxwell's [3] equations for free quantum fields.

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The Many Problems with Quantum Mechanics from the Earliest Days On and How the Theory of Quantum Mechanics on Phase Space Addresses Them

Franklin E. Schroeck, Jr.[*]

At the turn of the last century there were a number of questions concerning the atomic and molecular levels in the physical world. That was the motivation of Heisenberg, Schroedinger, Dirac, Weyl, Wigner, etc. in developing quantum mechanics, and they were duly "solved". The contributions fell into two categories: the analytic approach of Heisenberg, Schroedinger, Dirac, et al, and the group theory approach of Weyl, Wigner, et al. People in the first set even had a name for the second set - the "gruppenpest" (or the "pestilence of groups").

But the analysts had problems that compounded as time went on that led to the difficulty of obtaining relativistic formulations, to the concept of the Dirac sea of electrons, to a break between classical mechanics and quantum mechanics, to quantum field theory at a point, etc. We shall review the literature of the time showing what prominent physicists thought concerning these problems, as well as giving the good points of the theory.

On the other hand, the group theorists went their merry way with rigorous mathematical proofs of their various results. There weren't any arguments that I could find that were against what they did and were given by prominent physicists. This was either because putting physics in terms of group theory was too radical and abstract for them or because mistakes of the group theorists were not made. We shall review what the group theorists did.

We shall then take the various problems that have been around for the past 85 years and show how the phase space approach solves almost all of them. Examples of the difficuties that remain include solving the many body problem which predates the era of quantum theory, determining a number of constants, determining why there are apparently only Fermions and Bosons, etc.

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Substructural Logics and Residuated Lattices: The Central Role of l-Groups

*Constantine Tsinakis**

Algebra and proof theory traditionally represent two distinct approaches within logic: the former is concerned with semantic meaning and structures, the latter with syntactic and algorithmic aspects. In many intriguing cases, however, methods from one field are essential to obtaining proofs in the other. In particular, proof-theoretic techniques involving substructural logics have been used to establish important results for their algebraic counterparts, the so called residuated lattices, and vice versa. The purpose of my talk is to survey some of these developments. The talk also emphasizes the central role of lattice-ordered groups in the study of algebras of logic.

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Master Eqaution in Stochastic Phase Space

Djaafer Benrabia[*] *Rabah Oubagha*[†] *Mahmoud Hachemane*[‡]

We review briefly the stochastic quantum theory[1] and project the Born-Markov master equation[2] on this representation. This is a very simple derivation of a Master equation for a positive phase space density rather than the Wigner function[3].

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Axiomatising Probabilistic Logic of Quantum Programs

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Probabilistic Logic of Quantum Programs (PLQP) is a logic for describing probabilities of outcomes of quantum tests, effects of other quantum programs, as well as separation programs that characterize subsystems. In previous work, we have shown PLQP is decidable and expressed the correctness of a number of quantum protocols in PLQP [1]. The aim of this presentation is to present a sound proof system for PLQP and use this system to prove the correctness of one protocol called quantum leader election.

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Characterization of the Compatible Smearings of Incompatible Observables

*Roberto Beneduci**

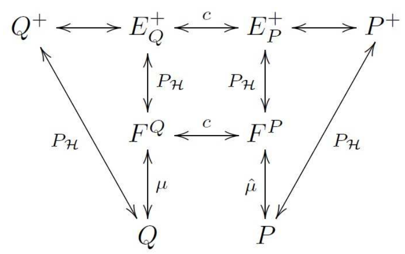
Let $\mathcal{H} = L_2(\mathbb{R})$ and Q, P be the position and momentum operators respectively. It is well known that Q, P are not jointly measurable since they do not commute but they can be smeared to two jointly measurable POVMs, $F^Q : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s^+(\mathcal{H})$, $F^P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s^+(\mathcal{H})$ where, $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of the reals and $\mathcal{L}_s^+(\mathcal{H})$ is the space of positive, linear, self-adjoint operators in $\mathcal{H} = L_2(\mathbb{R})$. The smearings are realized by means of two Markov kernels μ and $\hat{\mu}$. In particular, we have

$$F^Q(\Delta) = \int \mu_\Delta(q)\, dQ_q, \quad F^P(\Delta) = \int \hat{\mu}_\Delta(p)\, dP_p$$

where, dQ_q denotes integration with respect to the spectral measure corresponding to Q, dP_p denotes integration with respect to the spectral measure corresponding to P and μ_Δ (resp. $\hat{\mu}_\Delta$) is a measurable function for each $\Delta \in \mathcal{B}(\mathbb{R})$.

We recall that F^Q and F^P are jointly measurable (or compatible) if they are the marginals of a joint POVM F, i.e., there is a POVM $F : \mathcal{B}(\mathbb{R} \times \mathbb{R}) \to \mathcal{L}_s^+(\mathcal{H})$ such that $F^Q(\Delta_q) = F(\Delta_q \times \mathbb{R})$, $F(\mathbb{R} \times \Delta_p) = F^P(\Delta_p)$.

In the first part of the talk, we show that the existence of the jointly measurable (compatible) smearings F^Q and F^P is connected to the existence of two commuting dilations $Q^+ : \mathcal{H}^+ \to \mathcal{H}^+$ and $P^+ : \mathcal{H}^+ \to \mathcal{H}^+$ of Q and P respectively such that $P_\mathcal{H} Q^+ P_\mathcal{H} = Q$ and $P_\mathcal{H} P^+ P_\mathcal{H} = P$ where, $\mathcal{H}^+$ is an extended Hilbert space and $P_\mathcal{H} : \mathcal{H}^+ \to \mathcal{H}$ is the operator of projection onto $\mathcal{H}$. All that is illustrated in the following diagram.



where $\overset{P_\mathcal{H}}{\longleftrightarrow}$, denotes the relationship between a self-adjoint operator and its dilation ($P_\mathcal{H} Q^+ P_\mathcal{H} = Q$) as well as the relationship between a POVM and its dilation ($P_\mathcal{H} E_Q^+ P_\mathcal{H} = F^Q$), $\overset{c}{\longleftrightarrow}$ denotes joint measurability (compatibility), $\overset{\mu}{\longleftrightarrow}$ denotes smearing and $\longleftrightarrow$ denotes the correspondence between a self-adjoint operator and its spectral measure.

Next, we show that the scheme represented in the diagram can be generalized to the case of an arbitrary couple of self-adjoint operators. In particular, we show that the joint measurability of two POVMs F_1, F_2 which are smearings of two self-adjoint operators A_1 and A_2 is connected to the existence of two commuting self-adjoint dilations A_1^+ and A_2^+ of A_1 and A_2 respectively [5]. That is illustrated in the diagram below.

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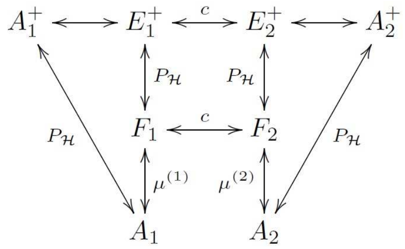

Note that the meaning of the arrows in the two diagrams is the same.

Then, we provide other relevant physical examples for which the scheme applies.
Finally, we give a sufficient condition for the compatibility of two effects.

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From Quantum Cellular Automata to Deformed Special Relativity

Alessandro Bisio[*] *Giacomo Mauro D'Ariano*[†] *Paolo Perinotti*[‡]
Alessandro Tosini[§] *Alexandre Bibeau-Delisle*[¶]

We consider a one-dimensional quantum cellular automaton that spawns the Dirac evolution in the limit of small wave-vectors and masses (in Planck units). The comparison between the automaton and the Dirac evolutions is rigorously set as a discrimination problem between unitary channels. We derive an exact lower bound for the probability of error in the discrimination as an explicit and function of the mass, the number and the momentum of the particles, and the duration of the evolution. We observe that, for values of these parameters compatible with current experiments, the probability of error approaches $\frac{1}{2}$, thus proving the emergence of the Dirac dynamics from the automaton model.

Then it is shown that, assuming the invariance of the dispersion relations for boosted observers, a model of Deformed Special Relativity emerges. Such a model is characterized by a non-linear representation of the Lorentz group on the momentum space, with an additional invariant given by the wave-vector $k = \pi/2$. We finally introduce a toy model for the emergent spacetime which exhibits the phenomenon of *relative locality.*

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Star Order and Jordan Structures

Martin Bohata[*] *Jan Hamhalter*[†]

We investigate a natural extension of star order to a nonassociative context of JBW algebras. The main attention is paid to study of certain continuous star order isomorphisms between JBW algebras. In particular, it is shown that a continuous star order isomorphism from a JBW factor of Type I_n, where $n \neq 2$, onto a JBW algebra is nothing but composition of uniquely determined Jordan isomorphism with functional calculus. This result generalizes the description of nonlinear preservers of Gudder order on self-adjoint parts of von Neumann factors of Type I [6]. Our arguments are based on a deep Gleason's type theorem for JBW algebras proved by Bunce and Wright [4, 5].

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Non-Locality as Phenomenon of Quantum Information

*Piotr Bulka**

The discussion over quantum enlargement and tunnelling of EM waves concerns quantum non-local effects (cf. Nimtz, 2003). Many open questions which have not got explanation yet are connected with the possibility of observation of non-local effects. The phenomenon of non-locality leads to a question concerning the possibility of faster than light communication. In my presentation, I would like to outline the recent state of the discussion concerning the phenomenon of non-locality, and to indicate the most important open questions waiting for explanatory models and their experimental confirmation connected with the principle of causality. My approach to non-local effects of QM is based on Gisin's (2013) interpretation of experimental results obtained by Bancal *et al* (2012).

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Coupled Right Orthosemirings Induced by Orthomodular Lattices

Ivan Chajda[*]

L. P. Belluce, A. Di Nola and B. Gerla established a connection between MV-algebras and (dually) lattice ordered semirings by means of the so-called coupled semirings. A similar approach was found for basic algebras and semilattice ordered right near semirings by the author and H. Laenger. Here we derive an analogous connection for orthomodular lattices and certain semilattice ordered near semirings via so-called coupled right orthosemirings.

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The Block Structure of Distributive D-lattices

Ferdinand Chovanec *

There is a close relationship between Boolean algebras and MV-algebras. Both structures are distributive lattices of pairwise compatible elements. Every Boolean algebra can be organized as an MV-algebra, but the converse assertion is not true, because an MV-algebra contains "unsharp" elements, i. e. (nonzero) elements which are covers by their orthosupplements.

A maximal Boolean subalgebra of an orthomodular poset (or an orthoalgebra) is called a block. In difference structures (D-posets) or effect algebras, blocks are maximal sub-MV-algebras of these structures. Riečanová [7] proved that every D-lattice (every lattice-ordered effect algebra) is the set-theoretical union of its maximal sub-MV-algebras (blocks).

There is a very usefull method of a construction of quantum logics (orthomodular posets and orthomodular lattices) called a pasting of Boolean algebras. This method was originally suggested by Greechie in 1971 [4] and later it has been generalized by many authors, above all by Dichtl [3], Navara and Rogalewicz [6], Navara [5]. A method of a construction of difference posets and lattices by means of an MV-algebra pasting was originally suggested in [2] and improved in [1]. Although pastings of of Boolean algebras are never distributive lattices, on the other hand there are pastings of MV-algebras (in the sense of [1]) which give distributive lattices.

We will investigate the block structure of finite distributive difference lattices and we show how look their Greechie and cluster Greechie diagrams.

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Dependency Orderings of Atomic Observables

*Jānis Cīrulis**

The notion of atomic observable on a logic (i.e., σ-orthomodular poset) is suggested by a similar notion for effect test spaces [4, 5]. The notion of functional dependency (of attributes of objects) arose in 70-ies in the theory of relational databases. It has been adapted for information systems in [1, 2] and some other structures of computer science (see [3]); here, we transfer the idea of dependency to the context of quantum logics (dependency of observables).

Let L be a logic. An *observable* is a σ-homomorphism from the Borel algebra $B(\mathbf{R})$ to L. The *point spectrum* of an observable x is the set $\sigma_p(x) := \{u \in \mathbf{R}\colon x(u) \neq 0\}$. The set $r(x) := \{x(u)\colon u \in \sigma_p(x)\}$ is an orthogonal subset of L; if it happens to be maximal, it is said to be the *support* of x. The restriction of x to its point spectrum maps the latter one-to-one onto $r(x)$.

An observable x is said to be *atomic* if, for every $A \in B(\mathbf{R})$,

$$x(A) = \bigvee(x(u)\colon u \in A \cap \sigma_p(x)).$$

It turns out that an observable x is atomic if and only if $r(x)$ is its support. Also, every maximal orthogonal subset of L is a support of an atomic observable. We denote by AO the set of all such observables.

Suppose that x and y are atomic observables interpreted as real-valued dynamical variables related to some physical system. Intuitively, x (functionally) depends on y if "every time" when y takes some value v, the variable x takes a value uniquely determined by v. In terms of the logic L, this requirement can be stated as follows: to every $v \in \sigma_p(y)$ there is an element $u \in \sigma_p(x)$ (necessarily unique!) such that $y(v) \leq x(u)$. This is the case if and only if $y(v) \leq x(d(v))$ for some function $d\colon \sigma_p(y) \to \sigma_p(x)$. If such a function exists, then it is surjective and uniquely defined.

These observations lead us to the following definition:

> given atomic observables x and y, we say that x *depends on* y (in symbols, $x \leftarrow y$) if there is a function $d\colon \sigma_p(y) \to \sigma_p(x)$ such that $y(v) \leq x(d(v))$ for all $v \in \sigma_p(y)$. If this is the case, we call d a *dependence* of x on y (in symbols, $d\colon x \leftarrow y$).

Thus, x takes a certain value u if y has an appropriate value, namely, one belonging to $d^{-1}(u)$. The intuitive idea of functional dependency also suggests (and the above definition implies) that x may take a value u *only if* y has such a value. This informal consequence could be written compactly as an inequality $x(u) \leq \bigvee(y(v)\colon v \in d^{-1}(u))$; however, existence of the joins involved in such inequalities (for various u) cannot be proved for an arbitrary dependence d without imposing additional restrictions on the logic L.

> We thus say that the dependence $d\colon x \leftarrow y$ is *admissible* (relatively to L) if every (orthogonal) subset $\{y(v)\colon v \in d^{-1}(u)\}$ with $u \in \sigma_p(x)$ has a join in L, and write $d\colon x \leftarrow_a y$ to mean that this is the case.

Where E is a subset of $\mathbf{R}$, we denote by $B(E)$ the set of relative Borel subsets of E, i.e., the set $\{A \cap E\colon A \in B(\mathbf{R})\}$.

> If $d\colon x \leftarrow y$ and d is a Borel function $\sigma_p(y) \to \sigma_p(x)$ (i.e., d^{-1} takes every set from $B(\sigma_p(x))$ into a set from $B(\sigma_p(y))$, then we say that d is a *Borel dependence* and write $d\colon x \leftarrow_b y$ to emphasize this.

Every Borel dependence is admissible.

The dependency relations $\leftarrow$, $\leftarrow_a$ and $\leftarrow_b$ are preorders on AO, i.e., are reflexive and transitive. They are naturally reflected into the set V of all maximal orthogonal subsets of L. Namely, for any $x, y \in AO$, let

$r(x) \leq r(y)$ iff $x \leftarrow y$,
$r(x) \leq_a r(y)$ iff $x \leftarrow_a y$,
$r(x) \leq_b r(y)$ iff $x \leftarrow_b y$.

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These definitions are correct (for example, $r(x) = r(x')$, $r(y) = r(y')$ and $x \leftarrow y$ imply $x' \leftarrow y'$), and all three relations $\leq, \leq_a, \leq_b$ are (partial) orders on V.

In the paper, we study, besides elementary properties of atomic observables, the order structure of V under these orders and draw consequences for AO.

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Galois Connections, P-Projections, Exposed Points, Quantum Logics, . . .

Thurlow Cook *Joanna Jeneralczuk*

Alfsen and Shultz constructed a spectral theory of operators using base-normed and order-unit normed spaces. They defined P-projections and postulated the properties of these projections on face lattices. Using the properties of predual and dual Banach spaces and Galois connections, we can construct these projections, establish a connection with the atoms in the logic, and connect these ideas with the seminal paper of Birkhoff and von Neumann.

A Quantum Approach to Vagueness and to the Semantics of Music

Maria Luisa Dalla Chiara *   *Eleonora Negri*†

The first-order quantum computational semantics represents a suitable logical environment for modelling linguistic situations where *vagueness* and *ambiguity* play a relevant role. In this framework *vague possible worlds* can be described as a kind of abstract *scenes*, where individual concepts, relations and events behave as quantum-like superpositions that ambiguously *allude* to a number of possibly antagonistic alternatives. The quantum semantics can be naturally applied to a formal analysis of musical compositions, where both *musical ideas* and *extra-musical meanings* seem to be characterized by some essentially vague and ambiguous features. As an example, we investigate the somewhat mysterious concept of *musical theme*, which often appears in a musical composition with different "masks", alluding to a (potentially) infinite set of possible variants. One is dealing with a vague musical idea that cannot be either played or written.

Some interesting examples of musical themes are the *Leitmotive* that play an essential role in Wagner's operas. Any appearance of a *Leitmotiv* involves: 1) a *musical idea* that corresponds to a theme; 2) a theatrical situation that can be represented as a *vague possible world*, described by the opera's libretto and evoked by the corresponding musical idea. The *Leitmotiv*-technique creates some very special effects, bringing into light the similarities between the developing of the musical ideas and the flowing of our (conscious or unconscious) thoughts. Of course, effects of this kind do not concern Wagner's music only. One is dealing with some deep and complicated cognitive phenomena that are today investigated in the framework of neuro-sciences. Perhaps a progress in the neuro-scientific research will allow us to understand (at least to a certain extent) some of the reasons why music represents a kind of universal thought, which is deeply connected with the sphere of our emotions.

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A First-order Epistemic Quantum Computational Semantics

Maria Luisa Dalla Chiara [*] *Roberto Giuntini*[†] *Roberto Leporini* [‡]

Quantum computational logics have been semantically characterized at a *sentential* level. In this framework, sentences are supposed to denote pieces of quantum information (density operators of convenient Hilbert spaces), while logical connectives correspond to quantum logical gates (unitary quantum operations that transform quantum information in a reversible way). The sentential quantum computational semantics can be extended to an *epistemic* semantics for a language expressing sentences like "Alice knows that Bob does not know that the spin-value in the x-direction is up". We investigate the possibility of generalizing this semantics to the case of a first-order language, where sentences like "All know that somebody does not know that the spin-value in the x-direction is up" can be formalized. Both knowledge operators and logical quantifiers can be represented as Hilbert-space operations that are generally irreversible. One is dealing with a kind of theoretic "jumps" that seem to be similar to quantum measurements. Unlike most first-order semantic approaches, the quantum computational semantics does not refer to any *domain of individuals* dealt with as a *closed set* (in a classical sense). The interpretation of a universal sentence (say, "All photons have null mass") does not require "ideal tests" that should be performed on *all* elements of a hypothetical domain. This way of thinking seems to be in agreement with a number of concrete semantic phenomena (even far from quantum physics), where the individual-domain appears highly indeterminate. Such situations, however, do not generally prevent a correct use of the logical quantifiers.

