# Weak Simplicial Bisimilarity for Polyhedral Models and $\mathrm{SLCS}_{\eta}{ }^{\star}$ - Extended version - 

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#### Abstract

In the context of spatial logics and spatial model checking for polyhedral models - mathematical basis for visualisations in continuous space - we propose a weakening of simplicial bisimilarity. We additionally propose a corresponding weak notion of $\pm$-bisimilarity on cell-poset models, discrete representation of polyhedral models. We show that two points are weakly simplicial bisimilar iff their representations are weakly $\pm$-bisimilar. The advantage of this weaker notion is that it leads to a stronger reduction of models than its counterpart that was introduced in our previous work. This is important, since real-world polyhedral models, such as those found in domains exploiting mesh processing, typically consist of large numbers of cells. We also propose $\operatorname{SLCS}_{\eta}$, a weaker version of the Spatial Logic for Closure Spaces (SLCS) on polyhedral models, and we show that the proposed bisimilarities enjoy the Hennessy-Milner property: two points are weakly simplicial bisimilar iff they are logically equivalent for $\operatorname{SLCS}_{\eta}$. Similarly, two cells are weakly $\pm$-bisimilar iff they are logically equivalent in the poset-model interpretation of $\mathrm{SLCS}_{\eta}$. This work is performed in the context of the geometric spatial model checker PolyLogicA and the polyhedral semantics of SLCS.


Keywords: Bisimulation relations. Spatial bisimilarity • Spatial logics . Logical equivalence • Spatial model checking • Polyhedral models.

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## 1 Introduction and Related Work

The notion of bisimulation is central in the theory of models for (concurrent) system behaviour, for characterising those systems which "behave the same". Properties of such behaviours are typically captured by formulas of appropriate logics, such as modal logics and variations/extensions thereof (e.g. temporal, deterministic-/stochastic-time temporal, probabilistic). A key notion, in this context, is the Hennessy-Milner property (HMP), that allows for logical characterisations of bisimilarity. Given a model $\mathcal{M}$, a bisimulation equivalence $E$ over $\mathcal{M}$, and a logic $\mathcal{L}$ intepreted on $\mathcal{M}$, we say that $E$ and $\mathcal{L}$ enjoy the HMP if the following holds: any two states in $\mathcal{M}$ are equivalent according to $E$ iff they satisfy the same formulas of $\mathcal{L}$. Besides the intrinsic theoretical value of the HMP, the latter is also of fundamental importance as mathematical foundation for safe model reduction procedures since it ensures that any formula $\mathcal{L}$ is satisfied by a state $s$ in $\mathcal{M}$ iff it is satisfied by the equivalence class $[s]_{E}$ of $s$, that is itself a state in the minimal model $\mathcal{M}_{/ E}$ - there are nowadays standard procedures for the effective and efficient computation of $\mathcal{M}_{/ E}$, for finite models $\mathcal{M}$. Model reduction is, in turn, extremely important for efficient model analysis via, for example, automatic model-checking of logic formulas.

Model-checking techniques have been developed for the analysis of models of space as well, with properties expressed in spatial logics, i.e. modal logics interpreted over such models, following the tradition going back to McKinsey and Tarski in 1940s [21] (see also [4] for an overview), where topological models are considered as models for space. In [12]13 the Spatial Logic for Closure Spaces (SLCS) has been proposed together with a model-checking algorithm for finite spaces and its implementation. Closure spaces are a generalisation of topological spaces that allow for a uniform treatment of continuous spaces and discrete spaces, such as general graphs. SLCS is interpreted on models whose carriers are closure spaces. Spatio-temporal versions of the logic and the model-checker have been presented in [10]16]. A version of the model-checker optimised for (2D and 3D) digital images - that can be seen as adjacency spaces, a subclass of closure spaces - has been proposed in [3] with SLCS enriched with a distance operator. Tools for spatial and spatio-temporal model-checking have been successfully used in several applications $15 / 9 / 2214112$ showing that the notions and techniques developed in the area of concurrency theory and formal methods can be extremely helpful in developing a foundational basis for the automatic analysis of spatial models in real applications.

As a natural step forward, and following classical developments in modal logic, in $7 \mid 17$ several notions of bisimulation for finite closure spaces have been studied. These cover a spectrum from CM-bisimilarity, an equivalence based on proximity - similar to and inspired by topo-bisimilarity for topological models (4) - to its specialisation for quasi-discrete closure models, CMC-bisimilarity, to CoPa-bisimilarity, an equivalence based on conditional reachability. Each of these bisimilarities has been equipped with its logical characterisation. In [11 an encoding from finite closure models to finite labelled transition systems (LTSs) has been defined and proven correct in the sense that two points in the space are

CoPa-bisimilar iff their corresponding states in the LTS are branching bisimilar. This makes it possible to perform minimisation of the spatial model w.r.t. CoPabisimilarity via minimisation of its LTS w.r.t. branching bisimulation. Very efficient tools are available for LTS minimisation w.r.t. branching equivalence 18 .

The spatial model-checking techniques mentioned above have been extended to polyhedral models 5|20, that we address in the present paper. Polyhedra are sets of points in $\mathbb{R}^{n}$ generated by simplicial complexes, i.e. certain finite collections of simplexes, where a simplex is the convex hull of a set of affinely independent points in $\mathbb{R}^{n}$. Given a set PL of proposition letters, a polyhedral model is obtained from a polyhedron in the usual way, i.e. by assigning a set of points to each proposition letter $p \in \mathrm{PL}$, namely those that "satisfy" $p$. Polyhedral models in $\mathbb{R}^{3}$ can be used for (approximately) representing objects in continuous 3D space. This is widely used in many 3D visual computing techniques, where an object is divided into suitable areas of different size. Such ways of division of an object are known as mesh techniques and include triangular surface meshes or tetrahedral volume meshes (see for example [19] and the example in Fig. 5a.

In [5] a version of SLCS, referred to as SLCS ${ }_{\gamma}$ in this paper, has been proposed for expressing spatial properties of points laying in polyhedral models, and in particular conditional reachability properties. Many other interesting properties, such as "being surrounded by" can be expressed using reachability (see [5]). Intuitively, a point $x$ in a polyhedral model satisfies the conditional reachability formula $\gamma\left(\Phi_{1}, \Phi_{2}\right)$ if there is a topological path starting from $x$, ending in a point $y$ satisfying $\Phi_{2}$, and such that all the intermediate points of the path between $x$ and $y$ satisfy $\Phi_{1}$, but note that neither $x$ nor $y$ is required to satisfy $\Phi_{1}$. A notion of bisimilarity between points has also been introduced in [5], namely simplicial bisimilarity, and it has been proven that the latter enjoys the HMP w.r.t. SLCS $_{\gamma}$. In addition, a representation $\mathbb{F}$ of polyhedral models as finite posets has been built and it has been shown that a point $x$ in a polyhedral model $\mathcal{P}$ satisfies a $\operatorname{SLCS}_{\gamma}$ formula $\Phi$ in $\mathcal{P}$ iff its representation $\mathbb{F}(x)$ in the poset model $\mathbb{F}(\mathcal{P})$ representing $\mathcal{P}$ satisfies $\Phi$ in $\mathbb{F}(\mathcal{P})$. An SLCS $_{\gamma}$ model-checking algorithm has been developed for finite poset models that has been implemented in the tool PolyLogicA thus achieving model-checking of continuous space that can be represented by polyhedral models (see [5] for details).

In [8] we addressed the issue of minimisation for polyhedral models and, more specifically, their poset representations. In particular, we defined $\pm$-bisimilarity, a notion based on $\pm$-paths, a subclass of undirected paths over poset models suitable for representing, in such models, topological paths over polyhedral models. We proved that $\pm$-bisimilarity enjoys the HMP w.r.t. SLCS $_{\gamma}$ interpreted on finite poset models and we showed that it can be used for poset model minimisation: for instance, the minimal model of the poset model of Fig. 1 C has only 10 elements.

In this paper we present $\operatorname{SLCS}_{\eta}$, another variant of SLCS for polyhedral models where the $\gamma$ modality has been replaced by $\eta$ so that a point $x$ satisfies $\eta\left(\Phi_{1}, \Phi_{2}\right)$ if there is a topological path starting from $x$, ending in a point $y$ satisfying $\Phi_{2}$, and such that all the intermediate points of the path between $x$ and $y$, and
including $x$ itself, satisfy $\Phi_{1}$ ( $y$ is not required to satisfy $\Phi_{1}$ ). Thus $\gamma$ and $\eta$ behave differently only in $\eta$ requiring that $x$ itself satisfies $\Phi_{1}$.

The result is that $\operatorname{SLCS}_{\eta}$ is weaker than $\operatorname{SLCS}_{\gamma}$ in the sense that it distinguishes less points than $\operatorname{SLCS}_{\gamma}$. Furthermore, SLCS $_{\gamma}$ can express proximity, here intended as topological closure - that boils down to the standard possibly modality $\diamond$ in the poset model interpretation - whereas $\operatorname{SLCS}_{\eta}$ cannot. Nevertheless, many interesting reachability properties can be expressed in $\operatorname{SLCS}_{\eta}$ and, perhaps most importantly, the latter characterises bisimilarities (in the polyhedral model and the associated poset model) that are coarser than simplicial bisimilarity and $\pm$-bisimilarity, respectively. This allows for a substantial model reduction. For instance, the minimal model, w.r.t. the new equivalence, of the poset of Fig. 1c, shown in Fig. 4 has only 4 states now. This greater reduction in model size is one of the main motivations for the study of $\operatorname{SLCS}_{\eta}$ presented in this paper.

In the remainder of this paper, we provide necessary background information in Sect. 2. Sect. 3 introduces $\mathrm{SLCS}_{\eta}$ and addresses its relationship with SLCS $_{\gamma}$. It is also shown that $\mathrm{SLCS}_{\eta}$ is preserved and reflected by the mapping $\mathbb{F}$ form polyhedral models to finite poset models. Weak simplicial bisimilarity and weak $\pm$-bisimilarity are defined in Sect. 4 where it is also shown that they enjoy the HMP w.r.t. the intepretation of $\operatorname{SLCS}_{\eta}$ on polyhedral models and on finite poset models, respectively. A larger example is given in Sect. 5, illustrating the reduction potential of weak $\pm$-bisimulation. Finally, conclusions and a discussion on future work are reported in Sect. 6. The proofs for all the results presented in Sections 2, 3 and 4 are provided in Appendix B whereas in Appendix A we recall background information and results, as well as notational details.

## 2 Background and Notation

In this section we recall the relevant details of the language $\mathrm{SLCS}_{\gamma}$, its polyhedral and poset models, and the truth-preserving map $\mathbb{F}$ between these models.

For sets $X$ and $Y$, a function $f: X \rightarrow Y$, and subsets $A \subseteq X$ and $B \subseteq Y$ we define $f(A)$ and $f^{-1}(B)$ as $\{f(a) \mid a \in A\}$ and $\{a \mid f(a) \in B\}$, respectively. The restriction of $f$ on $A$ is denoted by $f \mid A$. The powerset of $X$ is denoted by $2^{X}$. For a relation $R \subseteq X \times X$ we let $R^{-}=\{(y, x) \mid(x, y) \in R\}$ denote its converse and $R^{ \pm}$denote $R \cup R^{-}$. In the remainder of the paper we assume that a set PL of proposition letters is fixed. The sets of natural numbers and of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We use the standard interval notation: for $x, y \in \mathbb{R}$ we let $[x, y]$ be the set $\{r \in \mathbb{R} \mid x \leq r \leq y\},[x, y)=\{r \in \mathbb{R} \mid x \leq r<y\}$, and so on. Intervals of $\mathbb{R}$ are equipped with the Euclidean topology inherited from $\mathbb{R}$. We use a similar notation for intervals over $\mathbb{N}$ : for $n, m \in \mathbb{N}[m ; n]$ denotes the set $\{i \in \mathbb{N} \mid m \leq i \leq n\},[m ; n)=\{i \in \mathbb{N} \mid m \leq i<n\}$, and so on.

Below we recall, informally, some basic notions, assuming that the reader is familiar with topological spaces, Kripke models and posets. For all the details concerning basic notions and notation we refer the reader to [5]8.

