Some Considerations on FFT- Conjugate Gradient Method for Electromagnetic Scattering Problems

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ABSTRACT

An application of the conjugate gradient method to the electromagnetic scattering of plane waves from planar, perfectly conducting and material plates is presented. The method is combined with a Fast Fourier Transform algorithm to increase numerical efficiency. A detailed discussion on the basic features of the conjugate gradient method is also included.

1.0 Introduction

The description of Electromagnetic scattering from perfectly conducting or material structures is usually described in terms of the E-field, the H-field or combined field integral equations. By applying a Moment Method (MM) technique [1], integral equations can than be reduced to matrix equations. Many accurate and efficient MM formulations have been presented in the literature [2-5]. However, matrix solutions become more and more inefficient as the dimensions of the scatterer increase. These difficulties can be overcome by utilizing iterative methods. One of these methods, the Conjugate Gradient Method, has been widely used recently to solve both scattering and radiation problems [6]. In this context, the Conjugate Gradient - Fast Fourier Transform method (CG-FFT) is of particular interest, as it has revealed several features which enhance the efficiency [7].The purpose of this work is to discuss of the CG-FFT method in detail and to analyze the application of this method to the problem of the electromagnetic scattering of a plane wave by planar, perfectly conducting or material plates. In particular, in Section 2, the basic properties of the conjugate gradient method are rewied in order to provide a better understanding of the whole procedure. In Section 3, the electromagnetic problem of plane wave scattering from perfectly conducting and material plates is formulated in terms of the pertinent integral equations. In Section 4, the continuous problem is discretized and the FFT algorithm is introduced into the computations. Finally, the specific problems of the perfectly conducting and of the material plate are investigated in Sections 5 and 6, respectively.

2.0 The Conjugate Gradient Method

The Conjugate Gradient method belongs to the more general class of Gradient Methods which are used to minimize an arbitrary function $F(\mathbf{z})$ of the $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ variable. They are iterative methods; the k-th iteration consists of the computation of a search vector p_k from which a new estimation $\mathbf{z}_{k+1}$ of the solution can be obtained according to the rule:

(2.0.1) $$\mathbf{z}_{k+1} = \mathbf{z}_k + \alpha_k \mathbf{p}_k$$

where α_k is generally obtained from a linear search or a prior knowledge based upon the gradient method theory.

Process (2.0.1) involves a search along the direction of vector $\mathbf{p}_k$ from the current point $\mathbf{z}_k$. If the iterative method is to be stable at least α_k must be chosen so that:

(2.0.2) $$F(\mathbf{z}_{k+1}) < F(\mathbf{z}_k)$$

The gradient methods employ algorithms that use first and maybe second derivatives of the function $F(\mathbf{z})$ to compute $\mathbf{p}_k$. These derivatives may be available analytically or approximated in some way; the method is still efficient if the derivatives have not too numerous discontinuities.

The Conjugate Gradient method generally converges to a solution which requires fewer gradient evaluations per iteration with respect to other gradient methods; it will, however, have a quadratic termination.

Let us now consider the problem of minimizing a quadratic functional F. A possible finite set of search vectors $\mathbf{p}_k$, that satisfy the "conjugacy property", can be chosen . A set of vectors $\mathbf{p}_k$ is said to be mutually conjugate or B-orthogonal with respect to a linear operator **B** if and only if

(2.0.3) $$\langle \mathbf{B}\mathbf{p}_j, \mathbf{p}_k \rangle = 0 \qquad k \neq j$$

where $\langle \cdot, \cdot \rangle$ is a scalar inner product between two functions, which will be better defined in the following. For example, if **B** is a positive definite matrix, the eigenvectors of **B** form one such set of vectors, and it can be demonstrated that these vectors are linearly independent and lead to a quadratic termination [9]. It can be shown that in this case the solution can be reached in a finite number of steps.

For non-quadratic functionals, the exact minimum can not be located in N searches, but it can be approximated with good accuracy. In this case, there is no standard technique to choose the search vector, but experience can help with some useful suggestions.

2.1 Some definitions

Before describing the method, we recall some definitions. Let us define the scalar inner product between two functions f and g in one and two dimensions, as follows:

$$(2.1.1) \qquad \langle f,g\rangle = \int\int_{l} f(x)g^*(x)dl$$

$$(2.1.2) \qquad \langle f,g\rangle = \int\int_{s} f(x,y)g^*(x,y)ds$$

where the symbol " * " denotes the complex conjugate. In a quite similar, way a discrete inner product between two vectors $\mathbf{f} = (f_o, f_1, \ldots, f_{N-1})$ and $\mathbf{g} = (g_o, g_1, \ldots, g_{N-1})$ can be defined:

$$(2.1.3) \qquad \langle \mathbf{f},\mathbf{g}\rangle = \sum_{n=0}^{N-1} g_n^* f_n$$

Closely related to the inner product, the Euclidean norm can be established, defined as follows:

$$(2.1.4) \qquad \| \mathbf{f} \| = \sqrt{\langle \mathbf{f},\mathbf{f}\rangle}$$

Finally, if $\mathbf{B}$ is a linear operator, the adjoint operator of $\mathbf{B}$, denoted by $\mathbf{B}^a$, is defined by:

$$(2.1.5) \qquad \langle \mathbf{Bf},\mathbf{g}\rangle = \langle \mathbf{f},\mathbf{B}^a\mathbf{g}\rangle$$

for any function f and g.

2.2 Analytical derivation of the method

Let us consider the quadratic functional $F(\mathbf{z})$, defined on a Hilbert space:

(2.2.1) $$F(\mathbf{z}) = \langle \mathbf{Bz},\mathbf{z} \rangle - 2\langle \mathbf{z},\mathbf{h} \rangle$$

where **B** has the following properties:

i) it is self-adjointed such that $\mathbf{B}=\mathbf{B}^a$

ii) it is positive definite such that $\langle \mathbf{Bz},\mathbf{z} \rangle > 0$ for each $\mathbf{z} \neq 0$

iii) if **B** is a matrix the eigenvalues are real and positive and the eigenvectors are orthogonal.

The properties of the operator **B** guarantee that the $\hat{\mathbf{z}}$ minimizing the functional $F(\mathbf{z})$ defined by eq.(2.2.1) is the unique solution of the equation $\mathbf{Bz} = \mathbf{h}$.

The vector $\hat{\mathbf{z}}$ can be expanded in a Fourier series with respect to the set of vectors B-orthogonal $\mathbf{p}_k$:

(2.2.2) $$\mathbf{z} = \mathbf{z}_1 + \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 + \ldots + \alpha_k \mathbf{p}_k$$

If we denote by $\mathbf{z}_k$ the partial sum at the k-th term, then, for a known property of the Fourier series, the quantity $\|\mathbf{z}_k - \hat{\mathbf{z}}\|$ is minimized over the subspace $(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_k)$; this means that $F(\mathbf{z})$ decreases while k increases, then, if the sequence $\{\mathbf{p}_i\}$ is complete, the process converges to $\hat{\mathbf{z}}$.