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Demonstration of the Spin Statistics Theorem by Conformal Quantum Geometrodynamics

Francesco De Martini *Enrico Santamato*

Pauli's exclusion principle, one of the most important principles for physics and chemistry, is still not fully demonstrated in quantum mechanics. The connection between spin and statistics was established by Pauli using the methods of quantum field theory, but the existence of only two statistics, Bose and Fermi, still is based on experimental facts. Parastatistics cannot be excluded because we have no a priori information on the way the quantum state changes under the action of the two-particle permutation operator P12. Also the standard behaviour of the wave function as reported in many textbooks is not devoid of problems and applies only to the one-dimensional representation of the permutation group.

Moreover, Pauli's exclusion principle holds also in the nonrelativistic quantum mechanics based on the first quantization (Schrödinger's equation). It is then surprising that the proof of the spin-statistics connection requires full relativistic second quantization (quantum field) approach.

Many attempts were made to obtain Pauli's exclusion principle or at least the spin-statistics connection from Schrödinger nonrelativistic theory, but all these approaches nevertheless contain restrictions as, for example, the wave functions must have special invariance or symmetry properties under permutation, or must lie in special spin-component subspaces, or the exchange must be considered as continuous transportation of real objects.

It is expected that classical mechanics have a much simpler approach, e.g. since there all quantities are real valued, the only possible eigenvalues of the one-dimensional representation of the permutation group P12 are ± 1. Moreover, analytical mechanics provides powerful tools to see how classical objects change under a continuous path leading to the particle exchange.

The recently developed Conformal Quantum Geometrodynamics (CQG) provides a model mathematically equivalent to quantum mechanics entirely based on classical concepts and real valued quantities, so it is on no surprise that in the framework of CQG the study of Pauli's exclusion principle is enormously simplified and a proof of the spin-statistics connection which excludes parastatistics can be obtained even in the nonrelativistic approximation.

In the CQG, the spinning particle is modelled as a classical relativistic spherical top, described by six Euler angles, subjected to the potential given by the Weyl conformal curvature of the top configuration space. This geometric potential accounts for all quantum effects involving the quantum relativistic spin based on Dirac's equation.

In this work we present a derivation the spin-statistics connection in the nonrelativistic limit of CQG, leaving the relativistic case to further study. In this limit, the top is described by three Euler angles for orientation and three coordinates for space position and it can be show that the nonrelativistic CQG is mathematically equivalent to usual two-component spinor description based on Pauli-Schrödinger wave equation. A direct calculation shows that the nonrelativistic limit of the Weyl's curvature potential on the top is independent on the rotation angle γ of the top around its own axis. Therefore, the angle γ is an ignorable coordinate so that the dynamic of the top is better described by a Routhian Function rather than a Lagrangian. This result is in full agreement of the quantum interpretation of spin as a directed quantity proportional to the particle magnetic (or angular) moment. Only two angles are needed, in fact, to fix the orientation of the magnetic momentum. External fields can change the direction of the particle magnetic momentum but have no effect on the rotation of the top around its own axis. As a consequence, in absence of perturbations, the rotation of the particle around its axis is either clockwise or anticlockwise forever, any transition from clockwise to anticlockwise being forbidden. Without loss of generality, we may then assume that all particles with spin rotate anticlockwise around their own axis. The possible paths in the top configuration space are correspondently divided into two homotopy classes and particles must belong to one of these classes.

The main result of this work is that the spin statistics connection is a direct consequence of the invariance of the action integral in the exchange of the two particles, assimilated to identical spherical tops, and of the fact that the two tops can rotate only anticlockwise around their own axis. Parastatistics are automatically excluded. It is worth noting that this simple derivation of the spin-statistics connection cannot be carried out in the framework of standard Quantum and of its fixed rotation direction is extraneous to standard Quantum Mechanics and in Ref. 2 is simply assumed. In the present work, instead, the spin-statistics connection is derived from first principles Mechanics based on spinors and that only an approach based on classical dynamics as in CQG can provide a so simple derivation of such fundamental issue.

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8) This statement is in agreement with group theory. The rotation of around the intrinsic axis is the little group of the full SO(3) rotation group and is equivalent to SO(2) or, equivalently, to U(1). The two rotation direction correspond to the two different representations of the U(1) group.

9) Is then natural to suppose that clockwise rotation of the antiparticles.

10) A similar argument based on the single direction of the particle proper rotation was used in Ref. 2 to justify the spin-statistics theorem starting from Feynman’s quantum rule to add alternatives. However, the existence of the angle γ . . .

Contextuality and MBQC Through Algebraic Geometry

Raouf Dridi[*]

In (1; 2; 3) C. Isham and collaborators used the category of commutative von Neumann subalgebras as base category for their topos formulation of quantum mechanics. Here we present a different choice. Our choice is inspired by the practical definition of measurement based quantum computation (MBQC) and the common particularities of most (all) Kochen-Specker proofs available out there. Indeed, most Kochen-Specker proofs share the following property: Given a maximal context C (a basis) the set of all possible mea- surement outcomes s $= (s_1, s_2, \cdots, s_n)^T$ within the context is an *affine variety* (i.e., set of zeros of a system of polynomial equations). In fact, we assign to each maximal context an affine variety of the form

$$V(f(s_1, s_2, \cdots, s_n) - o(C)) := \{\text{zeros of } f(s_1, s_2, \cdots, s_n) - o(C) = 0\}$$

The polynomial function $f \in \mathbb{Z}_2[s_1, s_2, \cdots, s_n]$ is independent of the context C, an essential feature for MBQC. The binary number $o(C)$ does depend on C and it is the output which our MBQC assigns to the maximal context C. What we have is a poset of affine varieties which plays similar role played by the category of commutative von Neumann subalgebras for the constructions in Isham's work. We define the geometric analogue of the spectral presheaf through residue field. Beside capturing the main features of MBQC, with this considerations the mathematical constructions and proofs are simplified. In particular, verifying contextuality (although described in the language of sheaves) simplifies to solving simplified systems: For instance, for GHZ-3 we get exactly the well known Mermin's simple system (see be- low). Within this framework, powerful computational tools such as Groebner basis can be used, in particular, in solving (or constructing!) such systems. For GHZ-n family, Groebner basis simplifies to Gaussian elimination. The following table contrasts what we have done (right hand side) with the construction in Isham's.

Checking contextuality demands some intellectual gymnastic	Checking contextuality is straightforward (Mermin's type simple systems) and Groebner basis can be used for complicated polynomial systems
No quantum computation is apparent	Construction built around an MBQC
Maximal commutative von Neumann subalgebras	Affine schemes Spec A with coordinate rings A of the form $\mathbb{Z}_2[s_1, s_2, \cdots]/(f(s) - o)$ with same polynomial f and possibly different $o \in \{1, -1\}$
Category of commutative von Neumann subalgebras	Category of affine schemes
Morphisms = inclusions	Morphisms = Blow-ups
Valuation functions through Gelfand-Mazur theorem	Valuation functions through residue field
Spectral presheaf Σ	Geometric spectral presheaf Λ
Noncommutative von Neumann algebra	Noncommutative scheme

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On Kite Pseudo Effect Algebras

*Anatolij Dvurečenskij**

Effect algebras, [6], have been appeared in quantum structures twenty years as a counterpart to D-posets, where instead of difference of two comparable events, they introduced addition of two mutually excluding events. In many cases they are intervals in Abelian po-groups. A prototypical example of effect algebras is the set $\mathcal{E}(H)$ of Hermitian operators of a Hilbert space H that are among the zero and unit operator.

In [4, 5], we have dropped the commutativity of the partial addition $+$ and a non-commutative version of effect algebras, *pseudo effect algebras* were introduced. Also such algebras in many important cases are intervals in unital po-groups, not necessarily Abelian, satisfying a kind of the Riesz Decomposition Property.

We remind that a *pseudo effect algebra* is a partial algebra $E = (E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, such that for all $a, b, c \in E$, the following holds

(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$;

(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;

(iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;

(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

Let G be a po-group and I be a set. Define an algebra whose universe is the set $(G^+)^I \uplus (G^-)^I$, where $\uplus$ denotes a union of disjoint sets. We order its universe by keeping the original co-ordinatewise ordering within $(G^+)^I$ and $(G^-)^I$, and setting $x \leq y$ for all $x \in (G^+)^I$, $y \in (G^-)^I$. Then $\leq$ is a partial order on $(G^+)^I \uplus (G^-)^I$. Hence, the element $e^I := \langle e : i \in I\rangle$ appears twice: at the bottom of $(G^+)^I$ and at the top of $(G^-)^I$. To avoid confusion in the definitions below, we adopt a convention of writing $a_i^{-1}, b_i^{-1}, \ldots$ for co-ordinates of elements of $(G^-)^I$ and $f_j, g_j, \ldots$ for co-ordinates of elements of $(G^+)^I$. In particular, we will write e^{-1} for e as an element of G^-, e as an element of G^+, and without loss of generality, we will assume that formally $e^{-1} \neq e$. We also put 1 for the constant sequence $(e^{-1})^I := \langle e^{-1} : i \in I\rangle$ and 0 for the constant sequence $e^I := \langle e_j : j \in I\rangle$. Then 0 and 1 are the least and greatest elements of $(G^+)^I \uplus (G^-)^I$.

Using two injections $\lambda, \rho : I \to I$ we have constructed a pseudo effect algebra $K_I^{\lambda,\rho}(G) := ((G^+)^I \uplus (G^-)^I; +, 0, 1)$, called a *kite pseudo effect algebra* and the following construction of kite pseudo effect algebras was presented in [1, Thm 3.4]:

Theorem 0.1. *Let G be a po-group and $\lambda, \rho : I \to I$ be bijections. Let us endow the set $(G^+)^I \uplus (G^-)^I$ with $0 = e^I$, $1 = (e^{-1})^I$ and with a partial operation $+$ as follows,*

$$\langle a_i^{-1} : i \in I\rangle + \langle b_i^{-1} : i \in I\rangle \qquad (I)$$

is not defined;

$$\langle a_i^{-1} : i \in I\rangle + \langle f_j : j \in I\rangle := \langle a_i^{-1} f_{\rho^{-1}(i)} : i \in I\rangle \qquad (II)$$

whenever $f_{\rho^{-1}(i)} \leq a_i$ for all $i \in I$;

$$\langle f_j : j \in I\rangle + \langle a_i^{-1} : i \in I\rangle := \langle f_{\lambda^{-1}(i)} a_i^{-1} : i \in I\rangle \qquad (III)$$

whenever $f_{\lambda^{-1}(i)} \leq a_i$, for all $i \in I$,

$$\langle f_j : j \in I\rangle + \langle g_j : j \in I\rangle := \langle f_j g_j : j \in I\rangle \qquad (IV)$$

for all $\langle f_j : j \in I\rangle$ and $\langle g_j : j \in I\rangle$. Then the partial algebra $K_I^{\lambda,\rho}(G) := ((G^+)^I \uplus (G^-)^I; +, 0, 1)$ becomes a pseudo effect algebra.

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In our talk we present basic properties of kite pseudo effect algebras, irreducible elements, classification, as well as we indicate a generalization of our construction.

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Einstein’s Dream

*David A. Edwards**

I discuss Einstein’s dream of a **Final Theory** and recent attempts to realize it.

Einstein, the founder of modern cosmology, hoped to be able to explain **everything** by a geometrical theory of curved spacetime together with fields on it [Dongen]. Complex, dynamic patterns of bumps and ripples in the geometry and fields were to be the underlying reality for what we usually call light, matter, animals and people. His program had reasonable success with bulk light and matter. For instance he was able to obtain the Newtonian Gravitational Theory as a first approximation, and higher order approximations resolved the anomaly in the perihelion shift of the orbit of Mercury (a major open problem in celestial mechanics for over fifty years). Most of the models, simulations, and discussions of The **Big Bang Model** basically use this approach (see: [Weinberg-1]).

In Part 1 I will discuss a recent version of Einstein’s **Unified Field Theory Program**. If this theory had been empirically adequate, it would have provided a completely satisfactory **Final Theory**. The extreme irony is that one of the main places his **Unified Field Theory Program** fails is in Einstein’s own **EPR** type situations. In Part 2 I will discuss a quantization program for the theory introduced in Part 1. **If** this program could be worked out, it would provide a mathematically precise version of **The Standard Models of Particle Physics** and Cosmology and hence **The Final Theory**. In Part 3 I discuss my program of **Interlocking Worlds** and its relationship to **Pragmatic Pluralism**. I also mention the possibility that the Universe is **tangled**.

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[Dongen] Einstein’s Unification

[Weinberg-1] Cosmology

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The (Absent) Modal Structure of Topos Quantum Logic

Benjamin Eva[*]

The topos-theoretic reformulation of quantum theory was originally conceived over a decade ago, by Chris Isham and Jeremy Butterfield. In the intervening years, the project has made remarkable progress in providing new perspectives on many of the traditional interpretational problems of quantum theory.

One area in which this progress has been particularly striking concerns the logical structure of quantum theory. Unlike the orthodox Hilbert space formalism for quantum theory, whose natural logical structure is standardly represented by orthomodular lattices, the natural logical structure of topos quantum theory is intuitionistic and is standardly characterised by a suitably chosen Heyting algebra.

In his 2012 paper, 'Topos Based Logic for Quantum Systems and bi-Heyting Algebras', Andreas Doering showed that the logical structure of topos-quantum theory is actually richer than was previously thought. Specifically, he showed that the algebraic representation of this logical structure, as well as being a Heyting algebra, can actually be understood as a bi-Heyting algebra. Bi-Heyting algebras are characterised by the fact that they include two separate complementation operations, representing intuitionistic and paraconsistent negations, respectively.

In this talk, I will examine some of the technical and philosophical characteristics of the bi-Heyting algebraic representation of topos-quantum logic. Specifically, it is well known that any bi-Heyting algebra comes equipped with a canonically defined pair of modal operators that encode a great deal of information about the relationship between the two complementation operations of the bi-Heyting algebra, and usually have a natural geometric interpretation. I will show that the canonically defined modal operators on the bi-Heyting algebra of propositions in topos-quantum logic are generally trivial, and that this reflects the fact that the two complementation operations only coincide on the top and bottom elements of the algebra.

The situation here is in stark contrast to traditional quantum logic, where it is generally possible to define non-trivial modal structure on the orthomodular lattice of projections. I will talk about the philosophical significance of this modal structure, and how we can interpret its absence from topos quantum logic. I will conclude that the absence of any modal structure in topos quantum logic is symptomatic of the way that propositions are defined in topos-quantum theory, i.e. as inherently contextual. Specifically, modal operators in quantum logic serve to eliminate contextuality, but since the propositions of topos-quantum logic are inherently contextual, there is no non-trivial way to achieve this.

Finally, some further directions for the study of the bi-Heyting algebraic logical structure of topos quantum theory will be suggested and some related results will be sketched.

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Quantum Signaling Game

Piotr Frąckiewicz

We present a quantum approach to a signaling game; a special kind of extensive games of incomplete information. Our model is based on quantum schemes for games in strategic form where players perform unitary operators on their own qubits of some fixed initial state and the payoff function is given by a measurement on the resulting final state. We show that the quantum game induced by our scheme coincides with a signaling game as a special case and outputs nonclassical results in general. As an example, we consider a quantum extension of the signaling game in which the chance move is a three-parameter unitary operator whereas the players' actions are equivalent to classical ones. In this case, we study the game in terms of Nash equilibria and refine the pure Nash equilibria adapting to the quantum game the notion of a weak perfect Bayesian equilibrium.

An Algebraic Approach to Phase Spaces in Categorical Quantum Mechanics

Christian de Ronde[*] *Hector Freytes*[†] *Graciela Domenech*[‡]

In classical physics the state of the system is represented by a point (p, q) in the corresponding phase space Γ and its properties by subsets of Γ. Consequently, the propositional structure associated with the properties of a classical system follows the rules of classical logic. In the orthodox formulation of quantum mechanics a pure state of a system is represented by a ray in a Hilbert space $\mathcal{H}$ and its physical properties by closed subspaces of $\mathcal{H}$ which, with adequate definitions of meet and join operations, give rise to an orthomodular lattice. This lattice, denoted by $\mathcal{L}(\mathcal{H})$, is called the Hilbert lattice associated to $\mathcal{H}$ and motivates the standard quantum logic introduced in the thirties by Birkhoff and von Neumann [1]. Recently, several approaches have been attempting to develop an adequate and rigorous language for quantum systems using topos theory [2, 3, 4, 5]. In these approaches, the quantum analogue of classical phase space is captured by the notion of frame i.e. an intuitionistic structure. In this work we introduce an equational type system that describes the mentioned intuitionistic structure associated to phases spaces. In the paricular case in which the phase space is denumerable, a weak completeness theorem is given.

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Generalized Random Events

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Observables in fuzzy probability can send a crisp event to a genuine fuzzy event ([1]), hence posses a quantum quality. We study n-dimensional generalizations of fuzzy random events and n-dimensional variants of states. Simplex-valued functions represent n-dimensional generalizations of fuzzy random events. This leads to generalizations of probability on IF-random events ([4]) and simplex-valued probability ([3], [2]). We discuss some applications and mention related problems.

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Identities Valid in Orthomodular Lattices

Stephen M. Gagola III[*], *Jeannine J.M. Gabriëls*[†], *and Mirko Navara*[‡]

We investigated possibilities of finding tools reducing operations in orthomodular lattices (OMLs) into normal forms. Basic tools were already found by D. Foulis [3], S.S. Holland [9], R.J. Greechie [7, 8], M. Navara [13], and N.D. Megill and M. Pavičić [12]. Commuting elements play a fundamental role in each of these.

It is known that the free OML with two free generators has 96 elements [1], thus the word problem is solvable for it. The whole computation can be made automatically, e.g., by the use of a computer program. The idea of [13], described in [11], has been implemented by N.D. Megill [12] and M. Hyčko [10].

The absence of the distributivity law in OMLs is a great problem in simplification of complex expressions [6]; commuting elements simplify this task. Looking for alternative tools for normal forms, we first asked which OML operations are associative [5]. The conclusion is that associativity holds only for two binary operations (lattice meet and join), two unary operations (right and left projection) and two nulary operations (constants 0 and 1). This does not bring new tools except for the already well known lattice operations.