A simplex $\sigma$ is the convex hull of a set of $d+1$ affinely independent points in $\mathbb{R}^{m}$, with $d \leq m$, i.e. $\sigma=\left\{\lambda_{0} \mathbf{v}_{\mathbf{0}}+\ldots+\lambda_{d} \mathbf{v}_{\mathbf{d}} \mid \lambda_{0}, \ldots, \lambda_{d} \in[0,1]\right.$ and $\sum_{i=0}^{d} \lambda_{i}=$
$1\}$. For instance, a segment $A B$ together with its end-points $A$ and $B$ is a simplex in $\mathbb{R}^{m}$, for $m \geq 1$. Given a simplex $\sigma$ with vertices $\mathbf{v}_{\mathbf{0}}, \ldots, \mathbf{v}_{\mathbf{d}}$, any subset of $\left\{\mathbf{v}_{\mathbf{0}}, \ldots, \mathbf{v}_{\mathbf{d}}\right\}$ spans a simplex $\sigma^{\prime}$ in turn: we say that $\sigma^{\prime}$ is a face of $\sigma$, written $\sigma^{\prime} \sqsubseteq \sigma$. So, for instance, both $A$ and $B$ are simplexes in turn - with $A \sqsubseteq A B$ and $B \sqsubseteq A B-$ and $A B$ itself could be part of a larger simplex, e.g., a triangle $A B C$. Clearly, $\sqsubseteq$ is a partial order. The barycentre $b_{\sigma}$ of $\sigma$ is defined as follows: $b_{\sigma}=\sum_{i=0}^{d} \frac{1}{d+1} \mathbf{v}_{\mathbf{i}}$.

The relative interior $\widetilde{\sigma}$ of a simplex $\sigma$ is the same as $\sigma$ "without its borders", i.e. the set $\left\{\lambda_{0} \mathbf{v}_{\mathbf{0}}+\ldots+\lambda_{d} \mathbf{v}_{\mathbf{d}} \mid \lambda_{0}, \ldots, \lambda_{d} \in(0,1]\right.$ and $\left.\sum_{i=0}^{d} \lambda_{i}=1\right\}$. For instance, the open segment $\widetilde{A B}$, without the end-points $A$ and $B$ is the relative interior of segment $A B$. The relative interior of a simplex is often called a cell and is equal to the topological interior taken inside the affine hull of the simplex ${ }^{5}$ There is an obvious partial order between the cells of a simplex: $\widetilde{\sigma_{1}} \preceq \widetilde{\sigma_{2}}$ iff $\widetilde{\sigma_{1}} \subseteq \mathcal{C}_{T}\left(\widetilde{\sigma_{2}}\right)$, where $\mathcal{C}_{T}$ denotes the classical topological closure operator. So, in the above example, we have $\widetilde{A} \preceq \widetilde{A}, \widetilde{B} \preceq \widetilde{B}, \widetilde{A} \preceq \widetilde{A B}, \widetilde{B} \preceq \widetilde{A B}$, and $\widetilde{A B} \preceq \widetilde{A B}$. Note that for all simplexes $\sigma_{1}$ and $\sigma_{2}$ the following holds: $\sigma_{1} \sqsubseteq \sigma_{2}$ iff $\widetilde{\sigma_{1}} \preceq \widetilde{\sigma_{2}}$.

A simplicial complex $K$ is a finite collection of simplexes of $\mathbb{R}^{m}$ such that: (i) if $\sigma \in K$ and $\widetilde{\sigma}^{\prime} \preceq \widetilde{\sigma}$ then also $\sigma^{\prime} \in K$; (ii) if $\sigma, \sigma^{\prime} \in K$ then $\widetilde{\sigma \cap \sigma^{\prime}} \preceq \widetilde{\sigma} \cap \widetilde{\sigma}^{\prime}$. Given a simplicial complex $K$, the cell poset of $K$ is the poset $(\widetilde{K}, \preceq)$ where $\widetilde{K}$ is the set $\{\widetilde{\sigma} \mid \sigma \in K \backslash\{\emptyset\}\}$ and the polyhedron $|K|$ of $K$ is the set-theoretic union of the simplexes in $K$. Note that $|K|$ inherits the topological structure of $\mathbb{R}^{m}$.

A polyhedral model is a pair $(|K|, V)$ where $V: \mathrm{PL} \rightarrow \mathbf{2}^{|K|}$ maps every proposition letter $p \in \mathrm{PL}$ to the set of points of $|K|$ satisfying $p$. It is required that, for all $p \in \mathrm{PL}, V(p)$ is always a union of cells in $\widetilde{K}$. Similarly, a poset model $(W, \preceq, \mathcal{V})$ is a poset equipped with a valuation function $\mathcal{V}:$ PL $\rightarrow \mathbf{2}^{W}$. Given polyhedral model $\mathcal{P}=(|K|, V)$, we say that $(\widetilde{K}, \preceq, \mathcal{V})$ is the cell poset model of $\mathcal{P}$ iff $(\widetilde{K}, \preceq)$ is the cell poset of $K$ and, for all $\widetilde{\sigma} \in \widetilde{K}$, we have: $\widetilde{\sigma} \in \mathcal{V}(p)$ iff $\widetilde{\sigma} \subseteq V(p)$. We let $\mathbb{F}(\mathcal{P})$ denote the cell poset model of $\mathcal{P}$ and, with a little bit of overloading, for all $x \in|K|, \mathbb{F}(x)$ denotes the unique cell $\widetilde{\sigma}$ such that $x \in \widetilde{\sigma}$. Note that $\mathbb{F}:|K| \rightarrow \widetilde{K}$ is a continuous function [6, Corollary 3.4]. Furthermore, note that poset models are a subclass of Kripke models. In the sequel, when we say that $\mathcal{F}$ is a cell poset model, we mean that there exists a polyhedral model $\mathcal{P}$ such that $\mathcal{F}=\mathbb{F}(\mathcal{P})$.

Fig. 1 shows a polyhedral model. There are three proposition letters, red, green and gray, shown by different colours (1a). The model is "unpacked" into its cells in Fig. 1b. The latter are collected in the cell poset model, whose Hasse diagram is shown in Fig. 1c.

In a topological space $(X, \tau)$, a topological path from $x \in X$ is a total, continuous function $\pi:[0,1] \rightarrow X$ such that $\pi(0)=x$. We call $\pi(0)$ and $\pi(1)$ the starting and ending point of $\pi$, respectively, while $\pi(r)$ is an intermediate point of $\pi$, for all $r \in(0,1)$. Fig. 2a shows a path from a point $x$ in the open segment $\widetilde{A B}$ in the polyhedral model of Fig 1 a .

[^1]

Fig. 1: A polyhedral model $\mathcal{P}$ 1a with its cells 1 b and the Hasse diagram of the related cell poset 1 c .

Topological paths are represented in cell posets by so-called $\pm$-paths, a subclass of undirected paths [5]. For technical reasons ${ }^{6}$ in this paper we extend the definition given in [5] to general Kripke frames. Given a Kripke frame $(W, R)$, an undirected path of length $\ell \in \mathbb{N}$ from $w$ is a total function $\pi:[0 ; \ell] \rightarrow X$ such that $\pi(0)=x$ and, for all $i \in[0 ; \ell), R^{ \pm}(\pi(i), \pi(i+1))$. The starting and ending points are $\pi(0)$ and $\pi(\ell)$, respectively, while $\pi(i)$ is an intermediate point for all $i \in(0 ; \ell)$. The path is a $\pm$-path iff $\ell \geq 2, R(\pi(0), \pi(1))$ and $R^{-}(\pi(\ell-1), \pi(\ell))$.

The 土-path $(\widetilde{A B}, \widetilde{A B C}, \widetilde{B C}, \widetilde{B C D}, \widetilde{D})^{7}$. drawn in blue in Fig. 2b, faithfully represents the path from $x$ shown in Fig. 2a Note that a path $\pi$ such that, say, $\pi(0) \in \widetilde{C D}, \pi(1)=E$ and $\pi((0,1)) \subseteq \widetilde{C D E}$, i.e. a path that "jumps immediately" to $\widetilde{C D E}$ after starting in $\widetilde{C D}$ cannot be represented in the poset by any undirected path $\pi^{\prime}$, of some length $\ell \geq 2$ such that $\pi^{\prime}(0) \succ \pi^{\prime}(1)$ (or $\pi^{\prime}(\ell-1) \prec \pi^{\prime}(\ell)$, for symmetry reasons), while it is correctly represented by the士-path $(\widetilde{C D}, \widetilde{C D E}, \widetilde{E})$, where $\widetilde{C D} \prec \widetilde{C D E} \succ \widetilde{E}$.

In the context of this paper it is often convenient to use a generalisation of $\pm$-paths, so-called "down paths", $\downarrow$-paths for short: a $\downarrow$-path from $w$, of length $\ell \geq 1$, is an undirected path $\pi$ from $w$ of length $\ell$ such that $R^{-}(\pi(\ell-1), \pi(\ell))$. Clearly, every $\pm$-path is also a $\downarrow$-path. The following lemma ensures that in reflexive Kripke frames $\pm$ - and $\downarrow$-paths can be safely used interchangeably since for every $\downarrow$-path there is a $\pm$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order:

Lemma 1. Given a reflexive Kripke frame $(W, R)$ and $a \downarrow$-path $\pi:[0 ; \ell] \rightarrow W$, there is a $\pm$-path $\pi^{\prime}:\left[0 ; \ell^{\prime \prime}\right] \rightarrow W$, for some $\ell^{\prime}$, and a total, surjective, monotonic, non-decreasing function $f:\left[0 ; \ell^{\prime}\right] \rightarrow[0 ; \ell]$ with $\pi^{\prime}(j)=\pi(f(j))$ for all $j \in\left[0 ; \ell^{\prime}\right]$.

In [5], SLCS $_{\gamma}$, a version of SLCS for polyhedral models, has been presented that consists of predicate letters, negation, conjunction, and the single modal operator $\gamma$, expressing conditional reachability. The satisfaction relation for $\gamma\left(\Phi_{1}, \Phi_{2}\right)$,

[^2]

Fig. 2: (2a) A topological path from a point $x$ to vertex $D$ in the polyhedral model $\mathcal{P}$ of Figure 1a (2b) The corresponding $\pm$-path (in blue) in the Hasse diagram of the cell poset model $\mathbb{F}(\mathcal{P})$.
for polyhedral model $\mathcal{P}=(|K|, V)$ and $x \in|K|$, as defined in [5], is recalled below:
$\mathcal{P}, x \models \gamma\left(\Phi_{1}, \Phi_{2}\right) \Leftrightarrow$ a topological path $\pi:[0,1] \rightarrow|K|$ exists such that $\pi(0)=x$,

$$
\mathcal{P}, \pi(1) \models \Phi_{2} \text {, and } \mathcal{P}, \pi(r) \models \Phi_{1} \text { for all } r \in(0,1) .
$$

We also recall the interpretation of $\operatorname{SLCS}_{\gamma}$ on poset models. The satisfaction relation for $\gamma\left(\Phi_{1}, \Phi_{2}\right)$, for poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$ and $w \in W$, is as follows:
$\mathcal{F}, w \models \gamma\left(\Phi_{1}, \Phi_{2}\right) \Leftrightarrow$ a $\pm$-path $\pi:[0 ; \ell] \rightarrow W$ exists such that $\pi(0)=w$,

$$
\mathcal{F}, \pi(\ell) \mid=\Phi_{2}, \text { and } \mathcal{F}, \pi(i) \models \Phi_{1} \text { for all } i \in(0 ; \ell) .
$$

In [5] it has also been shown that, for all $x \in|K|$ and $\operatorname{SLCS}_{\gamma}$ formulas $\Phi$, we have: $\mathcal{P}, x \models \Phi$ iff $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$. In addition, simplicial bisimilarity, a novel notion of bisimilarity for polyhedral models, has been defined that uses a subclass of topological paths and it has been shown to enjoy the classical Hennessy-Milner property: two points $x_{1}, x_{2} \in|K|$ are simplicial bisimilar, written $x_{1} \sim_{\triangle}^{\mathcal{P}} x_{2}$, iff they satisfy the same $\operatorname{SLCS}_{\gamma}$ formulas, i.e. they are equivalent with respect to the logic SLCS $_{\gamma}$, written $x_{1} \equiv{ }_{\gamma}^{\mathcal{P}} x_{2}$.