It can be shown that, under the described hypotheses, for each $\mathbf{z}_1 \in H$ the sequence generated by the iterative process:

(2.2.3) $$\mathbf{z}_{k+1} = \mathbf{z}_k + \alpha_k \mathbf{p}_k$$

(2.2.4) $$\alpha_k = \frac{\langle \mathbf{p}_k,\mathbf{r}_k \rangle}{\langle \mathbf{p}_k,\mathbf{Bp}_k \rangle}$$

(2.2.5) $$\mathbf{r}_k = \mathbf{h} - \mathbf{Bz}_k = \mathbf{r}_{k-1} - \alpha_k \mathbf{Bp}_{k-1}$$

satisfies the condition $\langle \mathbf{r}_k,\mathbf{p}_i \rangle = 0$, i.e. $\mathbf{r}_k$ is orthogonal to $\mathbf{p}_i$ for $i=1,2,\ldots,k-1$ and $\mathbf{z}_k \rightarrow \hat{\mathbf{z}}$ which is the unique solution of $\mathbf{Bz} = \mathbf{h}$ [10]. It can be easily shown that α_k is real. In fact, eq.(2.2.4) can be deduced by substituting eq.(2.2.3) in eq.(2.2.1):

(2.2.6) $$\begin{aligned} F(\mathbf{z}_{k+1}) &= \langle \mathbf{B}(\mathbf{z}_k + \alpha_k \mathbf{p}_k),(\mathbf{z}_k + \alpha_k \mathbf{p}_k) \rangle - 2\langle (\mathbf{z}_k + \alpha_k \mathbf{p}_k),\mathbf{h} \rangle = \\ &= F(\mathbf{z}_k) + \alpha_k \langle \mathbf{Bz}_k,\mathbf{p}_k \rangle + \alpha_k \langle \mathbf{Bp}_k,\mathbf{z}_k \rangle + \\ &\quad + \alpha_k^2 \langle \mathbf{Bp}_k,\mathbf{p}_k \rangle - 2\alpha_k \langle \mathbf{p}_k,\mathbf{h} \rangle \end{aligned}$$

By means of property i) of operator **B**, the second and third term of the last expression in (2.2.6) can be written as :

(2.2.7)
$$\alpha_k\langle \mathbf{B}\mathbf{z}_k,\mathbf{p}_k \rangle + \alpha_k\langle \mathbf{B}\mathbf{p}_k,\mathbf{z}_k \rangle = \alpha_k\langle \mathbf{B}\mathbf{z}_k,\mathbf{p}_k \rangle + \alpha_k\langle \mathbf{p}_k,\mathbf{B}^a\mathbf{z}_k \rangle = \\ = 2\alpha_k \mathrm{Re}\{\langle \mathbf{p}_k,\mathbf{B}\mathbf{z}_k \rangle\}$$

then eq. (2.2.6) becomes:

(2.2.8)
$$F(\mathbf{z}_{k+1}) = F(\mathbf{z}_k) + 2\alpha_k \mathrm{Re}\{\langle \mathbf{p}_k,\mathbf{B}\mathbf{z}_k\rangle\} + \alpha_k^2 \langle \mathbf{B}\mathbf{p}_k,\mathbf{p}_k \rangle - 2\alpha_k\langle \mathbf{p}_k,\mathbf{h}\rangle = \\ = F(\mathbf{z}_k) - 2\alpha_k \mathrm{Re}\{\langle \mathbf{r}_k,\mathbf{p}_k\rangle\} + \alpha_k^2 \langle \mathbf{B}\mathbf{p}_k,\mathbf{p}_k \rangle$$

This functional has to be minimized with respect to α_k. To obtain this, α_k has to satisfy the following expression:

(2.2.9)
$$\frac{\partial}{\partial \alpha_k} F(\mathbf{z}_{k+1}) = 0$$

By developing this partial derivative, equation (2.2.4) is obtained.
A set of B-orthogonal vectors can be generated by the Gram-Schimdt process, applied to any sequence of vectors. Given a set of linear independent vectors $\mathbf{e}_1, \dots ,\mathbf{e}_N$, the B-orthogonal set $\mathbf{p}_k$ is given by:

(2.2.10)
$$\mathbf{p}_1 = \mathbf{e}_1$$
$$\mathbf{p}_{k+1} = \mathbf{e}_{k+1} - \sum_{s=1}^{k-1} \frac{\langle \mathbf{e}_{k+1},\mathbf{B}\mathbf{p}_s\rangle}{\langle \mathbf{p}_s,\mathbf{B}\mathbf{p}_s\rangle} \mathbf{p}_k \qquad \text{for } k>0$$

In particular, in the Conjugate Gradient method, $\mathbf{p}_1$ is chosen equal to $\mathbf{r}_1$, which is the direction of the negative gradient of $F(\mathbf{z})$. Then, after moving in this direction to $\mathbf{z}_2$, the new negative gradient direction $\mathbf{r}_2 = \mathbf{h} - \mathbf{B}\mathbf{z}_2$ is considered and $\mathbf{p}_2$ is chosen in the space spanned by $\mathbf{r}_1$, $\mathbf{r}_2$ and B-orthogonal to $\mathbf{p}_1$. The following $\mathbf{p}_i$ are chosen in a quite similar way. In other words, the sequence of $\mathbf{p}_i$'s is a B-orthogonalized version of the sequence of negative gradients $(\mathbf{r}_1,\mathbf{r}_2,\dots)$ generated as the descendant process progresses. The compact recursive form of the method will be:

(2.2.11) $\mathbf{z}_{k+1} = \mathbf{z}_k + \alpha_k\mathbf{p}_k$

(2.2.12) $\mathbf{p}_{k+1} = \mathbf{r}_{k+1} - \beta_k\mathbf{p}_k$

(2.2.13) $$\alpha_k = \frac{\langle \mathbf{p}_k, \mathbf{r}_k \rangle}{\langle \mathbf{p}_k, \mathbf{B}\mathbf{p}_k \rangle}$$

(2.2.14) $$\beta_k = \frac{\langle \mathbf{r}_{k+1}, \mathbf{B}\mathbf{p}_k \rangle}{\langle \mathbf{p}_k, \mathbf{B}\mathbf{p}_k \rangle}$$

with $\mathbf{p}_1 = \mathbf{r}_1$

Let us now re-arrange the above equations in a more useful form for the computation.

It can be shown that $\langle \mathbf{r}_{k+1}, \mathbf{B}\mathbf{p}_k \rangle = -\frac{1}{\alpha_k} \langle \mathbf{r}_{k+1}, \mathbf{r}_{k+1} \rangle$; in fact, by taking the above inner product and by using the eq. (2.2.5) we have:

(2.2.15) $$\langle \mathbf{r}_{k+1}, \mathbf{B}\mathbf{p}_k \rangle = \langle \mathbf{r}_{k+1}, -\frac{\mathbf{r}_{k+1} - \mathbf{r}_k}{\alpha_k} \rangle = -\frac{1}{\alpha_k} \{ \langle \mathbf{r}_{k+1}, \mathbf{r}_{k+1} \rangle - \langle \mathbf{r}_{k+1}, \mathbf{r}_k \rangle \}$$

The last term is zero, as can be easily seen by remembering that $\langle \mathbf{r}_{k+1}, \mathbf{p}_i \rangle = 0$ for i=1,2,...,k; in fact:

(2.2.16) $$\langle \mathbf{r}_{k+1}, \mathbf{p}_k \rangle = \langle \mathbf{r}_{k+1}, \mathbf{r}_k - \beta_{k-1} \mathbf{p}_{k-1} \rangle = \\ = \langle \mathbf{r}_{k+1}, \mathbf{r}_k \rangle - \beta_{k-1} \langle \mathbf{r}_{k+1}, \mathbf{p}_{k-1} \rangle = \langle \mathbf{r}_{k+1}, \mathbf{r}_k \rangle = 0$$

As a consequence, the equality $\langle \mathbf{r}_{k+1}, \mathbf{B}\mathbf{p}_k \rangle = -\frac{1}{\alpha_k} \langle \mathbf{r}_{k+1}, \mathbf{r}_{k+1} \rangle$ is established and the iterative equations (2.2.12) - (2.2.14) can be expressed as:

(2.2.17) $$\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \frac{\langle \mathbf{r}_k, \mathbf{r}_k \rangle}{\langle \mathbf{r}_{k+1}, \mathbf{r}_{k+1} \rangle} \mathbf{p}_k$$

(2.2.18) $$\alpha_k = \frac{1}{\langle \mathbf{p}_k, \mathbf{B}\mathbf{p}_k \rangle}$$

(2.2.19) $$\beta_k = \frac{\langle \mathbf{r}_k, \mathbf{r}_k \rangle}{\langle \mathbf{r}_{k+1}, \mathbf{r}_{k+1} \rangle}$$

