DIGITAL FOURIER TRANSFORM ON STAGGERED BLOCKS: RECURSIVE SOLUTION II+

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Relations (3),(6) and (5) represent the *recursive procedure* for evaluating the DFT on blocks of N elements, staggered of M. Such a procedure consists of:

1. constructing the set H^j;
2. computing the $GF^{s/L}$, s=0,1,...,L-1, on the ordered set H^j;
3. computing the N coefficients F^j_{s+Lr} by means of the recursive equation (5).

An assumption must be made that F^0_{s+Lr} is known. Such a procedure is graphically represented in Fig. 1.

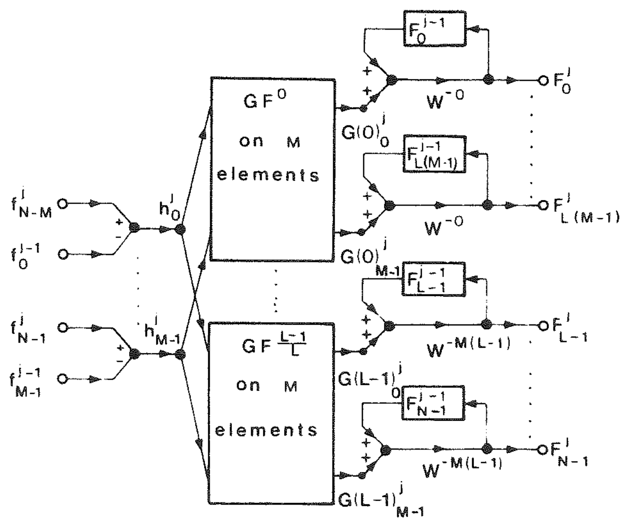

Fig. 1. Flow-graph of the recursive procedure

FGF Algorithm

Let

$$H = \left\{h_m \mid m=0,1,\ldots,M-1\right\}$$

be a block of M elements. The GF^t on block H is another block of M elements (*coefficients*)

$$K = \left\{G_r \mid G_r = \sum_{m=0}^{M-1} h_m V^{m(r+t)},\ r=0,1,\ldots,M-1\right\}, \qquad (7)$$

where $V=\exp(-2\pi\sqrt{-1}/M)$. The GF^t is reduced to the DFT for t=0.

We will show that for the GF there exists a *decimation in time* algorithm similar to the one holding for the DFT. In fact, the coefficients G_r can be expressed as

$$\begin{cases} G_{r'} = X_{r'} + V^{(r'+t)} Y_{r'} \\ G_{r'+M/2} = X_{r'} - V^{(r'+t)} Y_{r'} \end{cases} \qquad (8)$$

where r'=0,1,...,M/2 -1, and

$$\begin{cases} X_{r'} = \sum_{m'=0}^{M/2-1} h_{2m'} (V^2)^{m'(r'+t)} \\ Y_{r'} = \sum_{m'=0}^{M/2-1} h_{2m'+1} (V^2)^{m'(r'+t)} \end{cases}$$

are the coefficients of the GFs on the blocks constituted by the elements of H having even and odd index, respectively. Thus a GF on a block of M elements can be reduced to two GFs (having the same frequency parameter as the initial GF) on two blocks of M/2 elements each. This is the *decimation in time* algorithm for the GF.

This algorithm, if iterated, leads to a fast algorithm based on decimation in time for the GF, called *FGF algorithm*. Observe that, at the q-th iteration, the exponents of V can be obtained from those relative to the DFT by adding $2^{q-1}t$.