The result has been extended to $\pm$-bisimilarity on finite poset models, a notion of bisimilarity based on $\pm$-paths: $w_{1}, w_{2} \in W$ are $\pm$-bisimilar, written $x_{1} \sim_{ \pm}^{\mathcal{F}} x_{2}$, iff they satisfy the same $\operatorname{SLCS}_{\gamma}$ formulas, i.e. $x_{1} \equiv_{\gamma}^{\mathcal{F}} x_{2}$ (see [8] for details). In summary, we have:

$$
\begin{equation*}
x_{1} \sim_{\Delta}^{\mathcal{P}} x_{2} \text { iff } x_{1} \equiv{ }_{\gamma}^{\mathcal{P}} x_{2} \text { iff } \mathbb{F}\left(x_{1}\right) \equiv{ }_{\gamma}^{\mathbb{F}(\mathcal{P})} \mathbb{F}\left(x_{2}\right) \text { iff } \mathbb{F}\left(x_{1}\right) \sim_{ \pm}^{\mathbb{F}(\mathcal{P})} \mathbb{F}\left(x_{2}\right) \tag{1}
\end{equation*}
$$

As an illustration, with reference to Figure 1a, we have that no red point, call it $x$, in the open segment $C D$ is simplicial bisimilar to the red point $C$. In fact, although both $x$ and $C$ satisfy $\gamma($ green, true $)$, we have that $C$ satisfies also $\gamma($ gray, true $)$, which is not the case for $x$. Similarly, with reference to Figure 1c, cell $\widetilde{C}$ satisfies $\gamma$ (gray, true), which is not satisfied by $\widetilde{C D}$.

We aim to obtain a similar result as (1) above, for a weaker logic introduced in the next section.

## 3 SLCS $\eta_{\eta}$ : Weak SLCS on Polyhedral Models

In this section we introduce $\mathrm{SLCS}_{\eta}$, a logic for polyhedral models that is weaker than $\operatorname{SLCS}_{\gamma}$, yet is still capable of expressing interesting conditional reachability properties. We present also an interpretation of the logic on finite poset models.

Definition 1 (Weak SLCS on polyhedral models - SLCS $_{\eta}$ ). The abstract language of $\mathrm{SLCS}_{\eta}$ is the following:

$$
\Phi::=p|\neg \Phi| \Phi_{1} \wedge \Phi_{2} \mid \eta\left(\Phi_{1}, \Phi_{2}\right)
$$

The satisfaction relation of $\mathrm{SLCS}_{\eta}$ with respect to a given polyhedral model $\mathcal{P}=$ $(|K|, V), \operatorname{SLCS}_{\eta}$ formula $\Phi$, and point $x \in|K|$ is defined recursively on the structure of $\Phi$ as follows:

$$
\begin{array}{ll}
\mathcal{P}, x \models p & \Leftrightarrow x \in V(p) ; \\
\mathcal{P}, x \models \neg \Phi & \Leftrightarrow \mathcal{P}, x \models \Phi \text { does not hold; } \\
\mathcal{P}, x \models \Phi_{1} \wedge \Phi_{2} & \Leftrightarrow \mathcal{P}, x \models \Phi_{1} \text { and } \mathcal{P}, x \models \Phi_{2} ; \\
\mathcal{P}, x \models \eta\left(\Phi_{1}, \Phi_{2}\right) \Leftrightarrow & \text { a topological path } \pi:[0,1] \rightarrow|K| \text { exists such that } \pi(0)=x, \\
& \mathcal{P}, \pi(1) \models \Phi_{2}, \text { and } \mathcal{P}, \pi(r) \models \Phi_{1} \text { for all } r \in[0,1) .
\end{array}
$$

As usual, disjunction $(\vee)$ is derived as the dual of $\wedge$. Note that the only difference between $\eta\left(\Phi_{1}, \Phi_{2}\right)$ and $\gamma\left(\Phi_{1}, \Phi_{2}\right)$ is that the former requires that also the first element of a path witnessing the formula satisfies $\Phi_{1}$, hence the use of the left closed interval $[0,1)$ here. Although this might seem at first sight only a very minor difference, it has considerable consequences of both theoretical and practical nature, as we will see in what follows.

Definition $2\left(\mathrm{SLCS}_{\eta}\right.$ Logical Equivalence). Given polyhedral model $\mathcal{P}=$ $(|K|, V)$ and $x_{1}, x_{2} \in|K|$ we say that $x_{1}$ and $x_{2}$ are logically equivalent with respect to $\operatorname{SLCS}_{\eta}$, written $x_{1} \equiv_{\eta}^{\mathcal{P}} x_{2}$, iff, for all $\operatorname{SLCS}_{\eta}$ formulas $\Phi$, the following holds: $\mathcal{P}, x_{1} \models \Phi$ if and only if $\mathcal{P}, x_{2} \models \Phi$.

In the sequel, we will refrain from indicating the model $\mathcal{P}$ explicitly as a superscript of $\equiv{ }_{\eta}^{\mathcal{P}}$ when it is clear from the context. Below, we show that $\operatorname{SLCS}_{\eta}$ can be encoded into $\mathrm{SLCS}_{\gamma}$ so that the former is weaker than the latter.

Definition 3. We define the following encoding of $\mathrm{SLCS}_{\eta}$ into $\mathrm{SLCS}_{\gamma}$ :

$$
\begin{array}{ll}
\mathcal{E}(p)=p & \mathcal{E}\left(\Phi_{1} \wedge \Phi_{2}\right)=\mathcal{E}\left(\Phi_{1}\right) \wedge \mathcal{E}\left(\Phi_{2}\right) \\
\mathcal{E}(\neg \Phi)=\neg \mathcal{E}(\Phi) & \mathcal{E}\left(\eta\left(\Phi_{1}, \Phi_{2}\right)\right)=\mathcal{E}\left(\Phi_{1}\right) \wedge \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)
\end{array}
$$

The following lemma is easily proven by structural induction:
Lemma 2. Let $\mathcal{P}=(|K|, V)$ be a polyhedral model, $x \in|K|$ and $\Phi$ a $\operatorname{SLCS}_{\eta}$ formula. Then $\mathcal{P}, x \models \Phi$ iff $\mathcal{P}, x \models \mathcal{E}(\Phi)$.

A direct consequence of Lemma 2 is that $\operatorname{SLCS}_{\eta}$ is weaker than $\operatorname{SLCS}_{\gamma}$.

Proposition 1. Let $\mathcal{P}=(|K|, V)$ be a polyhedral model. For all $x_{1}, x_{2} \in|K|$ the following holds: if $x_{1} \equiv_{\gamma} x_{2}$ then $x_{1} \equiv_{\eta} x_{2}$.
Remark 1. The converse of Proposition 1 does not hold, as shown by the example polyhedral model $\mathcal{P}=(|K|, V)$ of Figure 3a. It is easy to see that, for all $x \in$ $\widetilde{A B C}$, we have $A \not 三_{\gamma} x$ and $A \equiv_{\eta} x$. Let $x \in \widetilde{A B C}$. Clearly, $A \not 三_{\gamma} x$ since $\mathcal{P}, A \models \gamma($ red, true $)$ whereas $\mathcal{P}, x \not \vDash \gamma($ red, true $)$. It can be easily shown, by induction on the structure of formulas, that $A \equiv_{\eta} x$ for all $x \in \widetilde{A B C}$ (see Sect. B.3.

(a)

(b)

Fig. 3: A polyhedral model 3 a and its cell poset model 3 b

Remark 2. The example of Figure 3 a is useful also for showing that the classical topological interpretation of the modal logic operator $\diamond$ cannot be expressed in $\operatorname{SLCS}_{\eta}$. We recall that

$$
\mathcal{P}, x \mid=\diamond \Phi \Leftrightarrow x \in \mathcal{C}_{T}\left(\left\{x^{\prime} \in|K| \mid \mathcal{P}, x^{\prime} \models \Phi\right\}\right) .
$$

Clearly, in the model of the figure, we have $\mathcal{P}, A \models \diamond$ red while $\mathcal{P}, x \models \diamond$ red for no $x \in \widetilde{A B C}$. On the other hand, $A \equiv_{\eta} x$ holds for all $x \in \widetilde{A B C}$, as we have just seen in Remark 1. So, if $\diamond$ were expressible in $\operatorname{SLCS}_{\eta}$, then $A$ and $x$ should have agreed on $\diamond$ red for each $x \in \widetilde{A B C}$. Note that $\diamond$ can be expressed in SLCS ${ }_{\gamma}$ as $\gamma(\Phi$, true $)$, see [5].

Below we re-interpret $\mathrm{SLCS}_{\eta}$ on finite poset models instead of polyhedral models. The only difference from Def. 11 is, of course, the fact that $\eta$-formulas are defined for $\pm$-paths instead of topological ones.

Definition $4\left(\mathrm{SLCS}_{\eta}\right.$ on finite poset models). The satisfaction relation of SLCS $_{\eta}$ with respect to a given finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$, SLCS $_{\eta}$ formula $\Phi$, and element $w \in W$ is defined recursively on the structure of $\Phi$ :

$$
\begin{array}{ll}
\mathcal{F}, w \models p & \Leftrightarrow w \in \mathcal{V}(p) ; \\
\mathcal{F}, w \models \neg \Phi & \Leftrightarrow \mathcal{F}, w \not \models \Phi ; \\
\mathcal{F}, w \models \Phi_{1} \wedge \Phi_{2} & \Leftrightarrow \mathcal{F}, w \models \Phi_{1} \text { and } \mathcal{F}, w \models \Phi_{2} ; \\
\mathcal{F}, w \models \eta\left(\Phi_{1}, \Phi_{2}\right) \Leftrightarrow & a \pm-p a t h \pi:[0 ; \ell] \rightarrow W \text { exists such that } \pi(0)=w, \\
& \mathcal{F}, \pi(\ell) \models \Phi_{2} \text { and } \mathcal{F}, \pi(i) \models \Phi_{1} \text { for all } i \in[0 ; \ell) .
\end{array}
$$

Definition 5 (Logical Equivalence). Given a finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$ and elements $w_{1}, w_{2} \in W$ we say that $w_{1}$ and $w_{2}$ are logically equivalent with respect to $\mathrm{SLCS}_{\eta}$, written $w_{1} \equiv_{\eta}^{\mathcal{F}} w_{2}$, iff, for all $\mathrm{SLCS}_{\eta}$ formulas $\Phi$, the following holds: $\mathcal{F}, w_{1} \models \Phi$ if and only if $\mathcal{F}, w_{2} \models \Phi$.

Again, in the sequel, we will refrain from indicating the model $\mathcal{F}$ explicitly in $\equiv{ }_{\eta}^{\mathcal{F}}$ when it is clear from the context. It is useful to define a "characteristic" $\operatorname{SLCS}_{\eta}$ formula $\chi(w)$ that is satisfied by all and only those $w^{\prime}$ with $w^{\prime} \equiv_{\eta} w$.

Definition 6. Given a finite poset model $(W, \preceq, \mathcal{V}), w_{1}, w_{2} \in W$, define SLCS $_{\eta}$ formula $\delta_{w_{1}, w_{2}}$ as follows: if $w_{1} \equiv_{\eta} w_{2}$, then set $\delta_{w_{1}, w_{2}}=$ true, otherwise pick some $\operatorname{SLCS}_{\eta}$ formula $\psi$ such that $\mathcal{F}, w_{1} \models \psi$ and $\mathcal{F}, w_{2} \models \neg \psi$, and set $\delta_{w_{1}, w_{2}}=\psi$. For $w \in W$ define $\chi(w)=\bigwedge_{w^{\prime} \in W} \delta_{w, w^{\prime}}$.

Proposition 2. Given a finite poset model $(W, \preceq, \mathcal{V})$, for $w_{1}, w_{2} \in W$, it holds that $\mathcal{F}, w_{2} \models \chi\left(w_{1}\right)$ if and only if $w_{1} \equiv_{\eta} w_{2}$.

The following lemma is the poset model counterpart of Lemma 2 .
Lemma 3. Let $\mathcal{F}=(W, \preceq, \mathcal{V})$ be a finite poset model, $w \in W$ and $\Phi$ a $\operatorname{SLCS}_{\eta}$ formula. Then $\mathcal{F}, w \models \Phi$ iff $\mathcal{F}, w \models \mathcal{E}(\Phi)$.

Thus we get, as for the interpretation on polyhedral models, that $\operatorname{SLCS}_{\eta}$ on finite poset models is weaker than $\operatorname{SLCS}_{\gamma}$ :

Proposition 3. Let $\mathcal{F}=(W, \preceq, \mathcal{V})$ be a finite poset model. For all $w_{1}, w_{2} \in W$ the following holds: if $w_{1} \equiv_{\gamma} w_{2}$ then $w_{1} \equiv_{\eta} w_{2}$.

Remark 3. As expected, the converse of Proposition 3 does not hold, as shown by the poset model $\mathcal{F}$ of Figure 3 b . Clearly, $\widetilde{A} \not \equiv_{\gamma} \widetilde{A B C}$. In fact $\mathcal{F}, \widetilde{A} \models \gamma($ red, true $)$ whereas $\mathcal{F}, \widetilde{A B C} \not \vDash \gamma($ red, true $)$. On the other hand, $\widetilde{A} \equiv_{\eta} \widetilde{A B C}$ can be easily proven by induction on the structure of formulas (see Sect. B.6.

