

A class of quasi-linear Allen–Cahn type equations with dynamic boundary conditions*

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Abstract. In this paper, we consider a class of coupled systems of PDEs, denoted by $(ACE)_\varepsilon$ for $\varepsilon \geq 0$. For each $\varepsilon \geq 0$, the system $(ACE)_\varepsilon$ consists of an Allen–Cahn type equation in a bounded spacial domain Ω , and another Allen–Cahn type equation on the smooth boundary $\Gamma := \partial\Omega$, and besides, these coupled equations are transmitted via the dynamic boundary conditions. In particular, the equation in Ω is derived from the non-smooth energy proposed by Visintin in his monography “Models of phase transitions”: hence, the diffusion in Ω is provided by a quasilinear form with singularity. The objective of this paper is to build a mathematical method to obtain meaningful L^2 -based solutions to our systems, and to see some robustness of $(ACE)_\varepsilon$ with respect to $\varepsilon \geq 0$. On this basis, we will prove two Main Theorems 1 and 2, which will be concerned with the well-posedness of $(ACE)_\varepsilon$ for each $\varepsilon \geq 0$, and the continuous dependence of solutions to $(ACE)_\varepsilon$ for the variations of $\varepsilon \geq 0$, respectively.

Key words and phrases: quasi-linear Allen–Cahn equation, dynamic boundary conditions, non-smooth energy functional, initial-boundary value problem, well-posedness, continuous dependence.

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Introduction

Let $0 < T < \infty$, $\kappa > 0$ and $N \in \mathbb{N}$ be fixed constants. Let $Q := (0, T) \times \Omega$ be a product set of a time-interval $(0, T)$ and a bounded spatial domain $\Omega \subset \mathbb{R}^N$. Let $\Gamma := \partial\Omega$ be the boundary of Ω with sufficient smoothness (when $N > 1$), and let n_Γ be the unit outer normal to Γ . Besides, we put $\Sigma := (0, T) \times \Gamma$.

In this paper, we fix a constant $\varepsilon \geq 0$ to consider the following system of PDEs, denoted by $(ACE)_\varepsilon$.

$(ACE)_\varepsilon$:

$$\partial_t u - \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} + \kappa^2 \nabla u \right) + \beta(u) + g(u) \ni \theta \text{ in } Q, \quad (0.1)$$

$$\begin{aligned} \partial_t u_\Gamma - \varepsilon^2 \Delta_\Gamma u_\Gamma + \left(\frac{\nabla u}{|\nabla u|} + \kappa^2 \nabla u \right) \Big|_\Gamma \cdot n_\Gamma + \beta_\Gamma(u_\Gamma) + g_\Gamma(u_\Gamma) \ni \theta_\Gamma \\ \text{and } u|_\Gamma = u_\Gamma \text{ on } \Sigma, \end{aligned} \quad (0.2)$$

$$u(0, \cdot) = u_0 \text{ in } \Omega, \text{ and } u_\Gamma(0, \cdot) = u_{\Gamma,0} \text{ on } \Gamma. \quad (0.3)$$

The system $(ACE)_\varepsilon$ is a modified version of an Allen–Cahn type equation, proposed in [36, Chapter VI], and the principal modifications are in the points that:

- the quasi-linear (singular) diffusion in (0.1) includes the regularization term $\kappa^2 \nabla u$ with a small constant $\kappa > 0$;
- the boundary data u_Γ is governed by the dynamic boundary condition (0.2).

In general, “Allen–Cahn type equation” is a collective term to call gradient flows (systems) of governing energies, which include some double-well type potentials to reproduce the bi-stability of different phases, such as solid-liquid phases. The governing energy is called *free-energy*, and in the case of $(ACE)_\varepsilon$, the corresponding free-energy is provided as follows.

$$\begin{aligned} [u, u_\Gamma] \in H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma) &\mapsto \mathcal{F}_\varepsilon(u, u_\Gamma) \\ &:= \int_\Omega \left(|\nabla u| + \frac{\kappa^2}{2} |\nabla u|^2 + B(u) + G(u) \right) dx \\ &\quad + \int_\Gamma \left(\frac{\varepsilon^2}{2} |\nabla_\Gamma u_\Gamma|^2 + B_\Gamma(u_\Gamma) + G_\Gamma(u_\Gamma) \right) d\Gamma \in (-\infty, \infty], \end{aligned} \quad (0.4)$$

with the effective domain:

$$D(\mathcal{F}_\varepsilon) := \left\{ [z, z_\Gamma] \left| \begin{array}{l} z \in H^1(\Omega), z_\Gamma \in H^{\frac{1}{2}}(\Gamma), \varepsilon z_\Gamma \in H^1(\Gamma), \\ \text{and } z|_\Gamma = z_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma) \end{array} \right. \right\}.$$

In the context, “ $|_\Gamma$ ” denotes the trace (boundary-value) on Γ for a Sobolev function, $d\Gamma$ denotes the area-element on Γ , ∇_Γ denotes the surface gradient on Γ , and Δ_Γ denotes the Laplacian on the surface, i.e., the so-called Laplace–Beltrami operator. $B : \mathbb{R} \rightarrow [0, \infty]$ and $B_\Gamma : \mathbb{R} \rightarrow [0, \infty]$ are given proper l.s.c. and convex functions, and $\beta = \partial B$ and $\beta_\Gamma = \partial B_\Gamma$ are the subdifferentials of B and B_Γ , respectively. $G : \mathbb{R} \rightarrow \mathbb{R}$ and $G_\Gamma : \mathbb{R} \rightarrow \mathbb{R}$ are C^1 -functions, that have locally Lipschitz differentials g and g_Γ , respectively. $\theta : Q \rightarrow \mathbb{R}$ and $\theta_\Gamma : \Sigma \rightarrow \mathbb{R}$ are given heat sources of (relative) temperature, and $u_0 : \Omega \rightarrow \mathbb{R}$ and $u_{\Gamma,0} : \Gamma \rightarrow \mathbb{R}$ are initial data for the components u and u_Γ , respectively.

In (0.4), the functions:

$$\sigma \in \mathbb{R} \mapsto B(\sigma) + G(\sigma) \in (-\infty, \infty] \text{ and } \sigma \in \mathbb{R} \mapsto B_\Gamma(\sigma) + G_\Gamma(\sigma) \in (-\infty, \infty],$$

correspond to the double-well potentials, and for instance, the setting:

$$B(\sigma) = B_\Gamma(\sigma) = I_{[-1,1]}(\sigma) \text{ and } G(\sigma) = G_\Gamma(\sigma) = -\frac{1}{2}\sigma^2, \text{ for } \sigma \in \mathbb{R},$$

with use of the indicator function:

$$\sigma \in \mathbb{R} \mapsto I_{[-1,1]}(\sigma) := \begin{cases} 0, & \text{if } \sigma \in [-1, 1], \\ \infty, & \text{otherwise,} \end{cases}$$

is known as one of representative choices of the components (cf. [36]).

Additionally, it should be noted that the presence or absence of the term

$$\frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma u_\Gamma|^2 d\Gamma$$

brings the gap of effective domains $D(\mathcal{F}_\varepsilon)$ between the cases when $\varepsilon > 0$ and $\varepsilon = 0$. More precisely, the domains $D(\mathcal{F}_\varepsilon)$ when $\varepsilon > 0$ will uniformly coincide with a convex subset in $H^1(\Omega) \times H^1(\Gamma)$, and this convex set will be a proper subset of the domain $D(\mathcal{F}_0)$ when $\varepsilon = 0$ which will be located in the wider space $H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$.

In the case when the diffusion in (0.1) is just given by the usual Laplacian, the corresponding Allen–Cahn equation has been studied by a number of researches (cf., e.g., [5, 7, 15, 16, 23]), and some qualitative results for L^2 -based solutions were obtained by means of the theories of parabolic PDEs, in [27, 29]. To investigate dynamic boundary conditions, our approach exploits techniques similar to those employed in [5] and resumed in other solvability studies and optimal control theories, the reader may see [7, 8, 9, 10, 11, 12, 13]. Still about dynamic boundary conditions, let us point out that there has been a recent growing interest about the justification and the study of phase field models, as well as systems of Allen–Cahn and Cahn–Hilliard type, including dynamic boundary conditions. Without trying to be exhaustive, let us mention at least the papers [6, 15, 17, 19, 20, 21, 22, 28, 31].

Nevertheless, the mathematical analysis for our system $(ACE)_\varepsilon$ will not be just an analogy work with the previous ones. In fact, due to the singularity of the diffusion $-\operatorname{div}(\frac{\nabla u}{|\nabla u|} + \kappa^2 \nabla u)$ in (0.1), it will not be so easy to apply the theories of [27, 29], and to see the L^2 -based expression of the first variation of the free-energy.

In view of this, we set the goal in this paper to show the following two Main Theorems, which are concerned with qualitative properties of the systems $(ACE)_\varepsilon$ for $\varepsilon \geq 0$.

Main Theorem 1: the well-posedness for $(ACE)_\varepsilon$, for all $\varepsilon \geq 0$.

Main Theorem 2: the continuous dependence of solutions to $(ACE)_\varepsilon$ with respect to the value of $\varepsilon \geq 0$, and especially the (right-hand) continuity at $\varepsilon = 0$.

The content of this paper is as follows. The Main Theorems 1 and 2 are stated in Section 2, and these results are discussed on the basis of the preliminaries prepared in Section 1, and Key-Lemmas 1–3. Based on this, we give the proofs of the Key-Lemmas and Main Theorems in the remaining Sections 4 and 5, respectively.

1 Preliminaries

In this Section, we outline some basic notations and known facts, as preliminaries of our study.

Notation 1 (Notations in real analysis) For arbitrary $a, b \in [-\infty, \infty]$, we define:

$$a \vee b := \max\{a, b\} \text{ and } a \wedge b := \min\{a, b\}.$$

Let $d \in \mathbb{N}$ be any fixed dimension. Then, we simply denote by $|x|$ and $x \cdot y$ the Euclidean norm of $x \in \mathbb{R}^d$ and the standard scalar product of $x, y \in \mathbb{R}^d$, respectively. Besides, we denote by \mathbb{B}^d and \mathbb{S}^{d-1} the d -dimensional unit open ball centered at the origin, and its boundary, respectively.

For any $d \in \mathbb{N}$, the d -dimensional Lebesgue measure is denoted by \mathcal{L}^d , and d -dimensional Hausdorff measure is denoted by \mathcal{H}^d . Unless otherwise specified, the measure theoretical phrases, such as ‘‘a.e.’’, ‘‘ dt ’’, ‘‘ dx ’’, and so on, are with respect to the Lebesgue measure in each corresponding dimension. Also, in the observations on a smooth surface S , the phrase ‘‘a.e.’’ is with respect to the Hausdorff measure in each corresponding Hausdorff dimension, and the area element on S is denoted by dS .

Additionally, we mention about the following elementary fact, which is used, frequently, in the proofs of Key-Lemmas and Main Theorems.

(Fact 0) Let $m \in \mathbb{N}$ be a fixed finite number. If $\{\alpha_1, \dots, \alpha_m\} \subset \mathbb{R}$ and $\{a_n^k\}_{n=1}^\infty$, $k = 1, \dots, m$, fulfill that

$$\varliminf_{n \rightarrow \infty} a_n^k \geq \alpha_k, \quad k = 1, \dots, m, \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} \sum_{k=1}^m a_n^k \leq \sum_{k=1}^m \alpha_k$$

then, it holds that

$$\lim_{n \rightarrow \infty} a_n^k = \alpha_k, \quad k = 1, \dots, m.$$

Notation 2 (Notations of functional analysis) For an abstract Banach space X , we denote by $|\cdot|_X$ the norm of X , and denote by ${}_X \langle \cdot, \cdot \rangle_X$ the duality pairing between X and the dual space X^* of X . Let $\mathcal{I}_X : X \rightarrow X$ be the identity map from X onto X . In particular, when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product in X .

Here and in the sequel, Ω denotes an open subset of \mathbb{R}^N , which we assume to be bounded and smooth. Moreover, Γ and n_Γ denote its boundary $\partial\Omega$ and the outward unit normal vector field on Γ , respectively. Let Δ_N be the Laplace operator, subject to the Neumann-zero boundary condition, which is defined as:

$$\begin{aligned} \Delta_N : v \in D(\Delta_N) &:= \left\{ z \in H^2(\Omega) \mid (\nabla z)|_\Gamma \cdot n_\Gamma = 0 \text{ in } H^{\frac{1}{2}}(\Gamma) \right\} \subset L^2(\Omega) \\ &\mapsto \Delta_N v := \Delta v \in L^2(\Omega). \end{aligned} \tag{1.1}$$

In this paper, we identify the unbounded closed operator $-\Delta_N$ with its linear and continuous extension from $H^1(\Omega)$ into $H^1(\Omega)^*$, by setting:

$${}_{H^1(\Omega)^*} \langle \Delta_N z, w \rangle_{H^1(\Omega)} = \int_\Omega \nabla z \cdot \nabla w \, dx, \quad \text{for all } [z, w] \in H^1(\Omega) \times H^1(\Omega).$$

Remark 1.1 Note that the boundary $\Gamma = \partial\Omega$ has no boundary as $(N - 1)$ -dimensional surface. Therefore, for any $s > 0$, the dual space $H^{-s}(\Gamma) := H^s(\Gamma)^*$ of the Sobolev space $H^s(\Gamma)$ coincides with the closure of the class of the smooth functions on Γ in the topology of $H^s(\Gamma)^*$.

Notation 3 (Notations of surface-differentials) Since Ω is bounded and smooth, there exists a function $d_\Gamma : \overline{\Omega} \rightarrow \mathbb{R}$ such that

$$d_\Gamma \text{ is smooth on } \overline{\Omega} \text{ and } d_\Gamma(x) = \inf_{y \in \Gamma} |x - y| \text{ for } x \text{ in a neighborhood of } \Gamma. \quad (1.2)$$

We notice that $n_\Gamma(x) = -\nabla d_\Gamma(x)$ for every $x \in \Gamma$. On this basis, let ∇_Γ be the operator of surface-gradient on Γ , which is defined as:

$$\nabla_\Gamma : \varphi \in C^1(\Gamma) \mapsto \nabla_\Gamma \varphi := \nabla \varphi^{\text{ex}} - (\nabla d_\Gamma \otimes \nabla d_\Gamma) \nabla \varphi^{\text{ex}} \in C^0(\Gamma)^N, \quad (1.3)$$

by using the extension $\varphi^{\text{ex}} \in C^1(\overline{\Omega})$ of each $\varphi \in C^1(\Gamma)$. Let div_Γ be the operator of surface-divergence, which is defined as:

$$\text{div}_\Gamma : \omega \in C^1(\Gamma)^N \mapsto \text{div}_\Gamma \omega := \text{div} \omega^{\text{ex}} - \nabla(\omega^{\text{ex}} \cdot \nabla d_\Gamma) \cdot \nabla d_\Gamma \in C^0(\Gamma), \quad (1.4)$$

by using the extension $\omega^{\text{ex}} \in C^1(\overline{\Omega})^N$ of each $\omega \in C^1(\Gamma)^N$.

It is known that the definition formulas (1.3)–(1.4) are well-defined, and the values $\nabla_\Gamma \varphi$ and $\text{div}_\Gamma \omega$ are settled independently of the choices of the extensions $\varphi^{\text{ex}} \in C^1(\overline{\Omega})$ and $\omega^{\text{ex}} \in C^1(\overline{\Omega})^N$ of $\varphi \in C^1(\Gamma)$ and $\omega \in C^1(\Gamma)^N$, respectively, and of the function d_Γ satisfying (1.2).

On the basis of (1.3)–(1.4), the Laplace–Beltrami operator Δ_Γ , i.e., the surface-Laplacian on Γ is defined as follows:

$$\Delta_\Gamma : \varphi \in C^2(\Gamma) \mapsto \Delta_\Gamma \varphi := \text{div}_\Gamma(\nabla_\Gamma \varphi) \in C^0(\Gamma),$$

by using the extension $\varphi^{\text{ex}} \in C^2(\overline{\Omega})$ of each $\varphi \in C^2(\Gamma)$.

Remark 1.2 Let us define a closed subspace $\mathbf{L}_{\text{div}}^2(\Omega)$ in $L^2(\Omega)^N$ and a closed subspace $\mathbf{L}_{\text{tan}}^2(\Gamma)$ in $L^2(\Gamma)^N$, by putting:

$$\mathbf{L}_{\text{div}}^2(\Omega) := \{ \omega \in L^2(\Omega)^N \mid \text{div } \omega \in L^2(\Omega) \}, \text{ and}$$

$$\mathbf{L}_{\text{tan}}^2(\Gamma) := \{ \omega \in L^2(\Gamma)^N \mid \omega \cdot n_\Gamma = 0 \text{ a.e. on } \Gamma \}, \text{ respectively.}$$

Then, on account of the general theories as in [25, 33], we can see the following facts (cf. [25]).

(Fact 1) The mapping $\nu \in H^1(\Omega)^N \mapsto \nu|_\Gamma \cdot n_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ can be extended as a linear and continuous operator $[(\cdot) \cdot n_\Gamma]_\Gamma$ from $\mathbf{L}_{\text{div}}^2(\Omega)$ into $H^{-\frac{1}{2}}(\Gamma)$, such that:

$$\begin{aligned} {}_{H^{-1/2}(\Gamma)} \langle [\nu \cdot n_\Gamma]_\Gamma, z|_\Gamma \rangle_{H^{1/2}(\Gamma)} &= \int_\Omega \text{div } \nu z \, dx + \int_\Omega \nu \cdot \nabla z \, dx, \\ &\text{for all } \nu \in \mathbf{L}_{\text{div}}^2(\Omega) \text{ and } z \in H^1(\Omega). \end{aligned}$$

(Fact 2) The surface gradient ∇_Γ can be extended as a linear and continuous operator from $H^1(\Gamma)$ into $\mathbf{L}_{\text{tan}}^2(\Gamma)$. The extension is derived in the definition process of the space $H^1(\Gamma)$ as the completion of $C^1(\Gamma)$. Then, the topology of the completion is taken with respect to the norm, induced by the following bi-linear form:

$$[\varphi, \psi] \in C^1(\Gamma)^2 \mapsto \int_\Gamma (\varphi\psi + \nabla_\Gamma\varphi \cdot \nabla_\Gamma\psi) d\Gamma.$$

The inner product in $(\cdot, \cdot)_{H^1(\Gamma)}$ is given as the extension of the above bi-linear form. Hence, in this paper, we identify the operator ∇_Γ with the extension from $H^1(\Gamma)$ into $\mathbf{L}_{\text{tan}}^2(\Gamma)$.