In order to save on computations, let us normalize the search directions $\mathbf{p}_k$ with respect to $\langle \mathbf{r}_k, \mathbf{r}_k \rangle$. The algorithm assumes the following form:

-) First step: initialize the residual and search vectors

(2.2.20) $$\mathbf{r}_1 = \mathbf{h} - \mathbf{B}\mathbf{z}_1$$

(2.2.21) $\beta_o = \frac{1}{\langle \mathbf{r}_k, \mathbf{r}_k \rangle}$

(2.2.22) $\mathbf{p}_1 = \beta_o \mathbf{r}_1$

-) Following steps: for k=1,...,N

(2.2.23) $\alpha_k = \frac{1}{\langle \mathbf{B}\mathbf{p}_k, \mathbf{p}_k \rangle}$

(2.2.24) $\mathbf{z}_{k+1} = \mathbf{z}_k + \alpha_k \mathbf{p}_k$

(2.2.25) $\mathbf{r}_{k+1} = \mathbf{r}_k - \alpha k \mathbf{B} \mathbf{p}_k$

(2.2.26) $\beta_k = \frac{1}{\langle \mathbf{r}_{k+1}, \mathbf{r}_{k+1} \rangle}$

(2.2.27) $\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k$

The algorithm stops when k=N or when

(2.2.28) $\frac{\|\mathbf{r}_{k+1}\|}{\|\mathbf{h}\|} < \text{tolerance.}$

3.0 The Problem of Scattering from a Conducting Plate.

The formulation of the plate scattering problem begins by postulating that the total solution is given by the superimposition of a known source function and a perturbation function. For a plate illuminated by a plane wave, the total electric and magnetic field at any point in space will be denoted by $\mathbf{E}_t$ and $\mathbf{H}_t$, respectively. The source function is the incident electric or magnetic field which is assumed to originate at an infinite distance from the scatterer. The perturbation function is the scattered electric or magnetic field, which will be denoted by $\mathbf{E}_s$ and $\mathbf{H}_s$, respectively; the electric or magnetic field radiates outward from the scatterer and must approaches zero as the distance from the object become infinite. At any point in space, the fields are postulated to satisfy the following relations:

(3.0.1) $\mathbf{E}_t = \mathbf{E}_s + \mathbf{E}_i$

(3.0.2) $\mathbf{H}_t = \mathbf{H}_s + \mathbf{H}_i$

In the case of a material plate, the total fields inside the plate are directly related to the volume currents induced by the incident radiation. For a better understanding, we will first study the case of a perfectly conducting plate, which is simpler both from the theoretical and the analytical point of view. Approach towards the problem of a material plate can be cosidered simply as an extension of the previous case.

Let us consider a plane sinusoidal wave which illuminates a thin plate (i.e. the thickness τ of the plate is negligible with respect to the wavelength λ_p inside the plate), oriented as in Fig.(3.1).

In the following, we will only refer to the electric field when the magnetic field can be treated similarly. Let us assume that λ_o and f_o are the wavelength and the frequency of the incident electromagnetic wave, respectively. The electrical incident field $\mathbf{E}_i$ can be expressed by means of its components:

(3.0.3) $$\mathbf{E}_i = (E_{ix}\mathbf{i}_x + E_{iy}\mathbf{i}_y + E_{iz}\mathbf{i}_z)\, e^{-j(\mathbf{k}_i \cdot \mathbf{R})}$$

where $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ are the unit vectors of the x,y,z axes, respectively; the components of the electrical incident field are:

(3.0.4) $$E_{ix} = \cos(\alpha_i)\cos(\theta_i)\cos(\phi_i) - \sin(\alpha_i)\sin(\phi_i)$$

(3.0.5) $$E_{iy} = \cos(\alpha_i)\cos(\theta_i)\sin(\phi_i) + \sin(\alpha_i)\cos(\phi_i)$$

(3.0.6) $$E_{iz} = -\cos(\alpha_i)\sin(\theta_i)$$

$\mathbf{R}$ is the position vector defined as $\mathbf{R} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$; $\mathbf{k}_i$ is the propagation vector

(3.0.7) $$\mathbf{k}_i = -k_o\,[\sin(\theta_i)\,(\cos(\phi_i)\mathbf{i}_x + \sin(\phi_i)\mathbf{i}_y) + \cos(\theta_i)\mathbf{i}_z]$$

where $k_o = 2\pi/\lambda_o$ is the wavenumber.

The fundamental assumption in the scattering problem is that the total field $\mathbf{E}_t$ is:

(3.0.8) $$\mathbf{E}_t = \mathbf{E}_i + \mathbf{E}_s$$

where $\mathbf{E}_s$ is the scattered field. If the plate is perfectly conducting, then the tangential field is zero, i.e. $\mathbf{E}_s^T = - \mathbf{E}_i^T$.

The scattered field can be represented by a magnetic vector potential $\mathbf{F}_m$ and a scalar potential Φ which satisfy the following differential equations, in which $\mathbf{K}_e$ is the surface current and ε_o the permittivity of the free space:

(3.0.9) $$\nabla^2 \mathbf{F}_m + k_o^2 \mathbf{F}_m = - \mathbf{K}_e$$

(3.0.10) $$\nabla^2 \Phi + k_o^2 \Phi = - \rho_{es}$$

on the plate and

(3.0.11) $$\nabla^2 \mathbf{F}_m + k_o^2 \mathbf{F}_m = 0$$

(3.0.12) $$\nabla^2 \Phi + k_o^2 \Phi = 0$$

in the exterior region[1]. The impulse response of these equations yields Green's function $G(|\mathbf{R}|)$:

(3.0.13) $$G(|\mathbf{R}|) = \frac{e^{-jk_o|\mathbf{R}|}}{4\pi|\mathbf{R}|}$$

where $|\mathbf{R}| = \sqrt{x^2 + y^2 + z^2}$. The vector and scalar potentials can be obtained as a convolution between the impulse response of the free space (Green's function) and the surface current and surface charge, respectively:

(3.0.14) $$\mathbf{F}_m = \iint_{s'} \mathbf{K}_e(\mathbf{R}') \, G(|\mathbf{R} - \mathbf{R}'|) \, ds'$$

(3.0.15) $$\Phi = \frac{1}{\varepsilon_o} \iint_{s'} \rho_{es}(\mathbf{R}') \, G(|\mathbf{R} - \mathbf{R}'|) \, ds'$$

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[1]the index "e" indicates the dependence on the electric field, while "m" indicates dependence on the magnetic field.

By using the equation of continuity:

$$\nabla \cdot K_e = -j \frac{k_o}{Z_o \varepsilon_o} \rho_{es} \tag{3.0.16}$$

where Z_o is the characteristic impedance of the free space and ρ_{es} is the charge density induced on the plate.
It can be shown that the scattered field can be expressed by means of the only vector potential $\mathbf{F}_m$:

$$\mathbf{E}_s = -j \frac{Z_o}{k_o} \left[k_o^2 \mathbf{F}_m + \nabla(\nabla \cdot \mathbf{F}_m) \right] \tag{3.0.17}$$

If condition $\mathbf{E}_t^T = 0$ is verified, then each component of the incident electric field can be expressed in terms of surface currents as follows:

$$E_{ix} = j \frac{Z_o}{k_o} \iint_{s'} \left[K_{ex} \left(k_o^2 + \frac{\partial^2}{\partial x^2} G \right) + K_{ey} \frac{\partial^2}{\partial x \partial y} G \right] ds' \tag{3.0.18}$$

$$E_{iy} = j \frac{Z_o}{k_o} \iint_{s'} \left[K_{ex} \frac{\partial^2}{\partial x \partial y} G + K_{ey} \left(k_o^2 + \frac{\partial^2}{\partial y^2} G \right) \right] ds' \tag{3.0.19}$$