In [4], we studied more general *alternative algebras*, where the following identities (generalizing associativity) hold:

$$x * (x * y) = (x * x) * y$$
$$(y * x) * x = y * (x * x)$$
$$(x * y) * x = x * (y * x)$$

We have found all OML operations satisfying these identities. Among them, there are 8 non-associative operations satisfying all three of these identities. Essentially, these are

expression	name
$x \wedge (x' \vee y)$	Sasaki projection
$(x \vee (x' \wedge y)) \wedge (x' \vee y)$	swapped projection
$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$	lower commutator

The remaining 5 operations are obtained from these by duality or interchanging the arguments.

We further ask for which of these operations the following expressions give the same results independently of the 5 possible ways of bracketing:

$x * x * y * z$	$x * x' * y * z$
$x * y * x * z$	$x * y * x' * z$
$x * y * z * x$	$x * y * z * x'$
$y * x * x * z$	$y * x * x' * z$
$z * x * y * x$	$z * x * y * x'$
$z * y * x * x$	$z * y * x * x'$

We solve these questions first without additional assumptions, then under the assumptions that some of the variables commute. We have found counterexamples showing that our proved results are as general as possible.

As a by-product, we have proved many properties of the Sasaki projection, e.g.:

Theorem 0.2. *Let L be an orthomodular lattice and let $*$ be the Sasaki projection. If $x, y, z \in L$ such that x and y commute then*

$$x * (y * z) = (x * y) * z\,.$$

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The Sasaki projection seems to be the most perspective candidate for future use and investigations. It has already been studied in [1] under the name *skew meet* (with the arguments interchanged), together with its dual operation, the *skew join*. Also Chevalier and Pulmannová [2] present interesting properties of Sasaki projections. Our results might also contribute to the understanding of the algebra based on this operation.

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The Calculus of Monoidal Quantaloids and Quantum Mechanics

*Emmanuel Galatoulas**

We unfold an approach to Quantum Mechanics inspired by and utilising the conceptual and formal tools of bicategories, in particular of quantaloids. We develop the basic framework, introducing the monoidal structure for quantaloids, a prerequisite to make sense of composite systems, and further taking the first steps towards a "semantics" of OM in this quantaloidal context by taking advantage of the calculus of monoidal quantaloids and its applicability to the fundamental issues of quantum mechanical entanglement and measurement, mostly in terms of universal categorical constructions (ie. limits and colimits).

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Outline of a Generalization and Reinterpretation of Quantum Mechanics Recovering Objectivity

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Most scholars concerned with the foundations of quantum mechanics (QM) think that *contextuality* and *nonlocality* (hence *nonobjectivit)* of physical properties are unavoidable features of QM which directly follow from the mathematical apparatus of QM, as proven by known "no-go" theorems. Moreover contextuality and nonlocality are usually considered as basic features and resources for quantum information processing. Nevertheless these features raise deep and still unsolved problems (most important, the *objectification problem* of the quantum theory of measurement). The *extended semantic realism (ESR) model* that we have proposed in several papers [1, 2, 3, 6, 9, 10] offers a possible way out from the difficulties following from nonobjectivity by embedding the mathematical formalism of QM into a broader mathematical formalism and reinterpreting quantum probabilities as *conditional on detection* rather than absolute. Indeed, the embedding allows us to recover the formal apparatus of QM within the ESR model, and the reinterpretation of QM allows us to construct a noncontextual hidden variables theory which justifies the assumptions introduced in the ESR model and proves its objectivity, circumventing the "no-go" theorems. In the updated and more general formulation of this model that has been recently produced [10] also time evolution has been discussed, showing that both linear and nonlinear evolution may occur, depending on the physical environment. When applied to special cases the ESR model settles long-standing conflicts. It reconciles Bell's inequalities with QM [2, 3, 4, 9], supplies different mathematical representations of proper and improper mixtures [3, 5, 6, 9, 10]), provides a general framework in which previous results obtained by other authors (as local interpretations of the GHZ experiment) are recovered and explained [7], and supports an interpretation of quantum logic which avoids the introduction of the problematic notion of quantum truth [8].

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Infinite Measures in Noncommutative Measure Theory

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In the applications to quantum physics, probability measures are not enough. We are confronted with problems involving type III von Neumann algebras that can only be tackled with well developed theory of weights, i.e. infinite measures. We compare infinite measures from classical measure theory with their counterparts in noncommutative measure theory. To get from classical to noncommutative, we add one intermediate step: measure algebras. We show how the noncommutative situation differs from the commutative one. In particular, we show how four different conditions of semifiniteness of a classical measure lead to four different classes of semifinite weights. We also consider various sets of 'finiteness' of the measures, indicating possible pitfalls to avoid. Some natural open problems are given.

Probabilistic Groupoids

Lidija Goračinova-Ilieva[*] *Smile Markovski*[†]

Algebraic structures are commonly used as a tool in treatments of various processes. But their exactness reduces the opportunity of their application in non-deterministic environment. On the other hand, probability theory and fuzzy logic do not provide convenient means for expressing the result of combining elements in order to produce new ones. Moreover, these theories are not developed to "measure" algebraic properties. Therefore, we propose a new concept which relies both on universal algebra and probability theory.

We introduce probabilistic mappings, whose special case is the notion of probabilistic algebra. Here we consider discrete sets with only one binary operation, additionally including the "possibility" of obtaining one particular element as a product, among all of the others. This leads to a structure that we call probabilistic groupoid. "Ordinary" groupoids are just a special type of probabilistic ones.

Let A and B be discrete non-empty sets, and let $\mathcal{D}_B$ be the set of all probability distributions on B. Probabilistic mapping from A to B is a mapping $h : A \to \mathcal{D}_B$.

Every probabilistic mapping from the n-th power of A to A is a probabilistic (n-ary) operation on A. A pair (A, F) of a set A and a family F of probabilistic operations on A is called probabilistic algebra. When $F = \{f\}$ has one binary operation, then the probabilistic algebra (A, f) is a probabilistic groupoid. Basic properties of such structures are considered in this paper.

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A Unified Approach to Discrete Quantum Gravity

Stan Gudder

This paper is based on a covariant causal set (c-causet) approach to discrete quantum gravity. A c-causet is a partially ordered set $(x, <)$ that is invariant under labeling. We first consider the microscopic picture which describes the detailed structure of c-causets. The unique labeling of a c-causet x enables us to define a natural metric $d(a, b)$ between comparable vertices a, b of x. The metric is then employed to define geodesics and curvatures on x. We next consider the macroscopic picture which describes the growth process $x \to y$ of c-causets. We propose that this process is governed by a quantum dynamics given by complex amplitudes. Denoting the set of c-causets by $\mathcal{P}$ we show that the growth process $(\mathcal{P}, \to)$ can be structured into a discrete 4-manifold. This 4-manifold presents a unified approach to a discrete quantum gravity for which we define discrete analogues of Einstein's field equations and Dirac's equation.

Orthogonal Decomposition of the Gaussian Measure

Samigulla Haliullin[*]

The ultrapower of real line, $\mathbf{R}_{\mathscr{U}}$, where $\mathscr{U}$ is a nontrivial ultrafilter in the set the $\mathbb{N}$ of natural integers, is some realizations of the "non-standard expansion"$^*\mathbf{R}$ of the set of real numbers. Due to "good"properties of the factorization of cartesian product with respect to ultrafilter, ultraproducts hold a number of considerable value properties from the algebraic point of view. At the same time it is not any good "natural" (i.e. determined by the topology of factors) topology.

For the proof of many statements of the work the technics of the ultraproducts developed in work ([1]) is used.

Definition 1. *Let a sequence (X_n) of nonempty sets, and a nontrivial ultrafilter $\mathscr{U}$ in the set $\mathbb{N}$ of natural integers are given. The ultraproduct $(X_n)_{\mathscr{U}}$ is the factorization of the product-set $\prod_{n=1}^{\infty} X_n$ by the equivalence relation:*

$$(x)_n \sim (y)_n \Leftrightarrow \{n : x_n = y_n\} \in \mathscr{U}.$$

It is known (see, for example, [2]) that if $(\mathbf{E}_n, \mathscr{E}_n, \mu_n)$ be a sequence of probability spaces,

$$\mathscr{E}_{\mathscr{U}} = \{(A_n)_{\mathscr{U}} : A_n \in \mathscr{E}_n, n \in \mathbb{N}\},$$

$$\mu_{\mathscr{U}}(A) = \lim_{\mathscr{U}} \mu_n(A_n),\ A = (A_n)_{\mathscr{U}},\ A_n \in \mathscr{E}_n,\ n \in \mathbb{N},$$

then $\mathscr{E}_{\mathscr{U}}$ is the algebra, the probability measure $\mu_{\mathscr{U}}$ is σ-additive on $\mathscr{E}_{\mathscr{U}}$ and has the unique extension μ on the σ-algebra $\mathscr{E}$, generated by $\mathscr{E}_{\mathscr{U}}$. The probability space $(\mathbf{E}, \mathscr{E}, \mu)$ is said to be ultraproduct of sequence of probability spaces, where $\mathbf{E} = (\mathbf{E}_n)_{\mathscr{U}}$. This construction of a countable-additive probability measure on ultraproduct in a more general situation was offered P. Loeb in work [3].

Definition 2. *(X. Fernique, [4]) The probability measure μ on $\mathbf{E}$ is said to be gaussian if $(t\cdot\mu)*(s\cdot\mu) = \mu$ for all $t, s \in (0, 1)$ such that $t^2 + s^2 = 1$. (Here $*$ denote a convolution of measures, $(t \cdot \mu)(A) =: \mu(t^{-1}A), t \in \mathbb{R}$).*

Theorem 1. *(D. Mushtari, S. Haliullin, [1]) The ultraproduct of sequence of gaussian measures is gaussian.*

We denote by μ_x the translation of measure μ by x: $\mu_x(A) = \mu(A - x)$.

Definition 3. *(see, for example, Y. Okazaki, [5]) (i) The probability measure μ on $(\mathbf{E}, \mathscr{E})$ is said to be H-quasiinvariant if measures μ and μ_x are equivalent for all $x \in H$.*

(ii) The probability measure μ on $(\mathbf{E}, \mathscr{E})$ is said to be H-ergodic if μ is H-quasiinvariant and for all $A \in \mathscr{E}$ such that $0 < \mu(A) < 1$ there exists $x_0 \in H$ such that $\mu(A\Delta(A - x_0)) > 0$.

Theorem 2. *Let $(\mathbf{E}, \mathscr{E})$ is measurable space, where $\mathbf{E}$ is lineal space. Then μ is H-ergodic measure on $\mathscr{E}$ iff μ is extreme measure on the class $\mathcal{M}$ of H-quasiinvariant measures on $\mathscr{E}$.*

It is well known that a Gaussian measure on locally convex Hausdorff measurable space $(\mathbf{E}, \mathscr{E})$ is extreme in the class of H-quasiinvariant measures, that can't to be presented in the form of the sum of two mutually orthogonal measures from this class (see, for example, [6]). Above we showed that on the ultraproduct of linear spaces a Gaussian measure is naturally specified. Unlike its properties on locally convex space the Gaussian measure on the ultraproduct of linear spaces has the following property.

Theorem 3. *Let $\mathbf{E}_n = \mathbf{R}^n$, $\mathscr{E}_n = \mathscr{B}(\mathbf{R}^n)$. We will set on $\mathscr{E}_n$ measure μ_n such that coordinates $x_1, x_2...x_n$ are independent and have normal distribution $\mathcal{N}(0, 1)$. Then Gaussian measure $\mu_{\mathscr{U}}$ on the ultraproduct $((\mathbf{E}_n)_{\mathscr{U}}, \sigma((\mathscr{E}_n)_{\mathscr{U}}))$ it is represented in the form of the infinite convex sum of mutually orthogonal measures.*

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Mackey-Gleason Problem for Jordan Type Maps and Orthoisomorphisms of Projection Lattices

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A map $\Phi : A \to B$ between C^*-algebras A and B is called quasi-linear if it is linear on each abelian subalgebra of A and respects decompositions of elements into real and imaginary part. Famous Mackey-Gleason problem asks: Is every quasi-linear map $\Phi : A \to B$, where A is a C^*-algebra with no quotient isomorphic to two by two matrices, linear? There is both mathematical and physical motivation of this question. From mathematical standpoint it is desirable to know to what extend the structure of "local" abelian subalgebras determines "global" linearity of important maps. On the other hand, according to Bohr's doctrine, any quantum phenomenon can be seen only through classical systems given by abelian subalgebras. In this picture one can see only quasi-linear maps representing states or dynamics of quantum system. Therefore it is vital to know when is a quasi-linear map linear in order to justify linear model of quantum theory. For this reason celebrated Gleason theorem [2] is considered to be one of the main principles of mathematical foundation of quantum theory. Far reaching generalization of Gleason's breakthrough is due to Bunce and Wright who established positive answer to Mackey-Gleson problem for Banach space valued maps on von Neumann algebras (see [3] and references therein). However, little is known about the statute of Mackey-Gleason problem outside von Neumann algebras. The aim of the present paper is to summarize recent results in this direction for important quasi-linear maps between general C^*-algebras.

First we study quasi-linear maps $\Phi : A \to B$, called transformation maps, preserving binary operation $(a, b) \to aba$ for self-adjoint elements. Let us remark that a linear map Φ is a transformation map if and only if it preserves the Jordan triple product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. We derive some structutal properties of these maps. Our main result says that if A is an infinite C^*-algebra, then any transformation map on it must be linear. For the proof no Gleason type theorem is needed.

In the second part we study orthoisomorphisms $\tau : P(A) \to P(B)$ between projection lattices of AW^*-algebras A and B. Famous Dye theorem [1] says that any orthoisomorphism between von Neumann projection lattices is a restriction of Jordan $*$-isomorphism between corresponding von Neumann algebras. We exend this result to AW^*-algebras and discuss various consequences for isomorphisms of unitary groups and ordered structure of abelian subalgebras.

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Beyond Para-quantization: Quantization and Quantum-like Phenomena

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Para-quantization as an approach to obtaining the commutation rules of second quantization without assuming canonical quantization procedures arose out of a question raised by Wigner in the late 1940s concerning whether or not the Hamiltonian equations of motion necessarily determined the canonical commutation relations. Wigner's own attempts to investigate this question were followed up by a number of authors in the 1950s. These studies showed that there were indeed more general commutation relations than the conventional one between the annihilation and creation operators of second quantization that were consistent with classical Hamiltonian formalism. In this paper we take a radically different approach than either the conventional quantization methods that involve an assumption of canonical commutation relation or the methods of para-quantization, in that not only do we not assume canonical quantization, we do not begin by assuming any relations based on classical Hamiltonian or Lagrangian mechanics either. It is in this sense that we regard our approach as going beyond para-quantization. In our approach, a crucial role is assumed by the existence of a vacuum. This leads to a novel, entirely algebraic method of quantization that not only does not depend on analogies with classical mechanics, nor the usual canonical commutation relations, but is also free from any preconceptions about space and time. Non-abelian variables are introduced via the Leibniz rule for the differentiation of a product and the basic rules of second quantization are derived from a single postulate, namely that the self-adjoint form of the derivative associated with a harmonic field, represented in a non-abelian, associative algebra, has a non-negative spectrum. This postulate is tantamount to the assumption that a vacuum exists and indicates the fundamental role of counting numbers, including zero. There are a number of motivations for this novel approach. First, deriving the equations of quantum fields without the aid of classical mechanics leaves us free to recover the Hamiltonian equations of classical mechanics and the Schrödinger wave equation from the fundamental quantum field equations. Thus we achieve an intellectually satisfying progression from the fundamental to the less fundamental. Second, by freeing the quantum formalism from any physical connotation, we are free to apply it to non-physical, so-called quantum-like systems. Over the past decade or so there has been a rapid growth of interest in such applications. These include, the use of the Schrödinger equation and Bohmian quantum dynamics in finance, second quantization and the number operator in social interactions, population dynamics and financial trading, and quantum probability models in cognitive processes and decision-making. The physics-free approach presented here provides a foundational underpinning of such applications.

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A novel approach to characterising quantum theory based on limited information and complementarity

*Philipp Höhn**

The last decade has seen a wave of reconstructions and characterizations of quantum theory using the formalism of generalized probability theory. In this talk, we shall outline a novel (operational) approach to characterizing and reconstructing quantum theory which gives primacy to limited information and complementarity rather than the probability structure. In particular, we consider an observer interrogating a system with binary questions and analyze the consequences of (1) a postulate asserting a limit on the (simultaneous) information the observer can acquire about the system, and (2) a postulate asserting the existence of complementarity on the set of possible questions. We explain how the ensuing compatibility and complementarity structure of the binary questions implies many features of qubit quantum theory in an elegant way (e.g. three-dimensionality of the Bloch sphere, the entanglement structure of two qubits, monogamy of entanglement and many other features). Time permitting, we shall also sketch how this program can be completed to a full reconstruction of quantum theory by adding further ingredients.

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Lorentz transformations from quantum communication

Philipp Höhn[*] *Markus Müller*[†] *Robert Hübener*[‡] *Oscar Dahlsten*[§]

In order to render quantum theory into a relativistic theory, one usually augments it by the Lorentz transformations. In this talk, we shall argue that the structure of abstract quantum theory, in fact, already contains Lorentz transformations if certain prejudices are given up. More precisely, we shall present a thought experiment of quantum communication among different observers which implies that the group translating among different observers' descriptions of the physical world is the orthochronous Lorentz group. This result is obtained without putting special relativity in by hand, without assuming a specific spatio-temporal structure and, instead, by only assuming finite dimensional abstract quantum theory and sensible operational conditions. Among other repercussions, this result also suggests a novel perspective on density matrices.

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Quantum Lévy Area

*Robin L Hudson**

Lévy's stochastic area is the signed area cut off by a chord joining two points on the trajectory of a two-dimensional Brownian motion. It can be defined rigorously either as a martingale limit or as an iterated stochastic integral. Here we imitate both constructions, replacing the two independent one dimensional component Brownian motions by the "momentum" and "position" Brownian motions P and Q of quantum stochastic calculus [1]. These do not commute with eachother, instead satisfying the commutation relation

$$[P(s), Q(t)] = -2is \wedge t,$$

but they are stochastically independent in a certain sense. We shall also investigate the relationship of the characteristic function for Lévy area, which has many interesting mathematical connections, with the causal quantum stochastic double product integral

$$\prod_{a<x<y<b} \left(1 + \frac{i\lambda}{2} \left(dP(x) \, dQ(y) - dQ(x) \, dP(y)\right)\right).$$

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Generalized Weak pre Pseudo-effect Algebras

*Marek Hyčko**

Continuing with generalizing results from [1] to a non-commutative setting, the notion of generalized (weak) pre pseudo-effect algebras is introduced. From the naming convention it follows that such algebras in general do not posses a top element and are a generalization of previously introduced generalized pre effect algebras. Some algebraic properties of these algebras will be presented. The so-called unitization construction ([2, 4]) leads to weak pre pseudo-effect algebras (finite models up to 11 elements can be found at [3]). On the contrary, not each weak pre pseudo-effect algebra is generalized weak pre pseudo-effect algebra. The necessary which turn out to be also sufficient conditions for weak pre pseudo-effect algebras to be generalized weak pre pseudo-effect algebras will be presented. The relations of weak Riesz decomposition property (RDP_0) and Riesz decomposition property (RDP) are studied and it is shown that unlike in the case of pseudo-effect algebras these are completely independent. Additional results, such as attempts to introduce congruences and possible generalization of RDP_0 and RDP will be mentioned.

