SIMULATION OF METAL-FORMING PROCESSES BY THE FINITE ELEMENT METHOD

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1 Introduction

The quantitative analysis of metal-forming processes is difficult because non-linearities of different kinds are present in the problem, such as, for example, the elastic-plastic flow in a finite deformation context and the progressive contact, with or without friction, between the workpiece and the dies, not to mention the problem arising from the heat generated by the dissipation of the plastic work.

In this paper we examine with the problem of the contact, without friction, between a deformable body, simulating the workpiece, and several moving rigid surfaces, representing the dies. We describe an algorithm appropriate to solve this problem and which can be used in a finite element scheme.

The algorithm has been implemented into the in house developed code NOSA, which can deal with elastoplasticity problems in the finite deformation range, and, at the end of the paper, the results obtained using NOSA in a bulge test case are compared with the experimental data and with the results from widely known general-purpose codes like ABAQUS and MARC.

The problem of the contact between an elastic body and a rigid wall was stated, early on, in a differential form, by Signorini in 1933 [1], and was numerically approached in the Seventies using the Lagrange multiplier method or the penalty-function scheme. The analysis of the contact in presence of finite deformations was carried out first by Oden, Cheng and Kikuchi [2, 3, 4, 5] in the middle of the Eighties, and, to get an idea of the state of the art in this area, the proceedings of NUMIFORM '89 and of the symposium recently held in Zürich can be consulted [6, 7].

The algorithm described in this paper differs from the most common methods used by commercial codes, namely:

1. the incremental equilibrium equation is calculated on the current configuration of the deformable body, thus directly giving the Cauchy stress tensor [8];

2. the elastic-plastic constitutive response of the material is completely described by an ordinary differential equation system where the yield condition is stated in the space of deformation, so no prediction of

the increase in stress is needed. Moreover, the system is numerically integrated, taking those at the beginning of the increment as initial conditions: in this way the artificial unloading, which can be produced by stress redistribution in a high stress gradient zone or by the motion of the node in establishing the right contact conditions, is avoided;

3. the contact algorithm is based on the direct application of boundary conditions, *i.e.* an automatic procedure has been set up that translates, at each iteration, the contact conditions between the workpiece and rigid dies into the appropriate boundary conditions applied to the boundary nodes.
This method avoids the drawbacks of the Lagrange multiplier approach, such as the increase in the number of degrees of freedom and the presence of zeroes on the diagonal of the evolution matrix, and the inaccurate representation of the contact, characteristic of the penalty-function scheme.
The algorithm described is similar to the one outlined by Nagtegaal and Rebelo [9], and implemented into the MARC program, but it is more accurate in the representation of rigid dies and is simpler because, due to the particulaties of the above mentioned stress increment calculation method, it avoids the increment being necessarily broken up, when a boundary node comes into contact with a die.

2 The equation of the motion

We describe the motion of a deformable body subjected to the action of external forces and kinematic constraints and deformed by contact with other rigid bodies; the friction forces are not considered in the present article.

Referring to [10, 8], let $\mathcal{B}_0$ be the initial configuration of the deformable body, $\mathcal{B}_\tau$ be the configuration at time $\tau \in [0, \overline{\tau}]$ and $\partial\mathcal{B}_\tau$ be its boundary, which is supposed to be regular. Denoting by p the points of $\mathcal{B}_0$ and by x those of $\mathcal{B}_\tau$, the motion of the body is given by class C^3 mapping

$$\chi(p,\tau) \,:\, \mathcal{B}_0 \times [0,\overline{\tau}] \rightarrow \mathcal{B}_\tau \,:\quad x = \chi(p,\tau), \tag{2.1}$$

and the displacement u from the initial configuration is defined by

$$u(x,\tau) = x - p = \chi(p,\tau) - p.$$

At every point of $\partial\mathcal{B}_\tau$ let three orthogonal unit vectors e_1, e_2, e_3 be chosen; for each $i = 1,2,3$, three disjoint subsets $\mathcal{S}_u^i(\tau)$, $\mathcal{S}_{f_i}^i(\tau)$, $\mathcal{S}_c(\tau)$ of $\partial\mathcal{B}_\tau$ can be singled out in such a way that $\mathcal{S}_u^i(\tau)$ and $\mathcal{S}_{f_i}^i(\tau)$ are the subsets of boundary $\partial\mathcal{B}_\tau$ where the components along the direction e_i of displacement and force, respectively, are prescribed and $\mathcal{S}_c(\tau)$ is the set of points in contact with the rigid bodies. Moreover, a particular reference triad is chosen on $\mathcal{S}_c(\tau)$ in such a way that e_1 is the outward normal unit vector n on $\mathcal{S}_c(\tau)$.

So on $\partial\mathcal{B}_\tau$ we have

$$\overline{\mathcal{S}}^i_u(\tau) \cup \overline{\mathcal{S}}^i_f(\tau) \cup \overline{\mathcal{S}}_c(\tau) = \partial\mathcal{B}_\tau, \qquad i = 1,2,3\,, \tag{2.2}$$

$$e_i(x,\tau)\cdot u(x,\tau) = \delta_i(x,\tau), \qquad x \in \mathcal{S}^i_u(\tau),\ i = 1,2,3\,, \tag{2.3}$$

$$e_i(x,\tau)\cdot T(x,\tau)n(x,\tau) = \sigma_i(x,\tau), \qquad x \in \mathcal{S}^i_f(\tau),\ i = 1,2,3\,, \tag{2.4}$$

$$n(x,\tau)\cdot(\dot u(x,\tau) - \dot u_c(x,\tau)) = 0, \qquad x \in \mathcal{S}_c(\tau)\,, \tag{2.5}$$

$$n(x,\tau)\cdot T(x,\tau)n(x,\tau) = \pi_c(x,\tau) \le 0, \qquad x \in \mathcal{S}_c(\tau)\,, \tag{2.6}$$

where δ_i is the i-th assigned component of the displacement with respect to the initial configuration $\mathcal{B}_0$ of the body, T is the Cauchy stress tensor, σ_i is the i-th assigned component of the external forces per unit surface, $\dot u_c$ is the assigned velocity of the rigid bodies and π_c is the contact pressure.

The equations (2.3) to (2.6) are the boundary conditions for the equilibrium equation

$$\operatorname{div} T(x,\tau) + f_b(x,\tau) = 0, \qquad x \in \mathcal{B}_\tau\,, \tag{2.7}$$

where $f_b(x,\tau)$ is the body force.

Let $w(x,\tau)$ be an arbitrary velocity field on $\mathcal{B}_\tau$ such that

$$e_i(x,\tau)\cdot w(x,\tau) = 0, \qquad x \in \mathcal{S}^i_u(\tau),\ i = 1,2,3\,, \tag{2.8}$$

$$n(x,\tau)\cdot w(x,\tau) = 0, \qquad x \in \mathcal{S}_c(\tau)\,. \tag{2.9}$$

Multiplying (2.7) by w and integrating over $\mathcal{B}_\tau$ we get

$$\int_{\mathcal{B}_\tau} (w\cdot \operatorname{div} T + w\cdot f_b)\, dV = 0\,. \tag{2.10}$$

By differentiating (2.10) with respect to time we get [8]

$$\begin{aligned} &\int_{\mathcal{B}_\tau} \{D^*\cdot(\,\mathbb{C}[D] - 2TD + \operatorname{tr}(D)T\,) + L^*\cdot LT\,\}\, dV\, + \\ &- \int_{\mathcal{B}_\tau} \{w\cdot(\dot f_b + \operatorname{tr}(D) f_b\,)\,\}\, dV\, + \\ &- \sum_{i=1}^{3}\int_{\mathcal{S}^i_f(\tau)} \{w\cdot(\,(\sigma_i e_i)^{\cdot} + (\,\operatorname{tr}(D) - n\cdot L^T n)\sigma_i e_i\,)\}\, dA\, + \\ &+ \int_{\mathcal{S}_c(\tau)} \{w\cdot(\pi_c L^T n)\,\}\, dA \;=\; 0, \end{aligned} \tag{2.11}$$

with

$$L \;:=\; \operatorname{grad}\dot u, \text{ the spatial velocity gradient,}$$
$$D \;:=\; \frac{1}{2}(L + L^T), \text{ the stretching,}$$

and with

$$L^* \;:=\; \operatorname{grad} w,$$
$$D^* \;:=\; \frac{1}{2}(L^* + L^{*T}),$$

the corresponding "virtual" quantities. In (2.11) $\mathbb{C}$ is the constitutive 4th order tensor which, applied to stretching D, gives the Jaumann derivative of the Cauchy stress *i.e.*:

$$\overset{\circ}{T} := \dot{T} + TW - WT = \mathbb{C}[D],$$

where $W := \frac{1}{2}(L - L^T)$ is the spin of the motion.