Remark 4. The example of Figure 3b is useful also for showing that the classical modal logic operator $\diamond$ cannot be expressed in $\operatorname{SLCS}_{\eta}$. We recall that

$$
\mathcal{F}, w \models \diamond \Phi \Leftrightarrow w^{\prime} \in W \text { exists such that } w \preceq w^{\prime} \text { and } \mathcal{F}, w^{\prime} \models \Phi \text {. }
$$

Clearly, in the model of the figure, we have $\mathcal{F}, \widetilde{A} \models \diamond$ red while $\mathcal{F}, \widetilde{A B C} \not \vDash \diamond$ red. On the other hand $\widetilde{A} \equiv_{\eta} \widetilde{A B C}$ holds, as we have just seen in Remark 3 . So, if $\diamond$ were expressible in $\operatorname{SLCS}_{\eta}$, then $\widetilde{A}$ and $\widetilde{A B C}$ should have agreed on $\diamond$ red. Note that $\diamond$ can be expressed in $\operatorname{SLCS}_{\gamma}$, as the following equality can be proven [5]: $\diamond \Phi \equiv \gamma(\Phi$, true $)$.

The following result is useful for setting a bridge between the continuous and the discrete interpretation of SLCS $_{\eta}$.

Lemma 4. Given a polyhedral model $\mathcal{P}=(|K|, V)$, for all $x \in|K|$ and formulas $\Phi$ of $\operatorname{SLCS}_{\eta}$ the following holds: $\mathcal{P}, x \models \Phi$ iff $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi)$.

As a direct consequence of Lemma 3 and Lemma 4 we get the bridge between the continuous and the discrete interpretation of $\operatorname{SLCS}_{\eta}$.

Theorem 1. Given a polyhedral model $\mathcal{P}=(|K|, V)$, for all $x \in|K|$ and formulas $\Phi$ of $\operatorname{SLCS}_{\eta}$ it holds that: $\mathcal{P}, x \models \Phi$ iff $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$.

This theorem allows one to go back and forth between the polyhedral model and the corresponding poset model without loosing anything expressible in $\operatorname{SLCS}_{\eta}$.

## 4 Weak Simplicial Bisimilarity

In this section, we introduce weak versions of simplicial bisimilarity and $\pm$ bisimilarity and we show that they coincide with logical equivalence induced by $\operatorname{SLCS}_{\eta}$ in polyhedral and poset models, respectively.

Definition 7 (Weak Simplicial Bisimulation). Given a polyhedral model $\mathcal{P}=(|K|, V)$, a symmetric relation $B \subseteq|K| \times|K|$ is a weak simplicial bisimulation if, for all $x_{1}, x_{2} \in|K|$, whenever $B\left(x_{1}, x_{2}\right)$, it holds that:

1. $V^{-1}\left(\left\{x_{1}\right\}\right)=V^{-1}\left(\left\{x_{2}\right\}\right)$;
2. for each topological path $\pi_{1}$ from $x_{1}$, there is topological path $\pi_{2}$ from $x_{2}$ such that $B\left(\pi_{1}(1), \pi_{2}(1)\right)$ and for all $r_{2} \in[0,1)$ there is $r_{1} \in[0,1)$ such that $B\left(\pi_{1}\left(r_{1}\right), \pi_{2}\left(r_{2}\right)\right)$.

Two points $x_{1}, x_{2} \in|K|$ are weakly simplicial bisimilar, written $x_{1} \approx_{\Delta}^{\mathcal{P}} x_{2}$, if there is a weak simplicial bisimulation $B$ such that $B\left(x_{1}, x_{2}\right)$.

For example, the open segments $A B, B C$, and $A C$ in Figure 3 are mutually weakly simplicial bisimilar and every point in set $\widetilde{A B C} \cup \widetilde{A} \cup \widetilde{B} \cup \widetilde{C}$ is weakly simplicial bisimilar to every other point in the same set.

Definition 8 (Weak $\pm$-bisimulation). Given a finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$, a symmetric binary relation $B \subseteq W \times W$ is a weak $\pm$-bisimulation if, for all $w_{1}, w_{2} \in W$, whenever $B\left(w_{1}, w_{2}\right)$, it holds that:

1. $\mathcal{V}^{-1}\left(\left\{w_{1}\right\}\right)=\mathcal{V}^{-1}\left(\left\{w_{2}\right\}\right)$;
2. for each $u_{1}, d_{1} \in W$ such that $w_{1} \preceq^{ \pm} \quad u_{1} \succeq d_{1}$ there is a $\pm$-path $\pi_{2}$ : $\left[0 ; \ell_{2}\right] \rightarrow W$ from $w_{2}$ such that $B\left(d_{1}, \pi_{2}\left(\ell_{2}\right)\right)$ and, for all $j \in\left[0 ; \ell_{2}\right)$, the following holds: $B\left(w_{1}, \pi_{2}(j)\right)$ or $B\left(u_{1}, \pi_{2}(j)\right)$.
We say that $w_{1}$ is weakly $\pm$-bisimilar to $w_{2}$, written $w_{1} \approx_{ \pm}^{\mathcal{F}} w_{2}$ if there is a weak $\pm$-bisimulation $B$ such that $B\left(w_{1}, w_{2}\right)$.

For example, all red cells in the Hasse diagram of Figure 3 b are weakly $\pm-$ bisimilar and all blue cells are weakly $\pm$-bisimilar.

The following lemma shows that, in a polyhedral model $\mathcal{P}$, weak simplicial bisimilarity $\approx_{\Delta}^{\mathcal{P}}$ (Def. 7 ) is stronger than $\equiv_{\eta}$ - logical equivalence w.r.t. $\operatorname{SLCS}_{\eta}$ :

Lemma 5. Given a polyhedral model $\mathcal{P}=(|K|, V)$, for all $x_{1}, x_{2} \in|K|$, the following holds: if $x_{1} \approx_{\Delta}^{\mathcal{P}} x_{2}$ then $x_{1} \equiv_{\eta} x_{2}$.

Proof. By induction on the structure of the formulas. We consider only the case $\eta\left(\Phi_{1}, \Phi_{2}\right)$. Suppose $x_{1} \approx_{\Delta} x_{2}$ and $\mathcal{P}, x_{1} \models \eta\left(\Phi_{1}, \Phi_{2}\right)$. Then there is a topological path $\pi_{1}$ from $x_{1}$ such that $\mathcal{P}, \pi_{1}(1) \models \Phi_{2}$ and $\mathcal{P}, \pi_{1}\left(r_{1}\right) \models \Phi_{1}$ for all $r_{1} \in[0,1)$. Since $x_{1} \approx_{\Delta} x_{2}$, then there is a topological path $\pi_{2}$ from $x_{2}$ such that $\pi_{1}(1) \approx_{\Delta}$ $\pi_{2}(1)$ and for each $r_{2} \in[0,1)$ there is $r_{1}^{\prime} \in[0,1)$ such that $\pi_{1}\left(r_{1}^{\prime}\right) \approx_{\Delta} \pi_{2}\left(r_{2}\right)$. By the Induction Hypothesis, we get $\mathcal{P}, \pi_{2}(1) \models \Phi_{2}$ and for each $r_{2} \in[0,1)$ $\mathcal{P}, \pi_{2}\left(r_{2}\right) \models \Phi_{1}$. Thus $\mathcal{P}, x_{2} \models \eta\left(\Phi_{1}, \Phi_{2}\right)$.

Furthermore, logical equivalence induced by $\operatorname{SLCS}_{\eta}$ is stronger than weak simplicial-bisimilarity, as implied by Lemma 8 below, which uses the following two auxiliary lemmas:

Lemma 6. Given a finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$ and weak $\pm$-bisimulation $B \subseteq W \times W$, for all $w_{1}, w_{2}$ such that $B\left(w_{1}, w_{2}\right)$, the following holds: for each $\downarrow$-path $\pi_{1}:\left[0 ; k_{1}\right] \rightarrow W$ from $w_{1}$ there is a $\downarrow$-path $\pi_{2}:\left[0 ; k_{2}\right] \rightarrow W$ from $w_{2}$ such that $B\left(\pi_{1}\left(k_{1}\right), \pi_{2}\left(k_{2}\right)\right)$ and for each $j \in\left[0 ; k_{2}\right)$ there is $i \in\left[0 ; k_{1}\right)$ such that $B\left(\pi_{1}(i), \pi_{2}(j)\right)$.

Lemma 7. Given a polyhedral model $\mathcal{P}=(|K|, V)$, and associated cell poset model $\mathbb{F}(\mathcal{P})=(W, \preceq, \mathcal{V})$, for any $\downarrow$-path $\pi:[0 ; \ell] \rightarrow W$, there is a topological path $\pi^{\prime}:[0,1] \rightarrow|K|$ such that: (i) $\mathbb{F}\left(\pi^{\prime}(0)\right)=\pi(0)$, (ii) $\mathbb{F}\left(\pi^{\prime}(1)\right)=\pi(\ell)$, and (iii) for all $r \in(0,1)$ there is $i<\ell$ such that $\mathbb{F}\left(\pi^{\prime}(r)\right)=\pi(i)$.

Lemma 8. In a given polyhedral model $(|K|, V), \equiv_{\eta}$ is a weak simplicial bisimulation.

Proof. Let $x_{1}, x_{2} \in|K|$ such that $x_{1} \equiv_{\eta} x_{2}$. The first condition of Definition 7 is clearly satisfied since $x_{1} \equiv_{\eta} x_{2}$. Suppose $\pi_{1}$ is a topological path from $x_{1}$. $\mathbb{F}\left(\pi_{1}([0,1])\right)$ is a connected subposet of $\widetilde{K}$. Thus, due to continuity of $\mathbb{F} \circ \pi_{1}$, a $\downarrow$ path $\hat{\pi}_{1}:\left[0 ; k_{1}\right] \rightarrow \widetilde{K}$ from $\mathbb{F}\left(\pi_{1}(0)\right)$ to $\mathbb{F}\left(\pi_{1}(1)\right)$ exists such that for all $i \in\left[0 ; k_{1}\right)$ there is $r_{1} \in[0,1)$ with $\hat{\pi}_{1}(i)=\mathbb{F}\left(\pi_{1}\left(r_{1}\right)\right)$. We also know that $\mathbb{F}\left(x_{1}\right) \equiv_{\eta} \mathbb{F}\left(x_{2}\right)$, as a consequence of Theorem 1, since $x_{1} \equiv_{\eta} x_{2}$. In addition, due to Lemma 10 below, we also know that $\mathbb{F}\left(x_{1}\right) \approx_{ \pm} \mathbb{F}\left(x_{2}\right)$. By Lemma 6 , we get that there is a $\downarrow$-path $\hat{\pi}_{2}:\left[0 ; k_{2}\right] \rightarrow \widetilde{K}$ such that $\hat{\pi}_{1}\left(k_{1}\right) \equiv_{\eta} \hat{\pi}_{2}\left(k_{2}\right)$ and for each $j \in\left[0 ; k_{2}\right)$ there is $i \in\left[0 ; k_{1}\right)$ such that $\hat{\pi}_{1}(i) \equiv_{\eta} \hat{\pi}_{2}(j)$. By Lemma 7 , it follows that there is topological path $\pi_{2}$ from $x_{2}$ satisfying the three conditions of the lemma and, again by Theorem 11 , we have that $\pi_{2}(1) \equiv_{\eta} \pi_{1}(1)$. In addition, for any $r_{2} \in[0,1)$, since $\mathbb{F}\left(\pi_{2}\left(r_{2}\right)\right)=\hat{\pi}_{2}(j)$ for $j \in\left[0 ; k_{2}\right)$ (condition (ii) of Lemma 7) there is $i \in\left[0 ; k_{1}\right)$ such that $\hat{\pi}_{1}(i) \equiv_{\eta} \hat{\pi}_{2}(j)$. Finally, by construction, there is $r_{1} \in[0,1)$ such that $\mathbb{F}\left(\pi_{1}\left(r_{1}\right)\right)=\hat{\pi}_{1}(i)$. By Theorem 1 , we finally get $\pi_{1}\left(r_{1}\right) \equiv{ }_{\eta} \pi_{2}\left(r_{2}\right)$.