The problem, now, is how to solve these equations with the unknowns K_{ex} and K_{ey}. This problem has been successfully approached by Sarkar et Al. [6,7] using the Conjugate Gradient Method. The scattering problem discussed above requires the solution of an equation, which can be written in the general form:

$$\mathbf{Az} = \mathbf{b} \tag{3.0.20}$$

where $\mathbf{z}$ are the unknowns and $\mathbf{A}$ is a generic non-singular operator defined on a Hilbert space and is such that $\mathbf{A}^{-1}\mathbf{Az}=\mathbf{z}$. The Conjugate Gradient method is used to minimize a quadratic functional, but it can be successfully applied to solve equation (3.0.20). In fact the solution of (3.0.20) also minimizes the following quadratic functional $F(\mathbf{z})$:

(3.0.21) $F(\mathbf{z}) = \langle \mathbf{b}-\mathbf{Az}, \mathbf{b}-\mathbf{Az} \rangle$

which can be easily reconducted to the form of the functional (2.2.1) by developing the inner product in (3.0.21):

(3.0.22) $$\begin{aligned} F(\mathbf{z}) &= \langle \mathbf{b},\mathbf{b} \rangle - \langle \mathbf{b},\mathbf{Az} \rangle - \langle \mathbf{Az},\mathbf{b} \rangle + \langle \mathbf{Az},\mathbf{Az} \rangle \\ &= \langle \mathbf{b},\mathbf{b} \rangle - 2\,\mathrm{Re}\{\langle \mathbf{Az},\mathbf{b} \rangle\} + \langle \mathbf{Az},\mathbf{Az} \rangle = \end{aligned}$$

Note that the solution minimizing $F(\mathbf{z})$ is independent of the constant $\langle \mathbf{b},\mathbf{b} \rangle$ and of the multiplicative factor 2; hence the solution $\mathbf{z} = \mathbf{A}^{-1}\mathbf{b}$, minimizing the expression (3.0.20) also minimizes the functional $F(\mathbf{z})$ given by

(3.0.23) $F(\mathbf{z}) = \langle \mathbf{Az},\mathbf{Az} \rangle - 2\mathrm{Re}\{\langle \mathbf{Az},\mathbf{b} \rangle\} = \langle \mathbf{A}^a\mathbf{Az},\mathbf{z} \rangle - 2\mathrm{Re}\{\langle \mathbf{z},\mathbf{A}^a\mathbf{b} \rangle\}$

Making the substitutions $\mathbf{B} = \mathbf{A}^a\mathbf{A}$ and $\mathbf{h} = \mathbf{A}^a\mathbf{b}$, the eq. (3.0.23) assumes the same form as expression (2.2.1)

4.0 The discrete approximation of the Continuous Problem.

In order to discretize the scattering problem, the surface of the material plate has to be subdivided into rectangular cells, along both the x and y directions with steps $\Delta x = h_x$ and $\Delta y = h_y$, respectively. In order to avoid certain types of errors when computing FFT, it is worthwhile chosing square cells, such that the length of the sides of each cell is $\Delta x = \Delta y = h$.

Any geometrical planar shape can be approximated by a square cell grid; the decision whether some cells along the boundaries belong to the approximated shape or not is taken according to prestablished criteria. For example if the centroid of a boundary cell falls into the perimeter of the original shape, the cell belongs to the discrete shape or if the area of the cell covered by the original shape is more than 50% of the total area of the cell, it belongs to the discrete figure. Of course this information is coded in oder to be stored in a computer. Binary coding can be used, e.g. 1 for cells belonging to the original shape and 0 for the others.

Some physical hypotheses must also be made if the formulation is to be valid: each cell will have a constant thickness, permeability and permittivity, i.e. the material distribution and characteristics are constant over each cell. Note that this hypothesis allows inhomogeneous plates to be considered, too; in fact characteristics can be vary from cell to cell, and, in addition, step h can be made as small as desired, compatibly with the computational effort required by the algorithm.

Some examples of digital generation of plates are shown in Fig. (4.1). Square NxN matrices have been considered, without any consequence in digitizing the plate. The cell-centroid technique has been used to decide on the boundaries. In Fig. (4.2) the binary coded NxN matrix, referring to Fig.(4.1a), is depicted.

The direct and inverse two-dimensional Discrete Fourier Transform pair (denoted by F and F^{-1} respectively) of a discrete function $z(x_i, y_j)$ is given by:

$$\tilde{z}(f_{xl}, f_{yn}) = F\,(z(x_i,y_j)) = \sum_{i=1}^{N}\sum_{j=1}^{N} z(x_i,y_j)\, e^{-j2\pi\frac{ij}{N^2}} \tag{4.0.1}$$

$$z(x_i,y_j) = F^{-1}\,(\tilde{z}(f_{xl}, f_{yn})) = \frac{1}{N^2}\sum_{l=1}^{N}\sum_{n=1}^{N} \tilde{z}(f_{xl}, f_{yn})\, e^{-j2\pi\frac{ln}{N^2}} \tag{4.0.2}$$

The next problem is to discretize the differential equations (3.0.18) and (3.0.19). This can be achieved by using the finite center differences, which are shown in the following table:

Continuous Symbol	Discrete Symbol	Discrete Expression
$\frac{\partial}{\partial x}$	d_x	$\frac{1}{2h_x}[d_{1,0} - d_{-1,0}] + O(h_x)^2$
$\frac{\partial}{\partial y}$	d_y	$\frac{1}{2h_y}[d_{0,1} - d_{0,-1}] + O(h_y)^2$
$\frac{\partial^2}{\partial x^2}$	d_{xx}	$\frac{1}{h_x^2}[d_{1,0} - 2d_{0,0} + d_{-1,0}] + O(h_x)^2$
$\frac{\partial^2}{\partial y^2}$	d_{yy}	$\frac{1}{h_y^2}[d_{0,1} - 2d_{0,0} + d_{0,-1}] + O(h_y)^2$
$\frac{\partial^2}{\partial x \partial y}$	d_{xy}	$\frac{1}{4h_xh_y}[d_{1,1} - d_{-1,1} + d_{-1,-1} - d_{1,-1}] + O(h_xh_y)$

Table 4.1: Finite differences

5.0 AN APPLICATION OF THE METHOD

The appropriate integral equations for the solution of the scattering from a perfectly conducting plate are given by:

(5.0.1) $$\omega_1 E_{ix} = \iint_{S'} (\Psi_1 K_{ex} + \Psi_2 K_{ey})\, ds'$$

(5.0.2) $$\omega_1 E_{iy} = \iint_{S'} (\Psi_2 K_{ex} + \Psi_3 K_{ey})\, ds'$$

where:

(5.0.3) $$\Psi_1 = \left(k_o^2 + \frac{\partial^2}{\partial x^2}\right) G$$

(5.0.4) $$\Psi_2 = \frac{\partial^2}{\partial x \partial y} G$$

(5.0.5) $$\Psi_3 = \left(k_o^2 + \frac{\partial^2}{\partial y^2}\right) G$$

(5.0.6) $$G = \frac{e^{-jk_o\sqrt{(x-x')^2 + (y-y')^2}}}{4\pi \sqrt{(x-x')^2 + (y-y')^2}}$$

(5.0.7) $$E_{ix} = [\cos(\alpha_i)\cos(\theta_i)\cos(\phi_i) - \sin(\alpha_i)\sin(\phi_i)]\, e^{-j(\mathbf{k}_i \cdot \mathbf{r})}$$

(5.0.8) $$E_{iy} = [\cos(\alpha_i)\cos(\theta_i)\sin(\phi_i) - \sin(\alpha_i)\cos(\phi_i)]\, e^{-j(\mathbf{k}_i \cdot \mathbf{r})}$$

(5.0.9) $$\mathbf{k}_i \cdot \mathbf{r} = -\, k_o \sin(\theta_i)\, [\cos(\phi_i)\, x + \sin(\phi_i)\, y]$$

(5.0.10) $$\omega_1 = -j \frac{k_o}{Z_o}$$

The plate is divided into NxN square cells of side length h, over which the surface current is assumed to be constant. A set of matrix equations are imposed by satisfying the equations at the centroid of each square cell. The matrix equation may be represented as:

(5.0.11) $$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} K_{ex} \\ K_{ey} \end{bmatrix} = \omega_1 \begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix}$$

where A,B,C are NxN complex matrices and K_{ex}, K_{ey}, E_{ix}, E_{iy} are NxN complex vectors.