(Fact 3) The surface divergence div_Γ can be extended as a linear and continuous operator from $\mathbf{L}_{\text{tan}}^2(\Gamma)$ into $H^{-1}(\Gamma)$ ($= H^1(\Gamma)^*$), via the following Green-type formula:

$${}_{H^{-1}(\Gamma)}\langle \text{div}_\Gamma w, z \rangle_{H^1(\Gamma)} = - \int_\Gamma w \cdot \nabla_\Gamma z d\Gamma, \text{ for all } z \in H^1(\Gamma) \text{ and } w \in \mathbf{L}_{\text{tan}}^2(\Gamma).$$

Hence, in this paper, we regard the Laplace–Beltrami operator $\Delta_\Gamma = \text{div}_\Gamma \circ \nabla_\Gamma$ as a linear and continuous operator from $H^1(\Gamma)$ into $H^{-1}(\Gamma)$. In particular, the operator $-\Delta_\Gamma$ forms a duality map between $H^1(\Gamma)$ and $H^{-1}(\Gamma)$.

Notation 4 (Notations in convex analysis) For any proper lower semi-continuous (l.s.c. from now on) and convex function Ψ defined on a Hilbert space X , we denote by $D(\Psi)$ its effective domain, and denote by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of Ψ , and it turns out to be a maximal monotone graph in the product space $X \times X$. More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as a set of all elements $z_0^* \in X$ which satisfy the following variational inequality:

$$(z_0^*, z - z_0)_X \leq \Psi(z) - \Psi(z_0), \text{ for any } z \in D(\Psi).$$

The set $D(\partial\Psi) := \{z \in X \mid \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$. We often use the notation “[z_0, z_0^*] $\in \partial\Psi$ in $X \times X$ ”, to mean that “[$z_0^* \in \partial\Psi(z_0)$ in X with $z_0 \in D(\partial\Psi)$ ”, by identifying the operator $\partial\Psi$ with its graph in $X \times X$. Let us refer to [2, 3, 4, 26] for definitions, properties, results about subdifferentials and maximal monotone operators.

Remark 1.3 As one of representatives of the subdifferentials, we exemplify the following set-valued function $\text{Sgn} : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$, given as:

$$\omega \in \mathbb{R}^N \mapsto \text{Sgn}(\omega) := \begin{cases} \frac{\omega}{|\omega|}, & \text{if } \omega \neq 0, \\ \overline{\mathbb{B}^N}, & \text{otherwise.} \end{cases}$$

It is known that the set-valued function Sgn coincides with the subdifferential of the Euclidean norm $|\cdot| : \omega \in \mathbb{R}^N \mapsto |\omega| = \sqrt{\omega \cdot \omega} \in [0, \infty)$. Also, it is known that (cf. [3, 4]) the operator $-\Delta_N$ defined in (1.1) coincides with the subdifferential of the proper l.s.c. and convex function Ψ_N on $L^2(\Omega)$, defined as:

$$z \in L^2(\Omega) \mapsto \Psi_N(z) := \begin{cases} \frac{1}{2} \int_\Omega |\nabla z|^2 dx, & \text{if } z \in H^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

More precisely, we have:

$$\partial\Psi_N(z) = \{-\Delta_N z\} \text{ in } L^2(\Omega), \text{ for any } z \in D(\partial\Psi_N) = D(\Delta_N).$$

Remark 1.4 (Time-dependent subdifferentials) It is often useful to consider the subdifferentials under time-dependent settings of convex functions. With regard to this topic, certain general theories were established by some researchers (e.g., Kenmochi [26] and Ôtani [32]). So, referring to, e.g., [26, Chapter 2] or [34, Remark 1.1 (Fact 1)], we can see the following fact.

(Fact 4) Let E_0 be a convex subset in a Hilbert space X , let $I \subset [0, \infty)$ be a time-interval, and for any $t \in I$, let $\Psi^t : X \rightarrow (-\infty, \infty]$ be a proper l.s.c. and convex function such that $D(\Psi^t) = E_0$ for all $t \in I$. Based on this, let us define a convex function $\hat{\Psi}^I : L^2(I; X) \rightarrow (-\infty, \infty]$, by putting:

$$\zeta \in L^2(I; X) \mapsto \hat{\Psi}^I(\zeta) := \begin{cases} \int_I \Psi^t(\zeta(t)) dt, & \text{if } \Psi^{(\cdot)}(\zeta) \in L^1(I), \\ \infty, & \text{otherwise.} \end{cases}$$

Here, if $E_0 \subset D(\hat{\Psi}^I)$, i.e., if the function $t \in I \mapsto \Psi^t(z)$ is integrable for any $z \in E_0$, then it holds that:

$$\begin{aligned} & [\zeta, \zeta^*] \in \partial\hat{\Psi}^I \text{ in } L^2(I; X) \times L^2(I; X), \text{ iff} \\ & \zeta \in D(\hat{\Psi}^I) \text{ and } [\zeta(t), \zeta^*(t)] \in \partial\Psi^t \text{ in } X \times X, \text{ a.e. } t \in I. \end{aligned}$$

Finally, we mention about notions of convergence for functionals.

Definition 1.1 (Mosco convergence: cf. [30]) Let X be an abstract Hilbert space. Let $\Psi : X \rightarrow (-\infty, \infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \rightarrow (-\infty, \infty]$, $n \in \mathbb{N}$. Then, it is said that $\Psi_n \rightarrow \Psi$ on X , in the sense of Mosco [30], as $n \rightarrow \infty$, iff the following two conditions are fulfilled.

(M1) The condition of lower-bound: $\liminf_{n \rightarrow \infty} \Psi_n(z_n^\dagger) \geq \Psi(z^\dagger)$, if $z^\dagger \in X$, $\{z_n^\dagger\}_{n=1}^\infty \subset X$, and $z_n^\dagger \rightarrow z^\dagger$ weakly in X as $n \rightarrow \infty$.

(M2) The condition of optimality: for any $z^\ddagger \in D(\Psi)$, there exists a sequence $\{z_n^\ddagger\}_{n=1}^\infty \subset X$ such that $z_n^\ddagger \rightarrow z^\ddagger$ in X and $\Psi_n(z_n^\ddagger) \rightarrow \Psi(z^\ddagger)$, as $n \rightarrow \infty$.

Remark 1.5 (cf. [2, Proposition 2.68 and Theorem 3.26]) For a proper l.s.c. and convex function $\Psi : H \rightarrow (-\infty, \infty]$ on a Hilbert space H , it is known that the sequence $\{\Psi^\lambda\}_{\lambda>0}$ of Moreau–Yosida regularizations:

$$z \in H \mapsto \Psi^\lambda(z) := \inf \left\{ \frac{1}{2\lambda} |\tilde{z} - z|_H^2 + \Psi(\tilde{z}) \mid \tilde{z} \in H \right\}, \text{ for } \lambda > 0,$$

converges to Ψ on H , in the sense of Mosco, as $\lambda \downarrow 0$.

Definition 1.2 (Γ -convergence: cf. [14]) Let X be an abstract Hilbert space, $\Psi : X \rightarrow (-\infty, \infty]$ be a proper functional, and $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper functionals $\Psi_n : X \rightarrow (-\infty, \infty]$, $n \in \mathbb{N}$. We say that $\Psi_n \rightarrow \Psi$ on X , in the sense of Γ -convergence [14], as $n \rightarrow \infty$ iff the following two conditions are fulfilled.

(G1) **The condition of lower-bound:** $\liminf_{n \rightarrow \infty} \Psi_n(z_n^\dagger) \geq \Psi(z^\dagger)$ if $z^\dagger \in X$, $\{z_n^\dagger\}_{n=1}^\infty \subset X$, and $z_n^\dagger \rightarrow z^\dagger$ (strongly) in X as $n \rightarrow \infty$.

(G2) **The condition of optimality:** for any $z^\ddagger \in D(\Psi)$, there exists a sequence $\{z_n^\ddagger\}_{n=1}^\infty \subset X$ such that $z_n^\ddagger \rightarrow z^\ddagger$ in X and $\Psi_n(z_n^\ddagger) \rightarrow \Psi(z^\ddagger)$ as $n \rightarrow \infty$.

Remark 1.6 Of course, the Γ -convergence recalled in Definition 1.2 is the one associated with the strong topology of X . Note that if the functionals are convex, then the Mosco convergence just introduced implies Γ -convergence, i.e., the Γ -convergence of convex functions can be regarded as a weak version of Mosco convergence. Additionally, as a basic matter of the Mosco convergence, we can see the following fact (see [26, Chapter 2], or [34, Remark 1.5 (Fact 7)], for example).

(Fact 5) Let X , Ψ and $\{\Psi_n\}_{n=1}^\infty$ be as in Definition 1.1. Besides, let us assume that:

$$\Psi_n \rightarrow \Psi \text{ on } X, \text{ in the sense of } \Gamma\text{-convergence, as } n \rightarrow \infty,$$

and

$$\begin{cases} [z, z^*] \in X \times X, & [z_n, z_n^*] \in \partial\Psi_n \text{ in } X \times X, n \in \mathbb{N}, \\ z_n \rightarrow z \text{ in } X \text{ and } z_n^* \rightarrow z^* \text{ weakly in } X, & \text{as } n \rightarrow \infty. \end{cases}$$

Then, it holds that:

$$[z, z^*] \in \partial\Psi \text{ in } X \times X, \text{ and } \Psi_n(z_n) \rightarrow \Psi(z), \text{ as } n \rightarrow \infty.$$

2 Statements of Main Theorems

First, we configure the base-space of solutions to the systems $(ACE)_\varepsilon$, for $\varepsilon \geq 0$. In any case of $\varepsilon \geq 0$, the base-space is settled by a product Hilbert space:

$$\mathcal{H} := L^2(\Omega) \times L^2(\Gamma),$$

endowed with the inner product:

$$([z_1, z_{\Gamma,1}], [z_2, z_{\Gamma,2}])_{\mathcal{H}} := (z_1, z_2)_{L^2(\Omega)} + (z_{\Gamma,1}, z_{\Gamma,2})_{L^2(\Gamma)},$$

$$\text{for any } [z_k, z_{\Gamma,k}], k = 1, 2.$$

Next, we prescribe the assumptions in our study.

(A0) $N \in \mathbb{N}$ and $0 < T < \infty$ are fixed constants, and Ω is a bounded domain in \mathbb{R}^N with a smooth boundary Γ . In particular, it fulfills the condition (1.2) in Notation 3.

(A1) $B : D(B) \subset \mathbb{R} \rightarrow [0, \infty]$ and $B_\Gamma : D(B_\Gamma) \subset \mathbb{R} \rightarrow [0, \infty]$ are proper l.s.c. and convex functions and $\beta = \partial B \subset \mathbb{R} \times \mathbb{R}$ and $\beta_\Gamma = \partial B_\Gamma \subset \mathbb{R} \times \mathbb{R}$ are the subdifferentials of B and B_Γ , respectively. Furthermore, the convex functions B and B_Γ , and the subdifferentials β and β_Γ fulfill the following conditions:

(a1) $B(0) = 0$ and $B_\Gamma(0) = 0$, and hence $[0, 0] \in \beta$ and $[0, 0] \in \beta_\Gamma$ on \mathbb{R}^2 ;

(a2) there exists an interval $I_B \subset \mathbb{R}$, such that:

$$\text{int} I_B \neq \emptyset, \quad D(\beta) = D(\beta_\Gamma) = I_B, \quad \text{and } B, B_\Gamma \in C(\overline{I_B}) \cap L^\infty(I_B);$$

(a3) there exist positive constants $a_k, b_k, k = 0, 1$, such that:

$$a_0 |[\beta_\Gamma]^\circ(\tau)| - b_0 \leq |[\beta]^\circ(\tau)| \leq a_1 |[\beta_\Gamma]^\circ(\tau)| + b_1, \quad \text{for any } \tau \in I_B,$$

where $[\beta]^\circ$ and $[\beta_\Gamma]^\circ$ are the minimal sections for β and β_Γ , respectively.

(A2) $G : \mathbb{R} \rightarrow \mathbb{R}$ and $G_\Gamma : \mathbb{R} \rightarrow \mathbb{R}$ are $W_{\text{loc}}^{2,\infty}$ -functions such that the differentials $g = G'$ and $g_\Gamma = G'_\Gamma$ are Lipschitz continuous on $\overline{I_B}$.

(A3) The forcing pair $[\theta, \theta_\Gamma]$ belongs to $L^2(0, T; \mathcal{H})$ and the initial pair $[u_0, u_{\Gamma,0}]$ belongs to a class \mathcal{D}_* , defined as:

$$\mathcal{D}_* := \left\{ [z, z_\Gamma] \in \mathcal{H} \mid z \in \overline{I_B} \text{ a.e. in } \Omega \text{ and } z_\Gamma \in \overline{I_B} \text{ a.e. on } \Gamma \right\}. \quad (2.1)$$

In addition, let us set:

$$\mathcal{V}_\varepsilon := \left\{ [z, z_\Gamma] \in \mathcal{H} \mid \begin{array}{l} z \in H^1(\Omega), z_\Gamma \in H^{\frac{1}{2}}(\Gamma), \varepsilon z_\Gamma \in H^1(\Gamma), \\ \text{and } z|_\Gamma = z_\Gamma \text{ a.e. on } \Gamma \end{array} \right\}, \quad \text{for any } \varepsilon \geq 0,$$

and let us define the projection function $\mathcal{T}_B : \mathbb{R} \rightarrow \overline{I_B}$, by putting:

$$r \in \mathbb{R} \mapsto \mathcal{T}_B r := (r \vee (\inf I_B)) \wedge (\sup I_B) \in \overline{I_B}. \quad (2.2)$$

Then, we easily check the following facts.

(Fact 6) If $\varepsilon > 0$, then the space \mathcal{V}_ε is a closed linear subspace in $H^1(\Omega) \times H^1(\Gamma)$. Otherwise (i.e., if $\varepsilon = 0$), the corresponding space \mathcal{V}_0 is a closed linear subspace in $H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$. Hence, when $\varepsilon > 0$ (resp. $\varepsilon = 0$), the space \mathcal{V}_ε (resp. \mathcal{V}_0) forms a Hilbert space endowed with the inner product in $H^1(\Omega) \times H^1(\Gamma)$ (resp. $H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$).

(Fact 7) For any $\varepsilon \geq 0$, let us put:

$$\mathcal{D}_\varepsilon := \left\{ [z, z_\Gamma] \in \mathcal{V}_\varepsilon \mid B(z) \in L^1(\Omega) \text{ and } B_\Gamma(z_\Gamma) \in L^1(\Gamma) \right\}. \quad (2.3)$$

Then, the closures of \mathcal{D}_ε , for $\varepsilon \geq 0$, in the topology of \mathcal{H} coincide with the class \mathcal{D}_* given in (2.1), i.e.

$$\mathcal{D}_* = \overline{\mathcal{D}_\varepsilon} \text{ in } \mathcal{H}, \quad \text{for any } \varepsilon \geq 0.$$

Based on the above (A1)–(A3) and (Fact 6)–(Fact 7), the solutions to $(ACE)_\varepsilon$, for $\varepsilon \geq 0$, are defined as follows.

Definition 2.1 (Definition of solutions) A pair $[u, u_\Gamma]$ of functions $u : [0, T] \rightarrow L^2(\Omega)$ and $u_\Gamma : [0, T] \rightarrow L^2(\Gamma)$ is called a solution to $(ACE)_\varepsilon$, iff $[u, u_\Gamma]$ fulfills the following conditions.

(S1) $[u, u_\Gamma] \in C([0, T]; \mathcal{H}) \cap W_{\text{loc}}^{1,2}((0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}_\varepsilon) \cap L_{\text{loc}}^\infty((0, T]; \mathcal{V}_\varepsilon)$,
 $[u(0), u_\Gamma(0)] = [u_0, u_{\Gamma,0}]$ in \mathcal{H} .

(S2) There exist functions $\nu_u : Q \rightarrow \mathbb{R}^N$, $\xi : (0, T) \rightarrow L^2(\Omega)$ and $\xi_\Gamma : (0, T) \rightarrow L^2(\Gamma)$ such that:

$$\nu_u \in L^\infty(Q) \text{ and } \nu_u \in \text{Sgn}(\nabla u) \text{ a.e. in } Q,$$

$$\xi \in L_{\text{loc}}^2((0, T]; L^2(\Omega)) \text{ and } \xi \in \beta(u) \text{ a.e. in } Q,$$

$$\xi_\Gamma \in L_{\text{loc}}^2((0, T]; L^2(\Gamma)) \text{ and } \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \text{ a.e. in } \Sigma,$$

and

$$\begin{aligned} & \int_\Omega \partial_t u(t) z \, dx + \int_\Omega (\nu_u(t) + \kappa^2 \nabla u(t)) \cdot \nabla z \, dx + \int_\Omega (\xi(t) + g(u(t))) z \, dx \\ & + \int_\Gamma \partial_t u_\Gamma(t) z_\Gamma \, d\Gamma + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma(t)) \cdot \nabla_\Gamma(\varepsilon z_\Gamma) \, d\Gamma + \int_\Gamma (\xi_\Gamma(t) + g_\Gamma(u_\Gamma(t))) z_\Gamma \, d\Gamma \\ & = \int_\Omega \theta(t) z \, dx + \int_\Gamma \theta_\Gamma(t) z_\Gamma \, d\Gamma, \text{ for any } [z, z_\Gamma] \in \mathcal{V}_\varepsilon. \end{aligned}$$

Now our Main Theorems are stated as follows.

Main Theorem 1 (well-posedness) Let us assume (A1)–(A3) and let us fix an arbitrary $\varepsilon \geq 0$. Then, the following items hold.