The incremental equation (2.11) is used to calculate the stiffness tangent matrix, an essential ingredient for the finite element schemes.

3 The constitutive law

We shall now introduce the ordinary differential equation system which describes the constitutive response of the material in order to obtain the expression of $\mathbb{C}$; in the following we refer mainly to [11].

Denoting by $F(\tau)$ the deformation gradient with respect to the initial configuration, at a fixed material point, we have $L(\tau) = \dot{F}(\tau)F^{-1}(\tau)$. Moreover we know [11, pag. 16] that $F(\tau)$ can be uniquely decomposed as

$$F(\tau) = V(\tau)P(\tau), \tag{3.1}$$

where $V(\tau)$ is a positive-definite symmetric tensor and $P(\tau)$ takes into account the plastic deformation and the rigid rotation which the material undergoes at the point being considered.

We indicate by $L^p := \dot{P}P^{-1}$ the plastic velocity gradient and by $D^p := \frac{1}{2}(L^p + L^{p^T})$ and $W^p := \frac{1}{2}(L^p - L^{p^T})$ the plastic stretching and spin, respectively.

For A, a second order tensor, A_0 denotes the deviatoric part of A, *i.e.*$A_0 = A - (\frac{1}{3}\operatorname{tr} A)I$, with I the identity tensor.

We assume that there is no plastic change of volume so that we have $\operatorname{tr} D^p = 0$, and we refer to $(D_0(\tau) - D^p(\tau))$ and $(D(\tau) - D^p(\tau))$ as the elastic shearing and the elastic stretching, respectively.

In what follows we hypothetize that the material undergoes only small elastic shearing (the most important case of small elastic stretching is derived as a particular case). Moreover we consider von Mises materials, so that denoting the Kirchhoff stress tensor by

$$K(\tau) := \det(F(\tau))T(\tau) = \theta^3(\tau)T(\tau), \tag{3.2}$$

the yield locus is the sphere $\mathcal{Y}_0(\tau)$,

$$\mathcal{Y}_0(\tau) = \{K_0 \in \mathrm{Sym}_0 \mid \| K_0(\tau) - C(\tau) \| = \rho(\tau)\}, \tag{3.3}$$

where $C(\tau)$ and $\rho(\tau)$ are the center and the radius of $\mathcal{Y}_0(\tau)$, respectively, and Sym_0 is the set of the deviatoric symmetric tensors.

In these conditions, the constitutive equations are [11, pag.48]:

$$K_0 = 2\mu V_0, \tag{3.4}$$

$$\overset{\circ}{K} := \dot{K} + KW - WK = 2\theta\mu(D - D^p) + \theta\lambda \operatorname{tr}(D)I + \phi \operatorname{tr}(D)\, V_0, \tag{3.5}$$

where μ, λ and ϕ are functions of θ, in particular $\mu_0 := \mu(1)$ and $\lambda_0 := \lambda(1)$ are the Lame moduli.

We introduce the deformation measures E and E^p defined by

$$\overset{\circ}{E} := \dot{E} + EW - WE = D, \tag{3.6}$$

$$\overset{\circ}{E}{}^p := \dot{E}^p + E^pW - WE^p = D^p, \tag{3.7}$$

with the initial condition

$$E(0) = 0, \tag{3.8}$$

$$E^p(0) = 0. \tag{3.9}$$

If we define

$$H := \theta E, \tag{3.10}$$

$$H^p := \theta E^p, \tag{3.11}$$

from (3.6) and (3.7) we get

$$\overset{\circ}{H}{}_0 - \overset{\circ}{H}{}^p = \theta\,(D_0 - D^p\,) + \frac{\dot{\theta}}{\theta}\,(H_0 - H^p\,),$$

and, because [11, pag. 36]

$$\frac{\dot{\theta}}{\theta} = \frac{1}{3}\,\mathrm{tr}(D) \tag{3.12}$$

we obtain

$$\overset{\circ}{H}{}_0 - \overset{\circ}{H}{}^p = \theta\,(D_0 - D^p\,) + \frac{1}{3}\,\mathrm{tr}(D\,)\,(H_0 - H^p\,). \tag{3.13}$$

On the other hand, we have [11, pag. 40]

$$\overset{\circ}{V}{}_0 = \theta\,(D_0 - D^p\,) + \frac{1}{3}\,\mathrm{tr}(D\,)\,V_0, \tag{3.14}$$

with the initial condition

$$V_0(0) = 0. \tag{3.15}$$

So, comparing (3.13) and (3.14), we can conclude that

$$V_0 = H_0 - H^p = \theta(E_0 - E^p\,). \tag{3.16}$$

Taking into account (3.4) and (3.16), we can write the yield condition as a function of the deformation measures E and E^p. Indeed we have

$$\|E_0 - C^*\,\| = \rho^*, \tag{3.17}$$

where

$$C^* = \frac{C}{2\theta\mu} + E^p, \tag{3.18}$$

$$\rho^* = \frac{\rho}{2\theta\mu}. \tag{3.19}$$

Relations (3.17-3.19) define a new yield locus in the strain space, whose unit outward normal N is

$$N = \frac{E_0 - C^*}{\rho^*}. \tag{3.20}$$

To describe the hardening properties of the material we introduce the Odqvist function $\zeta(\tau)$ defined by

$$\zeta(\tau) := \int_0^\tau \|D^p(\tau')\| \, d\tau'. \tag{3.21}$$

$\zeta(\tau)$ is an overall measure of plastic deformation up to time τ, whereas $\dot{\zeta}(\tau) = \|D^p(\tau)\|$ measures the current plastic flow.

Assuming the present hypotheses it can be proved [11] that the constitutive behaviour is governed by the following ordinary differential system:

$$\begin{cases}
\dot{\zeta}(\tau) = \begin{cases} 0 & \text{if} \quad \|E_0(\tau) - C^*(\tau)\| < \rho^*(\tau), \\ 0 & \text{if} \quad \|E_0(\tau) - C^*(\tau)\| = \rho^*(\tau), \ \gamma(\tau)/\xi(\tau) \leq 0, \\ \gamma(\tau)/\xi(\tau) & \text{if} \quad \|E_0(\tau) - C^*(\tau)\| = \rho^*(\tau), \ \gamma(\tau)/\xi(\tau) > 0; \end{cases} \\
\overset{\circ}{E}(\tau) = D(\tau), \\
\overset{\circ}{E^p}(\tau) = D^p(\tau) = \dot{\zeta}(\tau)\, N(\tau), \\
\overset{\circ}{C^*}(\tau) = \dot{\zeta}(\tau)\, M^*(\tau) + \\ \quad -\frac{1}{3}\,\mathrm{tr}(D(\tau))(1 + \mu'(\theta(\tau))\dot{\theta}(\tau)/\mu(\theta(\tau)))(C^*(\tau) - E^p(\tau)), \\
\dot{\rho}^*(\tau) = \dot{\rho}^*(\zeta(\tau)) = \dot{\zeta}(\tau)\, \rho^{*\prime}(\zeta(\tau)) + \\ \quad -\frac{1}{3}\,\mathrm{tr}(D(\tau))(1 + \mu'(\theta(\tau))\dot{\theta}(\tau)/\mu(\theta(\tau)))\rho^*(\tau) \\
\overset{\circ}{K}(\tau) = 2\theta(\tau)\mu(\theta(\tau))\,(D(\tau) - D^p(\tau)) + \theta(\tau)\lambda(\theta(\tau))\,\mathrm{tr}(D(\tau))(\tau)\, I + \\ \quad +\theta(\tau)\phi\,\mathrm{tr}(D(\tau))\,(E_0(\tau) - E^p(\tau)),
\end{cases} \tag{3.22}$$

where:

- $\rho^*(\tau)$, the radius of the yield locus, is a nondecreasing function of $\zeta(\tau)$ and

$$\rho^{*\prime}(\zeta) = \frac{d\rho^*}{d\zeta}(\zeta),$$

- $M^*(\tau)$ is an appropriate bounded tensor valued function of $\zeta(\tau)$, $C(\tau)$, $\theta(\tau)$ and $N(\tau)$, with the property $M^* \cdot N > 0$,

- $$\gamma(\tau) = D(\tau) \cdot N(\tau) + \frac{\phi(\theta(\tau))\ \mathrm{tr}(D(\tau))\,(E_0(\tau) - E^p(\tau)) \cdot N(\tau)}{2\mu(\theta(\tau))}, \tag{3.23}$$

- $\xi(\tau) = M^*(\tau) \cdot N(\tau) + \rho^{*'}(\zeta(\tau)).$ (3.24)

From (3.22) we deduce the constitutive tensor $\mathbb{C}$ which appears in the incremental equilibrium equation (2.11), namely we have

$$\overset{\circ}{K} = 2\theta\mu D - 2\theta\mu\alpha\dot{\zeta}N + \theta\lambda[I \otimes I][D] + \theta\phi[(E_0 - E^p) \otimes I][D],$$

where $\alpha = 1$ if $\dot{\zeta} > 0$ and zero otherwise.

Taking into account the expression of $\dot{\zeta}$ in (3.22) we obtain

$$\overset{\circ}{K} = 2\theta\mu D - \frac{2\theta\mu\alpha}{\xi}[N \otimes N][D] - \frac{\theta\phi\alpha}{\xi}(E_0 - E^p \cdot N)[N \otimes I][D] +$$

$$+\theta\lambda[I \otimes I][D] + \theta\phi[(E_0 - E^p) \otimes I][D],$$

and finally

$$\mathbb{C} = 2\theta\mu\mathbb{1} - \frac{2\theta\mu\alpha}{\xi}[N \otimes N] - \frac{\theta\phi\alpha}{\xi}(E_0 - E^p) \cdot N[N \otimes I] + \theta\lambda[I \otimes I] + \quad (3.25)$$

$$+\theta\phi[(E_0 - E^p) \otimes I],$$

where $\mathbb{1}$ is the 4th order identity tensor.

The particular case of small elastic stretching, which is the most frequently encountered in applications, implies that $\operatorname{tr} D$ is small, so we have the following approximations:

$$\rho_0^*(\tau) \approx \frac{\rho}{2\mu_0},$$

$$\rho^{*'}(\tau) \approx \rho_0^{*'}(1 - (\theta - 1)(1 + \frac{\mu_0'}{\mu_0})),$$

where $\mu_0' = \frac{d\mu}{d\theta}(1)$,

$$C^*(\tau) \approx \frac{C}{2\mu_0} + E^p,$$

$$N(\tau) \approx \frac{1}{\rho_0^*}(E_0 - C^*),$$

$$M^*(\tau) \approx M_0^*(1 - (\theta - 1)(1 + \frac{\mu_0'}{\mu_0})) + N,$$

$$\gamma(\tau) \approx D \cdot N,$$

$$\xi(\tau) \approx 1 + (N \cdot M_0^* + \rho_0^{*'})(1 - (\theta - 1)(1 + \frac{\mu_0'}{\mu_0})).$$

The differential system (3.22) becomes [11, pagg. 68-69]

$$
\left\{
\begin{array}{lcl}
\dot{\zeta}(\tau) & = & \left\{
\begin{array}{lll}
0 & \text{if} & \|E_0 - C^*\| < \rho_0^*, \\[2ex]
0 & \text{if} & \begin{array}{rcl} \|E_0 - C^*\| & = & \rho_0^*, \\ \gamma/\xi & \leq & 0, \end{array} \\[2ex]
\gamma/\xi & \text{if} & \begin{array}{rcl} \|E_0 - C^*\| & = & \rho_0^*, \\ \gamma/\xi & > & 0; \end{array}
\end{array}
\right. \\[2ex]
\overset{\circ}{E}(\tau) & = & D(\tau), \\[1ex]
\overset{\circ}{E^p}(\tau) & = & D^p(\tau) = \dot{\zeta}\, N, \\[1ex]
\overset{\circ}{C^*}(\tau) & = & \dot{\zeta}\,(M_0^* + N\,), \\[1ex]
\dot{\rho}_0^*(\tau) & = & \dot{\zeta}\,\rho_0^{*'}, \\[1ex]
\overset{\circ}{T}(\tau) & = & 2\mu_0\,(D - D^p\,) + \lambda_0(\text{tr } D)\, I,
\end{array}
\right. \tag{3.26}
$$

and we can give the constitutive tensor $\mathbb{C}$ the simple expression

$$
\mathbb{C} = 2\mu_0(\mathbb{1} - \frac{\alpha}{\xi} N \otimes N\,) + \lambda_0 I \otimes I. \tag{3.27}
$$

4 Application of the finite element method

As usual, the initial configuration $\mathcal{B}_0$ is discretized into a number p of elements for a total number q of nodes. Each element $\mathcal{E}_0$ is the image, by the mapping j_0, of a standard cube $\mathcal{R}$. Let $j_\tau = \chi \circ j_0$ be the isoparametric mapping, we write

$$
\mathcal{E}_\tau = \chi(j_0(\mathcal{R},\, \tau),
$$

for the image of $\mathcal{E}_0$ at time τ.

Moreover, we have

$$
x^e(\xi,\, \eta,\, \zeta\,) = \sum_{i=1}^{k} s_i(\xi,\, \eta,\, \zeta\,)\, x_i^e(\tau) = [S]\,\{x^e\}, \quad -1 \leq \xi,\, \eta,\, \zeta \leq 1, \tag{4.1}
$$

where k is the number of the nodes of the element, ξ, η, ζ are the local coordinates on $\mathcal{R}$, $s_i(\xi,\, \eta,\, \zeta\,)$ are the shape functions of the element and $x_i^e(\tau)$ is the position vector of the i-th node of the element at time τ.

Relation (4.1) can be put in matricial form where $[S]$ is the $(3\times3k)$ matrix of the shape functions and $\{x^e\}$ is the $3k$ array of the nodal positions.

The displacement u, the velocity $\dot{u}$ and the arbitrary velocity w, expressed as functions of the local coordinates $(\xi,\, \eta,\, \zeta\,)$, are approximated, on each element, by the same mapping j_τ:

$$
u^e(\xi,\, \eta,\, \zeta\,) = [S]\,\{u^e\}, \quad -1 \leq \xi,\, \eta,\, \zeta \leq 1, \tag{4.2}
$$

$$\dot{u}^e(\xi, \eta, \zeta) = [S]\{\dot{u}^e\}, \quad -1 \le \xi, \eta, \zeta \le 1, \tag{4.3}$$

$$w^e(\xi, \eta, \zeta) = [S]\{w^e\}, \quad -1 \le \xi, \eta, \zeta \le 1. \tag{4.4}$$

In this way, the equation (2.11) can be numerically evaluated and we get the following matricial equation for each element:

$$\{w^e\} \cdot ([K^e]\{\dot{u}^e\} - \{\dot{f}^e\}) = 0, \tag{4.5}$$

where the load array $\{\dot{f}^e\}$ represents the terms in (2.11) which do not depend on u, *i.e.* (assuming the hypothesis that the body forces f_b do not depend on u):

$$\{\dot{f}^e\} = \int_{\mathcal{E}_\tau} [S]^T \dot{f}_b \, dV + \sum_{i=1}^{3} \int_{\partial\mathcal{E}_\tau \cap \mathcal{S}_f^i(\tau)} [S]^T \dot{\sigma}_i \, e_i \, dA. \tag{4.6}$$

By assembling (4.5), in view of the arbitrariness of w, we have the non-linear algebraic system

$$[K]\{\dot{u}\} = \{\dot{f}\}, \tag{4.7}$$

where $[K]$ is the $3q \times 3q$) tangent stiffness matrix..