Equation (5.0.11) has a simple physical interpretation. The left hand side of (5.0.11) represents the scattered field, while the second member is related to the incident field. It is found that each row of the matrix equation stands for the position of the boundary conditions at the centroid of the pertinent cell. In particular, the term (m,n) in the coefficient matrix represents the x or y component of the field radiated in the centroid of the n-th cell by a unit surface current lying in the x or y direction at the centroid of the m-th cell.

The matrix elements are computed by letting the row index denotes the observation point or the unprimed coordinate and the column index represents the source point or primed coordinate. It can be shown that if simple midpoint integration is used, then, for all points where the observation point is not equal to the source point, the matrix elements are given by:

$$A_{mn} = h^2 \left[\left(\frac{3}{r_{mn}^2} - k_o^2 + j\frac{3k_o}{r_{mn}} \right) \left(\frac{x_m - x_n}{r_{mn}} \right)^2 + k_o^2 - \left(\frac{1}{r_{mn}} + jk_o \right) \frac{1}{r_{mn}} \right] G(r_{mn})$$

(5.0.12)

$$B_{mn} = h^2 \left[\left(\frac{3}{r_{mn}^2} - k_o^2 + j\frac{3k_o}{r_{mn}} \right) \left(\frac{(x_m - x_n)(y_m - y_n)}{r_{mn}^2} \right) \right] G(r_{mn})$$

(5.0.13)

$$C_{mn} = h^2 \left[\left(\frac{3}{r_{mn}^2} - k_o^2 + j\frac{3k_o}{r_{mn}} \right) \left(\frac{y_m - y_n}{r_{mn}} \right)^2 + k_o^2 - \left(\frac{1}{r_{mn}} + jk_o \right) \frac{1}{r_{mn}} \right] G(r_{mn})$$

(5.0.14)

where

(5.0.15) $$G(r_{mn}) = \frac{e^{-jk_o r_{mn}}}{4\pi r_{mn}}$$

(5.0.16) $$r_{mn} = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}$$

When the observation point is also the source point, then the cell integral must be performed analytically due to the singularity of Green's function. These cell elements are usually called the "self-cell" contributions. The two integrals computed for m=n have the form:

(5.0.16) $$I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} G(\mathbf{r},\mathbf{r}')\, ds'$$

(5.0.16) $$I_2 = \frac{\partial^2}{\partial x^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} G(\mathbf{r},\mathbf{r}')\, ds'$$

Integral I_1 may be computed directly as:

(5.0.17) $$I_1 \approx \frac{1}{4\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1 - jk_o|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|}\, ds'$$

$$= -j\frac{k_o h^2}{4\pi} + \frac{1}{4\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{|\mathbf{r} - \mathbf{r}'|}\, ds' = -j\frac{k_o h^2}{4\pi} + \frac{2}{\pi} \int_0^{\frac{\pi}{4}} \int_0^{\frac{h}{2\cos(\phi)}} d\rho\, d\phi =$$

$$= -j\frac{k_o h^2}{4\pi} + \frac{2}{\pi} \int_0^{\frac{\pi}{4}} \frac{1}{\cos(\phi)}\, d\phi = -j\frac{k_o h^2}{4\pi} + \frac{h}{\pi} \left[\ln \sqrt{\frac{1 + \sin(\phi)}{1 - \sin(\phi)}} \right]_0^{\pi/4} =$$

(5.0.18) $$= -j\frac{k_o h^2}{4\pi} + \frac{h}{\pi} \left[\ln \sqrt{\frac{\sqrt{2} + 1}{\sqrt{2} - 1}} \right]$$

Integral I_2 is taken from Miron [11] and is given by:

(5.0.19) $$I_2 \approx -\frac{\sqrt{2}}{\pi h} + j\frac{k_o^3 h^2}{8\pi}$$

This yields the self cell matrix elements:

(5.0.20) $$A_{mm} = C_{mm} = \frac{1}{\pi} \left[hk_o^2 \ln \sqrt{\frac{\sqrt{2} + 1}{\sqrt{2} - 1}} - \frac{\sqrt{2}}{h} - j\frac{k_o^3 h^2}{8} \right]$$

(5.0.21) $B_{mn} = 0.$

Note that the matrices are symmetric.

By using the finite differences scheme, shown in Table 4.1, the conjugate gradient method is given as follows:

-) First step: Initialize the residual and search vectors

(5.0.22) $\gamma_h = \|h_x\|^2 + \|h_y\|^2$

(5.0.23) $d_x = A\,K_x^1 + B\,K_y^1$

(5.0.24) $d_y = B\,K_x^1 + C\,K_y^1$

(5.0.25) $\mathbf{r}_{x,y}^1 = h_{x,y} - d_{x,y}$

(5.0.26) $d_x = A^*\,\mathbf{r}_x^1 + B^*\,\mathbf{r}_y^1$

(5.0.27) $d_y = B^*\,r_x^1 + C^*\,r_y^1$

(5.0.28) $\gamma_d = \|d_x\|^2 + \|d_y\|^2$

(5.0.29) $\beta_o = \gamma_d^{-1}$

(5.0.30) $\mathbf{p}_{xy}^1 = \beta_o\,d_{x,y}$

-) Following steps: for k=1,...,N

(5.0.31) $d_x = A^*\,\mathbf{p}_x^k + B^*\,\mathbf{p}_y^k$

(5.0.32) $d_y = B^*\,\mathbf{p}_x^k + C^*\,\mathbf{p}_y^k$

(5.0.33) $\gamma_d = \|d_x\|^2 + \|d_y\|^2$

(5.0.34) $\alpha_k = \gamma_d^{-1}$

(5.0.35) $K_{x,y}^{k+1} = K_{x,y}^k + \alpha_k \mathbf{p}_{x,y}^k$

(5.0.36) $\mathbf{r}_{x,y}^{k+1} = \mathbf{r}_{x,y}^k + \alpha_k d_{x,y}^k$

(5.0.37) $\gamma_r = \|\mathbf{r}_x^{k+1}\|^2 + \|\mathbf{r}_y^{k+1}\|^2$

(5.0.38) $$d_x = A^* \mathbf{r}_x^{k+1} + B^* \mathbf{r}_y^{k+1}$$

(5.0.39) $$d_y = B^* \mathbf{r}_x^{k+1} + C^* \mathbf{r}_y^{k+1}$$

(5.0.40) $$\gamma_d = \|d_x\|^2 + \|d_y\|^2$$

(5.0.41) $$\beta_k = \gamma_d^{-1}$$

(5.0.42) $$\mathbf{p}_{x,y}^{k+1} = \mathbf{p}_{x,y}^{k} + \beta_k d_{x,y}$$

The algorithm stops when k=N or when

(5.0.42) $$\sqrt{\frac{\gamma_r}{\gamma_h}} < \text{tolerance}$$

6.0 THE GENERAL CASE OF A MATERIAL PLATE

In the case of a material plate the permittivity ε and the permeability μ of the material have to be considered in the resolution equations; they are defined by:

(6.0.1) $$\varepsilon = \varepsilon_o \varepsilon_r$$

(6.0.2) $$\mu = \mu_o \mu_r$$

where both the relative permittivity and permeability ε_r and μ_r are complex quantities. The total internal fields may be expressed in terms of electric and magnetic volume currents $\mathbf{J}_e$ and $\mathbf{J}_m$ as