(I-1)(Existence and uniqueness) The system $(ACE)_\varepsilon$ admits a unique solution $[u, u_\Gamma]$ and there exists a constant $C_1 > 0$, independent of the initial value $[u_0, u_{\Gamma,0}]$ and the forcing term $[\theta, \theta_\Gamma]$, such that:

$$\begin{aligned} & \|[u, u_\Gamma]\|_{C([0,T]; \mathcal{H})}^2 + \|[\nabla u, \nabla_\Gamma(\varepsilon u_\Gamma)]\|_{L^2(0,T; \mathcal{H}^N)}^2 \\ & + \left| \sqrt{t} [\partial_t u, \partial_t u_\Gamma] \right|_{L^2(0,T; \mathcal{H})}^2 + \sup_{t \in (0,T)} \left| \sqrt{t} [\nabla u(t), \nabla_\Gamma(\varepsilon u_\Gamma(t))] \right|_{\mathcal{H}^N}^2 \\ & \leq C_1 \left(1 + \|[u_0, u_{\Gamma,0}]\|_{\mathcal{H}}^2 + \|[\theta, \theta_\Gamma]\|_{L^2(0,T; \mathcal{H})}^2 \right). \end{aligned} \quad (2.4)$$

Moreover, if $[u_0, u_{\Gamma,0}] \in \mathcal{D}_\varepsilon$, then there exists a constant $C_2 > 0$, independent of the initial value $[u_0, u_{\Gamma,0}]$ and the forcing term $[\theta, \theta_\Gamma]$, such that:

$$\begin{aligned} & \|[\partial_t u, \partial_t u_\Gamma]\|_{L^2(0,T; \mathcal{H})}^2 + \sup_{t \in (0,T)} \|[\nabla u(t), \nabla_\Gamma(\varepsilon u_\Gamma(t))]\|_{\mathcal{H}^N}^2 \\ & \leq C_2 \left(\begin{array}{l} 1 + \|[u_0, u_{\Gamma,0}]\|_{\mathcal{H}}^2 + \|[\nabla u_0, \nabla_\Gamma(\varepsilon u_{\Gamma,0})]\|_{\mathcal{H}^N}^2 \\ + |B(u_0)|_{L^1(\Omega)} + |B_\Gamma(u_{\Gamma,0})|_{L^1(\Gamma)} + \|[\theta, \theta_\Gamma]\|_{L^2(0,T; \mathcal{H})}^2 \end{array} \right). \end{aligned} \quad (2.5)$$

(I-2)(Continuous-dependence) For $k = 1, 2$, let $[u^k, u_\Gamma^k]$ denote two solutions to the problem $(ACE)_\varepsilon$ corresponding to the forcing pairs $[\theta^k, \theta_\Gamma^k] \in L^2(0, T; \mathcal{H})$ and initial pairs $[u_0^k, u_{\Gamma,0}^k] \in \mathcal{D}_*$, respectively. Then, there exists a positive constant C_3 , independent of the choices of $[\theta^k, \theta_\Gamma^k]$ and $[u_0^k, u_{\Gamma,0}^k]$, $k = 1, 2$, such that:

$$\begin{aligned} & \|[u^1 - u^2, u_\Gamma^1 - u_\Gamma^2]\|_{C([0,T]; \mathcal{H})}^2 + \|[\nabla(u^1 - u^2), \nabla_\Gamma(\varepsilon(u_\Gamma^1 - u_\Gamma^2))]\|_{L^2(0,T; \mathcal{H}^N)}^2 \\ & \leq C_3 \left(\|[u_0^1 - u_0^2, u_{\Gamma,0}^1 - u_{\Gamma,0}^2]\|_{\mathcal{H}}^2 + \|[\theta^1 - \theta^2, \theta_\Gamma^1 - \theta_\Gamma^2]\|_{L^2(0,T; \mathcal{H})}^2 \right). \end{aligned} \quad (2.6)$$

Main Theorem 2 (ε -dependence of solutions) Let $\varepsilon_0 \geq 0$ be a fixed constant. Let $\{[\theta^\varepsilon, \theta_\Gamma^\varepsilon]\}_{\varepsilon \geq 0} \subset L^2(0, T; \mathcal{H})$ be a sequence of forcing pairs, let $\{[u_0^\varepsilon, u_{\Gamma,0}^\varepsilon] \in \mathcal{D}_\varepsilon\}_{\varepsilon \geq 0} \subset \mathcal{H}$ be a sequence of initial pairs and for any $\varepsilon \geq 0$, let $[u^\varepsilon, u_\Gamma^\varepsilon]$ be a solution to $(ACE)_\varepsilon$ corresponding to the forcing pair $[\theta^\varepsilon, \theta_\Gamma^\varepsilon] \in L^2(0, T; \mathcal{H})$ and the initial pair $[u_0^\varepsilon, u_{\Gamma,0}^\varepsilon] \in \mathcal{D}_\varepsilon$. If:

$$\begin{cases} [\theta^\varepsilon, \theta_\Gamma^\varepsilon] \rightarrow [\theta^{\varepsilon_0}, \theta_\Gamma^{\varepsilon_0}] \text{ weakly in } L^2(0, T; \mathcal{H}), \\ [u_0^\varepsilon, u_{\Gamma,0}^\varepsilon] \rightarrow [u_0^{\varepsilon_0}, u_{\Gamma,0}^{\varepsilon_0}] \text{ in } \mathcal{H}, \end{cases} \quad \text{as } \varepsilon \rightarrow \varepsilon_0, \quad (2.7)$$

then:

$$\begin{aligned} [u^\varepsilon, u_\Gamma^\varepsilon] &\rightarrow [u^{\varepsilon_0}, u_\Gamma^{\varepsilon_0}] \quad \text{in } C([0, T]; \mathcal{H}) \text{ and} \\ &\quad \text{in } L^2(0, T; \mathcal{V}_0), \text{ as } \varepsilon \rightarrow \varepsilon_0. \end{aligned} \quad (2.8)$$

In particular, if $\varepsilon_0 > 0$, then:

$$u_\Gamma^\varepsilon \rightarrow u_\Gamma^{\varepsilon_0} \text{ in } L^2(0, T; H^1(\Gamma)) \text{ as } \varepsilon \rightarrow \varepsilon_0. \quad (2.9)$$

3 Key-Lemmas

In this Section, we specify the essential points in the proofs of Main Theorems, in forms of Key-Lemmas.

In any case of $\varepsilon \geq 0$, the keypoint will be to reformulate the system $(ACE)_\varepsilon$ as the following Cauchy problem $(CP)_\varepsilon$ for an evolution equation:

$(CP)_\varepsilon$:

$$\begin{cases} U'(t) + \partial\Phi_\varepsilon(U(t)) + \mathcal{G}(U(t)) \ni \Theta(t) \text{ in } \mathcal{H}, \text{ a.e. } t \in (0, T), \\ U(0) = U_0 \text{ in } \mathcal{H}. \end{cases}$$

In the context:

- the unknown $U \in C([0, T]; \mathcal{H})$ corresponds to the solution pair $[u, u_\Gamma]$ of the system $(ACE)_\varepsilon$, i.e., $U(t) = [u(t), u_\Gamma(t)]$ in \mathcal{H} , for any $t \in [0, T]$ with the initial pair $U_0 = [u_0, u_{\Gamma,0}]$ in \mathcal{H} ;
- $\partial\Phi_\varepsilon$ denotes the subdifferential of a proper l.s.c. and convex function $\Phi_\varepsilon : \mathcal{H} \rightarrow [0, \infty]$, defined as

$$\begin{aligned} U &= [u, u_\Gamma] \in \mathcal{H} \mapsto \Phi_\varepsilon(U) = \Phi_\varepsilon(u, u_\Gamma) \\ &:= \begin{cases} \int_\Omega \left(|\nabla u| + \frac{\kappa^2}{2} |\nabla u|^2 \right) dx + \int_\Omega B(u) dx \\ \quad + \frac{1}{2} \int_\Gamma |\nabla_\Gamma(\varepsilon u_\Gamma)|^2 d\Gamma + \int_\Gamma B_\Gamma(u_\Gamma) d\Gamma, \\ \quad \text{if } U = [u, u_\Gamma] \in \mathcal{D}_\varepsilon, \\ \infty, \quad \text{otherwise,} \end{cases} \quad (3.1) \end{aligned}$$

for $\varepsilon \geq 0$;

– $\mathcal{G} : \mathcal{H} \rightarrow \mathcal{H}$ is a Lipschitz continuous operator, defined as

$$U = [u, u_\Gamma] \in \mathcal{H} \mapsto \mathcal{G}(U) = \mathcal{G}(u, u_\Gamma) := [g(u), g_\Gamma(u_\Gamma)] \in \mathcal{H},$$

where g and g_Γ can be meant as Lipschitz continuous extensions outside $\overline{I_B}$ of the functions g and g_Γ defined in (A2);

– the forcing term Θ corresponds to the forcing pair $[\theta, \theta_\Gamma]$ of $(ACE)_\varepsilon$, i.e., $\Theta = [\theta, \theta_\Gamma]$ in $L^2(0, T; \mathcal{H})$.

Remark 3.1 For any $\varepsilon \geq 0$, we can see that the convex function Φ_ε , defined in (3.1), corresponds to the convex part of the free-energy, given in (0.4). In addition, the subdifferentials $\partial\Phi_\varepsilon$, for $\varepsilon \geq 0$, are maximal monotone graphs in $\mathcal{H} \times \mathcal{H}$. So, the well-posedness for the Cauchy problem $(CP)_\varepsilon$ will be verified, immediately, by applying general theories for evolution equations, e.g., [3, 4, 24].

In the light of Remark 3.1, the essential points in Main Theorem 1 will be to show a certain association between our system $(ACE)_\varepsilon$ and the Cauchy problem $(CP)_\varepsilon$ for any $\varepsilon \geq 0$. To this end, we need to prepare a class of relaxed convex functions

$$\{\Phi_{\varepsilon, \delta}^\lambda \mid \varepsilon \geq 0, 0 < \delta, \lambda \leq 1\},$$

defined as follows:

$$\begin{aligned} U &= [u, u_\Gamma] \in \mathcal{H} \mapsto \Phi_{\varepsilon, \delta}^\lambda(U) = \Phi_{\varepsilon, \delta}^\lambda(u, u_\Gamma) \\ &:= \begin{cases} \int_{\Omega} \left(f_\delta(\nabla u) + \frac{\kappa^2}{2} |\nabla u|^2 \right) dx + \int_{\Omega} B^\lambda(u) dx \\ \quad + \frac{1}{2} \int_{\Gamma} |\nabla_\Gamma(\varepsilon u_\Gamma)|^2 d\Gamma + \int_{\Gamma} B_\Gamma^\lambda(u_\Gamma) d\Gamma, \\ \text{if } U = [u, u_\Gamma] \in \mathcal{V}_\varepsilon, \\ \infty, \text{ otherwise,} \end{cases} \quad (3.2) \end{aligned}$$

for $\varepsilon \geq 0$ and $0 < \delta, \lambda \leq 1$.

In the context, $\{f_\delta\}_{0 < \delta \leq 1}$, $\{B^\lambda\}_{0 < \lambda \leq 1}$ and $\{B_\Gamma^\lambda\}_{0 < \lambda \leq 1}$ are sequences of functions, prescribed under the following assumptions.

(A4) $\{f_\delta\}_{0 < \delta \leq 1} \subset C^1(\mathbb{R}^N)$ is a sequence of convex functions and C^1 -regularizations for the Euclidean norm $|\cdot| \in W^{1, \infty}(\mathbb{R}^N)$, such that:

$$f_\delta(0) = 0 \text{ and } f_\delta(\omega) \geq 0, \text{ for any } \omega \in \mathbb{R}^N \text{ and any } 0 < \delta \leq 1,$$

$$\begin{cases} f_\delta(\omega) \rightarrow |\omega|, \text{ for any } \omega \in \mathbb{R}^N, \\ f_\delta \rightarrow |\cdot| \text{ on } \mathbb{R}^N, \text{ in the sense of Mosco,} \end{cases} \quad \text{as } \delta \downarrow 0,$$

and there exists a δ -independent constant $C_0 > 0$, satisfying:

$$|\nabla f_\delta(\omega)| \leq C_0(|\omega| + 1), \text{ for any } 0 < \delta \leq 1 \text{ and } \omega \in \mathbb{R}^N.$$

(A5) $\{B^\lambda\}_{0<\lambda\leq 1}$ and $\{B_\Gamma^\lambda\}_{0<\lambda\leq 1}$ are sequences of Moreau–Yosida regularizations of the convex functions B and B_Γ , respectively, i.e., $\{B^\lambda\}_{0<\lambda\leq 1} \subset C^1(\mathbb{R})$, $\{B_\Gamma^\lambda\}_{0<\lambda\leq 1} \subset C^1(\mathbb{R})$, and

$$\left\{ \begin{array}{l} \tau \in \mathbb{R} \mapsto B^\lambda(\tau) := \inf \left\{ \frac{1}{2\lambda} |\tilde{\tau} - \tau|^2 + B(\tilde{\tau}) \mid \tilde{\tau} \in \mathbb{R} \right\}, \\ \tau \in \mathbb{R} \mapsto B_\Gamma^\lambda(\tau) := \inf \left\{ \frac{1}{2\lambda} |\tilde{\tau} - \tau|^2 + B_\Gamma(\tilde{\tau}) \mid \tilde{\tau} \in \mathbb{R} \right\}, \end{array} \right. \quad \text{for any } 0 < \lambda \leq 1.$$

Now, the first Key-Lemma is concerned with the representations of the subdifferentials $\partial\Phi_{\varepsilon,\delta}^\lambda \subset \mathcal{H} \times \mathcal{H}$ of the relaxed convex functions $\Phi_{\varepsilon,\delta}^\lambda$, for $\varepsilon \geq 0$, $0 < \delta, \lambda \leq 1$.

Key-Lemma 1 *Let us fix $\varepsilon \geq 0$, $0 < \delta, \lambda \leq 1$. Let us put:*

$$\mathcal{D}_{\varepsilon,\delta}^\lambda := \left\{ [u, u_\Gamma] \in \mathcal{V}_\varepsilon \left[\begin{array}{l} \nabla f_\delta(\nabla u) + \kappa^2 \nabla u \in \mathbf{L}_{\text{div}}^2(\Omega) \\ -\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma \in L^2(\Gamma) \end{array} \right] \right\},$$

and let us define an operator $\mathcal{A}_{\varepsilon,\delta}^\lambda : \mathcal{D}_{\varepsilon,\delta}^\lambda \subset \mathcal{H} \rightarrow \mathcal{H}$, by putting:

$$\begin{aligned} [u, u_\Gamma] \in \mathcal{D}_{\varepsilon,\delta}^\lambda &\mapsto \mathcal{A}_{\varepsilon,\delta}^\lambda [u, u_\Gamma] \\ &:= \begin{bmatrix} -\text{div}(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) + \beta^\lambda(u) \\ -\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma + \beta_\Gamma^\lambda(u_\Gamma) \end{bmatrix} \text{ in } \mathcal{H}. \end{aligned}$$

Then, $\partial\Phi_{\varepsilon,\delta}^\lambda = \mathcal{A}_{\varepsilon,\delta}^\lambda$ in $\mathcal{H} \times \mathcal{H}$, i.e.

$$\begin{aligned} D(\partial\Phi_{\varepsilon,\delta}^\lambda) &= \mathcal{D}_{\varepsilon,\delta}^\lambda \text{ and } \partial\Phi_{\varepsilon,\delta}^\lambda(u, u_\Gamma) = \mathcal{A}_{\varepsilon,\delta}^\lambda [u, u_\Gamma] \text{ in } \mathcal{H}, \\ &\text{for any } [u, u_\Gamma] \in D(\partial\Phi_{\varepsilon,\delta}^\lambda). \end{aligned}$$

The second Key-Lemma is concerned with the continuous dependence between the convex functions Φ_ε for $\varepsilon \geq 0$, and the relaxations of those.

Key-Lemma 2 (Continuous dependence of the convex functions) *Let*

$$\{\varepsilon_n\}_{n=0}^\infty \subset [0, \infty), \quad \{\delta_n\}_{n=1}^\infty \subset (0, 1] \quad \text{and} \quad \{\lambda_n\}_{n=1}^\infty \subset (0, 1]$$

be arbitrary sequences such that:

$$\varepsilon_n \rightarrow \varepsilon_0, \quad \delta_n \downarrow 0 \quad \text{and} \quad \lambda_n \downarrow 0, \quad \text{as } n \rightarrow \infty.$$

Then, for the sequence $\{\Phi_{\varepsilon_n, \delta_n}^{\lambda_n}\}_{n=1}^\infty$ of convex functions, it holds that:

$$\Phi_{\varepsilon_n, \delta_n}^{\lambda_n} \rightarrow \Phi_{\varepsilon_0} \text{ on } \mathcal{H}, \text{ in the sense of Mosco, as } n \rightarrow \infty.$$

On the basis of Key-Lemmas 1–2, we prove the third Key-Lemma, concerned with representations of the subdifferentials $\partial\Phi_\varepsilon \subset \mathcal{H} \times \mathcal{H}$ of Φ_ε , for $\varepsilon \geq 0$.

Key-Lemma 3 *For any $\varepsilon \geq 0$, the following two items are equivalent.*

(Key 0) $U = [u, u_\Gamma] \in D(\partial\Phi_\varepsilon)$ and $U^* = [u^*, u_\Gamma^*] \in \partial\Phi_\varepsilon(U) = \partial\Phi_\varepsilon(u, u_\Gamma)$ in \mathcal{H} .

(Key 1) $U = [u, u_\Gamma] \in \mathcal{D}_\varepsilon$, and there exists $\nu_u \in L^\infty(\Omega)^N$ and $[\xi, \xi_\Gamma] \in \mathcal{H}$, such that:

$$\begin{cases} \nu_u \in \text{Sgn}(\nabla u) \text{ and } \xi \in \beta(u), \text{ a.e. in } \Omega, \\ \xi_\Gamma \in \beta_\Gamma(u_\Gamma), \text{ a.e. on } \Gamma, \end{cases} \quad (3.3)$$

$$\begin{cases} \nu_u + \kappa^2 \nabla u \in \mathbf{L}_{\text{div}}^2(\Omega), \\ -\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(\nu_u + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma \in L^2(\Gamma), \end{cases} \quad (3.4)$$

and

$$\begin{cases} u^* = -\text{div}(\nu_u + \kappa^2 \nabla u) + \xi \text{ in } L^2(\Omega), \\ u_\Gamma^* = -\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(\nu_u + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma + \xi_\Gamma \text{ in } L^2(\Gamma). \end{cases} \quad (3.5)$$

The last Key-Lemma 3 is useful to guarantee the association between $(\text{ACE})_\varepsilon$ and $(\text{CP})_\varepsilon$ for $\varepsilon \geq 0$, via the representations of subdifferentials.