From (8) it results that the complexity of an FGF algorithm of M elements is equal to that of an FFT algorithm on M elements. That is, the number $\phi_g(M)$ of complex multiplications and the number $\psi_g(M)$ of complex additions are:

$$\begin{cases} \phi_g(M) = \frac{M}{2}\log_2 M \\ \psi_g(M) = M \log_2 M \end{cases}$$

Complexity of the recursive procedure

Let us evaluate the complexity of the recursive procedure on blocks of N elements staggered of M. For obtaining the elements of H^j in (3), M complex additions are needed. For evaluating the L GFs, expressed by relation (6), by means of the FGF algorithm, $(N/2)\log_2 M$ complex multiplications and $N \log_2 M$ complex additions are needed. Finally, for evaluating relation (5), N complex multiplications and N complex additions are needed. Therefore, the numbers $\phi(N,M)$ of complex multiplications and $\psi(N,M)$ of complex additions of the recursive procedure are respectively:

$$\begin{cases} \phi(N,M) = \frac{N}{2}\log_2 M + N = \frac{N}{2}\log_2 4M \\ \psi(N,M) = M + N\log_2 M + N = N\log_2 2M + M \end{cases}$$

We observe that the computation of a DFT on each block can be performed by means of an FFT algorithm on N elements, no regard being made on the relation between successive blocks. The recursive procedure presents, over the FFT algorithm, a gain in number of complex multiplications and complex additions given, respectively, by:

$$\begin{cases} \Delta^{\phi}(N,M) = \frac{N}{2}\log_2 \frac{N}{4M} \\ \Delta^{\psi}(N,M) = N\log_2 \frac{N}{2M} - M \end{cases}$$

Observe that $\Delta^{\phi}(N,M)$ and $\Delta^{\psi}(N,M)$ are greater than zero for M<N/4 and M<N/2 respectively. For every fixed N, these gains increase with decreasing M.

A structure for mechanizing the recursive procedure

In this section we will present a structure for implementing the recursive procedure (see Fig. 2), that works on the hypothesis that elements arrive sequentially to the structure itself, at uniformly spaced time instants. Under this hypothesis, the M elements of H^j can be constructed sequentially, and the L GFs on such block can be performed by means of L/2 structures, called P2-structures, that are similar to "radix-2 pipeline structures" based on decimation in time, introduced elsewhere[6,7]. Let us consider a P2-structure. It is able to perform two GFs with different frequency parameters, say s_1/L and s_2/L, on two blocks of M elements, provided that 1) at every discrete time instant one element arrives for each block and 2) the second block is delayed by one half the block length. The q-th stage of such a structure, (q=1,2,...,$\log_2 M$ from the output side to the input side), is depicted in Fig. 3. The memory RE_q is of

2^{q-1} elements, and the commutator control switches every 2^{q-1} steps. The first stage has no memory on the lower path. The multiplicative factors to be presented at the multiplier can be obtained from those relative to the DFT by adding to the exponent of V the quantity $2^{q-1}s_1/L$ for M/2 steps (while the stage is working on the GF with frequency parameter s_1/L), and the quantity $2^{q-1}s_2/L$ for the next M/2 steps (while the stage is working on the GF with frequency parameter s_2/L).