(6.0.3) $$\mathbf{E}_t = \frac{-jZ_o}{(\varepsilon_r - 1)k_o} \mathbf{J}_e$$

(6.0.4) $$\mathbf{H}_t = \frac{-j}{(\mu_r - 1)k_o Z_o} \mathbf{J}_m$$

For a thin material of thickness τ, the volume currents are proportional to the surface currents $\mathbf{K}_e$ and $\mathbf{K}_m$ such that at the interior midpoint we have:

(6.0.5) $$\mathbf{E}_t = \frac{-jZ_o}{(\varepsilon_r - 1)k_o\tau} \mathbf{K}_e = Z_e \mathbf{K}_e$$

(6.0.6) $$\mathbf{H}_t = \frac{-j}{(\mu_r - 1)k_o Z_o \tau}\,\mathbf{K}_m = Y_m\,\mathbf{K}_m$$

where the electric impedance Z_e and the magnetic admittance Y_m are obviously defined by:

(6.0.7) $$Z_e = \frac{-jZ_o}{(\varepsilon_r - 1)k_o \tau}$$

(6.0.8) $$Y_m = \frac{-j}{(\mu_r - 1)k_o Z_o \tau}$$

and are often referred to as the resistivity and the conductivity of the layer.

The scattered fields may be written in terms of the electric $\mathbf{F}_e$ and the magnetic $\mathbf{F}_m$ vector potentials as:

(6.0.9) $$\mathbf{E}_s = -\nabla\times\mathbf{F}_e - j\frac{Z_o}{k_o}\left(\nabla\times\nabla\times\mathbf{F}_m - \frac{\mathbf{K}_e}{\tau}\right)$$

(6.0.10) $$\mathbf{H}_s = -\nabla\times\mathbf{F}_m - j\frac{1}{Z_o k_o}\left(\nabla\times\nabla\times\mathbf{F}_e - \frac{\mathbf{K}_m}{\tau}\right)$$

where $\mathbf{F}_e$ and $\mathbf{F}_m$ satisfy the inhomogeneous vector Helmholtz equations:

(6.0.11) $$\nabla^2\,\mathbf{F}_m + k_o^2\,\mathbf{F}_m = -\frac{\mathbf{K}_e}{\tau}$$

(6.0.12) $$\nabla^2\,\mathbf{F}_e + k_o^2\,\mathbf{F}_e = -\frac{\mathbf{K}_m}{\tau}$$

The vector potentials are then given by the convolution of the currents with the impulse response of free space, such that:

(6.0.13) $$\mathbf{F}_m = \iint_{S'}\mathbf{K}_e(\mathbf{R}')\,G(|\mathbf{R} - \mathbf{R}'|)\,ds'$$

(6.0.14) $$\mathbf{F}_e = \iint_{S'}\mathbf{K}_m(\mathbf{R}')\,G(|\mathbf{R} - \mathbf{R}'|)\,ds'$$

Note that these integral equations have the form of convolution integrals and they can be computed by means of some Fast Fourier Transform algorithm.

Let us now recall that the Fourier Transform of the Green function is given by:

$$\tilde{G}(f_x,f_y) = \begin{cases} -j\dfrac{1}{4\pi\sqrt{|f_x^2+f_y^2|}} & \text{for } f_x^2+f_y^2<1 \\ \dfrac{1}{4\pi\sqrt{|f_x^2+f_y^2|}} & \text{for } f_x^2+f_y^2>1 \end{cases} \tag{6.0.15}$$

The goal is to solve equations (6.0.9) and (6.0.10); it is then necessary to calculate currents $\mathbf{K}_e$ and $\mathbf{K}_m$, by means of eq. (6.0.11)÷(6.0.14).

The hypothesis that the plate is electrically thin (i.e. $\tau \ll \lambda_p$), implies that the internal field components are assumed to have a constant variation such that $\dfrac{\partial}{\partial z} = 0$. The normal components E_z and H_z are then found directly from eq. (6.0.9) and (6.0.10) and the tangential components E_x, E_y, H_x, H_y can be found from eq. (6.0.9) and (6.0.10) into which eqs. (6.0.11) and (6.0.12) have been substituted. The fundamental equations (3.0.1) and (3.0.2) can be split as follows

$$Z_e K_{ex} + j\frac{Z_o}{k_o}\left[\left(k_o^2 + \frac{\partial^2}{\partial x^2}\right)F_{mx} + \frac{\partial^2}{\partial x \partial y}F_{my}\right] + \frac{\partial}{\partial y}F_{ez} = E_{ix} \tag{6.0.16}$$

$$Z_e K_{ey} + j\frac{Z_o}{k_o}\left[\frac{\partial^2}{\partial x \partial y}F_{mx} + \left(k_o^2 + \frac{\partial^2}{\partial y^2}\right)F_{my}\right] - \frac{\partial}{\partial x}F_{ez} = E_{iy} \tag{6.0.17}$$

$$\varepsilon_r Z_e K_{ez} - \frac{\partial}{\partial y}F_{ex} + \frac{\partial}{\partial x}F_{ey} - j\frac{Z_o}{k_o}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F_{mz} = E_{iz} \tag{6.0.18}$$

$$Y_m K_{mx} + j\frac{1}{k_o Z_o}\left[\left(k_o^2 + \frac{\partial^2}{\partial x^2}\right)F_{ex} + \frac{\partial^2}{\partial x \partial y}F_{ey}\right] + \frac{\partial}{\partial x}F_{mz} = H_{ix} \tag{6.0.19}$$

$$Y_m K_{my} + j\frac{1}{k_o Z_o}\left[\frac{\partial^2}{\partial x \partial y}F_{ex} + \left(k_o^2 + \frac{\partial^2}{\partial y^2}\right)F_{ey}\right] - \frac{\partial}{\partial x}F_{mz} = H_{iy} \tag{6.0.20}$$

$$\mu_r Y_m K_{mz} + \frac{\partial}{\partial y}F_{mx} - \frac{\partial}{\partial x}F_{my} - j\frac{1}{k_o Z_o}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F_{ez} = H_{iz} \tag{6.0.21}$$

where, of course, F_{mx}, F_{my}, F_{mz}, F_{ex}, F_{ey}, F_{ez} are the components of $\mathbf{F}_m$ and $\mathbf{F}_e$ in the integral equations (6.0.13) and (6.0.14). At each step of the conjugate gradient method, an FFT algorithm is applied to the convolution integrals in order to calculate each component of the current (i.e. K_{ex}, K_{ey}, K_{ez}, K_{mx}, K_{my}, K_{mz}).

Equation (6.0.16)–(6.0.21) can be rewritten in a more compact form, by assuming that the electric currents and the incident magnetic field are multiplied by Z_o and the coordinates are divided by λ_o:

(6.0.22) $$\eta_{et}K_{ex} + c_oL_1F_{mx} + c_oL_2F_{my} + L_6F_{ez} = E_{ix}$$

(6.0.23) $$\eta_{et}K_{ey} + c_oL_2F_{mx} + c_oL_3F_{my} - L_5F_{ez} = E_{iy}$$

(6.0.24) $$\eta_{en}K_{ez} - L_6F_{ex} + L_5F_{ey} - c_oL_4F_{mz} = E_{iz}$$

(6.0.25) $$\eta_{mt}K_{mx} + c_oL_1F_{ex} + c_oL_2F_{ey} - L_6F_{mz} = H_{ix}$$

(6.0.26) $$\eta_{mt}K_{my} + c_oL_2F_{ex} + c_oL_3F_{ey} + L_5F_{ez} = H_{iy}$$

(6.0.27) $$\eta_{mn}K_{mz} + L_6F_{mx} - L_5F_{my} - c_oL_4F_{ez} = H_{iz}$$

where

(6.0.28) $$L_1 = 4\pi^2 + \frac{\partial^2}{\partial x^2}$$

(6.0.29) $$L_2 = \frac{\partial^2}{\partial x \partial y}$$

(6.0.30) $$L_3 = 4\pi^2 + \frac{\partial^2}{\partial y^2}$$

(6.0.31) $$L_4 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(6.0.32) $$L_5 = \frac{\partial}{\partial x}$$

(6.0.33) $$L_6 = \frac{\partial}{\partial y}$$

and

(6.0.34) $$\eta_{et} = \frac{Z_e}{Z_o}$$

(6.0.35) $$\eta_{en} = \varepsilon_r \frac{Z_e}{Z_o}$$

(6.0.36) $\eta_{mt} = Y_m Z_o$

(6.0.37) $\eta_{mn} = \mu_r Y_m Z_o$

(6.0.38) $c_o = j \frac{1}{2\pi}$

Equations (6.0.22)-(6.027) can be discretized by the finite differences shown in Table 4.1 and the complete FFT-Conjugate Gradient Method can be applied.