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A Hilbert Space Operator Representation of Abelian Po-groups of Bilinear Forms

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A notion of a singular bilinear form (B. Simon, 1978) yields that on an infinite-dimensional complex Hilbert space $\mathcal{H}$ a set of bilinear forms $\mathcal{F}(\mathcal{H})$ is richer than a set of linear operators $\mathcal{V}(\mathcal{H})$. We show that there exists structure preserving embedding from an abelian po-group $\mathcal{S}_D(\mathcal{H})$ of symmetric bilinear forms with a fixed domain D on a Hilbert space $\mathcal{H}$ to a po-group of linear symmetric operators which are densely defined on a specific dense linear subspace of an infinite dimensional complex Hilbert space $l_2(M)$. Moreover, if we restrict ourselves on positive parts of mentioned po-groups, an analogy holds in terms of generalized effect algebras.

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A Survey of Homogeneous Effect Algebras

Gejza Jenča

An effect $(E, \oplus, 0, 1)$ is called *homogeneous* if and only if it satisfies the following property

(H) For every u, v_1, v_2 such that $u \leq v_1 \oplus v_2 \leq u'$ there ar u_1, u_2 such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = v_1 \oplus v_2$.

Every orthoalgebra, every lattice-ordered effect algebra and every effect algebra satisfying the Riesz decomposition property is homogeneous.

A *block* in a homogeneous effect algebra is a maximal subalgebra satisfying the Riesz decomposition property. The most interesting property of homogeneous effect algebras is the following one:

Theorem 0.3. *[1] Every homogeneous effect algebra is a union of its blocks.*

In the talk, we will review some recent and some not-so-recent results concerning homogeneous effect algebras and their generalizations. We will focus on representations of homogeneous effect algebras and their connection to other classes of effect algebras.

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On the MacNeille Completion of Generalized Effect Algebras

Martin Kalina[*] *Ivica Marinová*[†] *Zdenka Riečanová*[‡]

Generalized effect algebras $(G; \oplus, \mathbf{0})$ are unbounded versions of effect algebras $(E; \oplus, \mathbf{0}, \mathbf{1})$ (introduced by D. Foulis and M.K. Bennett in 1994 [1]). Note that simultaneously an equivalent structure called D-poset was introduced by F. Kôpka and F. Chovanec (in 1992 and 1994 [2, 3]). In fact, effect algebras are bounded posets with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$ (called also the top element). A well known fact is that every effect algebra is a generalized effect algebra and a generalized effect with a top element is an effect algebra.

It can be easily seen that if we omit the element $\mathbf{1}$ of an effect algebra $(E; \oplus, \mathbf{0}, \mathbf{1})$ then $G = E \setminus \{\mathbf{1}\}$, with the operation $\oplus$ restricted to G, becomes a generalized effect algebra. But not conversely. There are even finite generalized effect algebras G for which the operation $\oplus$ defined on G cannot be extended to the set $E = G \cup \{\mathbf{1}\}$ making E the effect algebra with the top element $\mathbf{1}$.

We present a necessary and sufficient condition for this "Top element problem" of generalized effect algebras. As an application, we find a necessary and sufficient condition for generalized effect algebras $(G; \oplus, \mathbf{0})$ to have the MacNeille completion a complete effect algebra. More precisely, a condition under which the operation $\oplus$ can be extended onto $\mathcal{MC}(G) = \mathcal{MC}(E)$, where $E = G \cup \{\mathbf{1}\}$.

Thus we obtain a necessary and sufficient condition for some important families of generalized effect algebras G to have MacNeille completions complete effect algebras, using some known facts on MacNeille completions of effect algebras (see [4, 5, 6]).

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Quantum Theory and Semifields

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Quantum theory allows to derive the principles of classical mechanics by the process of dequantization as the Planck constant tends to zero. In a similar way (by the Maslov dequantization) arises, from the traditional mathematics, the tropical mathematics with idempotent semirings and semifields (like the max-plus algebras) as its main objects. In the talk we mention a few interesting correspondences [1] (in the spirit of N. Bohr's principle) and confirm that semifields of certain kind are idempotent.

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The C*-algebra Generated by Phase-Space Fuzzy Localization Operators: A Singular-Value-Based Decomposition

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We consider an effect algebra of phase space localization operators for a quantum mechanical Hilbert space that contains no non-trivial projections, and the C*-algebra generated by it. This C*-algebra forms an informationally complete set in the original Hilbert space. Its elements are shown to have singular-value-based decompositions that permits their characterization in terms of limits of sums of product pairs of localization operators. Through these results, it is shown that the informational completeness of the C*-algebra can be reduced to the informational completeness of the set of product pairs formed from elements of the effect algebra.

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On When a Semantics Is Not a Good Semantics: The Algebraisation of Orthomodular Logic

Roberto Giuntini *Tomasz Kowalski* *Francesco Paoli*

It is commonplace to maintain that the algebraic counterpart of orthomodular quantum logic (*OML*, the most standard and perused deductive system for quantum logic) is the variety $\mathcal{OML}$ of orthomodular lattices. Surprisingly enough, however, this claim has been questioned by Pavičic and Megill in a series of papers ([1], [2]). They consider a Hilbert-style calculus for *OML* and observe that this calculus is sound and complete w.r.t. a supervariety of $\mathcal{OML}$, the variety $\mathcal{WOML}$ of *weakly* orthomodular lattices. Furthermore, they observe a few additional facts:

- Orthomodular lattices are exactly those members of $\mathcal{WOML}$ which satisfy the quasiequation $(\alpha \to \beta) \wedge (\beta \to \alpha) \approx 1 \Rightarrow \alpha \approx \beta$;
- $\mathcal{WOML}$ and $\mathcal{OML}$ have the same quasiequational theory, if we restrict ourselves to equations of the form $\alpha \approx 1$;
- The soundness and completeness proof also works if we replace $\mathcal{WOML}$ with a single algebra, which they name **O6**.

Having remarked the above, they venture on such bold claims as:

- Deductive quantum logic is not orthomodular [2, p. 26].
- The proofs of completeness [of orthomodular quantum logic and classical logic] introduce hidden axioms of orthomodularity and distributivity in the respective Lindenbaum algebras of the logic [2, p. 27].
- In the syntactical structure of quantum logic there is nothing orthomodular. The orthomodularity appears through the definition of the equivalence relation [1, p. 14 of the preprint version].

We argue that the plausibility of these claims can be properly assessed by considering the distinction between the concepts of algebraic semantics and of *equivalent* algebraic semantics for a deductive system. It is not hard to show that $\mathcal{OML}$ is the *unique* equivalent variety semantics for *OML*. $\mathcal{WOML}$ falls short of the requirements needed for an equivalent algebraic semantics for *OML* precisely because it includes algebras (like **O6**) containing as members distinct but *OML*-indiscernible elements. And this, in turns, happens because $\mathcal{WOML}$ does not universally satisfy the quasiequation

$$(\alpha \to \beta) \wedge (\beta \to \alpha) \approx 1 \Rightarrow \alpha \approx \beta,$$

which, true to form, characterises orthomodular lattices within the class of all ortholattices. Therefore, although *OML* is strongly sound and complete w.r.t. $\mathcal{WOML}$, we do not have the kind of *inverse* soundness and completeness theorem that is the hallmark of a "good" semantics.

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Quasi-MV Algebras and the Commutators

Tomasz Kowalski

Quasi-MV algebras are a generalisation of MV algebras intended to capture some properties of the logic of quantum computation. In fact, they capture very few. But this limitation is sometimes an advantage: quasi-MV algebras are a useful testing ground for universal algebraic features that algebras modelling logic of quantum computation are likely to have.

Universal algebraic theory of (congruence) commutators is a vast generalisation of the notion of commutator subgroup from group theory. It proved useful in many contexts, in particular in analysing congruence lattices and Mal'cev conditions. However, the varieties of algebras associated with logics are usually congruence distributive, and for such varieties commutator theory is useless, as the commutator is trivial. The variety of quasi-MV algebras is an exception: it is located at the opposite end of the spectrum, satisfying no Mal'cev conditions and no congruence properties other than the variety of sets. This statement itself follows from an easy application of commutator theory to quasi-MV algebras.

I will present some results on the commutator in quasi-MV algebras. In fact, even the phrase "the commutator" hides a result: that all commutators coincide. The crucial fact is that the commutator of two congruences is the intersection of their "MV restrictions". It follows that commutator trivial quasi-MV algebras are precisely MV algebras, and Abelian quasi-MV algebras are precisely the flat ones. The congruence lattice of an quasi-MV algebra augmented by the commutator turns out to be a rather special residuated semigroup.

As an application, I will show that a version of Jonsson's Lemma that is known to hold for quasi-MV algebras follows easily from commutator theory.

On Reconstruction of a C*-algebra from Its Closed Right Ideals

*David Kruml**

The well known Gelfand–Naimark duality states that the category of commutative C*-algebras is dual to the category of locally compact Hausdorff spaces. For a given space, the commutative C*-algebra can be constructed as an alegebra of continuous bounded functions on it. Conversely, a commutative C*-algebra yields a frame of closed ideals and this gives a topological space in a standard way.

It is natural to ask whether the duality can be extended also for non-commutative C*-algebras using suitable generalized q-spaces or q-frames. The problem is studied in two main directions: One uses the concept of quantale [5] and considers the one of closed subspaces of the C*-algebra. Such a spectrum is a complete invariant for the C*-algebra but it seems that quantales fail to establish a categorical duality [3, 4]. The other way considers a lattice of closed right ideals with some additional structure. Such structures are still complete invariants of C*-algebras when we remember how the closed right idelas are associated to projections in an enveloping W*-algebra [1, 2], and they seem to have more promissing categorical properties.

I want to present that the map from closed right ideals to projections can be mostly reconstructed only from the lattice structure with a compatibility relation. The only exception occurs for C*-algebras with a complicated structure of 2-dimensional irreducible representations. In final, the problem reduces to the question whether we are able to reconstruct a certain fiber bundle over the space of 2-dimensional irreducible representations.

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On Some Properties of Directoids

Ivan Chajda[*] *José Gil-Férez*[†] *Roberto Giuntini*[‡] *Miroslav Kolařík*[§]

Antonio Ledda[¶] *Francesco Paoli*[‖]

It is superfluous to recall how important partially ordered sets, and in particular directed posets, are for the whole of mathematics. However, unlike other equally fundamental mathematical structures, such as groups or Boolean algebras, posets and directed posets are *relational structures*, not algebras, whence they do not lend themselves to be the objects of common algebraic constructions like quotients, products, subalgebras and the like. In fact, insofar as they exist at all for relational structures, these constructions admit of several competing variants, none of which enjoys a universal acclaim, and are generally recognised as more cumbersome and less efficient than in the algebraic case. In order to enable such algebraic constructions with ordered sets, J. Ježek and R. Quackenbush [5] — and, independently, V.M. Kopytov and Z.I. Dimitrov [6] and B.J. Gardner and M.M. Parmenter [4] — introduced the notion of *directoid*.

To every directed poset $\mathbf{A} = \langle A, \leq \rangle$ a groupoid $\mathcal{D}(\mathbf{A}) = \langle A, \sqcup \rangle$ can be associated in such a way that for all $a, b \in A$, $a \leq b$ if and only if $a \sqcup b = b \sqcup a = b$. In the terminology of [5], this groupoid is called a *commutative directoid*. Directoids were investigated in detail by several authors; for a survey, see [1].

We study some properties of directoids and their expansions by additional signature, including bounded involutive directoids and complemented directoids.

A case in point is given by *effect algebras*, which play a noteworthy role in quantum logic (see e.g. [2] and [3]) — in fact, they can be presented as bounded posets equipped with an antitone involution, such that the supremum exists for orthogonal elements.

Among other results, we provide a shorter proof of the direct decomposition theorem for bounded involutive directoids given in [1]. Using techniques from [7], we present a description of central elements of complemented directoids. Finally, we address an open problem showing that the variety of directoids, as well as its expansions mentioned above, all have the strong amalgamation property.

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Entanglement and Quantum Logical Gates

Enrico Beltrametti [*] *Roberto Leporini* [†] *Giuseppe Sergioli*[‡]

In quantum computation pieces of information are represented as density operators of convenient Hilbert spaces, while information is processed by reversible quantum logical gates. We investigate the following question: are there any interesting correlations between entanglement, entanglement-measures and the probabilistic behavior of gates? This question gives rise to different answers in the case of two different classes of gates. The first class contains some special examples of unitary quantum operations, called *unitary connective-gates*. The most common gates used in quantum computation (like the `NOT`-gate, the Toffoli-gate, the `XOR`-gate, the Hadamard-gate, ...) belong to this class. The second class contains the (reversible) *additive bounded operations*, including the class of all anti-unitary operations (like the *universal* `NOT`). While gates of the first class are generally unable to characterize entanglement, one can prove that such characterization is (to a certain extent) possible by means of gates that belong to the second class.

In the semantics of quantum computational logics the basic logical connectives can be interpreted as additive bounded operations, instead of unitary quantum operations (as is usually done). This choice gives rise to a suitable framework for a "logical" characterization of entanglement and of entanglement-measures.

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A Quantum Limit on the Information Retrievable from an Image

Lev B Levitin[*] *Tommaso Toffoli*[†]

A very general and most common way to store information is to encode it in the optical properties of an object. One can then retrieve information by viewing the object by reflected or transmitted light—or even by light emitted by the object itself—for a specified time interval. Such methods, using natural (i.e., thermalized) light, are universally used in photography, television, printing, etc. In the present paper we discuss the question of what are the fundamental limitations imposed on these methods by the wave and quantum nature of light and by the statistical properties of thermal radiation, and we determine the maximum amount of information that can be retrieved from the object by illuminating it in this way and observing its optical image during a read-out window of a prescribed time width. We exclude from consideration cases, like holography, which make use of coherent rather than thermal radiation.

The problem under consideration can be reduced to the evaluation of the capacity of a specific classical-quantum information channel (a channel where classical signals are encoded within quantum states).

Though the bulk of work in this direction was done throughout the last few decades (e.g., [1, 3, 5, 7, 8, 9, 10, 11]), the special case of thermal radiation deserves a detailed analysis. Note that thermal radiation has maximum entropy for a given average occupation number (the average number of photons per field oscillator); therefore, it is the most "noisy" information carrier.

For simplicity, but without loss of generality, let the external surface of the object be a plane of area S. The optical properties of this surface as far as reflected light is concerned may be specified by a *reflectivity* $a(x, y, \nu)$—a function of a point (x, y) of the surface and of the frequency ν of the radiation. (Here and below we take this to be the reflectivity in the direction to the observer; it is inessential for us whether the reflection is diffuse or specular.) As far as we are only interested in limitations owed to the physical nature of light, we can disregard the material structure of the surface. Even though, for any material, reflectivity is well-defined only for *areas* that are large with respect to interatomic distances, the idealization of reflectivity as a *point* function is reasonable when the wavelengths that most significantly contribute to the spectrum of natural light sources are large with respect to interatomic distances.

Similarly, for transmitted light, the function $a(x, y, \nu)$ should express a *transmittance.* Finally, in the case of self-luminous surface, $a(x, y, \nu)$ will be the ratio of the spectral intensity of the radiation at a given point of the surface to the maximum available spectral intensity.

We also assume that a function $a(x, y, \nu)$ can be assigned independently for the two polarization states of radiation.

Suppose the area S is viewed by an optical radiation detector whose entrance pupil is seen under a solid angle Ω from any point of that area. If the object is observed for a time interval τ, then the frequency uncertainty of the photon quantum states is $\Delta\nu = 1/\tau$. Taking into account the two possible polarization states, we obtain, for the total number of radiation degrees of freedom,

$$G = \frac{2}{c^2}\nu^2 \Omega S \tau \delta\nu \tag{1}$$

(under the assumption that $\nu^2 \Omega S \tau \delta\nu / c^2 \gg 1$, i.e., that the geometric optics approximation is valid); thus, we have a *finite* number of degrees of freedom.

As is well known (see [4]), in the radiation of thermal sources the states of field oscillators are statistically independent, and described by a Gibbs distribution

$$p(n) = \frac{1}{\overline{n}+1}\Big(\frac{\overline{n}}{\overline{n}+1}\Big)^n \tag{2}$$

where n is the occupation number—or the number of photons in a given quantum state—and $\overline{n}$ the mean occupation number.

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Thus, if a source of thermal radiation is used as an information transmitter, the expected value $\overline{n}$—rather than the exact number of photons n—is the input signal. In other words, the signal is the value of the effective temperature for a given degree of freedom of the electromagnetic field.

Since the density matrix of thermal radiation is diagonal in energy (or occupation number) representation, the optimal strategy to extract maximum information is measuring the energy, or, in other words, counting the number of photons reaching the receiver. This follows from the fact that the radiation quantum state is completely described by distribution (2) (cf. [7]).

Let $P(\nu)$ be the average energy of a field oscillator of the radiation source of frequency ν, and let $r(\nu)$ be the fraction of the energy which is recorded by the detector in the calibration case of reflectivity a equal to unity. Then, assigning a specific value of a means specifying the mean number $\overline{n}$ of recorded photons for each field oscillator, or $\overline{n} = \frac{ar(\nu)P(\nu)}{h\nu}$. The *maximum* mean number of recorded photons will of course be $\overline{n}_m = \frac{r(\nu)P(\nu)}{h\nu}$.