4 Proofs of Key-Lemmas

In this section, we prove three Key-Lemmas for our Main Theorems. To this end, we first prepare the following lemma.

Lemma 4.1 *Let (S, \mathcal{B}, μ) be a measure space with a σ -algebra \mathcal{B} and a finite Radon measure μ . Let X be a (real) Hilbert space. Let $\Psi : X \rightarrow (-\infty, \infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \rightarrow (-\infty, \infty]$, $n \in \mathbb{N}$, such that:*

$$\Psi_n \rightarrow \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \rightarrow \infty. \quad (4.1)$$

Then, the following two items hold.

(I) *There exist two constants $c_0, d_0 > 0$, independent of n , such that:*

$$\Psi_n(z) + c_0|z|_X + d_0 \geq 0, \text{ for any } z \in X \text{ and any } n \in \mathbb{N}. \quad (4.2)$$

(II) *The sequence $\{\hat{\Psi}_n\}_{n=1}^\infty$ of proper l.s.c. and convex functions*

$$\zeta \in L^2(S; X) \mapsto \hat{\Psi}_n(\zeta) := \begin{cases} \int_S \Psi_n(\zeta) d\mu, \text{ if } \Psi_n(\zeta) \in L^1(S), \\ \infty, \text{ otherwise,} \end{cases} \quad (4.3)$$

for $n = 1, 2, 3, \dots$,

converges to the convex function

$$\zeta \in L^2(S; X) \mapsto \hat{\Psi}(\zeta) := \begin{cases} \int_S \Psi(\zeta) d\mu, \text{ if } \Psi(\zeta) \in L^1(S), \\ \infty, \text{ otherwise,} \end{cases} \quad (4.4)$$

on the Hilbert space $L^2(S; X)$, in the sense of Mosco, as $n \rightarrow \infty$.

Proof. This lemma can be proved by means of similar demonstration techniques as in [18, Appendix]. However, we report the proof for the reader's convenience.

First, we show the item (I). To this end, let us assume that:

$$\begin{aligned} \Psi_{n_k}(y_k) + k^2(|y_k|_X + 1) &< 0, \\ \text{for some } \{n_k\}_{k=1}^\infty \subset \{n\}_{n=1}^\infty, \{y_k\}_{k=1}^\infty \subset X \end{aligned} \quad (4.5)$$

to derive a contradiction.

Let us fix any $z_0 \in D(\Psi)$. Then, by (4.1), we find a sequence $\{\hat{z}_n\}_{n=1}^\infty \subset X$, such that:

$$\hat{z}_n \rightarrow z_0 \text{ in } X \text{ and } \Psi_n(\hat{z}_n) \rightarrow \Psi(z_0) \text{ as } n \rightarrow \infty. \quad (4.6)$$

Here, we define:

$$z_k := \varepsilon_k y_k + (1 - \varepsilon_k) \hat{z}_{n_k} \text{ in } X, \text{ for } k = 1, 2, 3, \dots, \quad (4.7)$$

with

$$\varepsilon_k := \frac{1}{k(1 + |y_k|_X)} \in (0, 1), \text{ for } k = 1, 2, 3, \dots \quad (4.8)$$

Then, it follows from (4.5)–(4.8) that $|\varepsilon_k y_k| \leq 1/k$, whence:

$$z_k \rightarrow z_0 \text{ in } X \text{ as } k \rightarrow \infty,$$

and subsequently, it follows from (4.1) that:

$$\overline{\lim}_{k \rightarrow \infty} \Psi_{n_k}(z_k) \geq \underline{\lim}_{k \rightarrow \infty} \Psi_{n_k}(z_k) \geq \Psi(z_0). \quad (4.9)$$

In the meantime, in the light of (4.7)–(4.8), and the convexity of Ψ_{n_k} , for $k \in \mathbb{N}$,

$$\begin{aligned} \overline{\lim}_{k \rightarrow \infty} \Psi_{n_k}(z_k) &\leq \overline{\lim}_{k \rightarrow \infty} \varepsilon_k \Psi_{n_k}(y_k) + \lim_{k \rightarrow \infty} (1 - \varepsilon_k) \Psi_{n_k}(\hat{z}_{n_k}) \\ &\leq \overline{\lim}_{k \rightarrow \infty} (\varepsilon_k (-k^2(1 + |y_k|_X))) + \Psi(z_0) \\ &= - \lim_{k \rightarrow \infty} k + \Psi(z_0) = -\infty. \end{aligned}$$

This contradicts with (4.9).

Next, we show the item (II). According to [2, Theorem 3.26], it is sufficient (equivalent) to check the following two conditions:

(ii-1) $\zeta_n := (\mathcal{I}_{L^2(S;X)} + \lambda \partial \hat{\Psi}_n)^{-1} \xi \rightarrow \zeta := (\mathcal{I}_{L^2(S;X)} + \lambda \partial \hat{\Psi})^{-1} \xi$ in $L^2(S; X)$ as $n \rightarrow \infty$,
for any $\lambda > 0$ and any $\xi \in L^2(S; X)$;

(ii-2) there exists $[\zeta, \eta] \in \partial \hat{\Psi}$ in $L^2(S; X) \times L^2(S; X)$ and a sequence $\{[\zeta_n, \eta_n] \in \hat{\Psi}_n\}_{n=1}^\infty \subset L^2(S; X) \times L^2(S; X)$ such that $[\zeta_n, \eta_n] \rightarrow [\zeta, \eta]$ in $L^2(S; X) \times L^2(S; X)$ and $\hat{\Psi}_n(\zeta_n) \rightarrow \hat{\Psi}(\zeta)$ as $n \rightarrow \infty$.

For the verification of (ii-1), let us fix any $\lambda > 0$ and any $\xi \in L^2(S; X)$. Then, invoking [4, Proposition 2.16], [2, Theorem 3.26], (4.1) and (4.3)–(4.4), we infer that:

$$\begin{cases} (\xi - \zeta)(\sigma) \in \lambda \partial \Psi(\zeta(\sigma)) \text{ in } X, \\ (\xi - \zeta_n)(\sigma) \in \lambda \partial \Psi_n(\zeta_n(\sigma)) \text{ in } X, n = 1, 2, 3, \dots, \end{cases} \quad \text{for } \mu\text{-a.e. } \sigma \in S, \quad (4.10)$$

and

$$\zeta_n(\sigma) \rightarrow \zeta(\sigma) \text{ in } X \text{ as } n \rightarrow \infty, \text{ for } \mu\text{-a.e. } \sigma \in S. \quad (4.11)$$

Also, by using the sequence $\{\hat{z}_n\}_{n=1}^\infty \subset X$ as in (4.6), it is seen that:

$$\begin{aligned} ((\xi - \zeta_n)(\sigma), \zeta_n(\sigma) - \hat{z}_n)_X &\geq \lambda \hat{\Psi}_n(\zeta_n(\sigma)) - \lambda \hat{\Psi}_n(\hat{z}_n), \\ &\text{for any } n \in \mathbb{N} \text{ and } \mu\text{-a.e. } \sigma \in S. \end{aligned} \quad (4.12)$$

Additionally, by virtue of the item (I), (4.6), (4.12) and the Schwarz and Young inequalities, we can compute that:

$$\begin{aligned} |\zeta_n(\sigma)|_X^2 &\leq (\zeta_n(\sigma), \hat{z}_n)_X + (\xi(\sigma), \zeta_n(\sigma) - \hat{z}_n)_X + \lambda \hat{\Psi}_n(\hat{z}_n) - \lambda \hat{\Psi}_n(\zeta_n(\sigma)) \\ &\leq |\zeta_n(\sigma)|_X |\hat{z}_n|_X + |\zeta_n(\sigma)|_X |\xi(\sigma)|_X + |\xi(\sigma)|_X |\hat{z}_n|_X \\ &\quad + \lambda \hat{\Psi}_n(\hat{z}_n) + \lambda (c_0 |\zeta_n(\sigma)|_X + d_0) \\ &\leq \frac{3}{4} |\zeta_n(\sigma)|_X^2 + \frac{5}{4} |\xi(\sigma)|_X^2 + \lambda^2 c_0^2 + \lambda d_0 + \lambda \hat{\Psi}_n(\hat{z}_n) + 2 |\hat{z}_n|_X^2, \end{aligned}$$

and therefore,

$$|\zeta_n(\sigma)|_X^2 \leq 5 |\xi(\sigma)|_X^2 + \hat{M}_1, \text{ for } \mu\text{-a.e. } \sigma \in S \text{ and any } n \in \mathbb{N}, \quad (4.13)$$

where

$$\hat{M}_1 := 4(\lambda(c_0 + d_0) + 1)^2 + 4 \sup_{n \in \mathbb{N}} (\lambda \hat{\Psi}_n(\hat{z}_n) + 2 |\hat{z}_n|_X^2).$$

In view of these, the condition (ii-1) will be obtained as a consequence of (4.11), (4.13) and Lebesgue's dominated convergence theorem.

Finally, for the verification of (ii-2), we consider the class of functions $\{\zeta, \zeta_n | n \in \mathbb{N}\} \subset L^2(S; X)$ as in (ii-1) with fixed $\lambda > 0$ and $\xi \in L^2(S; X)$, and let us set:

$$\begin{cases} \eta := \frac{\xi - \zeta}{\lambda} \text{ in } L^2(S; X), \\ \eta_n := \frac{\xi - \zeta_n}{\lambda} \text{ in } L^2(S; X), n = 1, 2, 3, \dots \end{cases} \quad (4.14)$$

Also, let us denote by $\Psi^\lambda : X \rightarrow \mathbb{R}$, $\hat{\Psi}^\lambda : L^2(S; X) \rightarrow \mathbb{R}$, $\Psi_n^\lambda : X \rightarrow \mathbb{R}$ and $\hat{\Psi}_n^\lambda : L^2(S; X) \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, the Moreau-Yosida regularizations of convex functions Ψ , $\hat{\Psi}$, Ψ_n and $\hat{\Psi}_n$, $n \in \mathbb{N}$, respectively. Then, by [3, Theorem 2.9, p. 48], [4, Proposition 2.11], (ii-1) and (4.14), we immediately have:

$$\begin{cases} \eta = \partial \hat{\Psi}^\lambda(\xi) \in \partial \hat{\Psi}(\zeta) \text{ in } L^2(S; X), \\ \eta_n = \partial \hat{\Psi}_n^\lambda(\xi) \in \partial \hat{\Psi}_n(\zeta_n) \text{ in } L^2(S; X), n = 1, 2, 3, \dots, \end{cases} \quad (4.15)$$

$$\eta_n = \partial \hat{\Psi}_n^\lambda(\xi) = \frac{\xi - \zeta_n}{\lambda} \rightarrow \eta = \partial \hat{\Psi}^\lambda(\xi) = \frac{\xi - \zeta}{\lambda} \text{ in } L^2(S; X) \text{ as } n \rightarrow \infty. \quad (4.16)$$

In particular, [4, Proposition 2.16] and (4.16) enable us to say that:

$$\partial \Psi_n^\lambda(\xi(\sigma)) \rightarrow \partial \Psi^\lambda(\xi(\sigma)) \text{ in } X \text{ as } n \rightarrow \infty, \text{ for } \mu\text{-a.e. } \sigma \in S, \quad (4.17)$$

by taking a subsequence if necessary. Besides, from [2, Theorem 3.26] and (4.1), it follows that:

$$\Psi_n^\lambda(\xi(\sigma)) \rightarrow \Psi^\lambda(\xi(\sigma)) \text{ as } n \rightarrow \infty, \text{ for } \mu\text{-a.e. } \sigma \in S. \quad (4.18)$$

On account of [3, Theorem 2.9, p. 48], [4, Proposition 2.11] and (4.17)–(4.18), it is inferred that:

$$\begin{aligned} \Psi_n(\zeta_n(\sigma)) &= \Psi_n^\lambda(\xi(\sigma)) - \frac{1}{2\lambda} |(\xi - \zeta_n)(\sigma)|_X^2 \\ &\rightarrow \Psi^\lambda(\xi(\sigma)) - \frac{1}{2\lambda} |(\xi - \zeta)(\sigma)|_X^2 \\ &= \Psi(\zeta(\sigma)) \text{ as } n \rightarrow \infty, \text{ for } \mu\text{-a.e. } \sigma \in S. \end{aligned} \quad (4.19)$$

Furthermore, invoking (4.12)–(4.13), the item(I), and using the sequence $\{\hat{z}_n\}_{n=1}^\infty$ as in (4.6) and the Schwarz and Young inequalities, we obtain that:

$$\begin{aligned} |\Psi_n(\zeta_n(\sigma))| &\leq \Psi_n(\zeta_n(\sigma)) \vee (c_0 |\zeta_n(\sigma)|_X + d_0) \\ &\leq |\Psi_n(\hat{z}_n)| + \left| \left(\frac{(\xi - \zeta_n)(\sigma)}{\lambda}, \hat{z}_n - \zeta_n(\sigma) \right)_X \right| + c_0 |\zeta_n(\sigma)|_X + d_0 \\ &\leq |\Psi_n(\hat{z}_n)| + \frac{1}{\lambda} |\xi(\sigma)|_X |\hat{z}_n|_X + \frac{1}{\lambda} |\xi(\sigma)|_X |\zeta_n(\sigma)|_X \\ &\quad + \frac{1}{\lambda} |\zeta_n(\sigma)|_X |\hat{z}_n|_X + \frac{1}{\lambda} |\zeta_n(\sigma)|_X^2 + c_0 |\zeta_n(\sigma)|_X + d_0 \\ &\leq |\Psi_n(\hat{z}_n)| + \frac{3}{\lambda} |\zeta_n(\sigma)|_X^2 + \frac{1}{\lambda} |\xi(\sigma)|_X^2 + \frac{1}{\lambda} |\hat{z}_n|_X^2 + \frac{\lambda}{4} c_0^2 + d_0 \\ &\leq \frac{3}{\lambda} \left(5 |\xi(\sigma)|_X^2 + \hat{M}_1 \right) + \frac{1}{\lambda} |\xi(\sigma)|_X^2 \\ &\quad + |\Psi_n(\hat{z}_n)| + \frac{1}{\lambda} |\hat{z}_n|_X^2 + \frac{\lambda}{4} c_0^2 + d_0 \\ &\leq \frac{16}{\lambda} |\xi(\sigma)|_X^2 + \hat{M}_2, \text{ for } \mu\text{-a.e. } \sigma \in S \text{ and any } n \in \mathbb{N}, \end{aligned} \quad (4.20)$$

where

$$\hat{M}_2 := \frac{3}{\lambda} \hat{M}_1 + \frac{\lambda}{4} c_0^2 + d_0 + \sup_{n \in \mathbb{N}} \left(|\Psi_n(\hat{z}_n)| + \frac{1}{\lambda} |\hat{z}_n|_X^2 \right).$$

In the light of (4.19)–(4.20), we can apply Lebesgue's dominated convergence theorem, to derive that:

$$\hat{\Psi}_n(\zeta_n) \rightarrow \hat{\Psi}(\zeta) \text{ as } n \rightarrow \infty. \quad (4.21)$$

Then, (4.14)–(4.16), (4.21) and the previous (ii-1) imply the validity of (ii-2). \square

Now the Key-Lemmas 1–3 are proved as follows.