The $[K^e]$ matrix is calculated from (2.11) and it is the sum of six different terms $[K_j^e]$, namely:

$$\{w^e\} \cdot [K_1^e]\{\dot{u}^e\} = \int_{\mathcal{E}_\tau} D^* \cdot \mathbb{C}[D] \, dV, \tag{4.8}$$

$$\{w^e\} \cdot [K_2^e]\{\dot{u}^e\} = \int_{\mathcal{E}_\tau} L^* \cdot LT \, dV, \tag{4.9}$$

$$\{w^e\} \cdot [K_3^e]\{\dot{u}^e\} = -2 \int_{\mathcal{E}_\tau} D^* \cdot TD \, dV, \tag{4.10}$$

$$\{w^e\} \cdot [K_4^e]\{\dot{u}^e\} = \int_{\mathcal{E}_\tau} \{D^* \cdot T - w \cdot f_b\} \, \mathrm{tr}(D) \, dV, \tag{4.11}$$

$$\{w^e\} \cdot [K_5^e]\{\dot{u}^e\} = \sum_{i=1}^{3} \int_{\partial\mathcal{E}_\tau \cap \mathcal{S}_f^i(\tau)} w \cdot [n \cdot L^T n - \mathrm{tr}(D)] \, \sigma_i e_i \, dA, \tag{4.12}$$

$$\{w^e\} \cdot [K_6^e]\{\dot{u}^e\} = \int_{\partial\mathcal{E}_\tau \cap \mathcal{S}_c(\tau)} w \cdot (\pi_c L^T n) \, dA. \tag{4.13}$$

Moreover, other terms can appear in $[K^e]$ if the external forces depend on the displacement (follower forces). In particular, if a pressure is applied we have [8]:

$$f_s = \pi n$$

and

$$\dot{f}_s = \dot{\pi} n + \pi \dot{n} = \dot{\pi} n + \pi(n \cdot L^T n - L^T) n,$$

so the following term must be added to $[K^e]$:

$$\{w^e\} \cdot [K_7^e]\{\dot{u}^e\} = -\pi \int_{\partial\mathcal{E}_\tau \cap S_f^1(\tau)} w \cdot (n \cdot L^T n - L^T)\, n\, dA, \tag{4.14}$$

whereas, in this case, for the load term $\{\dot{f}^e\}$ we get

$$\{\dot{f}^e\} = \int_{\mathcal{E}_\tau} [S]^T \dot{f}_b\, dV + \dot{\pi} \int_{\partial\mathcal{E}_\tau \cap S_f^1(\tau)} [S]^T n\, dA. \tag{4.15}$$

The following equilibrium equation, which follows directly from (2.10) and from the divergence theorem, must be satisfied at each time τ:

$$\int_{\mathcal{B}_\tau} D^* \cdot T\, dV = \int_{\mathcal{B}_\tau} w \cdot f_b\, dV + \sum_{i=1}^{3} \int_{S_f^i(\tau)} \sigma_i\, w \cdot e_i\, dA. \tag{4.16}$$

In the remainder of this section, an explicit form of the $[K_j^e]$ terms and the equilibrium equation (4.16) is calculated, by generalizing and completing what has been done in [8]. We shall limit the description to the three-dimensional case, from which the particular cases of plane-stress, plane-strain and axisymmetry can be deduced quite easily.

Denoting by $x_j, u_j, j = 1,2,3$, respectively, the components of the position and the displacement with respect to the global cartesian reference system, respectively, the spatial gradient L of the velocity, whose components are

$$L_{ij} = \frac{\partial \dot{u}_i}{\partial x_j},$$

can be expressed as a function of the nodal values $\{\dot{u}^e\}$ of the velocity. We get

$$\{L\} = \begin{bmatrix} L_{11} \\ L_{21} \\ L_{31} \\ L_{12} \\ L_{22} \\ L_{32} \\ L_{13} \\ L_{23} \\ L_{33} \end{bmatrix} = [G_1, ..., G_i, ..., G_k] \begin{bmatrix} \dot{u}_1^e \\ \vdots \\ \dot{u}_k^e \end{bmatrix} = [G]\{\dot{u}^e\}, \tag{4.17}$$

where

$$[G_i] = \begin{bmatrix} s_{i,1} & 0 & 0 \\ 0 & s_{i,1} & 0 \\ 0 & 0 & s_{i,1} \\ s_{i,2} & 0 & 0 \\ 0 & s_{i,2} & 0 \\ 0 & 0 & s_{i,2} \\ s_{i,3} & 0 & 0 \\ 0 & s_{i,3} & 0 \\ 0 & 0 & s_{i,3} \end{bmatrix} \tag{4.18}$$

with

$$s_{i,j} = \frac{\partial s_i}{\partial x_j}.$$

Because $D = (L + L^T)/2$ and having put

$$\{D\}^T = \{D_{11}, D_{22}, D_{33}, 2D_{12}, 2D_{23}, 2D_{13}\} \tag{4.19}$$

we get, from (4.17),

$$\{D\} = [B_1, \ldots, B_i, \ldots, B_k]\{\dot{u}^e\} = [B]\{\dot{u}^e\}, \tag{4.20}$$

where

$$[B_i] = \begin{bmatrix} s_{i,1} & 0 & 0 \\ 0 & s_{i,2} & 0 \\ 0 & 0 & s_{i,3} \\ s_{i,2} & s_{i,1} & 0 \\ 0 & s_{i,3} & s_{i,2} \\ s_{i,3} & 0 & s_{i,1} \end{bmatrix}. \tag{4.21}$$

We indicate by

$$\{T\}^T = \{T_{11}, T_{22}, T_{33}, T_{12}, T_{23}, T_{13}\} \tag{4.22}$$

the vector of the stress components and by $[C]$ the symmetric matrix so that

$$[C]\{D\} = \begin{bmatrix} (\mathbb{C}[D])_{11} \\ (\mathbb{C}[D])_{22} \\ (\mathbb{C}[D])_{33} \\ (\mathbb{C}[D])_{12} \\ (\mathbb{C}[D])_{23} \\ (\mathbb{C}[D])_{13} \end{bmatrix}, \tag{4.23}$$

where $\{D\}$ is the vector defined by (4.19) and $(\mathbb{C}[D])_{ij}$ are the components of the tensor $\mathbb{C}[D]$. Matrix $[C]$ is called the matrix of the elasto-plastic moduli.

We are now in the position to calculate the matrices $[K_j^e]$.

From (4.8), (4.20) and (4.23), we get

$$\{w^e\} \cdot [K_1^e]\{\dot{u}^e\} = \int_{\mathcal{E}_\tau} D^* \cdot \mathbb{C}[D]\, dV =$$

$$= \{w^e\} \cdot \left(\int_{\mathcal{R}} [B]^T [C][B] \det J \, d\xi \, d\eta \, d\zeta\right)\{\dot{u}^e\},$$

where J is the determinant of the Jacobi matrix of the isoparametric mapping j_τ; therefore

$$[K_1^e] = \int_{\mathcal{R}} [B]^T [C][B] \det J \, d\xi \, d\eta \, d\zeta. \tag{4.24}$$