It can be shown[6] that, when the mean number of photons distributed according to (2) is limited by a maximum value $\overline{n}_m \leq 9$ (in the optical range, this corresponds to temperatures $T \leq 3 \cdot 10^5\,{}^\circ K$), optimal encoding is achieved by using as signal levels just *two* mean-photon-number values, namely, $\overline{n} = 0$ and $\overline{n} = \overline{n}_m$ (corresponding to reflectivity values $a = 0$ and $a = 1$). In this case, the maximum amount of information per field oscillator is, in nats,

$$I_m(\nu) = \ln\Big[1 + \frac{r(\nu)P(\nu)}{r(\nu)P(\nu)+h\nu}\big(\frac{h\nu}{r(\nu)P(\nu)+h\nu}\big)^{\frac{h\nu}{r(\nu)P(\nu)}}\Big]. \tag{3}$$

Given a spectral band from ν_0 to ν_1, assuming $(\nu_1 - \nu_0)\tau \gg 1$, with (1) and (3) taken into account, we obtain that the maximum amount of information that can be obtained by observation of an object illuminated by incoherent light is

$$J_m = \frac{2\tau\Omega S}{c^2}\int_{\nu_1}^{\nu_2}\nu^2 I_m(\nu)d\nu; \tag{4}$$

thus, the amount of information increases proportionally to the area S of the object and the time τ of observation.

It will be interesting to derive an explicit expression for the amount of information in the case when the radiation recorded by the detector obeys Planck's spectral distribution (for instance, when the illumination source is a black body of temperature T and $r(\nu) = 1$), i.e., when

$$r(\nu)P(\nu) = h\nu/\big(e^{\frac{h\nu}{kT}} - 1\big),$$

where k is Boltzmann's constant and h Planck's constant. Let the frequency band be infinite. Then

$$J_m = \frac{2\tau\Omega S}{c^2}\int_0^\infty \nu^2 \ln\big[1 + e^{-\frac{h\nu}{kT}}(1 - e^{-\frac{h\nu}{kT}})^{e^{\frac{h\nu}{kT}}-1}\big]d\nu = \frac{2\tau\Omega S(kT)^3}{c^3h^3}\sigma,$$

where σ is a numerical constant, given by $\sigma = \int_0^\infty x^2 \ln\big(1 + e^{-x}(1 - e^{-x})^{e^x-1}\big)\,dx \approx 0.772$. The total number of photons recorded by the receiver is in this case $\frac{2\tau\Omega S(kT)^3}{c^3h^3}\eta$, where η is another constant, namely, $\eta = \int_0^\infty \frac{x^2(1-e^{-x})^{e^x-1}}{e^x-1+(1-e^{-x})^{e^x}}dx \approx 0.909$. Thus, the amount of information per photon is

$$\frac{J_m}{N} = \frac{\sigma}{\eta} \approx 0.849\frac{\text{nats}}{\text{photon}} \approx 1.225\frac{\text{bits}}{\text{photon}}.$$

This is a universal constant, as it does not depend on the parameters τ, Ω, S of observation.

Now let the solid angle Ω take on the maximum possible value 2π. Then the maximum amount R_m of information per unit time per unit area is

$$R_m = \frac{4\pi(kT)^3}{c^2h^3}\sigma = \frac{\sigma\sqrt{2}}{\pi^2\sqrt{c}}\big(\frac{15P}{\pi h}\big)^{\frac{3}{4}},$$

where $P = \frac{4\pi}{c^2}\int_0^\infty \nu^2 \frac{h\nu}{e^{\frac{h\nu}{kT}}-1}d\nu = \frac{4\pi^5(kT)^4}{15c^2h^3}$ is the energy flux of reflected radiation at maximum signal level per unit area.

Thus, the maximum amount of retrieved information grows with the 3/4 power of radiation energy.

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Reconstructing a C*-algebra from Its Poset of Commutative C*-subalgebras

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Given a unital C*-algebra A, one could consider the following question: is it possible to recover the C*-structure of A from the set $\mathcal{C}(A)$ of commutative unital C*-subalgebras ordered by inclusion? More precisely, if A and B are C*-algebras such that $\mathcal{C}(A)$ and $\mathcal{C}(B)$ are isomorphic as posets, can we find an *-isomorphism between A and B?

Apart from its mathematical relevance, this problem is of great importance for the so called *quantum toposophy* program, where one tries to describe quantum mechanics in terms of topos theory (see e.g., [1], [3], [6]). In this program, the central objects of research are the topoi $\mathbf{Sets}^{\mathcal{C}(A)}$ and $\mathbf{Sets}^{\mathcal{C}(A)^{\mathrm{op}}}$. The motivation behind this approach is Niels Bohr's doctrine of classical concepts, which, roughly speaking, states that a measurement provides a "classical snapshot of quantum reality" corresponding with an element of $\mathcal{C}(A)$, and knowledge of all classical snapshots should provide a picture of quantum reality corresponding to A, that is as complete as (humanely) possible. The possiblility of reconstructing A from C(A) would assure the soundness of this doctrine.

For commutative C*-algebras, Mendivil [8] showed that the answer to the question is affirmative. For non-commutative C*-algebras however, the answer is negative, since Connes [2] showed the existence of a C*-algebra A_c that is not isomorphic to its oppositve algebra A_c^{op}. Here opposite algebra means the C*-algebra with the same underlying topological vector space, but with multiplication defined by $(a, b) \mapsto ba$, where $(a, b) \mapsto ab$ denotes the original multiplication. Since $\mathcal{C}(A)$ is always isomorphic as poset to $\mathcal{C}(A^{\mathrm{op}})$ for each C*-algebra A, the existence of this C*-algebra A_c shows that extra structure is needed in order to reconstruct arbitrary C*-algebras A from $\mathcal{C}(A)$.

Nevertheless, it might be interesting to investigate in which cases an order isomorphism $\mathcal{C}(A) \to \mathcal{C}(B)$ implies the existence of a *-isomorphism $A \to B$. Partial results are obtained in [5] and [4], the latter in the context of von Neumann algebras instead of C*-algebras. In these papers, one tries to find an explicit *-isomorphism that induces the order isomorphism.

Another approach, which we wil follow, is to investigate for which C*-algebras $\mathcal{C}(A)$ turns out to be a complete invariant. We will discuss finite-dimensional C*-algebras, which are completely classified by the Artin-Wedderburn Theorem. This theorem states that each finite-dimensional C*-algebra A is *-isomorphic to $\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$ for some $k, n_1, \ldots, n_k \in \mathbb{N}$. We will discuss how to decide from $\mathcal{C}(A)$ whether A is finite dimensional or not, and how to read off the numbers $k, n_1, \ldots, n_k \in \mathbb{N}$ if we are dealing with a finite-dimensional algebra.

It turns out that A is finite dimensional exactly when $\mathcal{C}(A)$ is an *Artinian poset*, i.e., a poset that satisfies an ascending chain condition. The Artinian condition is very strong, since it allows to prove a generalization of the principle of induction. Moreover, the Artinian condition is equivalent to several statements in locale theory and topos theory. Dual to the notion of an Artinian poset is the notion of a *Noetherian poset*, which is a poset that satisfies an ascending chain condition. It turns out that both notions are equivalent for $\mathcal{C}(A)$. This is remarkable, since both conditions are certainly not equivalent for all posets, i.e., the natural numbers $\mathbb{N}$ equipped with the usual order is an Artinian, but not a Noetherian poset.

In order to find the numbers occuring in the Artin-Wedderburn Theorem, it turns out that the *center* $Z(A) = \{a \in A : ab = ba\ \forall b \in A\}$ plays an important role, since it measures the number k if $A = \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$. It turns out that the center is an element of $\mathcal{C}(A)$. Moreover, the center can be defined purely in order theoretic terms, which has the implication that the center of B is isomorphic to the center of A when $\mathcal{C}(A) \cong \mathcal{C}(B)$. Finally, we identify subposets of $\mathcal{C}(A)$ order isomorphic to partition lattices, which allows us to find the numbers $n_1, \ldots, n_k$. We conclude with the discussion of possible generalizations as AF-algebras or infinite dimensional von Neumann algebras.

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Cloning and Broadcasting in Operator Algebras

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We investigate cloning and broadcasting by quantum channels (Schwarz maps) in the general operator algebra framework, generalizing thus results obtained so far in finite dimension, and for the full algebra of operators on a Hilbert space. Instead of the full algebra of all bounded operators on a finite-dimensional Hilbert space we consider an arbitrary von Neumann algebra on a Hilbert space of arbitrary dimension. We characterize sets of states that are broadcastable and cloneable. We present as well some results on cloning by positive operators in von Neumann algebras.

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2-sufficiency of Operator Algebras

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The notion of 2-sufficiency of a von Neumann subalgebra of the full algebra of all bounded linear operators on a Hilbert space has been introduced and analyzed by A. Jenčova in the case of a finite dimensional Hilbert space and faithful states. We aim at giving a characterization of 2-sufficiency for arbitrary von Neumann algebras and normal states in terms of statistical morphisms. This latter notion was defined and exploited by F. Buscemi in the analysis of subordination of quantum statistical models.

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States on Conjugation Logics

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Abstract. In the paper we describe states on the conjugation logics of projections.

Let H be a complex Hilbert space with the Hilbert product $(\cdot,\cdot)$. Let $B(H)$ be the set of all bounded operators on H. Let J be a *conjugation* operator in H [1], i.e. 1) $J^2 = I$, 2) $(Jx, Jy) = (y, x)$, for all x, $y \in H$. Note by 1), and 2), $J(\lambda x + \beta y) = \overline{\lambda} Jx + \overline{\beta} Jy$, for all λ, $\beta \in \mathcal{C}$ and for all x, $y \in H$.

Put $< x, y >:= (Jx, y)$. For any $A \in B(H)$ there exists unique $A^{\#} \in B(H)$ such that $< Ax, y >=< x, A^{\#}y >$ for all x, $y \in H$. An operator $A \in B(H)$ is said to be J-*self-adjoint*, if $< Ax, y >=< x, Ay >$, $\forall x$, $y \in H$. Any bounded J-self-adjoint projection (=idempotent) P is said to be J-*projection*.

Let $\mathcal{M}$ be a von Neumann algebra on H, let $\mathcal{M}' := \{B \in B(H) : AB = BA, \forall\ A \in \mathcal{M}\}$, and let $\mathcal{Z} := \mathcal{M} \cap \mathcal{M}'$ be the the center of $\mathcal{M}$. Let us denote by $\mathcal{M}^{Jc}$ ($\mathcal{M}^{or}$) the set of all J-self-adjoint (orthogonal, respectively) projections from $\mathcal{M}$. Any one-dimensional J-projection has the form $(\cdot, Jx)x$, where $(x, Jx) = 1$. With respect to the standard relations, namely: the ordering $P \leq_1 Q$ iff $PQ = QP = P$, the orthogonal relation $P \perp Q$ iff $PQ = QP = 0$, and the orthocomplementation $P^{\perp} := I - P$ for all P, Q the set $\mathcal{M}^{Jc}$ is a *quantum logic* of projections. The logic $\mathcal{M}^{Jc}$ is said to be *conjugation* logic.

A von Neumann algebra $\mathcal{M}$ on H is said to be a *von Neumann J-algebra* if $A \in \mathcal{M}$ implies $A^{\#} \in \mathcal{M}$. Let $\mathcal{M}$ be a von Neumann J-algebra. Then: *i*) its center $\mathcal{Z}$ and $\mathcal{M}'$ are von Neumann J-algebras also.

Well-known classification of von Neumann algebras in a Krein space [2]. A similar classification is possible in a space with conjugation operator. Namely [3],

A commutative von Neumann J-algebra $\mathcal{Z}$ is said to be a *type* (A) algebra if $P = P^{\#}$ for all $P \in \mathcal{Z}^{or}$. A commutative von Neumann J-algebra $\mathcal{Z}$ is said to be a *type* (B) algebra if $\mathcal{Z}$ contains a pair F, $F^{\#} \in \mathcal{Z}^{or}$ such that $F + F^{\#} = I$. A von Neumann J-algebra $\mathcal{M}$ is said to be of *type* (A) (*type* (B)) if its center $\mathcal{Z}$ is of type (A) algebra (of type (B) algebra, respectively).

Let B be a quantum logic of projections. The function $\mu : B \to \mathcal{R}$ is said to be the *measure*, if $\mu(p) = \sum_i \mu(p_i)$ for any representation $p = \sum_i p_i$, $p_i p_j = 0$, $i \neq j$. (The sum being understood in the strong sense). Non negative measure μ is said to be *probability measure* (=*state*) if $\mu(I) = 1$. Let μ be a non negative measure. It is clear that if $\mu(I) > 0$ then $\frac{\mu(\cdot)}{\mu(I)}$ is a probability measure.

We are starting with Lemma 1.

Lemma 1[3]. *Let $\mathcal{Z}$ be a commutative von Neumann J-algebra. Then there exists unique maximal projection $E \in \mathcal{Z}^{or}$ such that $P \leq_1 E$, $P \in \mathcal{Z}^{pr}$ implies $P = P^{\#}$. In addition there exists* (*non unique, in general!*) *projection $F \in \mathcal{Z}^{pr}$ such that $F + F^{\#} = I - E$.*

By Lemma 1, if the set of points of the spectrum of $\mathcal{Z}$ is equal to three, then $\mathcal{Z} = \{\lambda E + \beta F + \gamma F^{\#},\ \ \lambda, \beta, \gamma \in \mathcal{C}\}$. Here E, F from Lemma 1.

Let $\mathcal{M}$ be a von Neumann J-algebra of type (B). A pair F, $F^{\#}$ of orthogonal projections in $\mathcal{Z}$ satisfying $F + F^{\#} = I$ is assumed to be fixed. Set $\mathcal{B} := FB(H)F + F^{\#}B(H)F^{\#}$. It is clear that $\mathcal{B}$ is a von Neumann J-algebra of type (B) and $\mathcal{M} \subseteq \mathcal{B}$.

Theorem 2. *Let $\mathcal{Z}$ be a commutative von Neumann J-algebra and the set of points of the spectrum of $\mathcal{Z}$ is equal to three. Then the probability measure μ on the conjugation logic $(\mathcal{Z}')^{Jc}$ exists if and only if:*

1) *or* $\dim EH < \infty$ *and* $\dim E^{\perp}H = \infty$. *Then* $\mu(P) = \frac{1}{n}\mathrm{tr}(EP)$ *if* $n := \dim EH \geq 3$;

2) *or* $\dim EH = \infty$ *and* $\dim E^{\perp}H < \infty$. *Then* $\mu(P) = \frac{1}{m}\mathrm{tr}(E^{\perp}P)$ *if* $m := \dim E^{\perp}H \geq 6$;

3) *or* $\dim H < \infty$. *Let* $3 \leq \dim EH \equiv n$, $6 \leq \dim E^{\perp}H \equiv m$, *Then there exists unique number* $a \in [0, 1]$ *such that*

$$\mu(P) = \frac{a}{n}\mathrm{tr}(EP) + \frac{1-a}{m}\mathrm{tr}(E^{\perp}P) \tag{1}$$

for all P.

Note that formula (1) is reminiscent of the formula of the paper [4] for the Krein space.

The theorem can be extended to arbitrary von Neumann J-algebra.

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Fuzzy Geometry, Unsharpness and Quantum Dynamics

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It's argued that Dodson-Zeeman fuzzy geometry (FG)[1] represents the geometric counterpart for unsharp algebras[2]. As the result, quantum dynamics can be described with the minimal axiomatic[3,4]. As the example, the quantization of massive particles is considered, it's shown that the coordinate uncertainty is generic in FG. FG fundamental set D is Poset[3,4], so that some its element pairs in place of standard ordering relation $d_j \leq d_k$, can obey to incomparability relation: $d_l \sim d_m$. For illustration, consider discrete Poset $D = A^p \cup B$, which includes the subset of incomparable elements $A^p = \{a_j\}$, and the subset $B = \{b_i\}$ which is maximal totally ordered D subset. B indexes grow correspondingly to their ordering, i.e. $\forall\, i,\, b_i \leq b_{i+1}$. Suppose that for some a_j and B interval $\{b_l, b_n\}$, $a_j \sim b_i\,; \forall i : l \leq i \leq n$. In this case a_j is 'smeared' over $\{b_l, b_n\}$ interval, which is analogue of a_j coordinate uncertainty, if to regard B as D 'coordinate axe'. Analogously to it, 1-dimensional model Universe corresponds to Poset $U = A^p \cup X$ where A^p is the massive particle subset, X - continuous ordered subset R^1, which describes 1-dimensional euclidian geometry. If for some a_j and X interval $\{x_c, x_d\}$ the relation $a_j \sim x_b$ holds for all $x_b \in \{x_c, x_d\}$, then a_j possess x-coordinate uncertainty of the order $|x_d - x_c|$. To detalize a_j characteristics, the corresponding fuzzy weight $w_j(x) \geq 0$ introduced with the norm $||w_j|| = 1$, so that $w_j(x)$ value indicates where on X axe a_j is mainly concentrated[3]. In this framework, a_j is the fuzzy point and U is the fuzzy set[3].

In such approach massive particle m can be described as the evolving fuzzy point $a_i(t)$ of U. It's shown then that the corresponding normalized m density $w(x,t)$ evolves according to the flow continuity equation: $\frac{\partial w}{\partial t} = -\frac{\partial (wv)}{\partial x}$ where $v(x,t)$ is w flow local velocity. The independent m parameters $w(x), v(x)$, which characterize m state, can be unambiguously mapped to normalized complex function $\varphi(x)$. Assuming space-time shift invariance, it's proved that $\varphi(x,t)$ evolution obeys to Schroedinger equation for arbitrary m mass μ , such theory can be also extended for 3-dimensional case[4]. It's proved also that in relativistic case m evolution described by Dirac equation for spin $\frac{1}{2}$. Particle's interactions on such fuzzy manifold are shown to be gauge invariant, the interactions of fermion muliplets are performed by Yang-Mills fields[5].

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Information Transfer and Randomness in Quantum Measurements

Sergey Mayburov *

The information transfer in measuring system (MS) and its influence on measurement outcome is analysed. Formally, any MS can be described as the information channel between the measured object S and the information processing system (observer) O. Information theory demonstrates that the amount of information per event which can be transferred via the quantum channel (capacity) is severely restricted [1, 2]. In case of measurement such constraints can distort the measurement outcome. Earlier it was argued that due to them, the information about the purity of S state practically can't be transferred to O. If this the case, O wouldn't discriminate pure and mixed S ensembles which is equivalent to the collapse of S pure state [3]. Besides, the amount of information about purity carried by individual S state should be calculated also, because for quantum S it can turn out to be too small for effective discrimination of pure/mixed S state in single event.

To study both these effects, we considered the measurement of S dichotomic (spin$\frac{1}{2}$) observable S_z performed by MS, which includes the detector D and O, both of them are regarded as the quantum objects [4]. The measurement of two S ensembles $E_{1,2}$ is compared; E_1 includes the pure states which are the superposition of S_z eigenstates $|S_{1,2}\rangle$ with amplitudes $a_{1,2}$, another ensemble E_2 is the probabilistic mixture of such eigenstates with the same $\bar{S}_z$. In our model S, D, O interactions are tuned so that initial $|S_{1,2}\rangle$ induce the orthogonal O states $|O_{1,2}\rangle$ with 'pointer' eigenvalues $O_{1,2}$, for any other pure S state it results in entangled S, D, O state.