7.0 CONCLUSIONS

The FFT-Conjugate Gradient Method seems to be very attractive from a computational point of view, expecially compared with other more traditional methods employed for solving scattering problems. On the other side, it has a limit in digitizing the object and truncating the functions, because of the errors which are introduced by these type of processing.

The most interesting aspect, which is to be investigated, is the extension to the three-dimensional surfaces, in particular to surfaces with some discontinuities, as wedge. This argument will be object of future study.

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FIGURE CAPTIONS

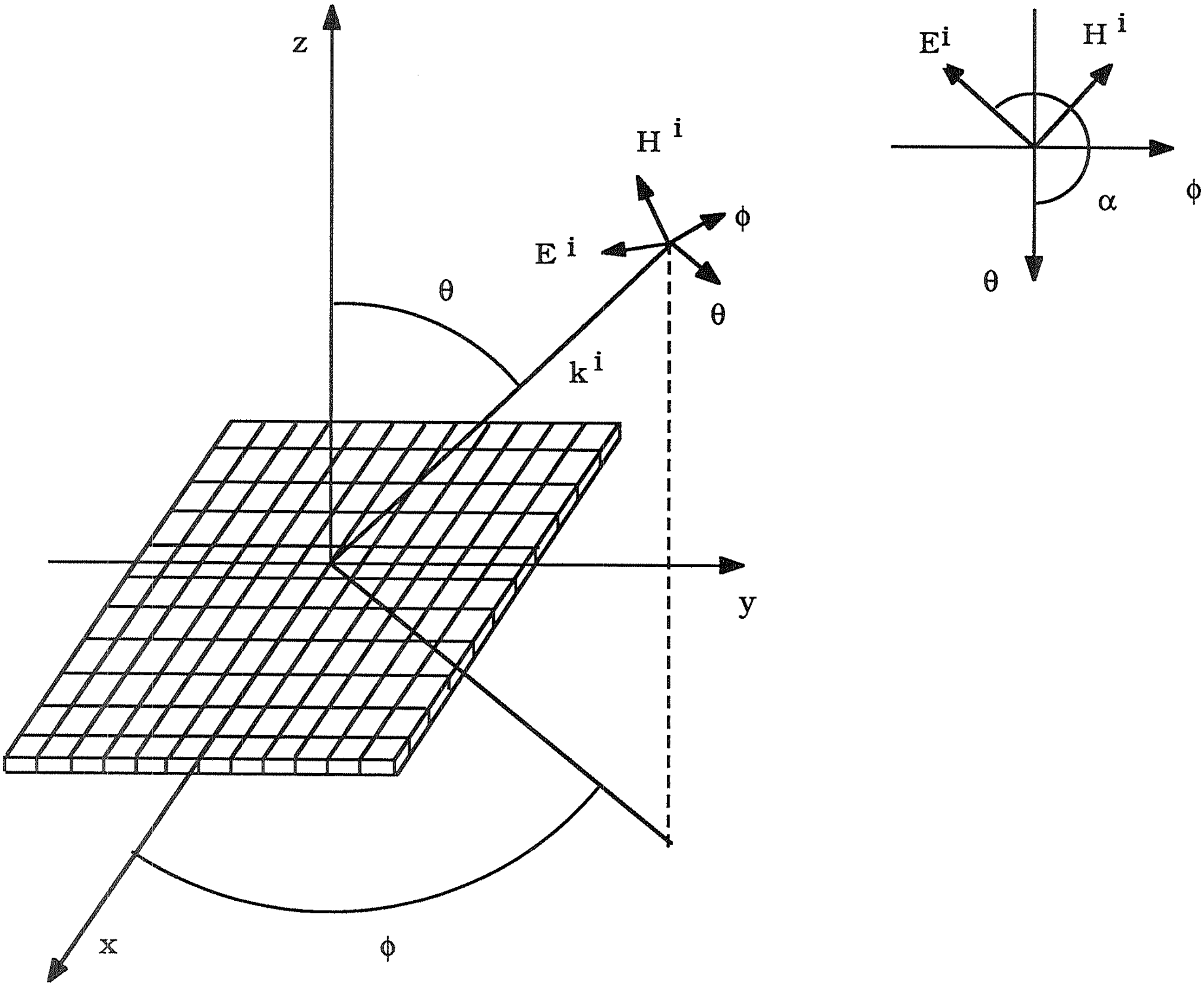

Fig. 3.1

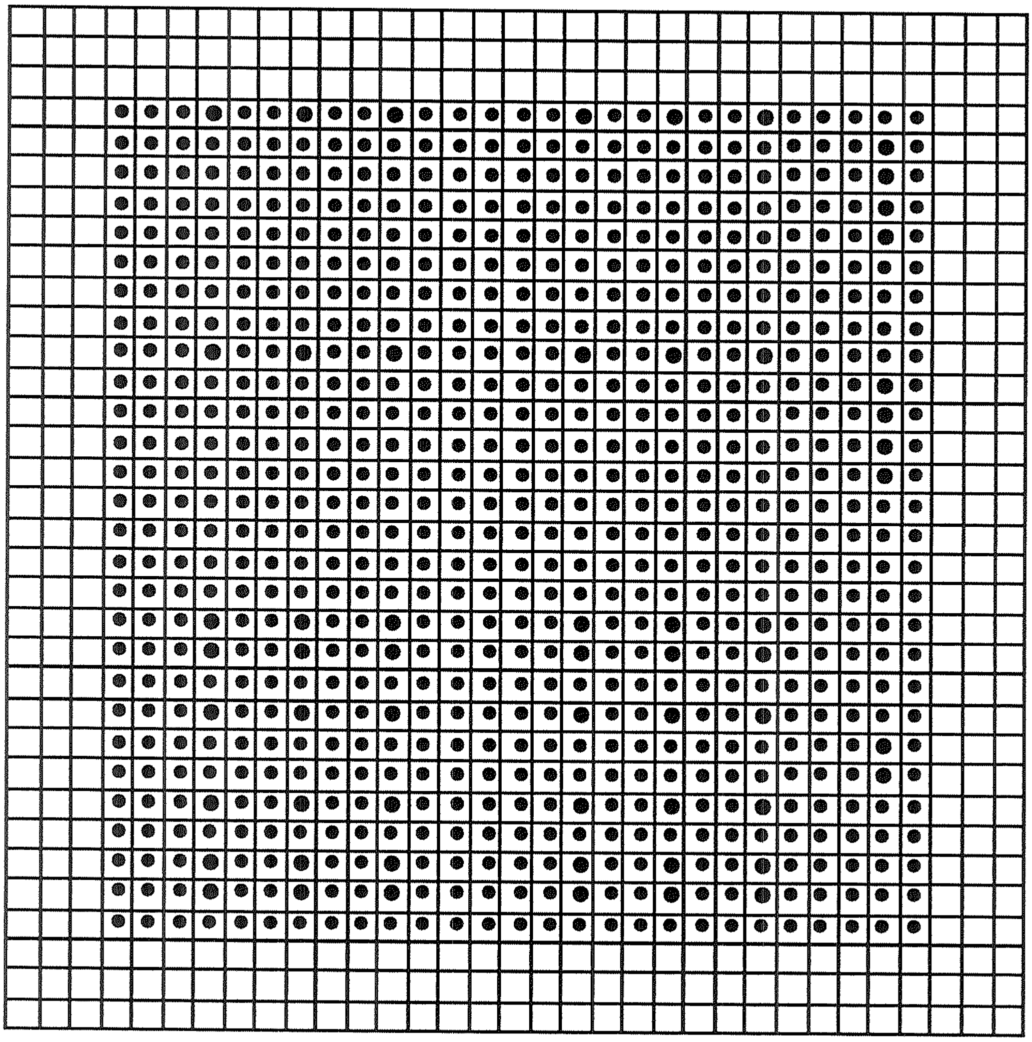

Fig. 4.1a

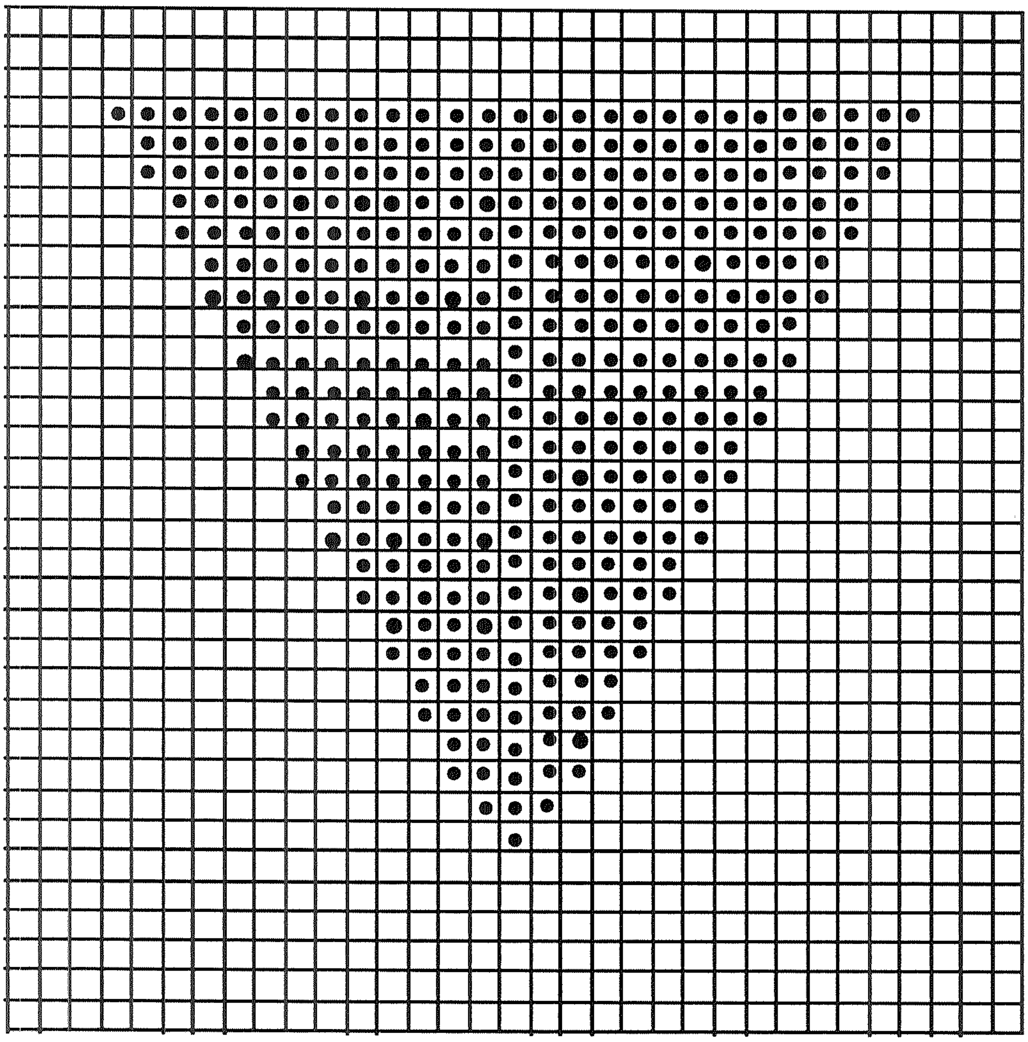

Fig. 4.1b

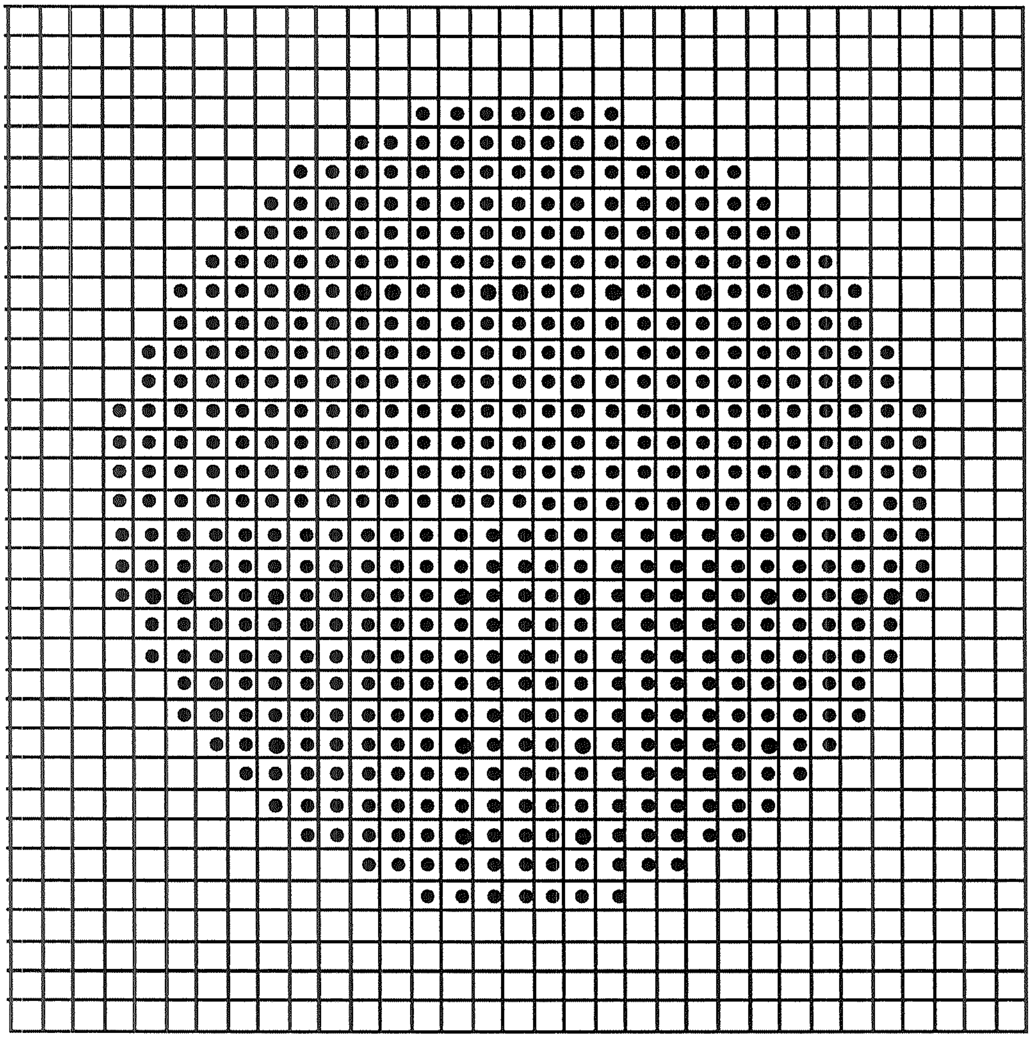

Fig. 4.1c

Fig. 4.2