A direct calculation proves that

$$L^* \cdot LT = \mathrm{tr}(TL^T L^*) = \{L^*\}^T [M_2]\{L\}$$

where

$$[M_2] = \begin{bmatrix} T_{11} & 0 & 0 & T_{12} & 0 & 0 & T_{13} & 0 & 0 \\ 0 & T_{11} & 0 & 0 & T_{12} & 0 & 0 & T_{13} & 0 \\ 0 & 0 & T_{11} & 0 & 0 & T_{12} & 0 & 0 & T_{13} \\ T_{12} & 0 & 0 & T_{22} & 0 & 0 & T_{23} & 0 & 0 \\ 0 & T_{12} & 0 & 0 & T_{22} & 0 & 0 & T_{23} & 0 \\ 0 & 0 & T_{12} & 0 & 0 & T_{22} & 0 & 0 & T_{23} \\ T_{13} & 0 & 0 & T_{23} & 0 & 0 & T_{33} & 0 & 0 \\ 0 & T_{13} & 0 & 0 & T_{23} & 0 & 0 & T_{33} & 0 \\ 0 & 0 & T_{13} & 0 & 0 & T_{23} & 0 & 0 & T_{33} \end{bmatrix}$$

from which, using (4.17), we get

$$\{w^e\} \cdot [K_2^e]\{\dot{u}^e\} = \int_{\mathcal{E}_\tau} \mathrm{L}^* \cdot LT \, dV =$$

$$= \{w^e\} \cdot \left(\int_{\mathcal{R}} [G]^T [M_2][G] \det J \, d\xi \, d\eta \, d\zeta\right) \{\dot{u}^e\},$$

and therefore

$$[K_2^e] = \int_{\mathcal{R}} [G]^T [M_2][G] \det J \, d\xi \, d\eta \, d\zeta. \tag{4.25}$$

Since, as we can easily verify,

$$2 D^* \cdot TD = 2\{D^*\} \cdot [M_3]\{D\}$$

where

$$[M_3] =$$

$$= \begin{bmatrix} T_{11} & 0 & 0 & T_{12}/2 & 0 & T_{13}/2 \\ 0 & T_{22} & 0 & T_{12}/2 & T_{23}/2 & 0 \\ 0 & 0 & T_{33} & 0 & T_{23}/2 & T_{13}/2 \\ T_{12}/2 & T_{12}/2 & 0 & (T_{11}+T_{22})/4 & T_{13}/4 & T_{23}/4 \\ 0 & T_{23}/2 & T_{23}/2 & T_{13}/4 & (T_{22}+T_{33})/4 & T_{12}/4 \\ T_{13}/2 & 0 & T_{13}/2 & T_{23}/4 & T_{12}/4 & (T_{11}+T_{33})/4 \end{bmatrix},$$

we obtain from (4.10) and (4.20)

$$\{w^e\} \cdot [K_3^e]\{\dot{u}^e\} = -2 \int_{\mathcal{E}_\tau} D^* \cdot TD \, dV =$$

$$= \{w^e\} \cdot \left(\int_{\mathcal{R}} [B]^T [M_3][B] \det J \, d\xi \, d\eta \, d\zeta\right) \{\dot{u}^e\},$$

from which we obtain

$$[K_3^e] = -2 \int_{\mathcal{R}} [B]^T [M_3][B] \det J \, d\xi \, d\eta \, d\zeta. \tag{4.26}$$

To calculate $[K_4^e]$, observing that

$$\mathrm{tr}(D)\, D^* \cdot T = \{D^*\} \cdot [M_4]\{D\}$$

and

$$\mathrm{tr}(D) = I \cdot D = \{I\} \cdot \{D\},$$

where

$$[M_4] = \begin{bmatrix} T_{11} & T_{11} & T_{11} & 0 & 0 & 0 \\ T_{22} & T_{22} & T_{22} & 0 & 0 & 0 \\ T_{33} & T_{33} & T_{33} & 0 & 0 & 0 \\ T_{12} & T_{12} & T_{12} & 0 & 0 & 0 \\ T_{23} & T_{23} & T_{23} & 0 & 0 & 0 \\ T_{13} & T_{13} & T_{13} & 0 & 0 & 0 \end{bmatrix},$$

moreover

$$\{I\}^T = \{1, 1, 1, 0, 0, 0\}, \tag{4.27}$$

and we get , from (4.11) and (4.20),

$$\{w^e\} \cdot [K_4^e]\{\dot{u}^e\} = \int_{\mathcal{E}_\tau} \{D^* \cdot T - w \cdot f_b\}\ \mathrm{tr}(D)\, dV =$$

$$= \{w^e\} \cdot (\int_{\mathcal{R}} ([B]^T [M_4][B] - [S]^T [f_b \otimes \{I\}][B])\ \det J\, d\xi\, d\eta\, d\zeta\,)\{\dot{u}^e\}.$$

Therefore

$$[K_4^e] = \int_{\mathcal{R}} ([B]^T [M_4][B] - [S]^T [f_b \otimes \{I\}][B])\ \det J\, d\xi\, d\eta\, d\zeta. \tag{4.28}$$

To calculate $[K_5^e]$, we observe that the quantity $n \cdot L^T n$, which appears in (4.12), can be written as follows

$$n \cdot L^T n = \{\overline{n}\} \cdot \{L\}$$

where

$$\{\overline{n}\}^T = (n_1^2,\ n_1 n_2,\ n_1 n_3,\ n_1 n_2,\ n_2^2,\ n_2 n_3,\ n_1 n_3,\ n_2 n_3,\ n_3^2).$$

So, we can write

$$w \cdot [n \cdot L^T n - \mathrm{tr}(D)]\sigma_i e_i = w \cdot \sigma_i e_i (\{\overline{n}\} \cdot \{L\} - \{I\} \cdot \{D\}) =$$

$$= w \cdot ([\sigma_i e_i \otimes \{\overline{n}\}]\{L\} - \sigma_i [e_i \otimes \{I\}\{D\}]),$$

from which, in view of (4.17) and (4.20),

$$\{w^e\} \cdot [K_5^e]\{\dot{u}^e\} =$$

$$= \{w^e\} \cdot (\sum_{i=1}^{3} \int_{\partial\mathcal{R} \cap J_\tau^{-1}(S_f^i(\tau))} (\sigma_i [S]^T [e_i \otimes \{\overline{n}\}][G] - [e_i \otimes \{I\}][B])$$

$\sqrt{J^{-T}m \cdot J^{-T}m}\, \det J\, dA_{\mathcal{R}})\{\dot{u}^e\},$

where m is the unit vector normal external to $\partial\mathcal{R}$ and the quantity $\det J\sqrt{J^{-T}m \cdot J^{-T}m} = dA/dA_{\mathcal{R}}$ is the area on $\partial\mathcal{E}_\tau$ per unit area on $\partial\mathcal{R}$.

Therefore

$$[K_5^e] = \sum_{i=1}^{3}\int_{\partial\mathcal{R}\cap J_\tau^{-1}(\mathcal{S}_f^i(\tau))}(\sigma_i\,[S]^T[e_i\otimes\{\overline{n}\}][G] - [e_i\otimes\{I\}][B])$$

$$\sqrt{J^{-T}m \cdot J^{-T}m}\, \det J\, dA_{\mathcal{R}}. \tag{4.29}$$

To calculate $[K_6^e]$, we observe that

$$w\cdot(\pi_c L^T n) = w\cdot[M_6]\{L\},$$

where

$$[M_6] = \begin{bmatrix} n_1 & n_2 & n_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_1 & n_2 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_1 & n_2 & n_3 \end{bmatrix}$$

and so

$$\{w^e\}\cdot[K_6^e]\{\dot{u}^e\} = \int_{\partial\mathcal{E}_\tau\cap\mathcal{S}_c(\tau)}\pi_c w\cdot[M_6]\{L\}\,dA =$$

$$= \{w^e\}\cdot\int_{\partial\mathcal{R}\cap J_\tau^{-1}(\mathcal{S}_c(\tau))}(\pi_c\,[S]^T[M_6][G])\sqrt{J^{-T}m \cdot J^{-T}m}\, \det J\, dA_{\mathcal{R}}\,\{\dot{u}^e\},$$

from which

$$[K_6^e] = \int_{\partial\mathcal{R}\cap J_\tau^{-1}(\mathcal{S}_c(\tau))}(\pi_c\,[S]^T[M_6][G])\sqrt{J^{-T}m \cdot J^{-T}m}\, \det J\, dA_{\mathcal{R}}. \tag{4.30}$$