For described ensembles S purity is characterized by the expectation value $\bar{\Lambda}$ of S observable Λ, conjugated to S_z; for $a_1 = a_2$ it's given by $\Lambda = S_x$. Yet for our MS final states there is no O observable Q which expectation value $\bar{Q}$ depends on $\bar{\Lambda}$. As the result, O can't discriminate statistically the pure and mixed S ensembles with the same $\bar{S}_z$ [4]. The calculation in the formalism of restrictive maps [5], indicate that for individual events these information losses induce the appearance of randomness in the measurement of S pure ensemble E_1, so that the observed O outcomes correspond to the random $O_{1,2}$ values, i.e. for O the collapse of S pure state occurs [4].

Our analysis shows that the purity information carried by S state is insufficient for the discrimination of pure/mixed S states in a single event. At its best, its average amount $\bar{I}_p$ is equal to $\frac{1}{2}$ bit per event, whereas for effective discrimination it should be $\bar{I}_p \geq 1$ bit [2]. It follows that such S information incompleteness is the main observer-independent mechanism of outcome randomness for pure S states, in this framework the described loss of S purity information in MS channel is auxiliary effect. Born rule for outcome probabilities is derived from the linearity and the invariant properties of MS Hilbert space. Concerning with MS decoherence effects, it's shown that by itself, due to the unitarity of decoherence interactions, it can't result in the appearance of randomness in the measurement outcomes, however, it's shown that MS interactions with its environment stabilize the obtained final MS states additionally.

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Qualitative Noise-Disturbance Relation – An order-theoretic approach

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The inherent connection between noise and disturbance is one of the most fundamental features of quantum measurements. (1) We derive a structural connection between certain order relations defined on observables and channels, and we explain how this connection properly explains the trade-off between noise and disturbance. A link to a quantitative noise-disturbance relation is also demonstrated. (2) The unavoidable disturbance implies that the realizable subsequent measurements are getting limited after one performs some measurement. The obvious general limitation that one cannot circumvent by sequential or any other method is that the actually implemented measurements must be jointly measurable. By using the partially ordered set (poset) structure, we show that any jointly measurable pair of observables can be obtained in a sequential measurement scheme, even if the second observable would be decided after the first measurement. That is, there exists a universal measurement scheme that does not spoil the possible measurements afterwards.

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Multidimensional States and Non-compatible Random Events

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Algebraic approach to uncertainty is based on the study of more general structures than Boolean algebras. We use an orthomodular lattice (OML) with sufficient number of states (a quantum logic).

On such OML has been defined multivariable states which represent measures of intersection (s-map), union (q-map) and symmetric difference (d-map) in the case of compatibility ([1, 2, 3]). These maps can be used e.g. for modeling of an unknown joint distribution for non-compatible observables or interactions between two non-compatible random events.

Definition 1. *Let L be an OML. Then G-map $Q : L^2 \to [0,1]$ is called a special bivariables map if satisfying the following conditions:*

(G1) $G(i,j) \in \{0,1\}$ *for* $i,j \in \{1_L, 0_L\}$ *and* $G(1_L, 0_L) = G(0_L, 1_L)$*;*

(G2) if $a \perp b$*, then* $G(a,b) = G(a,0_L) + G(0_L, b)$*;*

(G3) if $a \perp b$ *then for any* $c \in L$

$$G(a \vee b, c) = G(a,c) + G(b,c) - G(0_L, c)$$

$$G(c, a \vee b) = G(c,a) + G(c,b) - G(c, 0_L).$$

In the special case, if $G(1_L, 0_L) = G(0_L, 1_L) = 1$ and $G(1_L, 1_L) = G(0_L, 0_L) = 0$ we receive a measure of "symmetric difference" (d-map). In fact, a G-map is not invariant with respect to order in general.

- The properties of d-map induced by 2-dimensional s-map play crucial role for the existence of a 3-dimensional s-maps.
- d-maps allow to study various types "undistinguishable" elements on a quantum logic.

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New Results about Symmetric Difference in Quantum Structures

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The symmetric difference in Boolean algebras, $A \mathbin{\Delta} B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$, can be generalized to quantum structures in different ways. E.g., in orthomodular lattices, these expressions are two of six possible generalizations of the Boolean symmetric difference, see [12]. However, none of them has very convenient properties replacing the role of the symmetric difference in Boolean algebras. Therefore another symmetric difference has been introduced in [10] as a new binary operation which cannot be expressed by the original operations of an orthomodular lattice. We obtain the notion of an *orthocomplemented difference lattice* with interesting algebraic properties.

If an orthocomplemented difference lattice is set-representable (=concrete [7, 15]), the set-theoretical symmetric difference has a clear meaning which has been studied, e.g., in [13, 14]. Relaxing the lattice condition (assuming only a poset), we obtain a *symmetric logic* which is a set-representable poset less general than orthomodular posets, but more general than Boolean algebras.

We summarize the recent results on symmetric logics (see [1, 2, 3, 4, 5, 10]) and solutions to some open problems which were formulated there. We study extensions of states (=finitely additive probability measures), in particular the question when states defined on a subalgebra admit extensions to the whole symmetric logic. Further, we investigate when all states m are Δ-subadditive, i.e.,

$$m(A \mathbin{\Delta} B) \leq m(A) + m(B) \text{ for any pair } A, B\,.$$

We clarify under which conditions a symmetric logic becomes a Boolean algebra.

We show that symmetric logics, considered as metric spaces equipped with a natural metric based on the symmetric difference and a state, need not be complete.

Special attention is paid to the logics of idempotents in rings [6, 8, 9, 11], which are generalizations of set-representable logics and hence also of symmetric logics.

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On Probability Domains III

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Domains of generalized probability have been introduced in order to provide a general construction of random events, observables and states. It is based on the notion of a cogenerator and the properties of product (cf. [1], [2]).

We continue our previous study and show how some other quantum structures fit our approach.

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Tense Operators in Quantum Structures

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For effect algebras, the so-called tense operators were already introduced by Chajda and Paseka. They presented also a canonical construction of them using the notion of a frame.

Tense operators express the quantifiers "it is always going to be the case that" and "it has always been the case that" and hence enable us to express the dimension of time in the logic of quantum mechanics.

A crucial problem concerning tense operators is their representation. Having an effect algebra with tense operators, we can ask if there exists a frame such that each of these operators can be obtained by the canonical construction. Introducing the notion of a q-effect algebra we solve this problem for E-tense operators on E-representable E-Jauch-Piron q-effect algebras.

1 Preliminaries and basic facts

Effect algebras were introduced by Foulis and Bennett [7] as an abstraction of the Hilbert space effects which play an important role in the logic of quantum mechanics. However, this notion does not incorporate the dimension of time.

This means that effect algebras can serve to describe the states of effects in a given time but they cannot reveal what these effects expressed in the past or what they will reveal in the next time.

By an *effect algebra* is meant a structure $\mathcal{E} = (E; +, 0, 1)$ where 0 and 1 are distinguished elements of E, $0 \neq 1$, and $+$ is a partial binary operation on E satisfying the following axioms for $x, y, z \in E$:

(E1) if $x + y$ is defined then $y + x$ is defined and $x + y = y + x$

(E2) if $y + z$ is defined and $x + (y + z)$ is defined then $x + y$ and $(x + y) + z$ are defined and $(x + y) + z = x + (y + z)$

(E3) for each $x \in E$ there exists a unique $x' \in E$ such that $x + x' = 1$; x' is called a *supplement* of x

(E4) if $x + 1$ is defined then $x = 0$.

Having an effect algebra $\mathcal{E} = (E; +, 0, 1)$, we can introduce the *induced order* $\leq$ on E and the partial operatin $-$ as follows

$$\begin{array}{l} x \leq y \quad \text{if for some } z \in E \quad x + z = y, \\ \text{and in this case } z = y - x \end{array}$$

(see e.g. [6] for details). Then $(E; \leq)$ is an ordered set and $0 \leq x \leq 1$ for each $x \in E$.

It is worth noticing that $a + b$ exists in an effect algebra $\mathcal{E}$ if and only if $a \leq b'$ (or equivalently, $b \leq a'$). This condition is usually expressed by the notation $a \perp b$ (we say that a, b are orthogonal). Dually, we have a partial operation $\cdot$ on E such that $a \cdot b$ exists in an effect algebra $\mathcal{E}$ if and only if $a' \leq b$ in which case $a \cdot b = (a' + b')'$. This allows us to equip E with a dual effect algebraic operation such that $\mathcal{E}^{op} = (E; \cdot, 1, 0)$ is again an effect algebra, $'^{\mathcal{E}^{op}} = '^{\mathcal{E}} = '$ and $\leq_{\mathcal{E}^{op}} = \leq^{op}$.

Let $d, q : E \to E$ be maps such that, for all $x, y, z \in E$,

(Q1) $d(x') = q(x)'$,

(Q2) $d(0) = 0 = q(0)$,

(Q3) d is order-preserving,

(Q4) $x' \leq x$ implies $x \cdot x = d(x)$,

(Q5) $z \leq x, z \leq y \leq x'$ imply $d(z) \leq x \cdot y$,

(Q6) $d(x) \leq x$.

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We then say that $\mathcal{E} = (E; +, q, d, 0, 1)$ is a *q-effect algebra.* Note that a dual of $\mathcal{E} = (E; +, q, d, 0, 1)$ is a q-effect algebra $\mathcal{E}^{op} = (E; \cdot, d, q, 1, 0)$.

A *morphism of effect algebras* (*morphism of q-effect algebras*, respectively) is a map between them such that it preserves the partial operation $+$, (and the unary operations q and d, respectively), the bottom and the top elements. In particular, $' : \mathcal{E} \to \mathcal{E}^{op}$ is a morphism of effect algebras (morphism of q-effect algebras).

A map $s : E \to [0,1]$ is called a *state* (*an E-state*, respectively) on $\mathcal{E}$ if $s(0) = 0$, $s(1) = 1$, $(s(q(x)) = s(x) + \min(s(x'), s(x))$, $s(d(x)) = 1 - s(x') - \min(s(x'), s(x))$, respectively) and $s(x + y) = s(x) + s(y)$ whenever $x + y$ exists in $\mathcal{E}$.

A *morphism* $f : P_1 \to P_2$ *of bounded posets* is an order, top element and bottom element preserving map. Any morphism of effect algebras is a morphism of corresponding bounded posets. A morphism $f : P_1 \to P_2$ of bounded posets is *order reflecting* if ($f(a) \leq f(b)$ if and only if $a \leq b$) for all $a, b \in P_1$.

If, moreover $(E; \leq)$ is a lattice (with respect to the induced order), then $\mathcal{E}$ is called a *lattice effect algebra.* On any lattice effect algebra $\mathcal{E}$ we may introduce total operations $\oplus$ and $\odot$ as follows: $x \oplus y = x + (y \wedge x')$ and $x \odot y = (x' \oplus y')'$. Note that a lattice effect algebra $\mathcal{E}$ is an MV-algebra (see [4]) with respect to the operations $\oplus$ and $'$ if and only if $x \wedge y = 0$ implies $x \leq y'$. In this case the unary operations $q(x) = x \oplus x$ and $d(x) = x \odot x$ satisfy the conditions (Q1)-(Q5) and $\mathcal{E} = (E; +, q, d, 0, 1)$ is a q-effect algebra. Moreover, any morhism of MV-algebras is a morphism of q-effect algebras.

In what follows, motivated by the above situation, we will always use for q-effect algebras the notation $\mathcal{E} = (E; +, \oplus, \odot, 0, 1)$ such that $\oplus(x) = x \oplus x$ and $\odot(x) = x \odot x$.

Tense operators on q-effect algebras

Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra. Unary operators G and H on $\mathcal{E}$ are called *partial tense operators* if they are are partial mappings of E into itself satisfying the following axioms:

(T1) $G(0) = H(0) = 0$, $G(1) = H(1) = 1$,

(T2) $x \leq y$ implies $G(x) \leq G(y)$ whenever $G(x), G(y)$ exist and $H(x) \leq H(y)$ whenever $H(x), H(y)$ exist

(T3) if $x + y$ and $G(x), G(y), G(x + y)$ exist then $G(x) + G(y)$ exists and $G(x) + G(y) \leq G(x + y)$ and if $x + y$ and $H(x), H(y), H(x + y)$ exist then $H(x) + H(y)$ exists and $H(x) + H(y) \leq H(x + y)$

(T4) $x \leq GP(x)$ if $H(x')$ exists, $P(x) = H(x')'$ and $GP(x)$ exists, $x \leq HF(x)$ if $G(x')$ exists, $F(x) = G(x')'$ and $HF(x)$ exists.

If both G and H are total (i.e., G and H are mappings of E into itself defined for each $x \in E$) then G and H are called *tense operators* and P (or F) is a left adjoint to G (or H, respectively) (see [2]).

It is quite natural to ask that our (total) tense operators on q-effect algebras preserve unary operations $\oplus$ and $\odot$ (see [5]). This can be accomplished by the following axioms (see [3]):

(T5) $G(x \oplus x) = G(x) \oplus G(x)$,
$H(x \oplus x) = H(x) \oplus H(x)$,

(T6) $G(x \odot x) = G(x) \odot G(x)$,
$H(x \odot x) = H(x) \odot H(x)$.

We call such tense operators G and H *E-tense operators.* The main aim of our paper is to establish a representation theorem for E-tense operators.

E-semi-states on q-effect algebras

Definition 1.1. *Let $\mathcal{E} = (E; +, \oplus, \odot, 0, 1)$ be a q-effect algebra. A map $s : E \to [0,1]$ is called*

1. an E-semi-state on $\mathcal{E}$ *if*

(i) $s(0) = 0, s(1) = 1$,

(ii) $s(x) + s(y) \leq s(x + y)$ *whenever $x + y$ is defined,*

(iii) $s(x) \odot s(x) = s(x \odot x)$,

(iv) $s(x) \oplus s(x) = s(x \oplus x)$,

2. a Jauch-Piron E-semi-state on $\mathcal{E}$ *if s is an E-semi-state and* *(v)* $s(x) = 1 = s(y)$ *implies* $s(x \wedge y) = 1$.

Definition 1.2. *Let* $\mathcal{E} = (E; +, \oplus, \odot, 0, 1)$ *be a q-effect algebra.*

(a) If S *is an order reflecting set of E-states on* $\mathcal{E}$ *then* $\mathcal{E}$ *is said to be* E-representable*.*

(b) If S *is an order reflecting set of Jauch-Piron E-states on* $\mathcal{E}$ *then* $\mathcal{E}$ *is said to be* E-Jauch-Piron representable*.*

(c) If any E-state is E-Jauch-Piron then $\mathcal{E}$ *is called an* E-Jauch-Piron q-effect algebra*.*

2 The representation of E-tense operators

In this section we outline the problem of a representation of E-tense operators H and H and we solve it for E-representable E-Jauch-Piron q-effect algebras. This means that we get a procedure how to construct a corresponding time frame (it will be the set of all Jauch-Piron E-states equipped with an induced relation ρ_G) to be in accordance with the canonical construction from [2].

By a *frame* is meant a couple (S, R) where S is a non-void set and $R \subseteq S \times S$. For our sake, we will assume that for all $x \in S$ there are $y, z \in S$ such that xRy and zRx. Having a q-effect algebra $\mathcal{E} = (E; +, \oplus, \odot, 0, 1)$ and a non-void set T, we can produce the direct power $\mathcal{E}^T = (E^T; +, \oplus, \odot, o, j)$ where the operation $+$ and the induced operations $\vee$, $\wedge$, $\oplus$, $\odot$ are defined and evaluated on $p, q \in E^T$ componentwise. Moreover, o, j are such elements of E^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$. The direct power $\mathcal{E}^T$ is again a q-effect algebra.

The notion of frame allows us to construct E-tense operators on q-effect algebras.

Theorem 2.1. *Let* $\mathcal{M}$ *be a linearly ordered complete MV-algebra,* (S, R) *be a frame,* G^* *and* H^* *be maps from* M^S *into* M^S *defined by*

$$\begin{aligned} G^*(p)(s) &= \bigwedge\{p(t) \mid t \in S, sRt\}, \\ H^*(p)(s) &= \bigwedge\{p(t) \mid t \in S, tRs\} \end{aligned}$$

for all $p \in M^S$ *and* $s \in S$*. Then* G^* *(*H^**) is an E-tense operator on* M^S *which has a left adjoint* P^* *(*F^**). In this case, for all* $q \in M^S$ *and* $t \in S$*,*

$$\begin{aligned} P^*(q)(t) &= \bigvee\{q(s) \mid s \in S, sRt\} \\ F^*(q)(t) &= \bigvee\{q(s) \mid s \in S, tRs\}. \end{aligned}$$

Now we are able to establish our main result which is a generalization of the main results from [1, 3, 8].

Theorem 2.2. *Let* $\mathcal{E}$ *be an E-representable E-Jauch-Piron q-effect algebra with an order reflecting set* S *of E-states and with tense E-operators* G *and* H*. Then* $(\mathcal{E}, G, H)$ *can be embedded into the tense MV-algebra* $([0,1]^S, G^*, H^*)$ *induced by the frame* (S, ρ_G)*, where* S *is the set of all Jauch-Piron E-states from* $\mathcal{E}$ *to* $[0, 1]$ *and the relation* ρ_G *is defined by*

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in E.$$

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Some Ultimate Solutions of Old Problems on Representations of a Self-adjoint Operator, and Quantum Effects

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We give representations of any operator $x = x^* \in B(H)$ as a sum $x = \sum_{1 \leq i \leq n} Q_i P_i$ and a linear combination $x = \sum_{1 \leq j \leq n} \lambda_j R_j$ with Q_i, P_i, R_j being orthogonal projections, and with the smallest possible m and n. Then $n = 4$ ($n = 3$ is not enough). Some other new results on projections and quantum effects will also be given.

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Generalised Lüders Theorem and Sufficiency of the Fixed-point Algebra

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A generalisation of Lüders theorem is proved for an extended class of Lüders transformations. A similar result is obtained for quantum measurements represented by ideal instrument. For a given instrument $\mathcal{E}$ its observable is defined as a map e by the formula $e(\Delta) = \mathcal{E}_E^*(\mathbf{1})$, where e is a positive operator valued measure. The set of fixed points of its dual map $\mathcal{E}_\Omega^*$ is a relative commutant of algebra $\mathcal{N} = W^*\{e(\Delta)\colon \Delta \in \mathcal{F}\}$. It is shown that the fixed-point algebra of this quantum operation is sufficient for the family of states determined by the PVM defined by the instrument.