Proof of Key-Lemma 1. First, we show that $\partial \Phi_{\varepsilon, \delta}^\lambda \subset \mathcal{A}_{\varepsilon, \delta}^\lambda$ in $\mathcal{H} \times \mathcal{H}$. Let us assume

that $[u, u_\Gamma] \in D(\partial\Phi_{\varepsilon,\delta}^\lambda)$ and $[u^*, u_\Gamma^*] \in \partial\Phi_{\varepsilon,\delta}^\lambda(u, u_\Gamma)$ in \mathcal{H} . Besides, let us take arbitrary $\tau > 0$ and $[z, z_\Gamma] \in \mathcal{V}_\varepsilon$, to compute that:

$$\begin{aligned}
& (u^*, z)_{L^2(\Omega)} + (u_\Gamma^*, z_\Gamma)_{L^2(\Gamma)} \\
& \leq \frac{1}{\tau} \{ \Phi_{\varepsilon,\delta}^\lambda(u + \tau z, u_\Gamma + \tau z_\Gamma) - \Phi_{\varepsilon,\delta}^\lambda(u, u_\Gamma) \} \\
& = \frac{1}{\tau} \int_\Omega \left(f_\delta(\nabla(u + \tau z)) + \frac{\kappa^2}{2} |\nabla(u + \tau z)|^2 - f_\delta(\nabla u) - \frac{\kappa^2}{2} |\nabla u|^2 \right) dx \\
& \quad + \frac{1}{2\tau} \int_\Gamma (|\nabla_\Gamma(\varepsilon(u_\Gamma + \tau z_\Gamma))|^2 - |\nabla_\Gamma(\varepsilon u_\Gamma)|^2) d\Gamma \\
& \quad + \frac{1}{\tau} \int_\Omega (B^\lambda(u + \tau z) - B^\lambda(u)) dx + \frac{1}{\tau} \int_\Gamma (B_\Gamma^\lambda(u_\Gamma + \tau z_\Gamma) - B_\Gamma^\lambda(u_\Gamma)) d\Gamma \\
& \rightarrow \int_\Omega (\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma) \cdot \nabla_\Gamma(\varepsilon z_\Gamma) d\Gamma \\
& \quad + \int_\Omega \beta^\lambda(u) z dx + \int_\Gamma \beta_\Gamma^\lambda(u_\Gamma) z_\Gamma d\Gamma \text{ as } \tau \downarrow 0.
\end{aligned}$$

Since the choice of $[z, z_\Gamma] \in \mathcal{V}_\varepsilon$ is arbitrary, we have

$$\begin{aligned}
& (u^* - \beta^\lambda(u), z)_{L^2(\Omega)} + (u_\Gamma^* - \beta_\Gamma^\lambda(u_\Gamma), z_\Gamma)_{L^2(\Gamma)} \\
& = \int_\Omega (\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma) \cdot \nabla_\Gamma(\varepsilon z_\Gamma) d\Gamma, \quad (4.22) \\
& \quad \text{for any } [z, z_\Gamma] \in \mathcal{V}_\varepsilon.
\end{aligned}$$

Here, taking any $\varphi_0 \in H_0^1(\Omega)$ and putting $[z, z_\Gamma] = [\varphi_0, 0] \in \mathcal{V}_\varepsilon$ in (4.22),

$$\begin{aligned}
& (u^* - \beta^\lambda(u), \varphi_0)_{L^2(\Omega)} = \int_\Omega (\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot \nabla \varphi_0 dx, \\
& \quad \text{for any } \varphi_0 \in H_0^1(\Omega),
\end{aligned}$$

which implies

$$-\operatorname{div}(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) = u^* - \beta^\lambda(u) \in L^2(\Omega) \text{ in } \mathcal{D}'(\Omega). \quad (4.23)$$

Additionally, with Remark 1.2 (Fact 1)–(Fact 3) and (4.22)–(4.23) in mind, we can see that:

$$\begin{aligned}
& (u_\Gamma^* - \beta_\Gamma^\lambda(u_\Gamma), z_\Gamma)_{L^2(\Gamma)} \\
& = \int_\Omega (\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot \nabla z dx - (u^* - \beta^\lambda(u), z)_{L^2(\Omega)} + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma) \cdot \nabla_\Gamma(\varepsilon z_\Gamma) d\Gamma \\
& = {}_{H^{-\frac{1}{2}}(\Gamma)} \left\langle [(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma, z_\Gamma \right\rangle_{{}_{H^{\frac{1}{2}}(\Gamma)}} + {}_{H^{-1}(\Gamma)} \langle -\Delta_\Gamma(\varepsilon u_\Gamma), \varepsilon z_\Gamma \rangle_{{}_{H^1(\Gamma)}}, \\
& \quad \text{for any } [z, z_\Gamma] \in \mathcal{V}_\varepsilon.
\end{aligned}$$

This identity leads to:

$$-\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma = u_\Gamma^* - \beta_\Gamma^\lambda(u_\Gamma) \in L^2(\Gamma) \text{ in } H^{-1}(\Gamma). \quad (4.24)$$

As a consequence of (4.22)–(4.24), we obtain that:

$$[u, u_\Gamma] \in \mathcal{D}_{\varepsilon, \delta}^\lambda \text{ and } [u^*, u_\Gamma^*] \in \mathcal{A}_{\varepsilon, \delta}^\lambda[u, u_\Gamma] \text{ in } \mathcal{H}. \quad (4.25)$$

Secondly, we show that $\mathcal{A}_{\varepsilon, \delta}^\lambda \subset \partial\Phi_{\varepsilon, \delta}^\lambda$ in $\mathcal{H} \times \mathcal{H}$. Let us assume that $[u, u_\Gamma] \in \mathcal{D}_{\varepsilon, \delta}^\lambda$ and $[u^*, u_\Gamma^*] \in \mathcal{A}_{\varepsilon, \delta}^\lambda[u, u_\Gamma]$ in \mathcal{H} , and let us take an arbitrary $[z, z_\Gamma] \in \mathcal{V}_\varepsilon$. Then, taking into account Remark 1.2 (Fact 1)–(Fact 3), (A4)–(A5) and the convexity of the squared norm, we compute that:

$$\begin{aligned} & ([u^*, u_\Gamma^*], [z, z_\Gamma] - [u, u_\Gamma])_{\mathcal{H}} \\ &= \left(-\operatorname{div}(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) + \beta^\lambda(u), z - u \right)_{L^2(\Omega)} \\ & \quad + \left(-\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma + \beta_\Gamma^\lambda(u_\Gamma), z_\Gamma - u_\Gamma \right)_{L^2(\Gamma)} \\ &= \int_\Omega (\nabla f_\delta(\nabla u) + \kappa^2 \nabla u) \cdot \nabla(z - u) \, dx + \int_\Omega \beta^\lambda(u)(z - u) \, dx \\ & \quad + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma) \cdot \nabla_\Gamma(\varepsilon(z_\Gamma - u_\Gamma)) \, d\Gamma + \int_\Gamma \beta_\Gamma^\lambda(u_\Gamma)(z_\Gamma - u_\Gamma) \, d\Gamma \\ &\leq \int_\Omega \left(f_\delta(\nabla z) - f_\delta(\nabla u) + \frac{\kappa^2}{2} |\nabla z|^2 - \frac{\kappa^2}{2} |\nabla u|^2 \right) \, dx + \int_\Omega (B^\lambda(z) - B^\lambda(u)) \, dx \\ & \quad + \frac{1}{2} \int_\Gamma (|\nabla_\Gamma(\varepsilon z_\Gamma)|^2 - |\nabla_\Gamma(\varepsilon u_\Gamma)|^2) \, d\Gamma + \int_\Gamma (B_\Gamma^\lambda(z_\Gamma) - B_\Gamma^\lambda(u_\Gamma)) \, d\Gamma \\ &\leq \Phi_{\varepsilon, \delta}^\lambda(z, z_\Gamma) - \Phi_{\varepsilon, \delta}^\lambda(u, u_\Gamma), \text{ for any } [z, z_\Gamma] \in \mathcal{V}_\varepsilon. \end{aligned}$$

It implies that:

$$[u, u_\Gamma] \in D(\partial\Phi_{\varepsilon, \delta}^\lambda) \text{ and } [u^*, u_\Gamma^*] \in \partial\Phi_{\varepsilon, \delta}^\lambda(u, u_\Gamma) \text{ in } \mathcal{H}. \quad (4.26)$$

Thus, we conclude this lemma by (4.25)–(4.26). \square

Proof of Key-Lemma 2. First, we verify the condition of lower-bound. Let $[\check{u}, \check{u}_\Gamma] \in \mathcal{H}$ and $\{[\check{u}_n, \check{u}_{\Gamma, n}]\}_{n=1}^\infty \subset \mathcal{H}$ be such that:

$$[\check{u}_n, \check{u}_{\Gamma, n}] \rightarrow [\check{u}, \check{u}_\Gamma] \text{ weakly in } \mathcal{H} \text{ as } n \rightarrow \infty. \quad (4.27)$$

Then, we may suppose $\varliminf_{n \rightarrow \infty} \Phi_{\varepsilon_n, \delta_n}^{\lambda_n}(\check{u}_n, \check{u}_{\Gamma, n}) < \infty$, because the other case is trivial. Hence, there exists a subsequence $\{n_k\}_{k=1}^\infty \subset \{n\}_{n=1}^\infty$, such that:

$$\check{\Phi}_* := \varliminf_{n \rightarrow \infty} \Phi_{\varepsilon_n, \delta_n}^{\lambda_n}(\check{u}_n, \check{u}_{\Gamma, n}) = \lim_{k \rightarrow \infty} \Phi_{\varepsilon_{n_k}, \delta_{n_k}}^{\lambda_{n_k}}(\check{u}_{n_k}, \check{u}_{\Gamma, n_k}) < \infty. \quad (4.28)$$

Here, from (3.2), it can be seen that $\{[\check{u}_{n_k}, \check{u}_{\Gamma, n_k}]\}_{k=1}^\infty$ is bounded in \mathcal{V}_0 (resp. $\mathcal{V}_{\varepsilon_0}$), if $\varepsilon_0 = 0$ (resp. if $\varepsilon_0 > 0$). So, by invoking (Fact 6) and (4.27), and taking more subsequences if necessary, we can further suppose that:

$$\begin{cases} [\check{u}_{n_k}, \check{u}_{\Gamma, n_k}] \rightarrow [\check{u}, \check{u}_\Gamma] \text{ in } \mathcal{H} \text{ and weakly in } \mathcal{V}_0, \\ \check{u}_{n_k}(x) \rightarrow \check{u}(x) \text{ a.e. } x \in \Omega, \end{cases} \text{ as } k \rightarrow \infty, \quad (4.29)$$

and in particular, if $\varepsilon_0 > 0$, then

$$\check{u}_{\Gamma, n_k} \rightarrow \check{u}_\Gamma \text{ weakly in } H^1(\Gamma) \text{ as } k \rightarrow \infty. \quad (4.30)$$

Additionally, from (A4) and Lemma 4.1, we can infer that:

(Fact 8) The sequence of convex functions:

$$\left\{ \omega \in L^2(\Omega)^N \mapsto \int_{\Omega} f_{\delta_n}(\omega) dx \in [0, \infty] \right\}_{n=1}^{\infty}$$

converges to the convex function of L^1 -norm:

$$\omega \in L^2(\Omega)^N \mapsto \int_{\Omega} |\omega| dx \in [0, \infty),$$

on the Hilbert space $L^2(\Omega)^N$, in the sense of Mosco, as $n \rightarrow \infty$.

In the light of Remark 1.5, (4.27)–(4.30), Fatou’s lemma and the above (Fact 8), the condition of lower-bound is verified as follows:

$$\begin{aligned} \check{\Phi}_* &\geq \underline{\lim}_{k \rightarrow \infty} \int_{\Omega} f_{\delta_{n_k}}(\nabla \check{u}_{n_k}) dx + \frac{\kappa^2}{2} \underline{\lim}_{k \rightarrow \infty} \int_{\Omega} |\nabla \check{u}_{n_k}|^2 dx + \underline{\lim}_{k \rightarrow \infty} \int_{\Omega} B^{\lambda_{n_k}}(\check{u}_{n_k}) dx \\ &\quad + \frac{1}{2} \underline{\lim}_{k \rightarrow \infty} \int_{\Gamma} |\nabla_{\Gamma}(\varepsilon_{n_k} \check{u}_{\Gamma, n_k})|^2 d\Gamma + \underline{\lim}_{k \rightarrow \infty} \int_{\Gamma} B_{\Gamma}^{\lambda_{n_k}}(\check{u}_{\Gamma, n_k}) d\Gamma \\ &\geq \Phi_{\varepsilon_0}(\check{u}, \check{u}_{\Gamma}). \end{aligned}$$

Next, we verify the optimality condition. Let us fix any $[\hat{u}_0, \hat{u}_{\Gamma, 0}] \in D(\Phi_{\varepsilon_0})$, and let us take a sequence $\{\omega_i\}_{i=1}^{\infty} \subset H^1(\Omega)$ in the following way:

$$\left\{ \begin{array}{l} \bullet \text{ if } \varepsilon_0 > 0, \text{ then } \{\omega_i\}_{i=1}^{\infty} = \{\hat{u}_0\}; \\ \bullet \text{ if } \varepsilon_0 = 0, \text{ then } \{\omega_i\}_{i=1}^{\infty} \subset C^1(\bar{\Omega}) \text{ is such that } \omega_i \rightarrow \hat{u}_0 \text{ in } H^1(\Omega), \\ \quad \text{and pointwisely a.e. in } \Omega, \text{ as } i \rightarrow \infty. \end{array} \right. \quad (4.31)$$

Besides, let us define a sequence $\{\hat{\varphi}_i\}_{i=1}^{\infty} \subset H^1(\Omega)$, by putting:

$$\hat{\varphi}_i := \mathcal{T}_B \omega_i \text{ in } H^1(\Omega), \text{ for } i = 1, 2, 3, \dots,$$

where the projection function \mathcal{T}_B is defined by (2.2). Then, in view of (4.31), and taking a subsequence if necessary, we have that

$$\left\{ \begin{array}{l} \hat{\varphi}_i \rightarrow \hat{u}_0 \text{ in } H^1(\Omega), \text{ and pointwisely a.e. in } \Omega, \text{ as } i \rightarrow \infty \\ (\hat{\varphi}_i)|_{\Gamma} \rightarrow \hat{u}_{\Gamma, 0} \text{ in } H^{\frac{1}{2}}(\Gamma), \text{ and pointwisely a.e. on } \Gamma, \text{ as } i \rightarrow \infty \\ \{(\hat{\varphi}_i)|_{\Gamma}\}_{i=1}^{\infty} \subset H^1(\Gamma), \text{ and } (\varepsilon_0 \hat{\varphi}_i)|_{\Gamma} \rightarrow \varepsilon_0 \hat{u}_{\Gamma, 0} \text{ in } H^1(\Gamma), \text{ as } i \rightarrow \infty. \end{array} \right. \quad (4.32)$$

Also, invoking (A4) and Lebesgue’s dominated convergence theorem, we can configure a sequence $\{n_i\}_{i=0}^{\infty} \subset \mathbb{N}$, such that $1 =: n_0 < n_1 < n_2 < n_3 < \dots < n_i \uparrow \infty$, as $i \rightarrow \infty$, and for any $i \in \mathbb{N} \cup \{0\}$,

$$\left\{ \begin{array}{l} \sup_{n \geq n_i} |f_{\delta_n}(\nabla \hat{\varphi}_i) - |\nabla \hat{\varphi}_i||_{L^1(\Omega)} < \frac{1}{2^{i+1}}, \\ \sup_{n \geq n_i} \left(|B^{\lambda_n}(\hat{\varphi}_i) - B(\hat{\varphi}_i)|_{L^1(\Omega)} + |B_{\Gamma}^{\lambda_n}(\hat{\varphi}_i) - B_{\Gamma}(\hat{\varphi}_i)|_{L^1(\Gamma)} \right) < \frac{1}{2^i}, \\ \left(\sup_{n \geq n_i} (\varepsilon_n^2 - \varepsilon_0^2) \right) |\nabla \hat{\varphi}_i|_{L^2(\Omega)}^2 < \frac{1}{2^{i+1}}. \end{array} \right. \quad (4.33)$$

Based on these, let us define:

$$[\hat{u}_n, \hat{u}_{\Gamma,n}] := [\hat{\varphi}_i, (\hat{\varphi}_i)|_{\Gamma}], \text{ if } n_i \leq n < n_{i+1}, \text{ for some } i \in \mathbb{N} \cup \{0\}. \quad (4.34)$$

Then, with condition (a2) in (A1), (4.32) and Lebesgue's dominated convergence theorem in mind, one can see that:

$$\lim_{n \rightarrow \infty} \int_{\Omega} B(\hat{u}_n) dx = \int_{\Omega} B(\hat{u}_0) dx, \text{ and } \lim_{n \rightarrow \infty} \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,n}) d\Gamma = \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,0}) d\Gamma. \quad (4.35)$$

Taking into account (4.31)–(4.35), we obtain that:

$$\begin{aligned} & \left| \Phi_{\varepsilon_n, \delta_n}^{\lambda_n}(\hat{u}_n, \hat{u}_{\Gamma,n}) - \Phi_{\varepsilon_0}(\hat{u}_0, \hat{u}_{\Gamma,0}) \right| \\ & \leq \int_{\Omega} |f_{\delta_n}(\nabla \hat{u}_n) - |\nabla \hat{u}_n|| dx + \left| \int_{\Omega} (|\nabla \hat{u}_n| - |\nabla \hat{u}_0|) dx \right| \\ & \quad + \frac{\kappa^2}{2} \int_{\Omega} ||\nabla \hat{u}_n|^2 - |\nabla \hat{u}_0|^2| dx + \frac{1}{2} \left| \int_{\Gamma} (|\nabla_{\Gamma}(\varepsilon_n \hat{u}_{\Gamma,n})|^2 - |\nabla_{\Gamma}(\varepsilon_0 \hat{u}_{\Gamma,0})|^2) d\Gamma \right| \\ & \quad + \left| \int_{\Omega} B^{\lambda_n}(\hat{u}_n) dx - \int_{\Omega} B(\hat{u}_n) dx \right| + \left| \int_{\Omega} B(\hat{u}_n) dx - \int_{\Omega} B(\hat{u}_0) dx \right| \\ & \quad + \left| \int_{\Gamma} B_{\Gamma}^{\lambda_n}(\hat{u}_{\Gamma,n}) d\Gamma - \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,n}) d\Gamma \right| + \left| \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,n}) d\Gamma - \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,0}) d\Gamma \right| \\ & \leq \int_{\Omega} |f_{\delta_n}(\nabla \hat{u}_n) - |\nabla \hat{u}_n|| dx + |\nabla(\hat{u}_n - \hat{u}_0)|_{L^1(\Omega)^N} \\ & \quad + \frac{\kappa^2}{2} \|\nabla \hat{u}_n + |\nabla \hat{u}_0|\|_{L^2(\Omega)} \|\nabla(\hat{u}_n - \hat{u}_0)\|_{L^2(\Omega)^N} + \frac{1}{2} |\varepsilon_n^2 - \varepsilon_0^2| \|\nabla_{\Gamma} \hat{u}_{\Gamma,n}\|_{L^2(\Gamma)^N}^2 \\ & \quad + \frac{1}{2} \|\nabla_{\Gamma}(\varepsilon_0 \hat{u}_{\Gamma,n}) + |\nabla_{\Gamma}(\varepsilon_0 \hat{u}_{\Gamma,0})|\|_{L^2(\Gamma)} \|\nabla_{\Gamma}(\varepsilon_0(\hat{u}_{\Gamma,n} - \hat{u}_{\Gamma,0}))\|_{L^2(\Gamma)^N} \\ & \quad + \|B^{\lambda_n}(\hat{u}_n) - B(\hat{u}_n)\|_{L^1(\Omega)} + \|B_{\Gamma}^{\lambda_n}(\hat{u}_{\Gamma,n}) - B_{\Gamma}(\hat{u}_{\Gamma,0})\|_{L^1(\Gamma)} \\ & \quad + \left| \int_{\Omega} B(\hat{u}_n) dx - \int_{\Omega} B(\hat{u}_0) dx \right| + \left| \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,n}) d\Gamma - \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,0}) d\Gamma \right| \\ & \leq \frac{1}{2^{i-1}} + \left(1 + \mathcal{L}^N(\Omega)^{\frac{1}{2}} + \hat{\Phi}_*\right) \left(\|\hat{u}_n - \hat{u}_0\|_{H^1(\Omega)} + \|\varepsilon_0(\hat{u}_{\Gamma,n} - \hat{u}_{\Gamma,0})\|_{H^1(\Gamma)} \right. \\ & \quad \left. + \left| \int_{\Omega} B(\hat{u}_n) dx - \int_{\Omega} B(\hat{u}_0) dx \right| + \left| \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,n}) d\Gamma - \int_{\Gamma} B_{\Gamma}(\hat{u}_{\Gamma,0}) d\Gamma \right| \right), \\ & \quad \text{for any } i \in \mathbb{N} \cup \{0\} \text{ and } n \geq n_i, \end{aligned}$$

where

$$\hat{\Phi}_* := \sup_{n \in \mathbb{N}} \left(\frac{\kappa^2}{2} \|\nabla \hat{u}_n + |\nabla \hat{u}_0|\|_{L^2(\Omega)} + \frac{1}{2} \|\nabla_{\Gamma}(\varepsilon_0 \hat{u}_{\Gamma,n}) + |\nabla_{\Gamma}(\varepsilon_0 \hat{u}_{\Gamma,0})|\|_{L^2(\Gamma)} \right).$$

This implies that the sequence $\{[\hat{u}_n, \hat{u}_{\Gamma,n}]\} \subset H^1(\Omega) \times H^1(\Gamma)$ is the required sequence to verify the condition of optimality. \square

By a similar demonstration technique, we also see the following Corollary.