In the case that external loads like pressure are present, we have

$$w\cdot(n\cdot L^T n - L^T)n = w\cdot[n\otimes n - I]L^T n = w\cdot(n\otimes n - I)[M_6]\{L\},$$

and so

$$\{w^e\}\cdot[K_7^e]\{\dot{u}^e\} = -\int_{\partial\mathcal{E}_\tau\cap\mathcal{S}_f^1(\tau)}\pi\, w\cdot(n\otimes n - I)[M_6]\{L\}\,dA =$$

$$= \{w^e\}\cdot\int_{\partial\mathcal{R}\cap J_\tau^{-1}(\mathcal{S}_f^1(\tau))}(\pi\,[S]^T[I - n\otimes n][M_6][G])$$

$$\sqrt{J^{-T}m \cdot J^{-T}m}\, \det J\, dA_{\mathcal{R}}\,\{\dot{u}^e\},$$

from which

$$[K_7^e] = \int_{\partial\mathcal{R}\cap J_\tau^{-1}(\mathcal{S}_f^1(\tau))}(\pi\,[S]^T[I - n\otimes n][M_6][G])$$

$$\sqrt{J^{-T}m \cdot J^{-T}m}\, \det J\, dA_{\mathcal{R}}. \tag{4.31}$$

We observe that the terms $[K_4^e]$, $[K_5^e]$, $[K_6^e]$ and $[K_7^e]$ are nonsymmetric, so their use in a finite element code requires the use of solver routines that are appropriate for nonsymmetric matrices.

Lastly, equation (4.16) can be approximated on every element and we get

$$\{w^e\} \cdot \Big(\int_{\mathcal{R}} [B]^T \{T\} \det J \, d\xi \, d\eta \, d\zeta - \{f^e\} \Big) = 0, \tag{4.32}$$

where

$$\{f^e\} = \int_{\mathcal{R}} [S]^T [S] \{f_b\} \det J \, d\xi \, d\eta \, d\zeta +$$

$$+ \sum_{i=1}^{3} \int_{\partial\mathcal{R} \cap J_\tau^{-1}(S_f^i(\tau))} [S]^T [S] \{\sigma_i e_i\} \sqrt{J^{-T} m \cdot J^{-T} m} \det J \, dA_{\mathcal{R}}.$$

By assembling (4.32) over all elements, dropping $\{c^e\}$ for its arbitrarity, and adding the reactions forces for the constraints (2.3) and (2.5) where they belong, we obtain the equilibrium condition which must be satisfied, within a given tolerance, in order for the iterative scheme, which is used to solve problem (2.10), is to be considered as converged.

5 Numerical integration of the equation of motion

The equilibrium equation (2.10), with the constraints (2.3) and (2.5) and the constitutive law (3.26), has to be integrated in time in order to obtain the displacements $\{u\}$, the total deformation E and the Cauchy stress T.

To do that we use an iterative incremental scheme similar to the classical Newton-Raphson method but it has been designed in such a way as to overcome the difficulties arising from the irreversibilty character of the plasticity [13]. Moreover in every iteration the changes in the boundary conditions , due to the contact, and in the pressure loads, due to the motion, are taken into account [14].

The algorithm uses the direct application of the current boundary conditions set at the solver level: the remaining part of this section is devoted to the description of this integration scheme.

i) Let us assume that the body is in the equilibrium configuration $\mathcal{B}_\tau$ at time τ and we assign a load increment

$$\{\Delta f\} = \{f(\tau + \Delta\tau)\} - \{f(\tau)\}, \tag{5.1}$$

and a displacement increment at the points x_c of the rigid surfaces

$$\Delta u_c = x_c(\tau + \Delta\tau) - x_c(\tau); \tag{5.2}$$

because we suppose $\dot{u}_c(\tau)$ to be constant in the time interval $[\tau, \tau+\Delta\tau$, we have

$$\Delta u_c(\tau) = \dot{u}_c(\tau)\Delta\tau.$$

Note that $\{\Delta u_c\}$ is fixed for the whole increment, whereas, when follower forces are present, $\{\Delta f\}$ changes during the iterations process because it is a function of the displacement itself.

ii) To the nodes which are already in contact with a rigid surface (*i.e.* the nodes which at the end of the preceding increment belong to $\mathcal{S}_c(\tau)$) a fixed displacement $\Delta u_n = \Delta u_c \cdot n$ is applied along the normal direction in such a way that they are again brought into contact with the rigid surface at time $\tau + \Delta\tau$ (see fig. (1)), whereas they are free in the tangential directions.

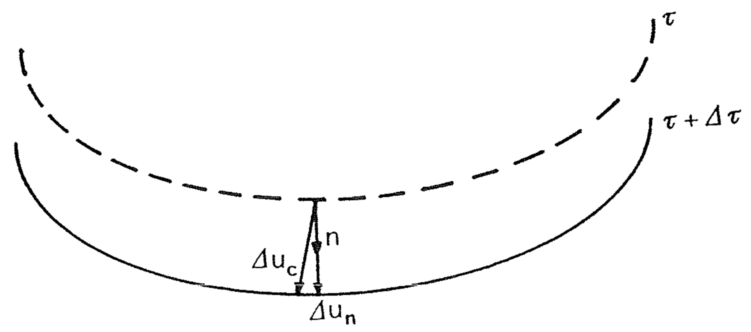

Figure 1: Displacement induced by the motion of a rigid surface

iii) The evolution system

$$[K]\{\Delta u\} = \{\Delta f\}, \tag{5.3}$$

with the constraints

$$\{\Delta u \cdot e_i\} = (\delta_i\,\Delta\tau\}, \quad \text{on } \mathcal{S}_u^i(\tau), i = 1, 2, 3, \tag{5.4}$$

$$\{\Delta u \cdot n\} = \{\Delta u_n\}, \quad \text{on } \mathcal{S}_c(\tau), \tag{5.5}$$

is solved to obtain an estimate $\{\Delta u\}$, $\{\Delta f_r\}$, $\{\Delta f_c \cdot n\}$ for the increment of the displacements, of the reaction forces and of the contact forces, respectively.
Care must be taken during this phase because constraint (5.5) is specified in the local reference system on $\mathcal{S}_c(\tau)$, so the equations containing these degrees of freedom need to be rotated in a suitable way, *i.e.* if the j-th node is on $\mathcal{S}_c(\tau)$, system (5.3) has to be changed to

$$[Q^j]^T\,[K]\,[Q^j] \begin{Bmatrix} \Delta u^1 \\ \vdots \\ \Delta u^j \\ \vdots \\ \Delta u^q \end{Bmatrix} = [Q^j]^T \begin{Bmatrix} \Delta f^1 \\ \vdots \\ \Delta f^j \\ \vdots \\ \Delta f^q \end{Bmatrix}, \tag{5.6}$$

where $[Q^j]$ is the rotation matrix which rotates the vector of the degrees of freedom of the j-th node from the local reference frame to the global Cartesian one, leaving the displacement of other nodes unchanged.
Moreover, once the system has been solved, the displacement $\{\Delta u\}$ and the contact forces $\{\Delta f_c \cdot n\}$ must be premultiplied by $[Q]$ in order to bring them into the global reference again.

iv) The differential system (3.26) is numerically integrated [8], using a 4th order Runge-Kutta formula, to get the increments ΔE, ΔT, $\Delta\zeta$, ΔE^p, and ΔC.
The initial conditions used to integrate system (3.26) are those at the beginning of the increment to avoid artificial unload. In order to understand this phenomenon better, let us consider the stress-strain diagram of a generic point (see fig. (2)).