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Lukasiewicz - Groupoid - Valued States

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Let $*$ be a Lukasiewicz t-norm. Let us define a new operation δ on $[0,1]$ obtained as a symmetric difference derived from $*$. Upon calling $([0,1],\delta,1)$ the Lukasiewicz groupoid, we investigate states on Boolean algebras with values in this groupoid. We show that these states are either standard states (= finitely additive probability measures) or $\mathbb{Z}_2$-valued states. We then comment on the consequences of this result for orthomodular posets (= for quantum logics).

Orthocomplemented Posets with a Symmetric Difference

Pavel Pták *Milan Matoušek*

A few years ago the authors iniciated a systematic study of the orthocomplemented posets with a symmetric difference, where the latter is assumed to be a primitive operation. They called such orthocomplemented posets ODPs. It is easily seen that the ODPs are orthomodular (and therefore the ODPs can be viewed as enriched quantum logics). If an ODP is a lattice, one denotes it by ODL. Thus the ODLs are certain algebras situated between orthomodular lattices and Boolean algebras. The talk will review the results, both in the algebraic and measure-theoretic line, which were obtained so far. It will also be indicated a potential research in this area and formulated some open problems.

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Synaptic Algebras as Models for Quantum Mechanics

Sylvia Pulmannová *

This lecture is based on a joint work with David Foulis [3, 4, 5]. A synaptic algebra, introduced by Foulis in [1], is both a special Jordan algebra and an order-unit normed space satisfying certain natural conditions suggested by the partially ordered Jordan algebra of bounded Hermitian operators on a Hilbert space. The adjective synaptic, borrowed from biology, is meant to suggest that such an algebra coherently ties together the notions of a Jordan algebra, a spectral order-unit space, a convex effect algebra and an orthomodular lattice. All these structures appeared in the mathematical foundations of quantum mechanics.

In this lecture, we first review some basic properties of a synaptic algebra. In particular, we focus on the interaction between a synaptic algebra and its orthomodular lattice of projections. Our aim is to show that a synaptic algebra can host the probability measures and it can serve as a value algebra for quantum observables. We show that each element determines and is determined by a one-parameter family of projections – its spectral resolution. It follows that elements of a synaptic algebras correspond to quantum mechanical observables, and as an (Archimedean) order-unit space, a synaptic algebra has a rich supply of states.

It is known that soon after introducing Hilbert-space based foundations for quantum mechanics [7], von Neumann had begun to focus on what is now called a type II1 factor as the appropriate mathematical basis for quantum mechanics. Later on, it was discovered that type III factors occur naturally in relativistic quantum field theory [6]. In this lecture we study the type I/II/III decomposition theory for a synaptic algebra. Whereas the orthomodular lattice (OML) of projections in a von Neumann algebra or a JW-algebra [8] is complete, the OML of projections in a synaptic algebra need not be complete. Our aim in this part is twofold. First we study equivalence of projections based on the symmetries in the synaptic algebra, and show that a synaptic algebra with a complete lattice of projections has sufficiently many properties in common with a JW-algebra to enable Topping's proof of his version of a type I/II/III decomposition theorem. Our second aim is to show how the type-theory developed in [2] for effect algebras applies to a synaptic algebra satisfying the much weaker central orthocompleteness condition. Our main tool in the study of type decompositions is the notion of a type determining (TD) subset of the projection lattice.

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Can Many-valued Logic Help to Comprehend Quantum Phenomena?

Jarosław Pykacz[*]

We argue, following Łukasiewicz, that future non-certain events should be described with the use of many-valued, not 2-valued logic. This concerns both macroscopic and microscopic phenomena. The GHZ 'paradox' is shown to be an artifact caused by unjustified use of 2-valued logic while considering results of future measurements. Basing on our previous results [1] - [3], concerning an isomorphism between quantum logics in Birkhoff - von Neumann sense and particular families of many-valued propositional functions, we propose a particular model of many-valued logic that should be used while discussing results of future experiments on quantum objects that are not certain at present. Truth values of propositions in this many-valued logic are numbers that are in the orthodox approach interpreted as probabilities of obtaining specific results of measurements.

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Multiply Degenerate Exceptional Points and Quantum Phase Transitions

*Géza Lévai** *František Růžička*† *Miloslav Znojil*‡

Several classes of finite-dimensional models of quantum systems exhibiting spectral degeneracies called quantum catastrophes will be reviewed and described in detail [1]. Abstract Krein-space techniques as well as computer-assisted symbolic manipulations will be shown unexpectedly efficient for the purpose.

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Bose-Einstein Condensation of Collective Cooper Pairs: A Linearized Richardson Approach

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Among quantum phenomena in solids at low temperatures, the superconductivity and Bose-Einstein condensation (BEC) are representatives of arising from coherent macroscopic quantum states. The superconductivity consists in a lack of electrical resistance and total expulsion of magnetic fields inside a material. It was first explained by Bardeen, Cooper and Schrieffer (BCS) in 1957 [1] as a consequence of the formation of electron pairs, named Cooper pairs, due to an attractive potential between them. The pairing Hamiltonian proposed by BCS has been also applied in fields like nuclear physics, to find the first nuclear energies of many atoms [2]. On the other hand, the BEC has been found in a wide range of systems, such as ^{4}He superfluid, atomic gases of rubidium, sodium and lithium, as well as exciton-polaritons in microcavities [3]. However, it is well known that the Cooper pairs are not true bosons as discussed in the original paper of BCS [1]. Recently, we have introduced the concept of collective Cooper pairs (CCP) [4] through a unitary transformation of Cooper pairs [5]. The CCP accomplish bosonic commutation relations at the dilute limit, being able to accumulate many of them at a single quantum state, in contrast to the standard Cooper pairs. Moreover, we have proven that in the narrow band limit, the BEC critical temperature coincides to the BCS one (T_c) and the superconducting gap [$\Delta(T)$] is proportional to the number of ground state pairs, founding that $2\Delta(0) = 4k_BT_c$, instead of $2\Delta(0) = 3.5k_BT_c$ in the BCS theory for the wide band case [6].

In this work, we find analytically the first order solutions of the BCS Hamiltonian with degenerate single electron energies, whose results are compared to the Richardson exact solutions calculated numerically showing good agreement in the weak interaction limit. Based on this first-order solution, it is determined the number of ground state pairs as a function of temperature. The BEC temperature occurs when the number of ground-state pairs becomes macroscopic, and it is compared to the BCS critical temperature providing a possible BEC explanation of the superconductivity at the weak coupling limit, which traditionally belongs to the BCS side of the BCS-BEC crossover picture.

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Boundariness and Minimum-error Discrimination

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We introduce the concept of boundariness capturing the most efficient way of expressing a given element of a convex set as a probability mixture of its boundary elements. In other words, this number measures (without the need of any explicit topology) how far the given element is from the boundary. It is shown that one of the elements from the boundary can be always chosen to be an extremal element. We focus on evaluation of this quantity for quantum sets of states, channels and observables. We show that boundariness is intimately related to (semi)norms that provide an operational interpretation of this quantity. In particular, the minimum error probability for discrimination of a pair of quantum devices is lower bounded by the boundariness of each of them. We prove that for states and observables this bound is saturated and conjectured this feature for channels. The boundariness is zero for infinite-dimensional quantum objects as in this case all the elements are boundary elements. The complete paper linked to this submission is arXiv:1401.7460 [quant-ph].

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Compositional and Holistic Conjunctions in Quantum Computation

Hector Freytes [*] *Roberto Giuntini* [†] *Giuseppe Sergioli* [‡]

In the general quantum computational framework [1] pieces of information are interpreted as density operators ρ (of convenient Hilbert spaces), which are processed by reversible quantum logical gates. Such gates are the natural extension of unitary operators to density operators. One can associate to any density operator ρ a probality value that represents the probability that the information encoded by ρ is true [2]. One of the most important quantum logical gate is the *Toffoli gate* that allows one to define a *compositional* and a *holistic* conjunction (AND_{Com}, AND_{Hol}, respectively), which are both reversible logical gates [2]. The different behavior of AND_{Com} and of AND_{Hol} only depends on the input. In the compositional case, inputs are always *factorized* states of bipartite systems, while the more "liberal" holistic conjunction can also be applied to non-factorized states, in particular to *entangled* states. We study some general algebraic and probabilistic properties of AND_{Com} and of AND_{Hol}. In particular, we investigate the behavior of such conjunctions in the case where inputs are represented by *Werner-states* and by *isotropic states*.

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On Monoidal (Co)Nuclei in Monoidal Topology

Sergejs Solovjovs[*]

In 1970, M. Barr [1] represented the category **Top** of topological spaces and continuous maps as the category of lax algebras and lax homomorphisms for the canonical extension of the ultrafilter monad on the category **Set** of sets and maps to the category **Rel** of sets and relations. In a series of papers, M. M. Clementino, D. Hofmann, and W. Tholen [2, 3, 4, 5, 6, 7] generalized this approach to an arbitrary monad $\mathbb{T}$ on **Set** and the category V-**Rel** of sets and V-relations, where V is an arbitrary unital quantale. They showed that many of the existing categories of topological structures (e.g., preordered sets, premetric spaces of F. W. Lawvere [10], approach spaces of R. Lowen [11], probabilistic metric spaces of B. Schweizer and A. Sklar [13]) can be represented as the categories $(\mathbb{T}, V)$-**Cat** of $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors with respect to a suitable monad $\mathbb{T}$ and a quantale V, initiating the so-called *monoidal topology* [8].

To continue, we recall from [8] that a *lax homomorphism of unital quantales* $(V_1, \otimes, k_1) \xrightarrow{\varphi} (V_2, \otimes, k_2)$ is a map $V_1 \xrightarrow{\varphi} V_2$, which satisfies the following three conditions:

(1) $\bigvee \varphi(S) \leqslant \varphi(\bigvee S)$ for every $S \subseteq V_1$;

(2) $\varphi(a) \otimes \varphi(b) \leqslant \varphi(a \otimes b)$ for every $a, b \in V_1$;

(3) $k_2 \leqslant \varphi(k_1)$.

It is easy to see that item (1) is equivalent to φ being order-preserving. Every lax homomorphism of unital quantales, which satisfies an additional property of compatibility with the lax extensions of the respective monad $\mathbb{T}$, gives rise to the so-called *change-of-base functor* $(\mathbb{T}, V_1)$-**Cat** $\xrightarrow{B_\varphi}$ $(\mathbb{T}, V_2)$-**Cat** [8]. This technique provides the following pairs of functors:

(1) **Ord** → **Set** → **Ord** (preordered sets and sets);

(2) **Met** → **Ord** → **Met** (metric spaces);

(3) **ProbMet** → **Met** → **ProbMet** (probabilistic metric spaces);

(4) **App** → **Top** → **App** (approach and topological spaces);

(5) $(\mathbb{U}, V)$-**Cat** → **Top** → $(\mathbb{U}, V)$-**Cat**, where $\mathbb{U}$ is the ultrafilter monad, and V is a completely distributive quantale with some additional properties.

In the next step, we recall (from, e.g., [9, 12]) that a *quantic (co)nucleus* on a quantale $(V, \otimes)$ is a (co)closure operator $V \xrightarrow{h} V$ such that $h(a) \otimes h(b) \leqslant h(a \otimes b)$ for every $a, b \in V$. Quantic (co)nuclei provide a convenient technique for constructing quotients (subquantales) of quantales. In particular, one can show the following well-known results (notice that given a set X, $\mathcal{P}(X)$ denotes the powerset of X).

Theorem 1. *Every quantic (co)nucleus $V \xrightarrow{h} V$ provides a quantale $V_h = \{u \in V \mid h(u) = u\}$ and a quantale homomorphism $V \xrightarrow{h} V_h$ ($V_h \xrightarrow{h} V$). Every surjective (injective) quantale homomorphism can be represented in this form.*

Theorem 2 (Quantale representation theorem)**.** *If V is a (unital) quantale, then there exists a semigroup (monoid) S and a quantic nucleus j on the free quantale $\mathcal{P}(S)$ over S such that $V \cong \mathcal{P}(S)_j$.*

Every quantic nucleus is a lax homomorphism of quantales. The same holds for *unital* quantic conuclei (preserving the quantale unit). A (unital) quantic (co)nucleus h, compatible with the lax extension of the monad $\mathbb{T}$, provides the change-of-base functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{B_h}$ $(\mathbb{T}, V)$-**Cat**. In this talk, we show a monoidal analogue of Theorem 1, in which the quantale V is replaced with the category $(\mathbb{T}, V)$-**Cat**, calling a compatible quantic (co)nucleus – *monoidal (co)nucleus*, and its respective quotient (subobject) – *monoidal quotient (subobject).* We arrive thus at a convenient technique for producing quotients and

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subcategories of the categories $(\mathbb{T}, V)$-**Cat**, thereby obtaining not only the above five examples, but also new ones, which are related to the categories of H-labeled graphs and multi-ordered sets of, e.g., [5, 14]. Based in the developed technique of monoidal nuclei, we provide a monoidal analogue of Theorem 2.

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Quantum Structure in Cognitive Processes and the Logical Foundations of Human Reasoning

*Diederik Aerts** *Sandro Sozzo*† *Tomas Veloz*‡

An increasing evidence in concept theory (conceptual vagueness, Guppy effect, overextension and underextension in concept combinations and borderline contradictions) reveals that classical (fuzzy set) logic is not capable to model the way concept combine in human thought [1, 2, 3, 4]. Similar deviations from classical logical structures in human reasoning have been observed in other domains of cognitive science, namely, decision making ('prisoner's dilemma', 'disjunction effect', 'conjunction fallacy') and behavioural economics ('Allais, Ellsberg, Machina paradoxes') [5, 6, 7, 8, 9]. We have recently worked out a theoretic modeling for conceptual combinations which employs the mathematical formalism of quantum theory. Our quantum modeling approach successfully describes a large amount of experimental data collected on conjunctions and disjunctions of two concepts, and puts forward a completely novel possible solution to the 'combination problem' [10, 11, 12, 13].

From our investigations follows that the presence of quantum structure in the mechanisms and dynamics of human concepts is systematic and paradigmatic, and that it is ubiquitous in cognitive, decision and reasoning processes. Moreover, we have meanwhile shown that these processes can be explained by considering human thought as a superposition of two layers, one consisting of 'logical thought' and the other one governed by 'conceptual thought'. Furthermore, these two layers are modeled by two different sectors of a Fock space within our quantum theoretic framework, where the conceptual thought dominates over the logical thought. Our investigation contrasts with old philosophical beliefs, that is, we argue that what has been often typically identified as a fallacy, an effect, a contradiction, or a deviation, is actually determined by the dominant dynamics within human thought, that one that takes place in this conceptual layer in our theory, and its nature is emergence. On the contrary, what has been historically considered as a default, namely logical reasoning, which takes place in the logical layer in our theory, is secondary within human thought, in general dominated by conceptual thought.

Starting from the above analysis, we inquire in the present work into the logical foundations of human reasoning as modeled in Fock space, i.e. we investigate which algebraic structures underlie the two layers of human thinking. We firstly prove that the logical thought cannot be cast in a classical (Boolean) framework. Then, we prove that the conceptual layer structure, and hence the emergent thought described in this layer, introduces an algebraic structure which is not similar to any type of logical structure imagined, e.g. it are not connectives that play a role, but superposition, hence the emergence of 'one' out of 'two'. In particular, the last result is, a priori, unexpected, even in our quantum mechanical modeling, and suggests that the assumption that human reasoning as a whole admits a definite logical structure – be it classical or quantum – is problematical.

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The Abstract-Concrete Dimension of the Conceptual Realm and Heisenberg's Uncertanty Principle

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The identification of quantum structures outside the micro world and the use of the mathematical formalisms of quantum theory to model situations in cognitive and social science has become a successful domain of research [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Applications range from concept theory [1, 2, 3, 8, 10, 11, 15] to decision making [4, 7, 13], natural language processing [17], information retrieval [18] and economics [5, 12, 19]. The motivation to use quantum models for conceptual entities is not only because the quantum formalism gives rise to better models for existing cognitive data. Also new experiments have been performed collecting data and demonstrating explicitly the presence of 'entanglement' in concept combination [16].

Despite the outstanding success of this quantum theoretic modeling scheme for conceptual entities, it is an ongoing effort of research to investigate what the depth is of this analogy, also with respect to developing a new interpretation for quantum theory [20, 21, 22]. In earlier work the common ground between both, quantum particles and conceptual entities, was identified with respect to 'contextuality', states of conceptual entities are indeed influenced by context in a very similar way as states of quantum particles are [1, 2, 10, 11]. Like the occurrence of entanglement [6, 16], also the presence of 'interference' was identified in combined concepts as an effect due to the role played by the combination as a newly emergent concept − modeled by a 'superposition' state −, and interference patterns for different combinations of concepts were elaborated [8, 10, 11].

With contextuality, interference, entanglement, and superposition, each of them very typical quantum aspects being identified in the dynamics of concepts and their combinations, we focus in the present work on one of the most characteristic aspect of quantum theory. We identify the 'Heisenberg uncertainty principle' [23] within the conceptual realm and hence provide an additional element deepening the analogy between concepts and quantum entities. More specifically, we show that the abstract-concrete dimension for conceptual entities (either in a decision, categorization, semantic-representation, or any other context) within the quantum modeling scheme we developed for concepts corresponds to the delocalisationlocalisation dimension of quantum entities. More concretely, a very abstract state of a concept ('thing', 'animal', 'furniture', etc.) corresponds to a very delocalised state of a quantum entity (sharp momentum, and spread out wave function), while a very concrete state of a concept ('that cat Felix', 'the chair he sits on', 'your finger') correspond to a very localised state of a quantum entity (spread out momentum, and sharp-delta-like wave function). An aspect of this Heisenberg uncertainty structure within concepts can be identified by observing that concepts indicating physical object are called 'instantiations' in psychology ('they instantiate, i.e. take time-place, or localise). In our approach such instantiations are the most localised states. An interesting aspect of our identification of the Heisenberg uncertainty in the conceptual realm, is that its ontological nature can be understood right away. Indeed, abstract and concrete cannot realise jointly without being confronted with a tradeoff, i.e. more abstract implies less concrete, and vice versa, if the ontology is conceptual. Additionally to the theoretical analysis of the Heisenberg uncertainty in concepts, we explore the extent at which this view is compatible with traditional approaches to concept representation, particularly with categorization graphs [24], and semantic (word) spaces [25].