On the basis of Lemma 5 and Lemma 8, we have that the largest weak simplicial bisimulation exists, it is a weak simplicial bisimilarity, it is an equivalence relation, and it coincides with logical equivalence in the polyhedral model induced by $\operatorname{SLCS}_{\eta}$, thus establishing the HMP for $\approx_{\Delta}^{\mathcal{P}}$ w.r.t. SLCS $_{\eta}$ :

Theorem 2. Given a polyhedral model $\mathcal{P}=(|K|, V), x_{1}, x_{2} \in|K|$, the following holds: $x_{1} \equiv{ }_{\eta}^{\mathcal{P}} x_{2}$ iff $x_{1} \approx^{\mathcal{P}}{ }_{\Delta} w_{2}$.

Similar results can be obtained for poset models. The following lemma shows that, in every finite poset model $\mathcal{F}$, weak $\pm$-bisimilarity (Def. 8) is stronger than logical equivalence with respect to $\operatorname{SLCS}_{\eta}$, i.e. $\approx_{ \pm}^{\mathcal{F}} \subseteq \equiv_{\eta}^{\mathcal{F}}$ :

Lemma 9. Given a finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$, for all $w_{1}, w_{2} \in W$, if $w_{1} \approx_{ \pm}^{\mathcal{F}} w_{2}$ then $w_{1} \equiv_{\eta}^{\mathcal{F}} w_{2}$.

Proof. By induction on formulas. We consider only the case $\eta\left(\Phi_{1}, \Phi_{2}\right)$. Suppose $w_{1} \approx_{ \pm} w_{2}$ and $\mathcal{F}, w_{1} \vDash \eta\left(\Phi_{1}, \Phi_{2}\right)$. Then, there is (a $\pm$-path and so) a $\downarrow$-path $\pi_{1}$ from $w_{1}$ of some length $k_{1}$ such that $\mathcal{F}, \pi_{1}\left(k_{1}\right) \models \Phi_{2}$ and for all $i \in\left[0 ; k_{1}\right)$ $\mathcal{F}, \pi_{1}(i) \models \Phi_{1}$ holds. By Lemma 6, we know that a $\downarrow$-path $\pi_{2}$ from $w_{2}$ exists of some length $k_{2}$ such that $\pi_{1}\left(k_{1}\right) \approx_{ \pm} \pi_{2}\left(k_{2}\right)$ and for all $j \in\left[0 ; k_{2}\right)$ there is $i \in\left[0 ; k_{1}\right)$ such that $\pi_{1}(i) \approx_{ \pm} \pi_{2}(j)$. By the Induction Hypothesis, we then get that $\mathcal{F}, \pi_{2}\left(k_{2}\right) \models \Phi_{2}$ and for all $j \in\left[0 ; k_{2}\right)$ we have $\mathcal{F}, \pi_{2}(j) \models \Phi_{1}$. This implies that $\mathcal{F}, w_{2} \models \eta\left(\Phi_{1}, \Phi_{2}\right)$.

Furthermore, logical equivalence induced by $\operatorname{SLCS}_{\eta}$ is stronger than weak $\pm$-bisimilarity, i.e. $\equiv_{\eta}^{\mathcal{F}} \subseteq \approx_{ \pm}^{\mathcal{F}}$, as implied by the following:

Lemma 10. In a finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V}), \equiv_{\eta}^{\mathcal{F}}$ is a weak $\pm$-bisimulation.
Proof. If $w_{1} \equiv_{\eta} w_{2}$, then the first requirement of Definition 8 is trivially satisfied. We prove that $\equiv_{\eta}$ satisfies the second requirement of Definition 8 . Suppose $w_{1} \equiv_{\eta} w_{2}$ and let $u_{1}, d_{1}$ as in the above mentioned requirement. This implies that $\mathcal{F}, w_{1} \models \eta\left(\chi\left(w_{1}\right) \vee \chi\left(u_{1}\right), \chi\left(d_{1}\right)\right)$, where, we recall, $\chi(w)$ is the 'characteristic formula' for $w$ as in Definition 6. Since $w_{1} \equiv_{\eta} w_{2}$, we have that also $\mathcal{F}, w_{2} \models \eta\left(\chi\left(w_{1}\right) \vee \chi\left(u_{1}\right), \chi\left(d_{1}\right)\right.$ holds. This in turn means that a $\downarrow$-path $\pi_{2}$ of some length $k_{2}$ from $w_{2}$ exists such that $\mathcal{F}, \pi_{2}\left(k_{2}\right) \models \chi\left(d_{1}\right)$ and for all $j \in\left[0 ; k_{2}\right)$ we have $\mathcal{F}, \pi_{2}(j) \models \chi\left(w_{1}\right) \vee \chi\left(u_{1}\right)$, i.e. $\mathcal{F}, \pi_{2}(j) \models \chi\left(w_{1}\right)$ or $\mathcal{F}, \pi_{2}(j) \models \chi\left(u_{1}\right)$. Consequently, by Proposition 2, we have: $\pi_{2}\left(k_{2}\right) \equiv_{\eta} d_{1}$ and, for all $j \in\left[0 ; k_{2}\right)$, $\pi_{2}(j) \equiv_{\eta} w_{1}$ or $\pi_{2}(j) \equiv_{\eta} u_{1}$, so that the second condition of the definition is fulfilled.

On the basis of Lemma 9 and Lemma 10 , we have that the largest weak $\pm$-bisimulation exists, it is a weak $\pm$-bisimilarity, it is an equivalence relation, and it coincides with logical equivalence in the finite poset induced by $\operatorname{SLCS}_{\eta}$ :

Theorem 3. For every finite poset model $(W, \preceq, \mathcal{V}), w_{1}, w_{2} \in W$, the following holds: $w_{1} \equiv_{\eta}^{\mathcal{F}} w_{2}$ iff $w_{1} \approx_{ \pm}^{\mathcal{F}} w_{2}$.

By this we have established the HMP for $\approx_{ \pm}$w.r.t. SLCS $_{\eta}$.
Finally, recalling that, by Theorem 11, given polyhedral model $\mathcal{P}=(|K|, V)$ for all $x \in|K|$ and $\operatorname{SLCS}_{\eta}$ formula $\Phi$, we have that $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$, we get the following final result:


Fig. 4: The minimal model, modulo weak $\pm$-bisimilarity, of the model of Fig. 1 .

Corollary 1. For all polyhedral models $\mathcal{P}=(|K|, V), x_{1}, x_{2} \in|K|$ :

$$
x_{1} \approx_{\Delta}^{\mathcal{P}} x_{2} \text { iff } x_{1} \equiv{ }_{\eta}^{\mathcal{P}} x_{2} \text { iff } \mathbb{F}\left(x_{1}\right) \equiv{ }_{\eta}^{\mathbb{F}(\mathcal{P})} \mathbb{F}\left(x_{2}\right) \text { iff } \mathbb{F}\left(x_{1}\right) \approx_{ \pm}^{\mathbb{F}(\mathcal{P})} \mathbb{F}\left(x_{2}\right)
$$

Saying that $\mathrm{SLCS}_{\eta}$-equivalence in a polyhedral model is the same as weak simplicial bisimilarity, which maps by $\mathbb{F}$ to the weak $\pm$-bisimilarity in the corresponding poset model, where the latter coincides with the $\operatorname{SLCS}_{\eta}$-equivalence.

Example 1. Fig. 4 shows the minimal model $\min (\mathbb{F}(\mathcal{P}))$, modulo $\approx_{ \pm}$, of $\mathbb{F}(\mathcal{P})$ (see Fig. 1c). We have the following equivalence classes: $\mathbb{C}_{1}^{\prime}=\{\widetilde{A}\}, \mathbb{C}_{2}^{\prime}=$ $\{\widetilde{B}, \widetilde{C}, \widetilde{A B}, \widetilde{A C}, \widetilde{B C}, \widetilde{B D}, \widetilde{C D}, \widetilde{A B C}, \widetilde{B C D}\}, \mathbb{C}_{3}^{\prime}=\{\widetilde{D}, \widetilde{E}, \widetilde{F}, \widetilde{C E}, \widetilde{D E}, \widetilde{D F}, \widetilde{E F}$, $\widetilde{D E F}\}$ and $\mathbb{C}_{4}^{\prime}=\{\widetilde{C D E}\}$. Note that the minimal model is not a poset model, but it is a reflexive Kripke model ${ }^{8}$

## 5 A Larger Example

As a proof-of-concept and feasibility we show a larger example of a 3D polyhedral structure composed of one white "room" and 26 green "rooms" connected by grey "corridors" as shown in Fig. 5a. In turn, each room is composed of 33 vertices, 122 edges, 150 triangles and 60 tetrahedra, i.e. it is composed of a total of 365 cells. Each corridor is composed of 8 edges, 12 triangles and 5 tetrahedra, i.e. it consists of 25 cells. The corridors are connected to rooms via the four points of the side of a room. In total, the structure consists of 11,205 cells. We have chosen a large, but symmetric structure on purpose. This makes it easy to interpret the various equivalence classes present in the minimal Kripke model of this structure shown in Fig. 5b. Observe that, for this example, the minimal model is also a poset model and, in particular, a cell poset model representing a polyhedron, as shown in Fig. 5c. The latter can be seen as a minimised version of the original polyhedral structure. Note also the considerable reduction that was obtained: from 11,205 cells to just 7 in the minimal model.

[^3]

Fig. 5: 5a A simplicial complex of a 3D structure composed of rooms and corridors. (5b Its minimal Kripke structure. (5c) Its minimal polyhedron.

In Fig. 5b we have indicated the various equivalence classes with a letter. Those indicated with a "C" correspond to classes of (cells of) corridors, those with an "R" correspond to classes of (cells of) rooms. For reasons of space and clarity, in the following we will not list all the individual cells that are part of a certain class, but instead we will indicate those cells by speaking about certain rooms and corridors, intending the cells that they are composed of.

There is one white class containing all white cells of the white room. Furthermore, there are three green classes corresponding to three types of green rooms, and three grey classes corresponding to three kinds of corridors. The green class R2 is composed of the (cells in) the six green rooms situated in the middle of each side of the cube structure. Those in R3 are the cells in the twelve green rooms situated in the middle of each 'edge' of the cube structure. Those in R4 are the cells in the eight green rooms situated at the corners of the cube structure. It is not difficult to find $\mathrm{SLCS}_{\eta}$ formulas that distinguish, for instance, the various green classes. For example, the cells in R2 satisfy $\Phi_{1}=\eta$ (green $\vee \eta$ (grey, white), white), whereas no cell in R3 or R4 satisfies $\Phi_{1}$. To distinguish class R3 from R4 one can observe that cells in R3 satisfy $\Phi_{2}=\eta\left(\right.$ green $\vee \eta\left(\right.$ grey,$\left.\left.\Phi_{1}\right), \Phi_{1}\right)$ whereas those in R4 do not satisfy $\Phi_{2}$.

In this symmetric case of this synthesised example, it was rather straightforward to find the various equivalence classes. In the general case it is much harder and one would need a suitable minimisation algorithm for $\operatorname{SLCS}_{\eta}$. We are currently working on an effective minimisation procedure based on encoding the cell poset model into a suitable LTS, exploiting behavioural equivalences for LTSs - strong bisimilarity and branching bisimulation equivalence - following an approach similar to that followed in [11] for finite closure spaces. The first results are promising. In fact, the large structure shown in this example can be handled that way and gives results as presented. Details and proofs of correctness of this approach and its potential efficiency gain will be the topic of future work, also for reasons of space limitations.

## 6 Conclusions

In [5 simplicial bisimilarity was proposed for polyhedral models - i.e. models of continuous space - while $\pm$-bisimilarity, the corresponding equivalence for cellposet models - discrete representations of polyhedral models - was introduced in 88. In order to support large model reductions, in this paper the novel notions of weak simplicial bisimilarity and weak $\pm$-bisimilarity have been proposed, and the correspondence between the two has been studied. We have proposed $\mathrm{SLCS}_{\eta}$, a weaker version of the Spatial Logic for Closure Spaces on polyhedral models, and we have shown that simplicial bisimilarity enjoys the Hennessy-Milner property (Thm. 2 ). We have also proven that the property holds for $\pm$-bisimilarity on poset models and the interpretation of SLCS $_{\eta}$ on such models (Thm. 3). SLCS $\eta_{\eta}$ can be used in the geometric spatial model checker PolyLogicA for checking spatial reachability properties of polyhedral models. Model checking results can be visualised by projecting them onto the original polyhedral structure in a colour. The results presented in this paper also have a practical value for the domain of visual computing where polyhedral models can be found in the form of surface meshes or tetrahedral volume meshes that are often composed of a huge number of cells.