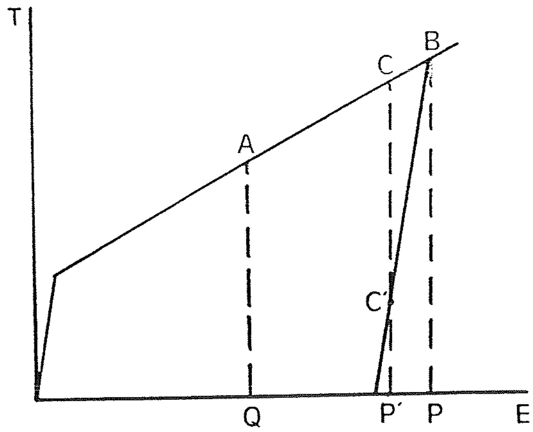

Figure 2: Stress-strain diagram

Let us assume that the deformation $\Delta E_1 = QP$ computed during the first iteration modifies the state from point A to point B. If the computed deformation $\Delta E_2 = P'P$ is negative at the second iteration, as is frequently the case due to the constrained repositioning of the contact nodes, and if state B at the end of the previous iteration is used as the initial condition to integrate the constitutive relations, the material reaches the unloaded state C'. Conversely, the correct state C is reached if state A, for every iteration, is used as the initial condition. Of course, in the latter case, a more sophisticated integration algorithm is needed and this explains the use of a Runge-Kutta formula to integrate system (3.26).

v) The residual forces

$$\{r^e\} = \int_{\mathcal{R}} [B]^T (\{T\} + \{\Delta T\}) \det J \, d\xi \, d\eta \, d\zeta + \tag{5.7}$$

$$-(\{f^e(\tau)\} + \{\Delta f^e\} + \{f^e_r(\tau)\} + \{\Delta f^e_r\} + \{f^e_c(\tau)\} + \{\Delta f^e_c\}),$$

are calculated for every element. Relation (5.7) is assembled over the whole structure to get the global vector of the residual forces $\{r\}$.
If the inequality

$$\frac{\|\{r\}\|}{\|\{f(\tau + \Delta\tau)\}\|} \leq \varepsilon \tag{5.8}$$

where ε is a tolerance factor, is verified the equilibrium equation (2.10) is considered to be satisfied.

vi) Each boundary node is checked against the contact conditions *i.e.* the new contact zone $\mathcal{S}_c(\tau + \Delta\tau)$ is determined.
A node already in contact will be considered to be a free node and its contact reaction will be zeroed in the next iteration if the contact force $\{f_c(\tau)\} + \{\Delta f_c\}$ is directed outwards from the deformable body, or if its current position $\{x(\tau)\} + \{\Delta u\}$ is found to be external to any rigid surface. Conversely a node will be considered to be constrained in the next iteration if it is already in contact and the contact conditions are still met or if it is a free node but compenetration with a rigid surface has occurred.

vii) If there are no changes in the contact conditions of the boundary nodes and the convergence relation (5.8) is satisfied, the procedure jumps to point (**x**), otherwise it proceeds with a new iteration.

viii) The boundary conditions for the contact zone $\mathcal{S}_c(\tau + \Delta\tau)$ are calculated, *i.e.* a displacement Δu^1_n is applied to the nodes in the contact area in the direction of the normal $n(\tau + \Delta\tau)$ in such a way that they are brought again onto the boundary of the rigid surface.

ix) The stiffness matrix $[K]$ is calculated in the new configuration and the system

$$[K]\{\Delta u^1\} = \{r\}, \tag{5.9}$$

with the constraints

$$\{\Delta u^1 \cdot e_i\} = 0 \quad \text{on } \mathcal{S}^i_u(\tau + \Delta\tau), i = 1, 2, 3, \tag{5.10}$$

$$\{\Delta u^1 \cdot n\} = \{\Delta u_n^1\}, \quad \text{on } \mathcal{S}_c(\tau + \Delta\tau), \tag{5.11}$$

is solved as in (**iii**) to calculate $\{\Delta u^1\}$, $\{\Delta f_r^1\}$ and $\{\Delta f_c^1 \cdot n\}$.
After that, the incremental quantities are updated

$$\begin{aligned} \{\Delta u\} &\leftarrow \{\Delta u\} + \{\Delta u^1\}, \\ \{\Delta f_r\} &\leftarrow \{\Delta f_r\} + \{\Delta f_r^1\}, \\ \{\Delta f_c\} &\leftarrow \{\Delta f_c\} + \{\Delta f_c^1\}, \end{aligned} \tag{5.12}$$

and the procedure jumps to point (**iv**) for the calculation of the deformation and stress increment by using the new estimate $\{\Delta u\}$of the displacement increment.

x) The state is updated, *i.e.* each incremental quantity is incorporated in the corresponding total variable to get the value at time $\tau + \Delta\tau$ of the position, total displacement, forces, deformation, stresses etc.
After that, the procedure jumps to point (**i**) and a new increment can be applied.

6 Numerical example

The contact algorithm has been implemented into an in house developed FEM code called NOSA. To check the performances of the method we required a problem where the friction had not affect the results, so we tried to simulate a bulge test case which has been performed at the C.S.M. (Centro Sviluppo Materiali) and for which the experimental data are available [15].

The test concerns a disc of steel, clamped on the boundary and subjected to pressure on a face. During its motion the disc leans onto a bearing which has a quarter-circle section.

The geometrical and mechanical data are the following:

radius of the disc	96.0	mm,
radius of the pressurized part	86.0	mm,
radius of the bearing section	11.0	mm,
thickness of the disc	0.8	mm,
maximum pressure	7.4	MPa,
Young's modulus	2.1×10^5	MPa,
Poisson's ratio	0.3	.

Because in the material of the disc there is a slight anisotropy, due to the lamination process, two simulations were performed using the hardening curves along the lamination direction (upper curve in fig. 3) and at 90° from this direction (lower curve in fig. 3), respectively, and we supposed there was a purely isotropic hardening.

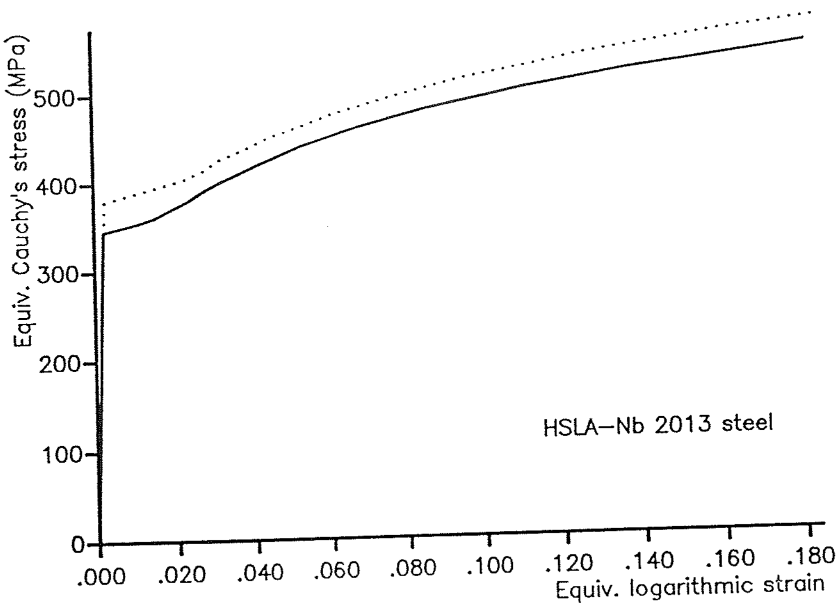

Figure 3: Work-hardening curve

The disc was schematized with a mesh composed by 170 axialsymmetic 4-node elements (2 elements in the thickness): the mesh is finer near the bearing to obtain a better modeling of the contact conditions.

The NOSA run required 78 load increments to reach the maximum pressure and 40 increments to unload until zero pressure. For purposes of comparison, MARC and ABAQUS codes were used for the same example, getting identical results to those obtained from the NOSA run, so, in the figures, only NOSA results are shown.

In fig. 4 the final configuration of a section of the disc is shown as compared with the initial one.

In fig. 5 the axial displacement of the points on the symmetry axis is plotted as a function of the pressure.

In fig. 6 the final thickness of the sheet is shown as a function of the undeformed radial distance.

In fig. 7 the final meridional strain is plotted versus the initial radial distance.

Lastly, in fig. 8, the final circumferential strain versus the radial distance is shown.

The uncertainty of the experimental data in the last two figures is twice the measured standard deviation.