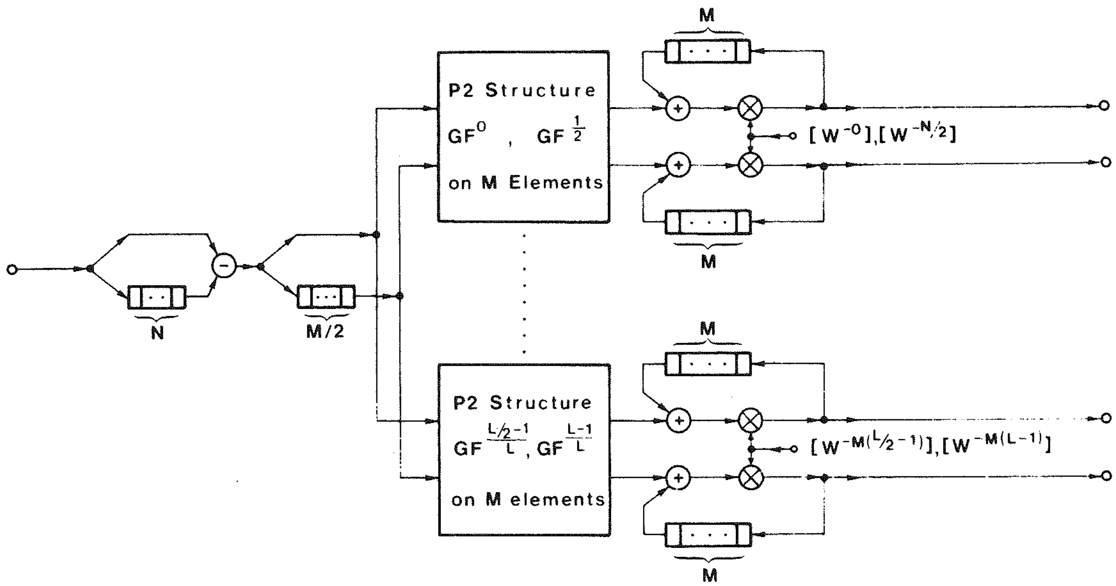

Fig. 2. A structure for mechanizing the recursive procedure

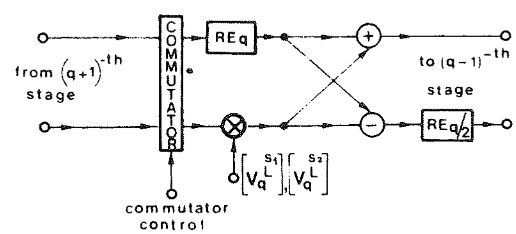

Fig. 3. The q-th stage of a P2-structure

That is, for the multiplier of the q-th stage, we have two sets $[V_q^{s_1/L}]$ and $[V_q^{s_2/L}]$ of multiplicative factors, having as elements, respectively, the quantities

$$\begin{cases} V^{(2^{q-1})(t_q+s_1/L)} \\ V^{(2^{q-1})(t_q+s_2/L)} \end{cases}$$

where $t_q=0,1,\ldots,(M/2^q)-1$ in bit reversed order. The set $[V_q^{s_1/L}]$ is repeated 2^{q-1} times, then the set $[V_q^{s_2/L}]$ is repeated 2^{q-1} times, and so on.

At the two outputs of a P2-structure we have sequentially the coefficients $G(s_1)^j_r$ for M/2 steps, and the coefficients $G(s_2)^j_r$ for the next M/2 steps. From these quantities and from the old coefficients $F^{j-1}_{s_1+Lr}$ and $F^{j-1}_{s_2+Lr}$ we obtain the new coefficients $F^j_{s_1+Lr}$ and $F^j_{s_2+Lr}$. This is illustrated in Fig. 2, where $[W^{-Ms_1}]$, $[W^{-Ms_2}]$ means that the multiplicative factor W^{-Ms_1} is repeated for M/2 steps, and then the multiplicative factor W^{-Ms_2} is repeated for M/2 steps, and so on.

We will now evaluate the number of multipliers, adders and memory elements required by the structure. Let us refer to the multipliers. We have $(L/2)\log_2 M$ of them inside and L of them at the outputs of the L/2 P2-structures. Therefore, their total number is:

$$MU(N,M) = \frac{N}{2M}\log_2 4M$$

Let us refer to the adders. We have 1 of them before, $L\log_2 M$ of them inside and L of them at the outputs of the L/2 P2-structures. Therefore, their total number is:

$$AD(N,M) = \frac{N}{M}\log_2 2M + 1$$

Let us refer to the memory elements. We have N+M/2 of them before and ML=N of them at the outputs of the L/2 P2-structures. As a P2-structure requires (3/2)M-2 memory elements, their total number is:

$$ME(N,M) = \frac{11}{4}N + \frac{M}{2} - \frac{N}{M}$$

Conclusions

A procedure for solving the DFT updating problem has been presented, that applies to the case in which successive blocks of N elements (N power of 2) are staggered of M<N elements (M dividing N).

The procedure evaluates the coefficients of the DFT on a new block as a function of 1) the coefficients of the DFT on the previous block, 2) the coefficients of proper GFs on differences between elements being not common to two successive blocks. Its complexity, in number of complex multiplications, is $(N/2)\log_2 4M$. For every fixed N, the complexity increases with increasing M.

A structure for mechanizing the procedure has been proposed that is able to execute one DFT on N elements every M steps. It works in real time, on the hypothesis that data arrive sequentially.

References

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