In future work, in line with our earlier work, we aim to develop an automatic, provably correct minimisation procedure so that model checking could potentially be performed on a much smaller model. We also intend to develop a procedure to translate results back to the original polyhedral model for their appropriate visualisation. Finally, the complexity and efficiency of such methods will be investigated.

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## A Background and Notation in Detail

For sets $X$ and $Y$, a function $f: X \rightarrow Y$, and subsets $A \subseteq X$ and $B \subseteq Y$, we define $f(A)$ and $f^{-1}(B)$ as $\{f(a) \mid a \in A\}$ and $\{a \mid f(a) \in B\}$, respectively. The restriction of $f$ on $A$ is denoted by $f \mid A$. The powerset of $X$ is denoted by $\mathbf{2}^{X}$. For relation $R \subseteq X \times X$ we let $R^{-}$denote its converse and $R^{ \pm}$denote $R \cup R^{-}$. In the sequel, we assume that a set PL of proposition letters is fixed. The set of natural numbers and that of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$
respectively. We use the standard interval notation: for $x, y \in \mathbb{R}$ we let $[x, y]$ be the set $\{r \in \mathbb{R} \mid x \leq r \leq y\},[x, y)=\{r \in \mathbb{R} \mid x \leq r<y\}$ and so on, where $[x, y]$ is equipped with the Euclidean topology inherited from $\mathbb{R}$. We use a similar notation for intervals over $\mathbb{N}$ : for $n, m \in \mathbb{N}[m ; n]$ denotes the set $\{i \in \mathbb{N} \mid m \leq i \leq n\},[m ; n)$ denotes the set $\{i \in \mathbb{N} \mid m \leq i<n\}$, and similarly for ( $m ; n$ ] and $(m ; n)$.

A topological space is a pair $(X, \tau)$ where $X$ is a set (of points) and $\tau$ is a collection of subsets of $X$ satisfying the following axioms: (i) $\emptyset, X \in \tau$, (ii) for any index set $I, \bigcup_{i \in I} A_{i} \in \tau$ if each $A_{i} \in \tau$, and (iii) for any finite index set $I$, $\bigcap_{i \in I} A_{i} \in \tau$ if each $A_{i} \in \tau$. We let $\mathcal{C}_{T}$ denote the topological closure operator.

A Kripke frame is a pair $(W, R)$ where $W$ is a set and $R \subseteq W \times W$, the accessibility relation on $W$. A Kripke model is a tuple $(W, R, \mathcal{V})$ where $(W, R)$ is a Kripke frame and $\mathcal{V}:$ PL $\rightarrow \mathbf{2}^{W}$ is the valuation function, assigning to each $p \in$ PL the set $\mathcal{V}(p)$ of elements of $W$ where $p$ holds.

In the context of the present paper, it is convenient to view a partially ordered set - poset, in the sequel - $(W, \preceq)$ as a Kripke frame where the relation $\preceq \subseteq W \times W$ is a partial order, i.e. it is reflexive, transitive and anti-symmetric. Similarly we define a poset model as a Kripke model where the accessibility relation is a partial order. For partial orders $(W, \preceq)$, we use the standard notation, i.e.: $\preceq^{-}$will be denoted by $\succeq, w_{1} \prec w_{2}$ denotes $w_{1} \preceq w_{2}$ and $w_{1} \neq w_{2}$, and similarly for $\succ$.

## A. 1 Paths

Paths play a crucial role in the present paper. In the sequel, we provide definitions for the different kinds of paths we will use later on in the paper and we prove some useful properties of theirs.

Definition 9 (Topological Path). Given a topological space $(X, \tau)$ and $x \in$ $X$, $a$ topological path from $x$ is a total, continuous function $\pi:[0,1] \rightarrow X$ such that $\pi(0)=x$. We call $x$ the starting point of $\pi$. The ending point of $\pi$ is $\pi(1)$, while for any $r \in(0,1), \pi(r)$ is an intermediate point of $\pi$.

Definition 10 (Paths Over Kripke Frames). Given a Kripke frame ( $W, R$ ) and $w \in W$ :

- An undirected path from $w$, of length $\ell \in \mathbb{N}$, is a total function $\pi:[0 ; \ell] \rightarrow W$ such that $\pi(0)=w$ and, for all $i \in[0 ; \ell), R^{ \pm}(\pi(i), \pi(i+1))$;
- $A \downarrow$-path (to be read as "down path") from $w$, of length $\ell \geq 1$, is an undirected path $\pi$ from $w$ of length $\ell$ such that $R^{-}(\pi(\ell-1), \pi(\ell))$;
$-A \pm$-path (to be read as "plus-minus path") from $w$, of length $\ell \geq 2$, is a $\downarrow$-path $\pi$ from $w$ of length $\ell$ such that $R(\pi(0), \pi(1))$;
- An $\uparrow \downarrow$-path (to be read as "up-down path") from $w$, of length $2 \ell$, for $\ell \geq 1$, is $a \pm$-path $\pi$ of length $2 \ell$ such that $R(\pi(2 i), \pi(2 i+1))$ and $R^{-}(\pi(2 i+1), \pi(2 i+$ $2)$ ), for all $i \in[0 ; \ell)$.

We call $w$ the starting point of $\pi$. The ending point of $\pi$ is $\pi(\ell)$, while for any $i \in(0 ; \ell), \pi(i)$ is an intermediate point of $\pi$.


Fig. 6: A simple finite Kripke frame. Arrows in the figure represent the accessibility relation $R$.

Below, we will show some facts regarding the relationship among $\uparrow \downarrow$-paths, $\pm-$ paths and $\downarrow$-paths, but first we need to introduce some notation and operations on paths over Kripke frames. For undirected path $\pi$ of length $\ell$ we often use the sequence notation $\left(w_{i}\right)_{i=0}^{\ell}$ where $w_{i}=\pi(i)$ for all $i \in[0 ; \ell]$.

Definition 11 (Operations on Paths). Given a Kripke frame ( $W, R$ ) and paths $\pi^{\prime}=\left(w_{i}^{\prime}\right)_{i=0}^{\ell^{\prime}}$ and $\pi^{\prime \prime}=\left(w_{i}^{\prime \prime}\right)_{i=0}^{\ell^{\prime \prime}}$, with $w_{\ell^{\prime}}^{\prime}=w_{0}^{\prime \prime}$, the sequentialisation $\pi^{\prime} \cdot \pi^{\prime \prime}:\left[0 ; \ell^{\prime}+\ell^{\prime \prime}\right] \rightarrow W$ of $\pi^{\prime}$ with $\pi^{\prime \prime}$ is the path from $w_{0}^{\prime}$ defined as follows:

$$
\left(\pi^{\prime} \cdot \pi^{\prime \prime}\right)(i)=\left\{\begin{array}{l}
\pi^{\prime}(i), \text { if } i \in\left[0 ; \ell^{\prime}\right], \\
\pi^{\prime \prime}\left(i-\ell^{\prime}\right), \text { if } i \in\left[\ell^{\prime} ; \ell^{\prime}+\ell^{\prime \prime}\right] .
\end{array}\right.
$$

For path $\pi=\left(w_{i}\right)_{i=0}^{\ell}$ and $k \in[0 ; \ell]$ we define the $k$-shift of $\pi$, denoted by $\pi \uparrow k$, as follows: $\pi \uparrow k=\left(w_{j+k}\right)_{j=0}^{\ell-k}$ and, for $0<m \leq \ell$, we let $\pi \leftarrow m$ denote the path obtained from $\pi$ by inserting a copy of $\pi(m)$ immediately before $\pi(m)$ itself. In other words, we have: $\pi \leftarrow m=(\pi \mid[0 ; m]) \cdot((\pi(m), \pi(m)) \cdot(\pi \uparrow m))$. Finally, any path $\pi \mid[0 ; k]$, for some $k \in[0 ; \ell]$, is a (non-empty) prefix of $\pi$.

Example 2. For Kripke frame $(\{a, b, c, d\}, R)$ with $R=\{(a, b),(b, c),(c, d)\}$ (see Figure 6), path ( $a, b, c$ ) of length 2 and path $(c, d)$ of length 1, we have that $(a, b, c) \cdot(c, d)=(a, b, c, d)$, of length $3,(a) \cdot(a, b)=(a, b),(a) \cdot(a)=(a)$. Note the difference between sequentialisation and concatenation ' ++ ': for instance, $(a, b)++(c)=(a, b, c)$ whereas $(a, b) \cdot(c)$ is undefined since $b \neq c,(a)++(a)$ is $(a, a)$ whereas $(a) \cdot(a)=(a)$. We have $(a, b, c) \uparrow 1=(b, c)$ and $(a, b, c) \uparrow 2=(c)$ while $(a, b, c) \leftarrow 1=(a, b, b, c)$. Paths $(a),(a, b),(a, b, c)$ are all the (non-empty) prefixes of $(a, b, c)$.

As it is clear from Def. 10, every $\uparrow \downarrow$-path is also a $\pm$-path, that is also a $\downarrow$-path. Furthermore, the three lemmas below ensure that, for reflexive Kripke frames:

- for every $\pm$-path there is a $\uparrow \downarrow$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 11 below);
- for every $\downarrow$-path there is a $\uparrow \downarrow$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 12 below);
- for every $\downarrow$-path there is a $\pm$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 1, proved in Sect. B.1 on page 23.

Lemma 11. Given a reflexive Kripke frame $(W, R)$ and $a \pm$-path $\pi:[0 ; \ell] \rightarrow W$, there is a $\uparrow \downarrow$-path $\pi^{\prime}:\left[0 ; \ell^{\prime}\right] \rightarrow W$, for some $\ell^{\prime}$, and a total, surjective, monotonic non-decreasing function $f:\left[0 ; \ell^{\prime}\right] \rightarrow[0 ; \ell]$ such that $\pi^{\prime}(j)=\pi(f(j))$ for all $j \in\left[0 ; \ell^{\prime}\right]$.

Proof. We proceed by induction on the length $\ell$ of $\pm$-path $\pi$.
Base case: $\ell=2$.
In this case, by definition of $\pm$-path, we have $R(\pi(0), \pi(1))$ and $R^{-}(\pi(1), \pi(2))$, which, by definition of $\uparrow \downarrow$-path, implies that $\pi$ itself is an $\uparrow \downarrow$-path and $f:[0 ; \ell] \rightarrow$ $[0 ; \ell]$ is just the identity function.