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The Essence of Nonclassicality: More Effect than Cause

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We consider a framework for general correlation theories in which multi-partite states are represented by correlations conditioned on measurements by two or more players, who chose between two or more measurement settings with multiple outcomes. There are a number of properties that are absent in classical probabilities, but are common to all general correlations that satisfy two conditions: no-signaling and nonlocality [1] . These nonclassical properties include intrinsic randomness, Heisenberg uncertainty, monogamy of nonlocal correlations, impossibility of perfectly cloning an unknown state, privacy of correlations which can serve as basis of establishing a cryptographic key, bounds on the shareability of some states, etc.

Here we point out that these nonclassical properties persist (possibly to a diminished extent) even when the above two assumptions are relaxed [2]. We quantify the degree of this relaxation by the *signal deficit*, defined as the excess of the communication cost (to establish the correlation) over the signal within the correlation. We can thus reduce nonclassicality of properties to a single assumption: non-vanishing signal deficit. This relaxed scenario is relevant to temporal quantum correlations [3], and also for clarifying the role of no-signaling in linking the local and global features of quantum mechanics. Intuitively, a signal in the correlation can be regarded as the efficient cause that propagates from spatial point A to produce an effect at point B. Nonclassicality thus betokens effects for which there is not enough cause available within the theory.

Our result suggests that nonclassicality so defined is analogous to incompleteness from the perspective of mathematical logic. Gödel's celebrated incompleteness theorem [4] proves that in a formal axiomatic system sufficiently complex to encompass arithmetic, there will be unprovable propositions that can nevertheless ascertained to be true by meta-theoretic reasoning. Incompleteness essentially arises because there are uncountably many true propositions, but only countably infinite proofs (or explanations). Given the violation of Bell-type inequalities, the signal in the correlation can be interpreted as the explanation or proof for the nonlocal effect within the base theory, whereas the communication cost as the meta-theoretic explanation. Nonclassicality, or a positive-definite signal deficit, is thus analogous to metamathematical incompleteness.

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A Process Model of Non-Relativistic Quantum Mechanics

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A process model of quantum mechanics utilizes a combinatorial game to generate a discrete and finite causal space $\mathcal{I}$, upon which can be defined a self-consistent (non-relativistic) quantum mechanics. An emergent space-time $\mathcal{M}$ and continuous wave function Ψ arise through a uniform interpolation process. Standard non-relativistic quantum mechanics (at least for integer spin particles) emerges on $\mathcal{M}$ under the idealized limit of infinite information (the causal space grows to infinity) and infinitesimal scale (the separation between points goes to zero). This process model is quasi-local, discontinuous, and quasi-non-contextual. The wave function reflects the strength of the process. The model gives rise to an emergent non-Kolmogorov probability structure. The bridge between process and wave function is through the process covering map (PCM). If Π is the process space and Θ the set of causal spaces, then the PCM is a set valued-map Σ from $\Pi \times \Theta$ to the power set on $\mathcal{H}(\mathcal{M})$, the Hilbert space on $\mathcal{M}$. For fixed causal set $\mathcal{I}$, the restriction $\Sigma(-, \mathcal{I})$ is a set valued algebraic homomorphism. Using the PCM, each process can be thought of as a generalized operator on $\mathcal{H}(\mathcal{M})$. The PCM reveals that the standard wave function formalism lacks important dynamical information related to the generation of the causal space. Reformulating several classical conundrums such as wave particle duality, Schrodinger cat, hidden variable results, the model offers potential resolutions to all, while retaining a high degree of locality and contextuality at the local level, yet nonlocality and contextuality at the emergent level. The model remains computationally powerful.

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Space and Time in a Quantized World

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Rather than consider space-time as an *a priori* arena in which events take place, it is a construction of our mind making possible a particular kind of ordering of events. As quantum entanglement is a property of states independent of classical distances, the notion of space and time has to be revised to represent the holistic interconnectedness of quanta.

Physical space and time appear to be *ordering events* by quantifying top-bottom, left-right, front-back, as well as before-after. In that function, space-time relates to actual physical events, such as clicks in particle detectors. Without such events, space-time would be metaphysical at best, because there would be no operational basis that gave meaning to the aforementioned categories. Intrinsic space-time is tied to, or rather based upon, physical events; and is bound to operational means available to observers "located inside" the physical system.

In acknowledging this empirical foundation, Einstein's centennial paper on space-time, and to a certain extent Poincare's thoughts, introduced *conventions and operational algorithmic procedures* that allow the generation of space-time frames by relying on intrinsically feasible methods and techniques alone. This renders a space-time (in terms of clocks, scales and conventions for the definition of space-time frames, as well as their transformations) which is *means relative* with respect to physical devices (such as clocks and scales), as well as to procedures and conventions (such as for *defining* simultaneity employing round-trip time, which is nowadays even used by *Cristian's Algorithm* for computer networks). These unanimously executable measurements and "algorithmic" physical procedures need not rely upon any kind of absolute metaphysical knowledge (such as "absolute space or time").

This approach is characterized by constructing operational, intrinsic space-time frames based on physical events alone; rather then by staging physical events in a Kantian *a priori* "space-time theatre." One step in this direction is, for instance, the determination of the *dimensionality* of space and of space-time from empirical evidence.

We also speculate about various forms of reprogramming, or reconfiguring, the propagation of information for multipartite statistics and in quantum field theory [1].

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Distributivity and Associativity in Effect Algebras

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We present an overview of distributivity-like properties of partial operations in effect algebras with respect to suprema and infima and vice versa generalizing several previous results. Moreover, the large associativity of the partial sum in effect algebras is studied.

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Completeness of Gelfand-Neumark-Segal Inner Product Space of Positive Maps on Jordan and C^*-algebras

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It is more than 70 years after ground breaking result of Gelfand and Neumark [2] saying that any abstract C^*-algebra can be represented by ring of operators acting on a Hilbert space. Since that this theorem has become everyday tool for every operator algebraist. From mathematical point of view Gelfand-Neumark-Segal (in short GNS) construction provides a bridge between states on abstract C^*-algebras and Hilbert spaces. This principle has also deep physical meaning of placing physical systems into various Hilbert spaces with different special properties, which is used often in investigating ground states in quantum statistical mechanics, quantum field theory, and elsewhere.

Substantial ingredient of GNS representation is a construction of an inner product space $(A_\varphi, \langle\cdot,\cdot\rangle_\varphi)$, where φ is a state on a C^*-algebra A and $\langle a,b\rangle_\varphi = \varphi(b^*a)$ (we take out zero vectors). Then one passes to completion $H_\varphi = \overline{A_\varphi}$. A natural question arises in this connection: When is the space A_φ complete? Positive answer makes the passage to completion in GNS procedure redundant. The goal of this paper is to summarize recent results concerning this problem. It was proved first by Halpern that A_φ is complete if and only if φ is a convex combination of pure states [3]. In [1] we proved that the same holds if we consider symmetric product $(a,b) \to \varphi(b^*a) + \varphi(a^*b)$. It requires subtle technique and delicate use of Kadison Transitivity Theorem. The main interest of this paper is to consider states on Jordan algebras that are far reaching generalizations of C^*-algebras. For a state ϱ on a JB (Jordan Banach) algebra A we study an inner product space A_ϱ resulting from semi inner product $(a,b) \to \varrho(a \circ b)$. We generalize Halpern's result by showing that if A_ϱ is complete, then ϱ is a convex combination of pure states. The proof requires deep structure theory of Jordan algebras and leads to new phenomena. For example, we exhibit new class of states, called spin factor states, that have infinite dimensional support and still provide complete GNS space. This cannot happen in the framework of C^*-algebras. In general, unlike C^*-algebras we cannot use irreducible representations and have to replace them by second dual Banach space arguments which allow to reduce the problem to normal states. On the other hand, we describe in detail spacial GNS representations of states that are of Type I_n, $n \geq 4$, in terms of real, complex and quaternion structures on Hilbert spaces which they induce. Finally, we study more general situation resulting by replacing states by completely positive maps in Stinespring's extension of the GNS construction.

We believe that our investigation shows interaction between various layers of theory of operator algebras: convexity properties of states, Banach space properties of quotients by kernels of states, structure of irreducible representations, and properties of extensions of states to enveloping JBW algebras.

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Measures on Projections with Probability Kernel

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An operator $\mathcal{J}$ on a complex Hilbert space H with the Hilbert product $(\cdot, \cdot)$ is said to be *conjugation* operator if: $\mathcal{J}^2 = I$, $(\mathcal{J}x, \mathcal{J}y) = (y, x)$ and $\mathcal{J}(\lambda x + \beta y) = \overline{\lambda}\mathcal{J}x + \overline{\beta}\mathcal{J}y$, $\forall x$, $y \in H$, $\forall \lambda$, $\beta \in \mathcal{C}$. The set $H_{\Re} := \{x \in H : \ \mathcal{J}x = x\}$ is the real Hilbert space, $\dim H = \dim H_{\Re}$. Let $\mathcal{P} := \{p \in B(H) : \ p = p^2, \ p = \mathcal{J}p^*\mathcal{J}\}$. Note that $p \in \mathcal{P} \Leftrightarrow p^* \in \mathcal{P}$. $\mathcal{P}$ is an analogue of the lattice $B^{or}(H)$ of all orthogonal projections on H. With respect to the *ordering*: $p \le q \Leftrightarrow p = qp$ and *orthocomplementation*: $p \to p^{\perp} := I - p$ the set $\mathcal{P}$ is a *quantum logic*. A projection $p \in \mathcal{P}$ is one-dimensional $\Leftrightarrow p = p_x := (\cdot, \mathcal{J}x)x$, where $(x, \mathcal{J}x) = 1$. A projection p_x is an orthogonal projection $(p_x = p_x^*) \Leftrightarrow x \in H_{\Re}$. Let $\Pi := \mathcal{P} \cap B^{or}(H)$. Note that $p \in \Pi \Leftrightarrow \mathcal{J}p\mathcal{J} = p$ and $p \in \mathcal{P}$. The set Π is isomorphic to the set $B^{or}(H_{\Re})$. For any projection $e \in \Pi$, $0 < e < I$ the set

$$H_{\Re}^e := \mathrm{lin}_R\{eH_{\Re} + ie^{\perp}H_{\Re}\}$$

is a real Hilbert space. Let

$$\mathcal{P}_e := \{p \in P : \ pH_{\Re}^e \subseteq H_{\Re}^e\}.$$

By $\mathcal{J}H_{\Re}^e = H_{\Re}^e$, $p \in \mathcal{P}_e \Leftrightarrow p^* \in \mathcal{P}_e$. The set $\mathcal{P}_e$ is said to be *hyperbolic* logic [1]. It is seams to be clear that $\mathcal{P} = \cup_{e \in \Pi}\mathcal{P}_e$. This underlines the importance of logic $\mathcal{P}$.

A function $\mu : \mathcal{P} \to \mathcal{R}$ is said to be a *measure* if $\mu(p) = \sum \mu(p_i)$ for any decomposition $p = \sum p_i$.

The nonnegative measure μ is said to be *probability* measure if $\mu(I) = 1$; a measure is *bounded* if there exists a number c such that $|\mu(p)| \le c\|p\|$, $\forall p \in \mathcal{P}$; a measure is said to bo *self adjoint*, if $\mu(p) = \mu(p^*)$, $\forall p \in \mathcal{P}$. The reduction of a measure μ on Π is said to be *kernel* of a measure. It is known that: 1) $\mu : \mathcal{P} \to [0, 1]$ is *probability measure* $\Leftrightarrow$ $n := dimH < \infty$ *and* $\mu(\cdot) = \frac{1}{n}tr(\cdot)$. 2) *For any probability measure* ν *on* $B^{or}(\mathcal{H})$, $dim\mathcal{H} \ge 3$ *there exists unique nonnegative trace-class operator* A *such that* $\nu(p) = \mathrm{tr}(Ap)$. The main result

Theorem 1. *Let* $dimH = \infty$ *and let* $\mu : \mathcal{P} \to \mathcal{R}$ *be a self adjoint measure with probability kernel. Then* $\mu(p) = \Re\mathrm{tr}(Ap)$, $\forall p \in \mathcal{P}$. *Here* A *is nonnegative* $\mathcal{J}$*-real trace-class operator generated by the restriction of* μ *on to* Π.

Let $\dim H < \infty$. There exists non bounded measure: 1) with zero kernel, 2) with probability kernel.

Theorem 2 is interesting to compare with a similar statement for the hyperbolic case.

Theorem 2. *Let* $3 \le \dim H < \infty$ *and let* $\mu : \mathcal{P} \to \mathcal{R}$ *be a self adjoint measure. The following conditions are equivalent:*

1) *there exists an operator* T *such that* $\mu(p) = \Re\mathrm{tr}(Tp)$, $\forall p \in P$.

2) *a measure* μ *is bounded.*

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1D Tight-Binding Models Render Quantum First Passage Time *Speakable*

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The calculation of *First Passage Time* (moreover, even its probability density) has been generally viewed as an ill-posed problem in the domain of quantum mechanics. The reasons can be summarized as the violation of *Kolmogorov sum rule for probabilities* in quantum mechanics: the probability for entering and non-entering of Feynman paths into a given region of space-time do not in general add up to 1, much owing to the interference between alternative paths [See, e.g., [1])].

In the present work, it is pointed out that a special case exists (within quantum framework), in which, by design, there is one and only one available path (*door-way*) which mediates the passage and therefore no interference of alternative paths (also applicable for the results of [2]). Further, it is identified that a popular family of quantum systems – namely the 1d tight binding Hamiltonian systems – falls under this special category. For these model quantum systems, the first passage time distributions are obtained analytically by suitably applying a method originally devised for classical stochastic systems (by Schroedinger in 1915)[See, e.g., [3])]. This result is interesting especially given the fact that the tight binding models are extensively used in describing everyday phenomena in condense matter physics.

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Dependence of Decoherence and Quantum Measurement Processes on the Number of Environmental Dynamical Variables

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Decoherence is one of the main obstacles placed in the way of the correct functioning of quantum devices. It is an ubiquitous phenomenon, due to the unavoidable interaction between a quantum principal system and its environment, which becomes particularly disruptive when quantum properties are to be exploited and controlled. Despite being an ordinary effect, decoherence is not easily describable in a general framework, as it depends on several details of the physical setup. However, it has at least two key features: *i)* it is a dynamical process and *ii)* it is due to the interaction with an environment, which set the topic into the domain of the open quantum systems (OQS) dynamics. In this work, we make use of a recently proposed method[1] for studying OQS, in order to analyze the dynamical evolution of a generic quantum system subject to decoherence. From such treatment, an analytical expression for a consistent measure of the coherence time emerges, and formally shows how, and why, decoherence depends on the number of dynamical variables of the environment. Based on this result we propose a strategy for extending the coherence time which also enlighten the relation between decoherence and the quantum measurement process.

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Representation of Concrete Logics and Concrete Generalized Orthomodular Posets

*Sylvia Pulmannová**　*Zdenka Riečanová*†　*Elena Vinceková*‡

Recently, it has been shown in [2] that every effect algebra with an order determining set of states can be embedded into an effect algebra of quantum effects - if E is an effect algebra and $\mathcal{S}$ is an order determining set of states on E, then there is an injective effect algebra morphism from E into the effect algebra of multiplication operators between the zero and identity operator on the complex Hilbert space $\ell_2(\mathcal{S})$.

In [1], the question of representability of a generalized effect algebra in a generalized effect algebra of operators densely defined on a complex Hilbert space was studied. It was shown that a generalized effect algebra is representable in the operator generalized effect algebra $\mathcal{G}_D(\mathcal{H})$ if and only if it has an order determining set of generalized states.

In this talk, we examine the conditions, under which an effect algebra, or a generalized effect algebra, can be represented in the projection lattice of a Hilbert space.

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States on Symmetric Logics: Conditional Probability and Independence

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We continue the first author's study begun in [1] and study the notions of conditional probabilities, independence and ε-independence for states on symmetric logics [2]. We prove that a non-atomic state on the logic with the Lyapunov's property is determined by its specification of independent events. We present the examples of 1) Δ-subadditive but is not subadditive and 2) two-valued non Δ-subadditive states on symmetric logic. We investigate the independence relation transitivity for a Δ-subadditive state.

We also study continuity properties of conditional probabilities and ε-independence relation with respect to natural pseudometric for Δ-subadditive state. We prove that in this pseudometric space any "triangle" possesses a "perimeter" less than or equal to 2.

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On the Form of the Optimal Measurement for the Probability of Detection

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We consider the problem of maximizing the probability of detection for an infinite number of mixed states. We show that for lineary independent states there exists a uniquely simple optimal measurement. We also obtain a special form of the optimal measurement without lineary independent assumption.

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States on Effect Algebras, Their Products and Horizontal Sums

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We introduce basic definitions of effect algebras, horizontal sums of effect algebras and products of effect algebras. We define various types of states (two-valued, extremal, Jauch–Piron) on effect algebras and various types of sets of such states (empty, unital, order determining, strongly order determining) and we furthermore study those states and properties of their sets. Our results consist of theorems on connection between sets of states on effect algebras and sets of states on products and horizontal sums of effect algebras and various examples.

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Ad hoc Physical Hilbert Spaces in Quantum Mechanics

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The overall principles of what is mostly known under the nickname of PT-symmetric quantum mechanics will be reviewed and illustrated via a few examples. In particular, models based on an elementary local interaction V(x) will be discussed as motivated, i.a., by the emergent possibility of an efficient regularization of an otherwise inacceptable presence of a strongly singular repulsive core in the origin. The emphasis will be put on the constructive aspects of the models. Besides the overall outline of the formalism[1] we shall show how the low-lying energies of bound states may be found in closed form in certain dynamical regimes[2]. Finally, once these energies are found real we explain that, paradoxically, in spite of a manifest non-Hermiticity of the Hamiltonian the time-evolution of the system becomes unitary in a properly amended physical Hilbert space [3].

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Ward Identities from Recursion Formulas for Correlation Functions in Conformal Field Theory

Alexander Zuevsky[*]

Algebraic methods in computation of the partition and n-point functions in Conformal Field Theory/Vertex Opearator Super Algebras proved their effectiveness. The Zhu reduction formula allowing to express $n+1$-point correlation functions as finite sums of n-point functions constitute the main algebraic tool for calculations. In this paper we give a conformal block formulation for the Zhu recursion procedure for vertex operator algebras considered on genus zero and higher genus Riemann surfaces. First we recall the notion of a vertex operator algebra, introduce correlation functions on Riemann surfaces, and review the Zhu recursion formulas for correlation functions. Then an appropriate definition for conformal blocks related to chiral vertex operator algebra correlation functions is constructed. We then define the Zhu reduction operators acting on a tensor product of VOA modules. By means of these operators we show that the Zhu reduction procedure generate explicit forms of Ward identities for vertex operator algebras. Finally we supply examples of explicit genus two Ward identities for the Heisenberg and free fermionic vertex operator algebras.

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