Corollary 4.1 *Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$ be arbitrary sequence such that $\varepsilon_n \rightarrow \varepsilon_0$ as $n \rightarrow \infty$. Then, for the sequence $\{\Phi_{\varepsilon_n}\}_{n=1}^\infty$ of convex functions, it holds that:*

$$\Phi_{\varepsilon_n} \rightarrow \Phi_{\varepsilon_0} \text{ on } \mathcal{H}, \text{ in the sense of Mosco, as } n \rightarrow \infty.$$

Proof of Key-Lemma 3. Let us fix any $\varepsilon \geq 0$, and let us define a set-valued map $\mathcal{A}_\varepsilon : \mathcal{H} \rightarrow 2^{\mathcal{H}}$, by putting:

$$D(\mathcal{A}_\varepsilon) := \left\{ [u, u_\Gamma] \in \mathcal{V}_\varepsilon \left| \begin{array}{l} \text{there exists } \nu_u \in L^\infty(\Omega)^N \text{ and } [\xi, \xi_\Gamma] \in \mathcal{H} \\ \text{such that (3.3)–(3.4) hold.} \end{array} \right. \right\}, \quad (4.36)$$

and

$$\begin{aligned} [u, u_\Gamma] &\in D(\mathcal{A}_\varepsilon) \subset \mathcal{H} \mapsto \mathcal{A}_\varepsilon[u, u_\Gamma] \\ &:= \left\{ [u^*, u_\Gamma^*] \in \mathcal{H} \left| \begin{array}{l} \text{(3.5) holds, for some } \nu_u \in L^\infty(\Omega)^N \text{ and} \\ [\xi, \xi_\Gamma] \in \mathcal{H}, \text{ satisfying (3.3)–(3.4)} \end{array} \right. \right\}. \end{aligned} \quad (4.37)$$

Then, the assertion of Key-Lemma 3 can be rephrased as follows:

$$\partial\Phi_\varepsilon = \mathcal{A}_\varepsilon \text{ in } \mathcal{H} \times \mathcal{H}. \quad (4.38)$$

This coincidence will be obtained as a consequence of the following *Claims #1–#2*.

Claim #1: $\mathcal{A}_\varepsilon \subset \partial\Phi_\varepsilon$ in $\mathcal{H} \times \mathcal{H}$.

Let us assume that $[u, u_\Gamma] \in D(\mathcal{A}_\varepsilon)$ and $[u^*, u_\Gamma^*] \in \mathcal{A}_\varepsilon[u, u_\Gamma]$ in \mathcal{H} . Then, from (A1), Remark 1.2 (Fact 1)–(Fact 3) and Remark 1.3, and (4.36)–(4.37), it is inferred that:

$$\begin{aligned} &([u^*, u_\Gamma^*], [z, z_\Gamma] - [u, u_\Gamma])_{\mathcal{H}} \\ &= (-\operatorname{div}(\nu_u + \kappa^2 \nabla u) + \xi, z - u)_{L^2(\Omega)} \\ &\quad + \left(-\Delta_\Gamma(\varepsilon^2 u_\Gamma) + [(\nu_u + \kappa^2 \nabla u) \cdot n_\Gamma]_\Gamma + \xi_\Gamma, z_\Gamma - u_\Gamma \right)_{L^2(\Gamma)} \\ &= \int_\Omega (\nu_u + \kappa^2 \nabla u) \cdot \nabla(z - u) \, dx + \int_\Omega \xi(z - u) \, dx \\ &\quad + \int_\Gamma \nabla_\Gamma(\varepsilon u_\Gamma) \cdot \nabla_\Gamma(\varepsilon(z_\Gamma - u_\Gamma)) \, d\Gamma + \int_\Gamma \xi_\Gamma(z_\Gamma - u_\Gamma) \, d\Gamma \\ &\leq \int_\Omega \left(|\nabla z| - |\nabla u| + \frac{\kappa^2}{2} |\nabla z|^2 - \frac{\kappa^2}{2} |\nabla u|^2 \right) \, dx + \int_\Omega (B(z) - B(u)) \, dx \\ &\quad + \frac{1}{2} \int_\Gamma (|\nabla_\Gamma(\varepsilon z_\Gamma)|^2 - |\nabla_\Gamma(\varepsilon u_\Gamma)|^2) \, d\Gamma + \int_\Gamma (B_\Gamma(z_\Gamma) - B_\Gamma(u_\Gamma)) \, d\Gamma \\ &\leq \Phi_\varepsilon(z, z_\Gamma) - \Phi_\varepsilon(u, u_\Gamma), \text{ for any } [z, z_\Gamma] \in \mathcal{V}_\varepsilon. \end{aligned}$$

Thus, we have:

$$[u, u_\Gamma] \in D(\partial\Phi_\varepsilon) \text{ and } [u^*, u_\Gamma^*] \in \partial\Phi_\varepsilon(u, u_\Gamma) \text{ in } \mathcal{H},$$

and we can say that:

$$\mathcal{A}_\varepsilon \subset \partial\Phi_\varepsilon(u, u_\Gamma) \text{ in } \mathcal{H} \times \mathcal{H}.$$

Claim #2: $(\mathcal{A}_\varepsilon + \mathcal{I}_{\mathcal{H}})\mathcal{H} = \mathcal{H}$.

Since, $(\mathcal{A}_\varepsilon + \mathcal{I}_{\mathcal{H}})\mathcal{H} \subset \mathcal{H}$ is trivial, it is sufficient to prove the converse inclusion. Let us take any $[w, w_\Gamma] \in \mathcal{H}$. Then, by applying Minty's theorem, Key-Lemma 1 and Remark 1.2 (Fact 1)–(Fact 3), we can configure a class of functions $\{[u_\delta^\lambda, u_{\Gamma,\delta}^\lambda] \mid 0 < \delta, \lambda \leq 1\} \subset \mathcal{V}_\varepsilon$, by setting:

$$\{[u_\delta^\lambda, u_{\Gamma,\delta}^\lambda] := (\mathcal{A}_{\varepsilon,\delta}^\lambda + \mathcal{I}_{\mathcal{H}})^{-1}[w, w_\Gamma], 0 < \delta, \lambda \leq 1\} \text{ in } \mathcal{H},$$

i.e.

$$[w - u_\delta^\lambda, w_\Gamma - u_{\Gamma,\delta}^\lambda] = \partial\Phi_{\varepsilon,\delta}^\lambda(u_\delta^\lambda, u_{\Gamma,\delta}^\lambda) \text{ in } \mathcal{H}, \text{ for any } 0 < \delta, \lambda \leq 1, \quad (4.39)$$

and we can see that:

$$\begin{aligned} & \int_{\Omega} (\nabla f_\delta(\nabla u_\delta^\lambda) + \kappa^2 \nabla u_\delta^\lambda) \cdot \nabla z \, dx + \int_{\Gamma} \nabla_{\Gamma}(\varepsilon u_{\Gamma,\delta}^\lambda) \cdot \nabla_{\Gamma}(\varepsilon z_{\Gamma}) \, d\Gamma \\ & \quad + \int_{\Omega} \beta^\lambda(u_\delta^\lambda) z \, dx + \int_{\Gamma} \beta_{\Gamma}^\lambda(u_{\Gamma,\delta}^\lambda) z_{\Gamma} \, d\Gamma \\ & = \int_{\Omega} (w - u_\delta^\lambda) z \, dx + \int_{\Gamma} (w_{\Gamma} - u_{\Gamma,\delta}^\lambda) z_{\Gamma} \, d\Gamma, \end{aligned} \quad (4.40)$$

for any $[z, z_{\Gamma}] \in \mathcal{V}_\varepsilon$ and any $0 < \delta, \lambda \leq 1$.

In the variational form (4.40), let us first put $[z, z_{\Gamma}] = [u_\delta^\lambda, u_{\Gamma,\delta}^\lambda] \in \mathcal{V}_\varepsilon$ in (4.40). Then, with (A1), (A4)–(A5) and Young's inequality in mind, we deduce that:

$$\begin{aligned} \frac{1}{2} |[u_\delta^\lambda, u_{\Gamma,\delta}^\lambda]|_{\mathcal{H}}^2 + \kappa^2 |\nabla u_\delta^\lambda|_{L^2(\Omega)^N}^2 + |\nabla_{\Gamma}(\varepsilon u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)^N}^2 & \leq \frac{1}{2} |[w, w_{\Gamma}]|_{\mathcal{H}}^2, \\ & \text{for any } 0 < \delta, \lambda \leq 1. \end{aligned} \quad (4.41)$$

Next, let us take $[z, z_{\Gamma}] = [\beta^\lambda(u_\delta^\lambda), \beta_{\Gamma}^\lambda(u_{\Gamma,\delta}^\lambda)]$ in \mathcal{V}_ε . Then, by applying (A1), (A4)–(A5), [1, Theorem 3.99], [5, Lemma 4.4] and Schwarz's inequality,

$$\begin{aligned} & |\beta^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 \leq (w, \beta^\lambda(u_\delta^\lambda))_{L^2(\Omega)} + (w_{\Gamma}, \beta_{\Gamma}^\lambda(u_{\Gamma,\delta}^\lambda))_{L^2(\Gamma)} \\ & \leq |w|_{L^2(\Omega)}^2 + \frac{1}{4} |\beta^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 + a_1^2 |w_{\Gamma}|_{L^2(\Gamma)}^2 + \frac{1}{4a_1^2} |\beta_{\Gamma}^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 \\ & \leq \left(|w|_{L^2(\Omega)}^2 + a_1^2 |w_{\Gamma}|_{L^2(\Gamma)}^2 \right) + \frac{1}{4} |\beta^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 + \frac{1}{4a_1^2} |a_1 \beta_{\Gamma}^\lambda(u_{\Gamma,\delta}^\lambda)| + b_1 |_{L^2(\Gamma)}^2 \\ & \leq (1 + a_1^2) |[w, w_{\Gamma}]|_{\mathcal{H}}^2 + \frac{1}{4} |\beta^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 + \frac{1}{2} |\beta_{\Gamma}^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 + \frac{b_1^2}{2a_1^2} \mathcal{H}^{N-1}(\Gamma), \end{aligned}$$

so that

$$\begin{aligned} \frac{3}{4} |\beta^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 - \frac{1}{2} |\beta_{\Gamma}^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 & \leq (1 + a_1^2) |[w, w_{\Gamma}]|_{\mathcal{H}}^2 + \frac{b_1^2}{2a_1^2} \mathcal{H}^{N-1}(\Gamma), \\ & \text{for any } 0 < \delta, \lambda \leq 1. \end{aligned} \quad (4.42)$$

Similarly, putting $[z, z_\Gamma] = [\beta_\Gamma^\lambda(u_\delta^\lambda), \beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda)] \in \mathcal{V}_\varepsilon$ in (4.40), and applying (A1), (A4)–(A5), [1, Theorem 3.99], [5, Lemma 4.4] and Schwarz’s inequality,

$$\begin{aligned}
& |\beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 \leq (w, \beta_\Gamma^\lambda(u_\delta^\lambda))_{L^2(\Omega)} + (w_\Gamma, \beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda))_{L^2(\Gamma)} \\
& \leq \frac{1}{a_0^2} |w|_{L^2(\Omega)}^2 + \frac{a_0^2}{4} |\beta_\Gamma^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 + |w_\Gamma|_{L^2(\Gamma)}^2 + \frac{1}{4} |\beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 \\
& \leq \left(\frac{1}{a_0^2} |w|_{L^2(\Omega)}^2 + |w_\Gamma|_{L^2(\Gamma)}^2 \right) + \frac{a_0^2}{4} \left| \frac{1}{a_0} |\beta^\lambda(u_\delta^\lambda)| + \frac{b_0}{a_0} \right|_{L^2(\Omega)}^2 + \frac{1}{4} |\beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 \\
& \leq \left(1 + \frac{1}{a_0^2} \right) |[w, w_\Gamma]|_{\mathcal{H}}^2 + \frac{1}{2} |\beta^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 + \frac{1}{4} |\beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 + \frac{b_0^2}{2} \mathcal{L}^N(\Omega),
\end{aligned}$$

so that

$$\begin{aligned}
-\frac{1}{2} |\beta^\lambda(u_\delta^\lambda)|_{L^2(\Omega)}^2 + \frac{3}{4} |\beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda)|_{L^2(\Gamma)}^2 & \leq \left(1 + \frac{1}{a_0^2} \right) |[w, w_\Gamma]|_{\mathcal{H}}^2 + \frac{b_0^2}{2} \mathcal{L}^N(\Omega), \\
& \text{for any } 0 < \delta, \lambda \leq 1.
\end{aligned} \tag{4.43}$$

Taking the sum of (4.42)–(4.43), it follows that:

$$\begin{aligned}
\frac{1}{4} |[\beta^\lambda(u_\delta^\lambda), \beta_\Gamma^\lambda(u_{\Gamma,\delta}^\lambda)]|_{\mathcal{H}}^2 & \leq \left(2 + \frac{1}{a_0^2} + a_1^2 \right) |[w, w_\Gamma]|_{\mathcal{H}}^2 + \frac{b_0^2}{2} \mathcal{L}^N(\Omega) + \frac{b_1^2}{2a_1^2} \mathcal{H}^{N-1}(\Gamma), \\
& \text{for any } 0 < \delta, \lambda \leq 1.
\end{aligned} \tag{4.44}$$

On account of (4.41) and (4.44), we find pairs of functions $[u, u_\Gamma] \in \mathcal{V}_\varepsilon$ and $[\xi, \xi_\Gamma] \in \mathcal{H}$ and sequences

$$\begin{cases} \delta_1 > \delta_2 > \delta_3 > \cdots > \delta_n & \downarrow & 0, \\ \lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n & \downarrow & 0, \end{cases} \quad \text{as } n \rightarrow \infty,$$

such that:

$$\begin{cases} [u_n, u_{\Gamma,n}] := [u_{\delta_n}^{\lambda_n}, u_{\Gamma,\delta_n}^{\lambda_n}] \rightarrow [u, u_\Gamma] \text{ in } \mathcal{H} \text{ and weakly in } \mathcal{V}_\varepsilon, \\ [\beta^{\lambda_n}(u_n), \beta_{\Gamma}^{\lambda_n}(u_{\Gamma,n})] \rightarrow [\xi, \xi_\Gamma] \text{ weakly in } \mathcal{H}, \end{cases} \quad \text{as } n \rightarrow \infty. \tag{4.45}$$

Here, in the light of Key-Lemma 2, (4.39) and (4.45), we can apply Remark 1.6 (Fact 5) to see that:

$$[w - u, w_\Gamma - u_\Gamma] \in \partial\Phi_\varepsilon(u, u_\Gamma) \text{ in } \mathcal{H},$$

and

$$\Phi_{\varepsilon, \delta_n}^{\lambda_n}(u_n, u_{\Gamma,n}) \rightarrow \Phi_\varepsilon(u, u_\Gamma) \text{ as } n \rightarrow \infty. \tag{4.46}$$

Also, by (A1), (A5), Remark 1.5 and Remark 1.6 (Fact 5), we see that:

$$\xi \in \beta(u) \text{ a.e. in } \Omega, \text{ and } \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \text{ a.e. on } \Gamma. \tag{4.47}$$

By virtue of (4.45)–(4.46), (A4), Remark 1.5, Lemma 4.1, and (Fact 8), we further compute that:

$$\begin{aligned}
& \frac{\kappa^2}{2} \int_{\Omega} |\nabla u|^2 dx \leq \frac{\kappa^2}{2} \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 dx \leq \frac{\kappa^2}{2} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 dx \\
& \leq \lim_{n \rightarrow \infty} \Phi_{\varepsilon, \delta_n}^{\lambda_n}(u_n, u_{\Gamma, n}) - \liminf_{n \rightarrow \infty} \int_{\Omega} f_{\delta_n}(\nabla u_n) dx - \liminf_{n \rightarrow \infty} \int_{\Gamma} |\nabla_{\Gamma}(\varepsilon u_{\Gamma, n})|^2 d\Gamma \\
& \quad - \liminf_{n \rightarrow \infty} \left(\int_{\Omega} B^{\lambda_n}(u_n) dx + \int_{\Gamma} B_{\Gamma}^{\lambda_n}(u_{\Gamma, n}) d\Gamma \right) \\
& \leq \Phi_{\varepsilon}(u, u_{\Gamma}) - \int_{\Omega} |\nabla u|^2 dx - \int_{\Gamma} |\nabla_{\Gamma}(\varepsilon u_{\Gamma})|^2 d\Gamma \\
& \quad - \left(\int_{\Omega} B(u) dx + \int_{\Gamma} B_{\Gamma}(u_{\Gamma}) d\Gamma \right) = \frac{\kappa^2}{2} \int_{\Omega} |\nabla u|^2 dx.
\end{aligned} \tag{4.48}$$

Having in mind (4.45), (4.48) and the above calculation and the uniform convexity of L^2 -based topologies, it is deduced that:

$$\begin{cases} u_n \rightarrow u \text{ in } H^1(\Omega), \\ \nabla u_n \rightarrow \nabla u \text{ in } L^2(\Omega)^N, \end{cases} \quad \text{as } n \rightarrow \infty. \tag{4.49}$$

In the meantime, from (A4) and (4.49),

$$|\nabla f_{\delta_n}(\nabla u_n)|_{L^2(\Omega)^N}^2 \leq 2C_0 \left(\sup_{n \in \mathbb{N}} |\nabla u_n|_{L^2(\Omega)^N}^2 + \mathcal{L}^N(\Omega) \right), \text{ for any } n \in \mathbb{N},$$

which enables us to say

$$\nabla f_{\delta_n}(\nabla u_n) \rightarrow \nu_u \text{ weakly in } L^2(\Omega)^N \text{ as } n \rightarrow \infty, \text{ for some } \nu_u \in L^2(\Omega)^N, \tag{4.50}$$

by taking a subsequence if necessary.