Induction step. We assume the assert holds for all $\pm$-paths of length $\ell$ and we prove it for $\ell+1$. Let $\pi:[0 ; \ell+1] \rightarrow W$ be a $\pm$-path. Then $R^{-}(\pi(\ell), \pi(\ell+1))$, since $\pi$ is a $\pm$-path. We consider the following cases:
Case A: $R^{-}(\pi(\ell-1), \pi(\ell))$ and $R^{-}(\pi(\ell), \pi(\ell+1))$.
In this case, consider the prefix $\pi_{1}=\pi \mid[0 ; \ell]$ of $\pi$, noting that $\pi_{1}$ is a $\pm$-path of length $\ell$. By the Induction Hypothesis there is an $\uparrow \downarrow$-path $\pi_{1}^{\prime}$ of some length $\ell_{1}^{\prime}$ and a total, surjective, monotonic non-decreasing function $g:\left[0 ; \ell_{1}^{\prime}\right] \rightarrow[0 ; \ell]$ such that $\pi_{1}^{\prime}(j)=\pi_{1}(g(j))=\pi(g(j))$ for all $j \in\left[0 ; \ell_{1}^{\prime}\right]$. Note that $\pi_{1}^{\prime}\left(\ell_{1}^{\prime}\right)=\pi(\ell)$ so that the sequentialisation of $\pi_{1}^{\prime}$ with the two-element path $(\pi(\ell), \pi(\ell+1))$ is well-defined. Consider path $\pi^{\prime}=\left(\pi_{1}^{\prime} \cdot(\pi(\ell), \pi(\ell+1))\right) \leftarrow \ell_{1}^{\prime}$, of length $\ell_{1}^{\prime}+2$ consisting of $\pi_{1}^{\prime}$ followed by $\pi(\ell)$ followed in turn by $\pi(\ell+1)$. In other words, $\pi^{\prime}=$ $\left(\pi_{1}^{\prime}(0) \ldots \pi_{1}^{\prime}\left(\ell_{1}^{\prime}\right), \pi(\ell), \pi(\ell+1)\right)$, with $\pi_{1}^{\prime}\left(\ell_{1}^{\prime}\right)=\pi(\ell)$ - recall that $R$ is reflexive. It is easy to see that $\pi^{\prime}$ is an $\uparrow \downarrow$-path and that function $f:\left[0 ; \ell_{1}^{\prime}+2\right] \rightarrow[0 ; \ell+1]$, with $f(j)=g(j)$ for $j \in\left[0 ; \ell_{1}^{\prime}\right], f\left(\ell_{1}^{\prime}+1\right)=\ell$ and $f\left(\ell_{1}^{\prime}+2\right)=\ell+1$, is total, surjective, and monotonic non-decreasing.
Case B: $R(\pi(\ell-1), \pi(\ell))$ and $R^{-}(\pi(\ell), \pi(\ell+1))$.
In this case the prefix $\pi \mid[0 ; \ell]$ of $\pi$ is not a $\pm$-path. We then consider the path consisting of prefix $\pi \mid[0 ; \ell-1]$ where we add a copy of $\pi(\ell-1)$, i.e. the path $\pi_{1}=(\pi \mid[0 ; \ell-1]) \leftarrow(\ell-1)$ - we can do that because $R$ is reflexive. Note that $\pi_{1}$ is a $\pm$-path and has length $\ell$. By the Induction Hypothesis there is an $\uparrow \downarrow$-path $\pi_{1}^{\prime}$ of some length $\ell_{1}^{\prime}$ and a total, surjective, monotonic non-decreasing function $g:\left[0 ; \ell_{1}^{\prime}\right] \rightarrow[0 ; \ell]$ such that $\pi_{1}^{\prime}(j)=\pi_{1}(g(j))=\pi(g(j))$ for all $j \in$ $\left[0 ; \ell_{1}^{\prime}\right]$. Consider path $\pi^{\prime}=\pi_{1}^{\prime} \cdot(\pi(\ell-1), \pi(\ell), \pi(\ell+1))$, of length $\ell_{1}^{\prime}+2$, that is well defined since $\pi_{1}^{\prime}\left(\ell_{1}^{\prime}\right)=\pi(\ell-1)$ by definition of $\pi_{1}$. In other words, $\pi^{\prime}=$ $\left(\pi_{1}^{\prime}(0), \ldots, \pi_{1}^{\prime}\left(\ell_{1}^{\prime}\right), \pi(\ell), \pi(\ell+1)\right)$, with $\pi_{1}^{\prime}\left(\ell_{1}^{\prime}\right)=\pi(\ell-1)$. Path $\pi^{\prime}$ is an $\uparrow \downarrow$-path. In fact $\pi^{\prime} \mid\left[0 ; \ell_{1}^{\prime}\right]=\pi_{1}^{\prime}$ is an $\uparrow \downarrow$-path. Furthermore, $\pi^{\prime}\left(\ell_{1}^{\prime}\right)=\pi(\ell-1), R(\pi(\ell-1), \pi(\ell))$, $R^{-}(\pi(\ell)), \pi(\ell+1)$ and $\pi(\ell+1)=\pi^{\prime}\left(\ell_{1}^{\prime}+2\right)$. Finally, function $f:\left[0 ; \ell_{1}^{\prime}+2\right] \rightarrow$ $[0 ; \ell+1]$, with $f(j)=g(j)$ for $j \in\left[0 ; \ell_{1}^{\prime}\right], f\left(\ell_{1}^{\prime}+1\right)=\ell$ and $f\left(\ell_{1}^{\prime}+2\right)=\ell+1$, is total, surjective, and monotonic non-decreasing.

Lemma 12. Given a reflexive Kripke frame $(W, R)$ and $a \downarrow$-path $\pi:[0 ; \ell] \rightarrow W$, there is a $\uparrow \downarrow$-path $\pi^{\prime}:\left[0 ; \ell^{\prime \prime}\right] \rightarrow W$, for some $\ell^{\prime}$, and a total, surjective, monotonic non-decreasing function $f:\left[0 ; \ell^{\prime}\right] \rightarrow[0 ; \ell]$ such that $\pi^{\prime}(j)=\pi(f(j))$ for all $j \in\left[0 ; \ell^{\prime}\right]$.

Proof. The proof is carried out by induction on the length $\ell$ of $\pi$.
Base case. $\ell=1$. Suppose $\ell=1$, i.e. $\pi:[0 ; 1] \rightarrow W$ with $R^{-}(\pi(0), \pi(1))$. Then let $\pi^{\prime}:[0 ; 2] \rightarrow W$ be such that $\pi^{\prime}(0)=\pi^{\prime}(1)=\pi(0)$ and $\pi^{\prime}(2)=\pi(1)-$ we can do that since $R$ is reflexive - and $f:[0 ; 2] \rightarrow[0 ; 1]$ be such that $f(0)=f(1)=0$ and $f(2)=1$. Clearly $\pi^{\prime}$ is an $\uparrow \downarrow$-path and $\pi^{\prime}(j)=\pi(f(j))$ for all $j \in[0 ; 2]$.
Induction step. We assume the assert holds for all $\downarrow$-paths of length $\ell$ and we prove it for $\ell+1$. Let $\pi:[0 ; \ell+1] \rightarrow W$ a $\downarrow$-path and suppose the assert holds for all $\downarrow$-paths of length $\ell$. In particular, it holds for $\pi \uparrow 1$, i.e., there is a $\uparrow \downarrow$-path $\pi^{\prime \prime}$ of some length $\ell^{\prime \prime}$ with $\pi^{\prime \prime}(0)=\pi(1)$, and total, monotonic non-decreasing surjection $g:\left[0 ; \ell^{\prime \prime}\right] \rightarrow W$ such that $\pi^{\prime \prime}(j)=\pi(g(j))$ for all $j \in\left[0 ; \ell^{\prime \prime}\right]$. Suppose $R(\pi(0), \pi(1))$ does not hold. Then, since $R$ is reflexive, we let $\pi^{\prime}=(\pi(0), \pi(0), \pi(1)) \cdot \pi^{\prime \prime}$ and $f:\left[0 ; \ell^{\prime \prime}+2\right] \rightarrow[0 ; \ell+1]$ with $f(0)=f(1)=0$ and $f(j)=g(j-2)$ for all $j \in\left[2 ; \ell^{\prime \prime}+2\right]$. If instead $R(\pi(0), \pi(1))$, then we let $\pi^{\prime}=(\pi(0), \pi(1), \pi(1)) \cdot \pi^{\prime \prime}$ and $f:\left[0 ; \ell^{\prime \prime}+2\right] \rightarrow[0 ; \ell+1]$ with $f(0)=0, f(1)=1$ and $f(j)=g(j-2)$ for all $j \in\left[2 ; \ell^{\prime \prime}+2\right]$.

The following result states that to evaluate an $\operatorname{SLCS}_{\eta}$ formula $\eta\left(\Phi_{1}, \Phi_{2}\right)$ in a poset model, it does not matter whether one considers $\pm$-paths or $\downarrow$-paths.

Proposition 4. Given a finite poset $\mathcal{F}=(W, \preceq, \mathcal{V})$, $w \in W$ and an $\operatorname{SLCS}_{\eta}$ formula $\eta\left(\Phi_{1}, \Phi_{2}\right)$ the following statements are equivalent:

1. There exists $a \pm$-path $\pi:[0 ; \ell] \rightarrow W$ for some $\ell$ with $\pi(0)=w, \mathcal{F}, \pi(\ell) \models \Phi_{2}$ and $\mathcal{F}, \pi(i) \mid=\Phi_{1}$ for all $i \in(0 ; \ell)$.
2. There exists $a \downarrow$-path $\pi:[0 ; \ell] \rightarrow W$ for some $\ell$ with $\pi(0)=w, \mathcal{F}, \pi(2 \ell) \models \Phi_{2}$ and $\mathcal{F}, \pi(i)=\Phi_{1}$ for all $i \in(0 ; \ell)$.

Proof. The equivalence of statements (1) and (2) follows directly from Lemma 1 and the fact that $\pm$-paths are also $\downarrow$-paths.

Lemma 13. Given a finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$ and a weak $\pm$-bisimulation $B \subseteq W \times W$, for all $w_{1}, w_{2}$ such that $B\left(w_{1}, w_{2}\right)$, the following holds: for each $\uparrow \downarrow$-path $\pi_{1}:[0 ; 2 h] \rightarrow W$ from $w_{1}$ there is a $\downarrow$-path $\pi_{2}:[0 ; k] \rightarrow W$ from $w_{2}$ such that $B\left(\pi_{1}(2 h), \pi_{2}(k)\right)$ and for each $j \in[0 ; k)$ there is $i \in[0 ; 2 h)$ such that $B\left(\pi_{1}(i), \pi_{2}(j)\right)$.

Proof. We prove the assert by induction on $h$.
Base case. $h=1$.
If $h=1$, the assert follows directly from Definition 8 on page 11 where $w_{1}=$ $\pi(0), u_{1}=\pi(1)$ and $d_{1}=\pi(2)$.
Induction step. We assume the assert holds for $\uparrow \downarrow$-paths of length $2 h$ or less and we prove it for $\uparrow \downarrow$-paths of length $2(h+1)$.
Suppose $\pi_{1}$ is a $\uparrow \downarrow$-path of length $2 h+2$ and consider $\uparrow \downarrow$-path $\pi_{1}^{\prime}=\pi_{1} \mid[0 ; 2 h]$. By the Induction Hypothesis, we know that there is a $\downarrow$-path $\pi_{2}^{\prime}:\left[0 ; k^{\prime}\right] \rightarrow W$ from $w_{2}$ such that $B\left(\pi_{1}^{\prime}(2 h), \pi_{2}^{\prime}\left(k^{\prime}\right)\right)$ and for each $j \in\left[0 ; k^{\prime}\right)$ there is $i \in[0 ; 2 h)$ such that $B\left(\pi_{1}^{\prime}(i), \pi_{2}^{\prime}(j)\right)$. Clearly, this means that $B\left(\pi_{1}(2 h), \pi_{2}^{\prime}\left(k^{\prime}\right)\right)$ and for each $j \in\left[0 ; k^{\prime}\right)$ there is $i \in[0 ; 2 h)$ such that $B\left(\pi_{1}(i), \pi_{2}^{\prime}(j)\right)$. Furthermore, since $B\left(\pi_{1}(2 h), \pi_{2}^{\prime}\left(k^{\prime}\right)\right)$ and $B$ is a weak $\pm$-bisimulation, we also know that there is
a $\downarrow$-path $\pi_{2}^{\prime \prime}:\left[0 ; k^{\prime \prime}\right] \rightarrow W$ from $\pi_{2}^{\prime}\left(k^{\prime}\right)$ such that $B\left(\pi_{1}(2 h+2), \pi_{2}^{\prime \prime}\left(k^{\prime \prime}\right)\right)$ and for each $j \in\left[0 ; k^{\prime \prime}\right)$ there is $i \in[2 h ; 2 h+2)$ such that $B\left(\pi_{1}(i), \pi_{2}^{\prime}(j)\right)$. Let $\pi_{2}$ : $\left[0 ; k^{\prime}+k^{\prime \prime}\right] \rightarrow W$ be defined as $\pi_{2}=\pi_{2}^{\prime} \cdot \pi_{2}^{\prime \prime}$. Clearly $\pi_{2}$ is a $\downarrow$-path, since so is $\pi_{2}^{\prime \prime}$. Furthermore $B\left(\pi_{1}(2 h+2), \pi_{2}\left(k^{\prime}+k^{\prime \prime}\right)\right)$ since $B\left(\pi_{1}(2 h+2), \pi_{2}^{\prime \prime}\left(k^{\prime \prime}\right)\right)$ and $\pi_{2}^{\prime \prime}\left(k^{\prime \prime}\right)=\pi_{2}\left(k^{\prime}+k^{\prime \prime}\right)$. Finally, it is straightforward to check for all $j \in\left[0 ; k^{\prime}+k^{\prime \prime}\right)$ there is $i \in[0 ; 2 h+2)$ such that $B\left(\pi_{1}(i), \pi_{2}(j)\right)$.

## B Detailed Proofs

## B. 1 Proof of Lemma 1

Lemma 1. Given a reflexive Kripke frame $(W, R)$ and $a \downarrow$-path $\pi:[0 ; \ell] \rightarrow W$, there is a $\pm$-path $\pi^{\prime}:\left[0 ; \ell^{\prime \prime}\right] \rightarrow W$, for some $\ell^{\prime}$, and a total, surjective, monotonic, non-decreasing function $f:\left[0 ; \ell^{\prime}\right] \rightarrow[0 ; \ell]$ with $\pi^{\prime}(j)=\pi(f(j))$ for all $j \in\left[0 ; \ell^{\prime}\right]$.