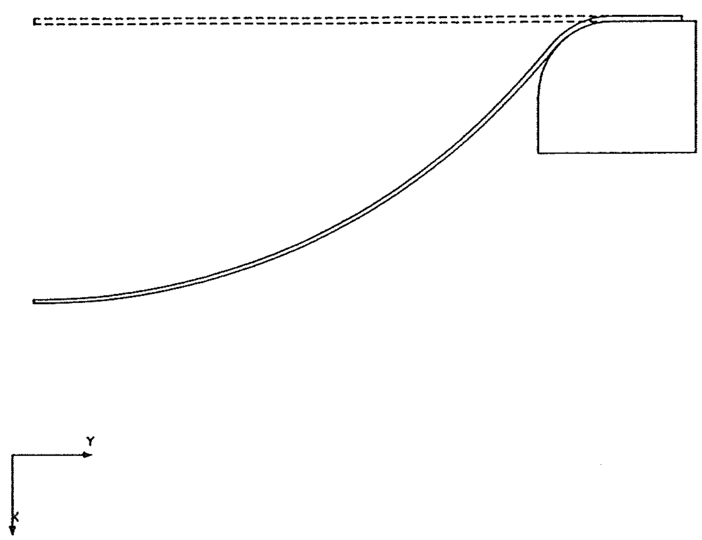

Figure 4: Final configuration of the disc

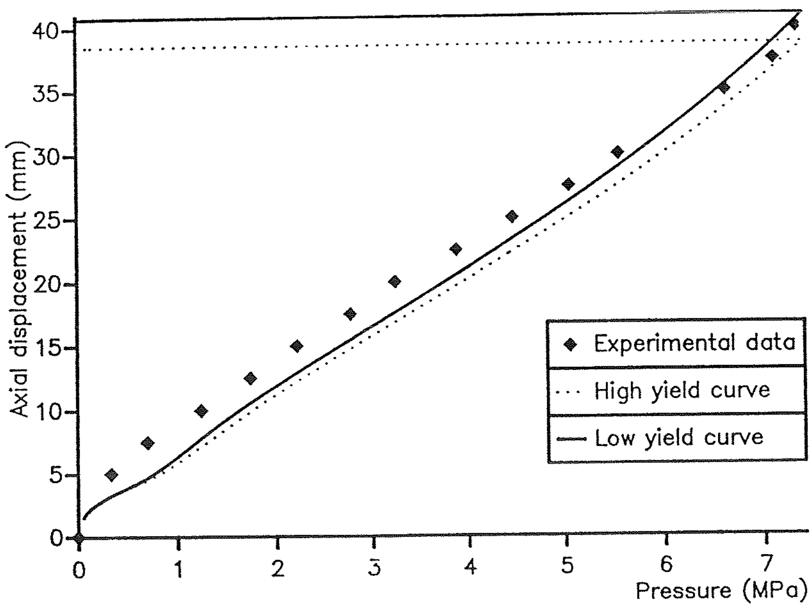

Figure 5: Axial displacement vs. pressure

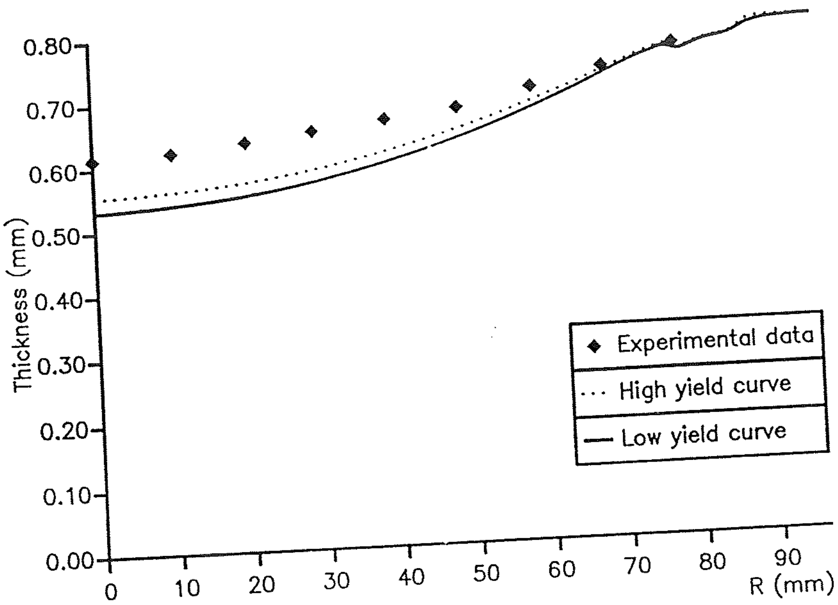

Figure 6: Thickness vs. radial distance

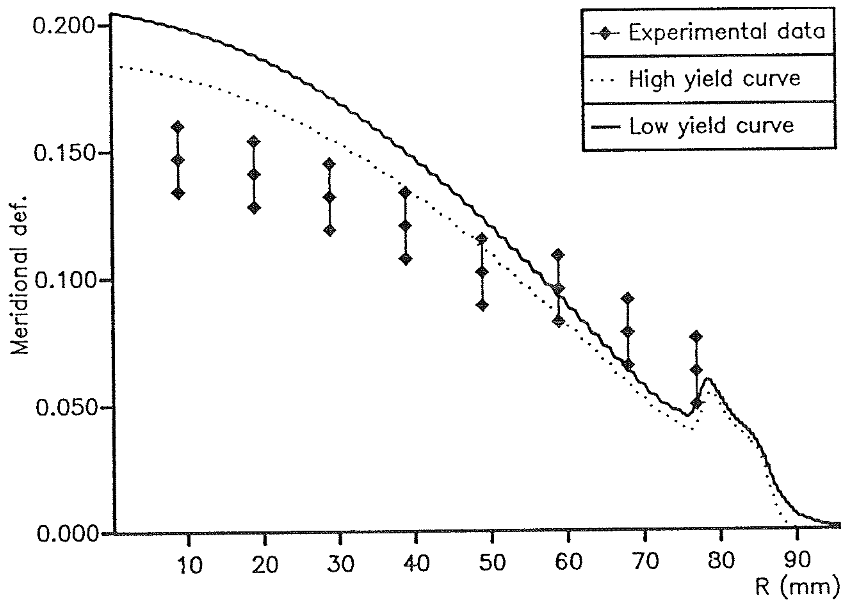

Figure 7: Meridional strain vs. radial distance

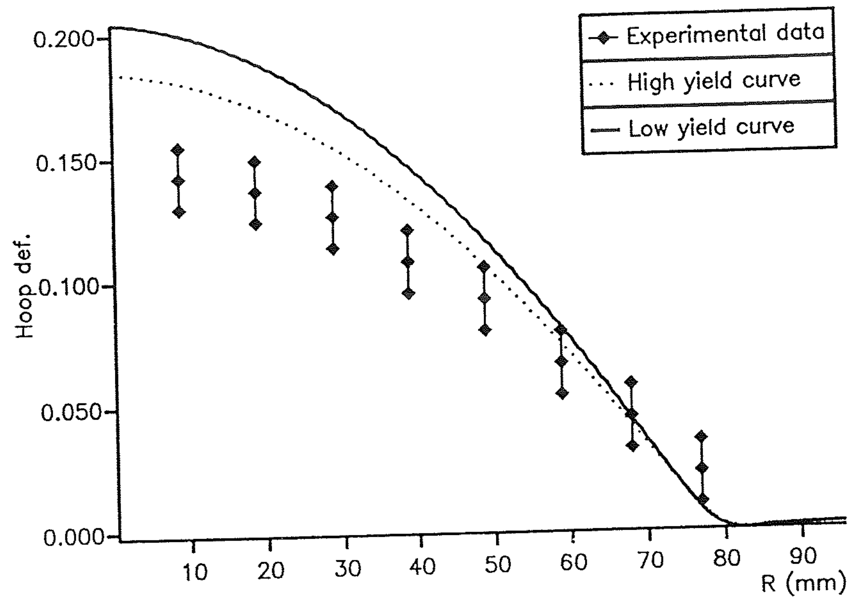

Figure 8: Circumferential strain vs. radial distance

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