In view of (4.49)–(4.50), (Fact 8), Remark 1.3, and Remark 1.6 (Fact 5), one can see that:

$$\nu_u \in \text{Sgn}(\nabla u) \text{ a.e. in } \Omega. \tag{4.51}$$

Hence, letting $n \rightarrow \infty$ in (4.40) yields that:

$$\begin{aligned}
& \int_{\Omega} (\nu_u + \kappa^2 \nabla u) \cdot \nabla z dx + \int_{\Gamma} \nabla_{\Gamma}(\varepsilon u_{\Gamma}) \cdot \nabla_{\Gamma}(\varepsilon z_{\Gamma}) d\Gamma + \int_{\Omega} \xi z dx + \int_{\Gamma} \xi_{\Gamma} z_{\Gamma} d\Gamma \\
& = \int_{\Omega} (w - u) z dx + \int_{\Gamma} (w_{\Gamma} - u_{\Gamma}) z_{\Gamma} d\Gamma, \text{ for any } [z, z_{\Gamma}] \in \mathcal{V}_{\varepsilon}.
\end{aligned} \tag{4.52}$$

In particular, taking any $\varphi_0 \in H_0^1(\Omega)$ and putting $[z, z_{\Gamma}] = [\varphi_0, 0]$ in $\mathcal{V}_{\varepsilon}$,

$$(w - u - \xi, \varphi_0)_{L^2(\Omega)} = \int_{\Omega} (\nu_u + \kappa^2 \nabla u) \cdot \nabla \varphi_0 dx, \text{ for any } \varphi_0 \in H_0^1(\Omega).$$

which implies:

$$-\text{div}(\nu_u + \kappa^2 \nabla u) = w - u - \xi \in L^2(\Omega) \text{ in } \mathcal{D}'(\Omega). \tag{4.53}$$

Furthermore, with Remark 1.2 (Fact 1)–(Fact 3), (4.52)–(4.53) in mind, we can see that:

$$\begin{aligned}
& (w_\Gamma - u_\Gamma - \xi_\Gamma, z_\Gamma)_{L^2(\Gamma)} \\
&= \int_{\Omega} (\nu_u + \kappa^2 \nabla u) \cdot \nabla z \, dx - (w - u - \xi, z)_{L^2(\Omega)} + \int_{\Gamma} \nabla_{\Gamma}(\varepsilon u_{\Gamma}) \cdot \nabla_{\Gamma}(\varepsilon z_{\Gamma}) \, d\Gamma \\
&= {}_{H^{-\frac{1}{2}}(\Gamma)} \left\langle [(\nu_u + \kappa^2 \nabla u) \cdot n_{\Gamma}]_{\Gamma}, z_{\Gamma} \right\rangle_{{}_{H^{\frac{1}{2}}(\Gamma)}} + {}_{H^{-1}(\Gamma)} \langle -\Delta_{\Gamma}(\varepsilon u_{\Gamma}), \varepsilon z_{\Gamma} \rangle_{{}_{H^1(\Gamma)}}, \\
&\quad \text{for any } [z, z_{\Gamma}] \in \mathcal{V}_{\varepsilon}.
\end{aligned}$$

This identity leads to:

$$-\Delta_{\Gamma}(\varepsilon^2 u_{\Gamma}) + [(\nu_u + \kappa^2 \nabla u) \cdot n_{\Gamma}]_{\Gamma} = w_{\Gamma} - u_{\Gamma} - \xi_{\Gamma} \in L^2(\Gamma) \text{ in } H^{-1}(\Gamma). \quad (4.54)$$

As a consequence of (4.47), (4.51), (4.53)–(4.54), we obtain *Claim #2*.

Now, by using *Claims #1–#2* and the maximality of $\partial\Phi_{\varepsilon}$ in $\mathcal{H} \times \mathcal{H}$, we can show the coincidence $\partial\Phi_{\varepsilon} = \mathcal{A}_{\varepsilon}$ in $\mathcal{H} \times \mathcal{H}$, and we conclude this Key-Lemma 3. \square

5 Proofs of Main Theorems

In this section, we will prove two Main Theorems by using the results of the previous sections.

Proof of Main Theorem 1. First, we show the item (I-1). In the Cauchy problem $(\text{CP})_{\varepsilon}$, we see from (A3) and (Fact 7) that:

$$\Theta = [\theta, \theta_{\Gamma}] \in L^2(0, T; \mathcal{H}) \text{ and } U_0 = [u_0, u_{\Gamma,0}] \in \overline{D(\Phi_{\varepsilon})}.$$

Hence, by applying the general theories of evolution equations, e.g., [3, Theorem 4.1, p. 158], [4, Theorem 3.6 and Proposition 3.2], [24, Section 2] and [26, Theorem 1.1.2], we immediately have the existence and uniqueness of solution $U = [u, u_{\Gamma}] \in L^2(0, T; \mathcal{H})$ to $(\text{CP})_{\varepsilon}$, such that:

$$U \in C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}_{\varepsilon}) \cap W_{\text{loc}}^{1,2}((0, T]; \mathcal{H}) \text{ and } \Phi_{\varepsilon}(U) \in L^1(0, T) \cap L_{\text{loc}}^{\infty}((0, T]).$$

Also, there exists a positive constant C_1 , independent of U_0 and Θ , such that:

$$\begin{aligned}
& |U|_{C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V}_{\varepsilon})}^2 + \int_0^T \Phi_{\varepsilon}(U(t)) \, dt + |\sqrt{t}U'|_{L^2(0,T;\mathcal{H})}^2 + \sup_{t \in (0,T)} t\Phi_{\varepsilon}(U(t)) \\
& \leq C_1 \left(1 + |U_0|_{\mathcal{H}}^2 + |\Theta|_{L^2(0,T;\mathcal{H})}^2 \right).
\end{aligned} \quad (5.1)$$

Moreover, if $U_0 \in D(\Phi_{\varepsilon})$, there exists a positive constant C_2 , independent of U_0 and Θ , such that:

$$|U'|_{L^2(0,T;\mathcal{H})}^2 + \sup_{t \in (0,T)} \Phi_{\varepsilon}(U(t)) \leq C_2 \left(1 + |U_0|_{\mathcal{H}}^2 + |\Theta|_{L^2(0,T;\mathcal{H})}^2 + \Phi_{\varepsilon}(U_0) \right). \quad (5.2)$$

Now, Key-Lemma 3 guarantees that the solution $U = [u, u_{\Gamma}]$ to $(\text{CP})_{\varepsilon}$ coincides with that to the system $(\text{ACE})_{\varepsilon}$. Besides, in the light of (3.1) and (A1), the inequalities (2.4) and (2.5) directly follows from (5.1) and (5.2), respectively.

Next, we show the item (I-2). For $k = 1, 2$, let $U^k := [u^k, u_\Gamma^k]$ be the two solutions to $(\text{CP})_\varepsilon$ corresponds to forcing term $\Theta^k := [\theta^k, \theta_\Gamma^k] \in L^2(0, T; \mathcal{H})$ and initial term $U_0^k = [u_0^k, u_{\Gamma,0}^k] \in D(\Phi_\varepsilon)$, respectively. Then, we can obtain the inequality (2.6) by using standard method: more precisely, by taking the difference between the two evolution equations, multiplying it by $U^1(t) - U^2(t)$, and applying Gronwall's lemma. \square

Proof of Main Theorem 2. The Main Theorem 2 is proved by referring to the demonstration technique as in [26, Theorem 2.7.1].

Let us set $J_0 := [\varepsilon_0 - 1, \varepsilon_0 + 1] \cap [0, \infty)$. On this basis, we divide the proof in the following two steps.

(Step 1) *The case when $\{\Phi_\varepsilon(U_0^\varepsilon)\}_{\varepsilon \in J_0}$ is bounded.*

If $\{\Phi_\varepsilon(U_0^\varepsilon)\}_{\varepsilon \in J_0}$ is bounded, then the estimate (2.5) imply the following facts:

$$\begin{cases} \{U^\varepsilon\}_{\varepsilon \in J_0} \text{ is bounded in } W^{1,2}(0, T; \mathcal{H}) \cap L^\infty(0, T; \mathcal{V}_0), \\ \{\varepsilon u_\Gamma^\varepsilon\}_{\varepsilon \in J_0} \text{ is bounded in } W^{1,2}(0, T; L^2(\Gamma)) \cap L^\infty(0, T; H^1(\Gamma)). \end{cases}$$

Therefore, applying general theories of compactness, such as Ascoli's theorem, we find a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset J_0$ and a limit point $U = [u, u_\Gamma] \in W^{1,2}(0, T; \mathcal{H}) \cap L^\infty(0, T; \mathcal{V}_0)$, such that:

$$\begin{aligned} U^{\varepsilon_n} \rightarrow U \quad & \text{in } C([0, T]; \mathcal{H}), \text{ weakly in } W^{1,2}(0, T; \mathcal{H}) \\ & \text{and weakly-* in } L^\infty(0, T; \mathcal{V}_0) \text{ as } n \rightarrow \infty, \end{aligned} \quad (5.3)$$

and

$$\begin{cases} \varepsilon_0 u_\Gamma \in W^{1,2}(0, T; L^2(\Gamma)) \cap L^\infty(0, T; H^1(\Gamma)), \\ \varepsilon_n u_\Gamma^{\varepsilon_n} \rightarrow \varepsilon_0 u_\Gamma \text{ weakly-* in } L^\infty(0, T; H^1(\Gamma)), \text{ as } n \rightarrow \infty. \end{cases} \quad (5.4)$$

Also, by Corollary 4.1 and Lemma 4.1 with $S = (0, T)$, we have that

$$\hat{\Phi}_{\varepsilon_n} \rightarrow \hat{\Phi}_{\varepsilon_0} \text{ on } L^2(0, T; \mathcal{H}), \text{ in the sense of Mosco, as } n \rightarrow \infty.$$

From (5.3), (A2), Remark 1.4 and Remark 1.6 (Fact 5), it is seen that

$$[-U' - \mathcal{G}(U) + \Theta^{\varepsilon_0}, U] \in \partial \hat{\Phi}_{\varepsilon_0} \text{ in } L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{H}), \quad (5.5)$$

and

$$\hat{\Phi}_{\varepsilon_n}(U^{\varepsilon_n}) \rightarrow \hat{\Phi}_{\varepsilon_0}(U) \text{ as } n \rightarrow \infty. \quad (5.6)$$

Note that (2.7), (5.3)–(5.5) and Remark 1.4 (Fact 4) enable us to say that $U = [u, u_\Gamma]$ is a solution to the Cauchy problem $(\text{CP})_{\varepsilon_0}$. So, due to the uniqueness of solutions, it must hold that:

$$U = [u, u_\Gamma] = U^{\varepsilon_0} = [u^{\varepsilon_0}, u_\Gamma^{\varepsilon_0}] \text{ in } L^2(0, T; \mathcal{H}). \quad (5.7)$$

Furthermore, since (A1), (5.3)–(5.4) and (5.7) imply:

$$\begin{cases} \liminf_{n \rightarrow \infty} \frac{\kappa^2}{2} \int_0^T \int_\Omega |\nabla u^{\varepsilon_n}|^2 dx dt \geq \frac{\kappa^2}{2} \int_0^T \int_\Omega |\nabla u^{\varepsilon_0}|^2 dx dt, \\ \liminf_{n \rightarrow \infty} \frac{1}{2} \int_0^T \int_\Gamma |\nabla_\Gamma(\varepsilon u_\Gamma^{\varepsilon_n})|^2 d\Gamma dt \geq \frac{1}{2} \int_0^T \int_\Gamma |\nabla_\Gamma(\varepsilon u_\Gamma^{\varepsilon_0})|^2 d\Gamma dt, \\ \liminf_{n \rightarrow \infty} \int_0^T \left(\int_\Omega (|\nabla u^{\varepsilon_n}| + B(u^{\varepsilon_n})) dx + \int_\Gamma B_\Gamma(u_\Gamma^{\varepsilon_n}) d\Gamma \right) dt \\ \geq \int_0^T \left(\int_\Omega (|\nabla u^{\varepsilon_0}| + B(u^{\varepsilon_0})) dx + \int_\Gamma B_\Gamma(u_\Gamma^{\varepsilon_0}) d\Gamma \right) dt, \end{cases} \quad (5.8)$$

one can see that:

$$\begin{cases} \lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} |\nabla u^{\varepsilon_n}|^2 dxdt = \int_0^T \int_{\Omega} |\nabla u^{\varepsilon_0}|^2 dxdt, \\ \lim_{n \rightarrow \infty} \int_0^T \int_{\Gamma} |\nabla_{\Gamma}(\varepsilon_n u_{\Gamma}^{\varepsilon_n})|^2 d\Gamma dt = \int_0^T \int_{\Gamma} |\nabla_{\Gamma}(\varepsilon_0 u_{\Gamma}^{\varepsilon_0})|^2 d\Gamma dt, \end{cases} \quad (5.9)$$

by applying (Fact 0) with (5.6)–(5.8) in mind.

Now, taking into account (5.3)–(5.4), (5.7) and (5.9), and applying the uniform convexity of L^2 -based topologies and the continuity of the trace operators, we obtain that:

$$\begin{cases} u^{\varepsilon_n} \rightarrow u^{\varepsilon_0} \text{ in } L^2(0, T; H^1(\Omega)), \\ \varepsilon_n u_{\Gamma}^{\varepsilon_n} \rightarrow \varepsilon_0 u_{\Gamma}^{\varepsilon_0} \text{ in } L^2(0, T; H^1(\Gamma)), \\ u_{\Gamma}^{\varepsilon_n} = u_{\Gamma}^{\varepsilon_n} \rightarrow u_{\Gamma}^{\varepsilon_0} = u_{\Gamma}^{\varepsilon_0} \text{ in } L^2(0, T; H^{\frac{1}{2}}(\Gamma)), \end{cases} \quad \text{as } n \rightarrow \infty. \quad (5.10)$$

Therefore, (5.3), (5.7) and (5.10) are sufficient to verify (2.8)–(2.9).

(Step 2) *The case when $\{\Phi_{\varepsilon}(U_0^{\varepsilon})\}_{\varepsilon \in J_0}$ is unbounded.*

Let $\rho \in (0, 1)$ be an arbitrary constant. Then, by the assumption for $\{U_0^{\varepsilon}\}_{\varepsilon \geq 0}$, for any sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset J_0$ converging to ε_0 we find a large number $n_1(\rho) \in \mathbb{N}$, such that:

$$|U_0^{\varepsilon_n} - U_0^{\varepsilon_0}|_{\mathcal{H}} \leq \rho, \text{ for any } n \geq n_1(\rho) \quad (5.11)$$

Also, since $U_0^{\varepsilon_0} \in \overline{D(\Phi_{\varepsilon_0})}$, we find a function $W_{0,\rho} \in D(\Phi_{\varepsilon_0})$, such that:

$$|U_0^{\varepsilon_0} - W_{0,\rho}|_{\mathcal{H}} \leq \rho. \quad (5.12)$$

Additionally, by Corollary 4.1, there exists a sequence $\{W_{0,\rho}^n \in D(\Phi_{\varepsilon_n})\}_{n=1}^{\infty} \subset \mathcal{H}$, such that:

$$\begin{cases} W_{0,\rho}^n \rightarrow W_{0,\rho} \text{ in } \mathcal{H}, \\ \Phi_{\varepsilon_n}(W_{0,\rho}^n) \rightarrow \Phi_{\varepsilon_0}(W_{0,\rho}), \end{cases} \quad \text{as } n \rightarrow \infty, \quad (5.13)$$

and in particular, there exists a large number $n_2(\rho) \in \mathbb{N}$, with $n_2(\rho) \geq n_1(\rho)$, such that:

$$|W_{0,\rho}^n - W_{0,\rho}|_{\mathcal{H}} \leq \rho, \text{ for any } n \geq n_2(\rho). \quad (5.14)$$

From (5.11)–(5.12) and (5.14), it follows that:

$$\begin{aligned} |U_0^{\varepsilon_n} - W_{0,\rho}^n|_{\mathcal{H}} &\leq |U_0^{\varepsilon_n} - U_0^{\varepsilon_0}|_{\mathcal{H}} + |U_0^{\varepsilon_0} - W_{0,\rho}|_{\mathcal{H}} + |W_{0,\rho} - W_{0,\rho}^n|_{\mathcal{H}} \\ &\leq 3\rho, \text{ for any } n \geq n_2(\rho). \end{aligned} \quad (5.15)$$

Based on this, let $W_{\rho} \in W^{1,2}(0, T; \mathcal{H}) \cap L^{\infty}(0, T; \mathcal{V}_{\varepsilon_0})$ be the solution to $(\text{CP})_{\varepsilon_0}$, corresponding to the forcing term $\Theta^{\varepsilon_0} \in L^2(0, T; \mathcal{H})$ and the initial data $W_{0,\rho} \in D(\Phi_{\varepsilon_0})$. As well as, for any $n \in \mathbb{N}$, let $W_{\rho}^n \in W^{1,2}(0, T; \mathcal{H}) \cap L^{\infty}(0, T; \mathcal{V}_{\varepsilon_n})$ be the solution to $(\text{CP})_{\varepsilon_n}$, corresponding to the forcing term $\Theta^{\varepsilon_n} \in L^2(0, T; \mathcal{H})$ and the initial data $W_{0,\rho}^n \in D(\Phi_{\varepsilon_n})$. Then, by applying the result of the previous (Step 1), we have:

$$W_{\rho}^n \rightarrow W_{\rho} \text{ in } C([0, T]; \mathcal{H}) \text{ as } n \rightarrow \infty. \quad (5.16)$$

Besides, from (2.6) and (5.15)–(5.16), one can see that:

$$\begin{aligned}
& \overline{\lim}_{n \rightarrow \infty} |U^{\varepsilon_n} - U^{\varepsilon_0}|_{C([0,T];\mathcal{H})} \\
& \leq \overline{\lim}_{n \rightarrow \infty} \left(|U^{\varepsilon_n} - W_\rho^n|_{C([0,T];\mathcal{H})} + |W_\rho^n - W_\rho|_{C([0,T];\mathcal{H})} + |W_\rho - U^{\varepsilon_0}|_{C([0,T];\mathcal{H})} \right) \\
& \leq \sqrt{C_3} \left(\overline{\lim}_{n \rightarrow \infty} |U_0^{\varepsilon_n} - W_{0,\rho}^n|_{\mathcal{H}} + |W_{0,\rho} - U_0^{\varepsilon_0}|_{\mathcal{H}} \right) + \lim_{n \rightarrow \infty} |W_\rho^n - W_\rho|_{C([0,T];\mathcal{H})} \\
& \leq 4\sqrt{C_3}\rho.
\end{aligned}$$

Since $\rho \in (0, 1)$ is arbitrary, the above inequality implies that:

$$U^{\varepsilon_n} \rightarrow U^{\varepsilon_0} \text{ in } C([0, T]; \mathcal{H}) \text{ as } n \rightarrow \infty. \quad (5.17)$$

Now, our remaining task will be to verify the convergences (2.8)–(2.9) under the unbounded situation of $\{\Phi_{\varepsilon_n}(U_0^{\varepsilon_n})\}_{n=1}^\infty$. To this end, we first invoke (2.5) and (5.13) to check the existence of a constant $K(\rho)$, depending on $\rho \in (0, 1)$, such that:

$$\begin{aligned}
& |(W_\rho)'|_{L^2(0,T;\mathcal{H})}^2 + |(W_\rho^n)'|_{L^2(0,T;\mathcal{H})}^2 + \sup_{t \in [0,T]} \Phi_{\varepsilon_0}(W_\rho(t)) + \sup_{t \in [0,T]} \Phi_{\varepsilon_n}(W_\rho^n(t)) \\
& \leq K(\rho), \text{ for any } n \in \mathbb{N}.
\end{aligned} \quad (5.18)$$

On this basis, let us consider a sequence $\{\mathcal{P}_n\}_{n=1}^\infty \subset C([0, T])$ of functions, given as:

$$\varsigma \in [0, T] \mapsto \mathcal{P}_n(\varsigma) := \int_0^\varsigma \Phi_{\varepsilon_n}(W_\rho^n(t)) dt \in [0, \infty), \text{ for any } n \in \mathbb{N}.$$

Then, by applying a similar method to show (5.6), we have:

$$\mathcal{P}_n(\varsigma) \rightarrow \mathcal{P}(\varsigma) := \int_0^\varsigma \Phi_{\varepsilon_0}(W_\rho(t)) dt \text{ as } n \rightarrow \infty, \text{ for any } \varsigma \in [0, T]. \quad (5.19)$$

Also, from (5.18), it is seen that:

$$\left\{ \frac{d}{d\varsigma} \mathcal{P}_n \right\}_{n=1}^\infty = \{\Phi_{\varepsilon_n}(W_\rho^n)\}_{n=1}^\infty \text{ is bounded in } L^\infty(0, T). \quad (5.20)$$

By (5.19)–(5.20) and Ascoli's theorem, we may suppose that:

$$\mathcal{P}_n \rightarrow \mathcal{P} \text{ in } C([0, T]) \text{ as } n \rightarrow \infty,$$

by taking a subsequence if necessary, and more precisely, we find a large number $n_*(\rho)$, independent of $\varsigma \in [0, T]$, such that:

$$\begin{aligned}
& n_*(\rho) \geq n_2(\rho) \text{ and } \left| \int_0^\varsigma \Phi_{\varepsilon_n}(W_\rho^n(t)) dt - \int_0^\varsigma \Phi_{\varepsilon_0}(W_\rho(t)) dt \right| < \rho, \\
& \text{for any } n \geq n_*(\rho) \text{ and any } \varsigma \in [0, T].
\end{aligned} \quad (5.21)$$

In the meantime, for the sequence of solutions $\{U^{\varepsilon_n}\}_{n=1}^\infty$, it is easily seen that:

$$\begin{aligned}
& ((U^{\varepsilon_n} - W_\rho^n)', U^{\varepsilon_n} - Z)_{L^2(0,\varsigma;\mathcal{H})} + \int_0^\varsigma \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt \\
& \leq \int_0^\varsigma \Phi_{\varepsilon_n}(Z(t)) dt + (\Theta^{\varepsilon_n} - \mathcal{G}(U^{\varepsilon_n}) - (W_\rho^n)', U^{\varepsilon_n} - Z)_{L^2(0,\varsigma;\mathcal{H})}, \\
& \text{for any } Z \in L^2(0, \varsigma; \mathcal{V}_{\varepsilon_n}) \text{ and any } n \in \mathbb{N}.
\end{aligned}$$

So, putting $Z = W_\rho^n$, and using (2.6)–(2.7), (5.15), (5.17)–(5.18), (5.21) and (A2), we infer that:

$$\begin{aligned}
& \int_0^\varsigma \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt \leq \frac{1}{2}|U_0^{\varepsilon_n} - W_{0,\rho}^n|_{\mathcal{H}}^2 + \int_0^\varsigma \Phi_{\varepsilon_n}(W_\rho^n(t)) dt \\
& \quad + |\Theta^{\varepsilon_n} - \mathcal{G}(U^{\varepsilon_n}) - (W_\rho^n)'|_{L^2(0,\varsigma;\mathcal{H})}|U^{\varepsilon_n} - W_\rho^n|_{L^2(0,\varsigma;\mathcal{H})} \\
& \leq \frac{9}{2}\rho^2 + \left(\int_0^\varsigma \Phi_{\varepsilon_0}(W_\rho(t)) dt + \rho \right) \\
& \quad + \sqrt{C_{3\varsigma}}|U_0^{\varepsilon_n} - W_{0,\rho}^n|_{\mathcal{H}} \left(|\Theta^{\varepsilon_n} - \mathcal{G}(U^{\varepsilon_n})|_{L^2(0,\varsigma;\mathcal{H})} + |(W_\rho^n)'|_{L^2(0,\varsigma;\mathcal{H})} \right) \quad (5.22) \\
& \leq \frac{11}{2}\rho + \int_0^\varsigma \Phi_{\varepsilon_0}(W_{0,\rho}(t)) dt \\
& \quad + 3\rho\sqrt{C_{3\varsigma}} \left(\sup_{n \in \mathbb{N}} |\Theta^{\varepsilon_n} - \mathcal{G}(U^{\varepsilon_n})|_{L^2(0,\varsigma;\mathcal{H})} + \sqrt{K(\rho)} \right), \\
& \quad \text{for any } \varsigma \in [0, T] \text{ and any } n \in \mathbb{N}.
\end{aligned}$$

Here, let us take a constant $\varsigma_*(\rho) \in (0, T)$, so small to satisfy that:

$$\begin{aligned}
& \int_0^{\varsigma_*(\rho)} \Phi_{\varepsilon_0}(W_{0,\rho}(t)) dt \\
& \quad + 3\rho\sqrt{C_{3\varsigma_*(\rho)}} \left(\sup_{n \in \mathbb{N}} |\Theta^{\varepsilon_n} - \mathcal{G}(U^{\varepsilon_n})|_{L^2(0,\varsigma_*(\rho);\mathcal{H})} + K(\rho) \right) < \frac{\rho}{2}. \quad (5.23)
\end{aligned}$$

Then, having in mind (5.17), (5.22)–(5.23), Corollary 4.1 and Fatou's lemma, it is observed that:

$$\begin{aligned}
\int_0^{\varsigma_*(\rho)} \Phi_{\varepsilon_0}(U^{\varepsilon_0}(t)) dt & \leq \underline{\lim}_{n \rightarrow \infty} \int_0^{\varsigma_*(\rho)} \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt \\
& \leq \sup_{n \in \mathbb{N}} \int_0^{\varsigma_*(\rho)} \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt \leq 6\rho. \quad (5.24)
\end{aligned}$$

Finally, by (2.4), one can see that the sequence $\{U^{\varepsilon_n}\}_{n=1}^\infty$ is bounded in $W^{1,2}(\varsigma_*(\rho), T; \mathcal{H}) \cap L^\infty(\varsigma_*(\rho), T; \mathcal{V}_0)$. So, we can apply a similar arguments to obtain (5.6), and we can show that:

$$\int_{\varsigma_*(\rho)}^T \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt \rightarrow \int_{\varsigma_*(\rho)}^T \Phi_{\varepsilon_0}(U^{\varepsilon_0}(t)) dt \text{ as } n \rightarrow \infty. \quad (5.25)$$

In view of (5.17), (5.24)–(5.25), we can say that:

$$\begin{cases} U^{\varepsilon_0} \in L^2(0, T; \mathcal{V}_0), \{U^{\varepsilon_n}\}_{n=1}^\infty \text{ is bounded in } L^2(0, T; \mathcal{V}_0), \\ U^{\varepsilon_n} \rightarrow U^{\varepsilon_0} \text{ weakly in } L^2(0, T; \mathcal{V}_0) \text{ as } n \rightarrow \infty, \end{cases} \quad (5.26)$$

and

$$\begin{aligned}
& \overline{\lim}_{n \rightarrow \infty} \left| \int_0^T \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt - \int_0^T \Phi_{\varepsilon_0}(U^{\varepsilon_0}(t)) dt \right| \\
& \leq \sup_{n \in \mathbb{N}} \int_0^{\varsigma_*(\rho)} \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt + \int_0^{\varsigma_*(\rho)} \Phi_{\varepsilon_0}(U^{\varepsilon_0}(t)) dt \\
& \quad + \lim_{n \rightarrow \infty} \left| \int_{\varsigma_*(\rho)}^T \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt - \int_{\varsigma_*(\rho)}^T \Phi_{\varepsilon_0}(U^{\varepsilon_0}(t)) dt \right| \\
& \leq 12\rho.
\end{aligned}$$

Since $\rho \in (0, 1)$ is arbitrary, the above inequality implies:

$$\lim_{n \rightarrow \infty} \int_0^T \Phi_{\varepsilon_n}(U^{\varepsilon_n}(t)) dt = \int_0^T \Phi_{\varepsilon_0}(U^{\varepsilon_0}(t)) dt. \quad (5.27)$$

By virtue of (5.26)–(5.27), we can apply a similar method to derive (5.10), and we obtain that:

$$\begin{cases} u^{\varepsilon_n} \rightarrow u^{\varepsilon_0} \text{ in } L^2(0, T; H^1(\Omega)), \\ \varepsilon_n u_{\Gamma}^{\varepsilon_n} \rightarrow \varepsilon_0 u_{\Gamma}^{\varepsilon_0} \text{ in } L^2(0, T; H^1(\Gamma)), \\ u_{\Gamma}^{\varepsilon_n} = u_{\Gamma}^{\varepsilon_n} \rightarrow u_{\Gamma}^{\varepsilon_0} = u_{\Gamma}^{\varepsilon_0} \text{ in } L^2(0, T; H^{\frac{1}{2}}(\Gamma)), \end{cases} \quad \text{as } n \rightarrow \infty. \quad (5.28)$$

Hence, (5.17) and (5.28) imply the conclusive convergences (2.8)–(2.9). \square

References

- [1] Ambrosio, L.; Fusco, N.; Pallara, D.: *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press, New York (2006).
- [2] Attouch, H.: *Variational Convergence for Functions and Operators*. Applicable Mathematics Series, Pitman, Massachusetts (1984).
- [3] Barbu, V.: *Nonlinear Differential Equations of Monotone Type in Banach Spaces*. Springer Monographs in Mathematics. Springer Springer, New York (2010).
- [4] Brézis, H.: *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*. North-Holland Mathematics Studies, **5**, Notas de Matemática (50), North-Holland Publishing and American Elsevier Publishing (1973).
- [5] Calatroni, L.; Colli, P.: Global solution to the Allen-Cahn equation with singular potentials and dynamic boundary conditions. *Nonlinear Anal.* **79** (2013), 12–27.
- [6] Cherfilis, L.; Gatti, S.; Miranville, A.: A variational approach to a Cahn-Hilliard model in a domain with nonpermeable walls. *J. Math. Sci. (N.Y.)* **189** (2013), 604–636.
- [7] Colli, P.; Fukao, T.: The Allen-Cahn equation with dynamic boundary conditions and mass constraints. *Math. Methods Appl. Sci.* **38** (2015), 3950–3967.
- [8] Colli, P.; Fukao, T.: Cahn-Hilliard equation with dynamic boundary conditions and mass constraint on the boundary. *J. Math. Anal. Appl.* **429** (2015), 1190–1213.
- [9] Colli, P.; Fukao, T.: Equation and dynamic boundary condition of Cahn-Hilliard type with singular potentials. *Nonlinear Anal.* **127** (2015), 413–433.
- [10] Colli, P.; Gilardi, G.; Sprekels, J.: On the Cahn-Hilliard equation with dynamic boundary conditions and a dominating boundary potential. *J. Math. Anal. Appl.* **419** (2014), 972–994.
- [11] Colli, P.; Gilardi, G.; Sprekels, J.: A boundary control problem for the pure Cahn-Hilliard equation with dynamic boundary conditions. *Adv. Nonlinear Anal.* **4** (2015), 311–325.

- [12] Colli, P.; Gilardi, G.; Sprekels, J.: A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions. *Appl. Math. Optim.* **73** (2016), 195–225.
- [13] Colli, P.; Sprekels, J.: Optimal control of an Allen–Cahn equation with singular potentials and dynamic boundary condition. *SIAM J. Control Optim.* **53** (2015), 213–234.
- [14] Dal Maso, G.: *An Introduction to Γ -convergence*. Progress in Nonlinear Differential Equations and their Applications, **8**. Birkhäuser Boston, Inc., Boston, Ma (1993).
- [15] Gal, C. G.; Grasselli, M.: The non-isothermal Allen-Cahn equation with dynamic boundary conditions. *Discrete Contin. Dyn. Syst.* **22** (2008), no. 4, 1009–1040.
- [16] Gal, C. G.; Grasselli, M.; Miranville, A.: Nonisothermal Allen-Cahn equations with coupled dynamic boundary conditions. *Nonlinear phenomena with energy dissipation*, 117–139, GAKUTO Internat. Ser. Math. Sci. Appl., 29, Gakktōsho, Tokyo, 2008.
- [17] Gal, C. G.; Warma, M.: Well posedness and the global attractor of some quasi-linear parabolic equations with nonlinear dynamic boundary conditions. *Differential Integral Equations* **23** (2010), 327–358.
- [18] Giga, Y., Kashima, Y., Yamazaki, N.: Local solvability of a constrained gradient system of total variation. *Abstr. Appl. Anal.* (2004), no. 8, 651–682.
- [19] Gilardi, G.; Miranville, A.; Schimperna, G.: On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions. *Commun. Pure. Appl. Anal.* **8** (2009), 881–912.
- [20] Gilardi, G.; Miranville, A.; Schimperna, G.: Long-time behavior of the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions. *Chin. Ann. Math. Ser. B* **31** (2010), 679–712.
- [21] Goldstein, G. R.; Miranville, A.: A Cahn-Hilliard-Gurtin model with dynamic boundary conditions. *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013), 387–400.
- [22] Goldstein, G. R.; Miranville, A.; Schimperna, G.: Cahn–Hilliard model in a domain with non-permeable walls. *Phys. D* **240** (2011), 754–766.
- [23] Israel, H.: *Long time behavior of an Allen-Cahn type equation with a singular potential and dynamic boundary conditions*, *J. Appl. Anal. Comput.* **2** (2012), 29–56.
- [24] Ito, A.; Yamazaki, N.; Kenmochi, N.: Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces. *Dynamical systems and differential equations*, Vol. I (Springfield, MO, 1996). *Discrete Contin. Dynam. Systems 1998, Added Volume I*, 327–349.
- [25] Kenmochi, N.: Pseudomonotone operators and nonlinear elliptic boundary value problems. *J. Math. Soc. Japan*, **27**, no. 1 (1975), 121–149.
- [26] Kenmochi, N.: Solvability of nonlinear evolution equations with time-dependent constraints and applications. *Bull. Fac. Education, Chiba Univ.*, **30** (1981), 1–87.
<http://ci.nii.ac.jp/naid/110004715232>

- [27] Ladyženskaja, O. A.; Solonnikov, V. A.; Ural'ceva, N. N.: *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith., Translations of Mathematical Monographs, **23**, American Mathematical Society, Providence, R.I. (1968).
- [28] Liero, M.: Passing from bulk to bulk-surface evolution in the Allen-Cahn equation. NoDEA Nonlinear Differential Equations Appl. **20** (2013), 919–942.
- [29] Lions, J.-L.; Magenes, E.: *Non-homogeneous boundary value problems and applications. Vol. II*. Translated from the French by P. Kenneth., Die Grundlehren der mathematischen Wissenschaften, **182**, Springer-Verlag, New York-Heidelberg (1972).
- [30] Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Advances in Math. **3**, 510–585 (1969).
- [31] Miranville, A.; Rocca, E.; Schimperna, G., Segatti, A.: The Penrose-Fife phase-field model with coupled dynamic boundary conditions. Discrete Contin. Dyn. Syst. **34** (2014), 4259–4290.
- [32] Ôtani, M.: Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators: Cauchy problems. J. Differential Equations, **46** (1982), no. 2, 268–299.
- [33] Savaré, G.; Visintin, A.: Variational convergence of nonlinear diffusion equations: applications to concentrated capacity problems with change of phase. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **8**, (1997), no. 1, 49–89.
- [34] Shirakawa, K., Watanabe, H., Yamazaki, N.: Phase-field systems for grain boundary motions under isothermal solidifications. Adv. Math. Sci. Appl., **24** (2014), 353–400.
- [35] Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. (4) **146**, 65–96 (1987).
- [36] Visintin, A.: *Models of phase transitions*. Progress in Nonlinear Differential Equations and their Applications, **28**, Birkhäuser Boston (1996).