Proof. The assert follows directly from Lemma 12 on page 21 since every $\uparrow \downarrow$-path is also a $\pm$-path.

## B. 2 Proof of Lemma 2

Lemma 2, Let $\mathcal{P}=(|K|, V)$ be a polyhedral model, $x \in|K|$ and $\Phi$ a $\operatorname{SLCS}_{\eta}$ formula. Then $\mathcal{P}, x \models \Phi$ iff $\mathcal{P}, x \models \mathcal{E}(\Phi)$.

Proof. By induction on the structure of $\Phi$. We consider only the case $\eta\left(\Phi_{1}, \Phi_{2}\right)$. Suppose $\mathcal{P}, x=\eta\left(\Phi_{1}, \Phi_{2}\right)$. By definition there is a topological path $\pi$ such that $\mathcal{P}, \pi(1) \models \Phi_{2}$ and $\mathcal{P}, \pi(r) \models \Phi_{1}$ for all $r \in[0,1)$. By the Induction Hypothesis this is the same to say that $\mathcal{P}, \pi(1) \models \mathcal{E}\left(\Phi_{2}\right)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}\left(\Phi_{1}\right)$ for all $r \in[0,1)$, i.e. $\mathcal{P}, x \models \mathcal{E}\left(\Phi_{1}\right), \mathcal{P}, \pi(1) \models \mathcal{E}\left(\Phi_{2}\right)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}\left(\Phi_{1}\right)$ for all $r \in(0,1)$. In other words, we have $\mathcal{P}, x \models \mathcal{E}\left(\Phi_{1}\right) \wedge \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$ that, by Definition 3 on page 8 means $\mathcal{P}, x=\mathcal{E}\left(\eta\left(\Phi_{1}, \Phi_{2}\right)\right)$.

Suppose now $\mathcal{P}, x \models \mathcal{E}\left(\eta\left(\Phi_{1}, \Phi_{2}\right)\right)$, i.e. $\mathcal{P}, x \models \mathcal{E}\left(\Phi_{1}\right) \wedge \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$, by Definition 3 on page 8 Since $\mathcal{P}, x \models \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$, there is a path $\pi$ such that $\mathcal{P}, \pi(1) \models \mathcal{E}\left(\Phi_{2}\right)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}\left(\Phi_{1}\right)$ for all $r \in(0,1)$. Using the Induction Hypothesis we know the following holds: $\mathcal{P}, x \models \Phi_{1}, \mathcal{P}, \pi(1) \models \Phi_{2}$, and $\mathcal{P}, \pi(r) \mid=$ $\Phi_{1}$ for all $r \in(0,1)$, i.e. $\mathcal{P}, \pi(1) \models \Phi_{2}$ and $\mathcal{P}, \pi(r) \models \Phi_{1}$ for all $r \in[0,1)$. So, we get $\mathcal{P}, x \models \eta\left(\Phi_{1}, \Phi_{2}\right)$.

## B. 3 Proof concerning the example of Remark 1

The assert can be proven by induction on the structure of formulas. The case for proposition letters, negation and conjunction are straightforward and omitted.

Suppose $\mathcal{P}, A \models \eta\left(\Phi_{1}, \Phi_{2}\right)$. Then there is a topological path $\pi_{A}:[0,1] \rightarrow|K|$ from $A$ such that $\mathcal{P}, \pi_{A}(1) \models \Phi_{2}$ and $\mathcal{P}, \pi_{A}(r) \models \Phi_{1}$ for all $r \in[0,1)$. Since $\mathcal{P}, A \models \Phi_{1}$, by the Induction Hypothesis, we have that $\mathcal{P}, x \models \Phi_{1}$ for all $x \in$
$\widetilde{A B C}$. For each $x \in \widetilde{A B C}$, define $\pi_{x}:[0,1] \rightarrow|K|$ as follows, for arbitrary $v \in(0,1)$ :

$$
\pi_{x}(r)=\left\{\begin{array}{l}
\frac{r}{v} A+\frac{v-r}{v} x, \text { if } r \in[0, v), \\
\pi_{A}\left(\frac{r-v}{1-v}\right), \text { if } r \in[v, 1]
\end{array}\right.
$$

Function $\pi_{x}$ is continuous. Furthermore, for all $y \in[0, v)$, we have that $\mathcal{P}, \pi_{x}(y) \models$ $\Phi_{1}$, since $\pi_{x}(y) \in \widetilde{A B C}$. Also, for all $y \in[v, 1)$ we have that $\mathcal{P}, \pi_{x}(y) \models \Phi_{1}$, since $\pi_{x}(y)=\pi_{A}\left(\frac{y-v}{1-v}\right), 0 \leq \frac{y-v}{1-v}<1$ and for $y \in[0,1)$ we have that $\mathcal{P}, \pi_{A}(y), \models \Phi_{1}$. Thus $\mathcal{P}, \pi_{x}(r) \models \Phi_{1}$ for all $r \in[0,1)$. Finally, $\pi_{x}(1)=\pi_{A}(1)$ and $\mathcal{P}, \pi_{A}(1) \models \Phi_{2}$ by hypothesis. Thus, $\pi_{x}$ is a topological path that witnesses $\mathcal{P}, x \models \eta\left(\Phi_{1}, \Phi_{2}\right)$.

The proof of the converse is similar, using instead function $\pi_{A}:[0,1] \rightarrow|K|$ defined as follows, for arbitrary $v \in(0,1)$ :

$$
\pi_{A}(r)=\left\{\begin{array}{l}
\frac{r}{v} p+\frac{v-r}{v} A, \text { if } r \in[0, v) \\
\pi_{p}\left(\frac{r-v}{1-v}\right), \text { if } r \in[v, 1]
\end{array}\right.
$$

## B. 4 Proof of Proposition 2

Proposition 2, Given a finite poset model $(W, \preceq, \mathcal{V})$, for $w_{1}, w_{2} \in W$, it holds that

$$
\mathcal{F}, w_{2} \models \chi\left(w_{1}\right) \text { if and only if } w_{1} \equiv_{\eta} w_{2} .
$$

Proof. Suppose $w_{1} \not \equiv \equiv_{\eta} w_{2}$, then we have $\mathcal{F}, w_{2} \not \vDash \delta_{w_{1}, w_{2}}$, and so $\mathcal{F}, w_{2} \not \vDash$ $\bigwedge_{w \in W} \delta_{w_{1}, w}$. If, instead, $w_{1} \equiv_{\eta} w_{2}$, then we have: $\delta_{w_{1}, w_{1}} \equiv \delta_{w_{1}, w_{2}} \equiv$ true by definition, since $w_{1} \equiv_{\eta} w_{1}$ and $w_{1} \equiv_{\eta} w_{2}$. Moreover, for any other $w$, we have that, in any case, $\mathcal{F}, w_{1} \models \delta_{w_{1}, w}$ holds and since $w_{1} \equiv_{\eta} w_{2}$, also $\mathcal{F}, w_{2} \models \delta_{w_{1}, w}$ holds. So, in conclusion, $\mathcal{F}, w_{2} \models \bigwedge_{w \in W} \delta_{w_{1}, w}$.

## B. 5 Proof of Lemma 3

Lemma 3, Let $\mathcal{F}=(W, \preceq, \mathcal{V})$ be a finite poset model, $w \in W$ and $\Phi$ a $\operatorname{SLCS}_{\eta}$ formula. Then $\mathcal{F}, w \models \Phi$ iff $\mathcal{F}, w \models \mathcal{E}(\Phi)$.

Proof. Similar to that of Lemma 2, but with reference to the finite poset intepretation of the logic.

## B. 6 Proof concerning the example of Remark 3

We prove the assert by induction on the structure of formulas. The case for atomic proposition letters, negation and conjunction are straightforward and omitted. Suppose $\mathcal{F}, \widetilde{A} \models \eta\left(\Phi_{1}, \Phi_{2}\right)$. Then, there is a $\pm$-path $\pi$ of some length $\ell \geq 2$ such that $\pi(0)=\widetilde{A}, \pi(\ell) \models \Phi_{2}$ and $\pi(i) \models \Phi_{1}$ for all $i \in[0 ; \ell)$. Since $\mathcal{F}, \widetilde{A}=$ $\Phi_{1}$, by the Induction Hypothesis, we have that $\mathcal{F}, \widetilde{A B C} \models \Phi_{1}$. Consider then
path $\pi^{\prime}=(\widetilde{A B C}, \widetilde{A B C}, \widetilde{A}) \cdot \pi$. Path $\pi^{\prime}$ is a $\pm$-path and it witnesses $\mathcal{F}, \widetilde{A B C} \models$ $\eta\left(\Phi_{1}, \Phi_{2}\right)$.

Suppose now $\mathcal{F}, \widetilde{A B C} \models \eta\left(\Phi_{1}, \Phi_{2}\right)$ and let $\pi$ be a $\pm$-path witnessing it. Then, path $(\widetilde{A}, \widetilde{A B C}, \widetilde{A B C}) \cdot \pi$ is a $\pm$-path witnessing $\mathcal{F}, \widetilde{A} \models \eta\left(\Phi_{1}, \Phi_{2}\right)$.

## B. 7 Proof of Lemma 4

Lemma 4. Given a polyhedral model $\mathcal{P}=(|K|, V)$, for all $x \in|K|$ and formulas $\Phi$ of $\operatorname{SLCS}_{\eta}$ the following holds: $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi)$.

Proof. The proof is by induction on the structure of $\Phi$. We consider only the case $\eta\left(\Phi_{1}, \Phi_{2}\right)$. Suppose $\mathcal{P}, x \models \eta\left(\Phi_{1}, \Phi_{2}\right)$. By Lemma 2 on page 8 we get $\mathcal{P}, x \models \mathcal{E}\left(\eta\left(\Phi_{1}, \Phi_{2}\right)\right)$ and then by Definition 3 on page 8 , we have $\mathcal{P}, x \mid=$ $\mathcal{E}\left(\Phi_{1}\right) \wedge \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$, that is $\mathcal{P}, x \models \mathcal{E}\left(\Phi_{1}\right)$ and $\mathcal{P}, x \models \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$. Again by Lemma 2 on page 8, we get also $\mathcal{P}, x \models \Phi_{1}$ and so, by the Induction Hypothesis, we have $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}\left(\Phi_{1}\right)$. Furthermore, by Theorem 4.4 of [5] we also get $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$. Thus we get $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \vDash$ $\mathcal{E}\left(\Phi_{1}\right) \wedge \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$, that is $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}\left(\eta\left(\Phi_{1}, \Phi_{2}\right)\right)$.
Suppose now $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}\left(\eta\left(\Phi_{1}, \Phi_{2}\right)\right)$. This means $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}\left(\Phi_{1}\right) \wedge$ $\gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$, that is $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}\left(\Phi_{1}\right)$ and $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$. By the Induction Hypothesis we get that $\mathcal{P}, x \models \Phi_{1}$. Furthermore, by Theorem 4.4 of [5] we also get $\mathcal{P}, x \models \gamma\left(\mathcal{E}\left(\Phi_{1}\right), \mathcal{E}\left(\Phi_{2}\right)\right)$. This means that there is topological path $\pi$ such that $\mathcal{P}, \pi(1) \models \mathcal{E}\left(\Phi_{2}\right)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}\left(\Phi_{1}\right)$ for all $r \in(0,1)$. Using Lemma 2 on page 8 we also get $\mathcal{P}, \pi(1) \models \Phi_{2}$ and $\mathcal{P}, \pi(r) \models \Phi_{1}$ for all $r \in(0,1)$ and since also $\mathcal{P}, x=\Phi_{1}$ (see above), we get $\mathcal{P}, \pi(1) \models \Phi_{2}$ and $\mathcal{P}, \pi(r) \models \Phi_{1}$ for all $r \in[0,1)$, that is $\mathcal{P}, x \models \eta\left(\Phi_{1}, \Phi_{2}\right)$.

## B. 8 Proof of Theorem 1

Theorem 1. Given a polyhedral model $\mathcal{P}=(|K|, V)$, for all $x \in|K|$ and formulas $\Phi$ of $\operatorname{SLCS}_{\eta}$ it holds that: $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$.

Proof. Using Lemma 4 on page 10 , we know that $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi)$. Moreover, by Lemma 3 on page 10 , we know that $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models$ $\mathcal{E}(\Phi)$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$, which brings to the result.

## B. 9 Proof of Lemma 6

Lemma 6. Given a finite poset model $\mathcal{F}=(W, \preceq, \mathcal{V})$ and weak $\pm$-bisimulation $B \subseteq W \times W$, for all $w_{1}, w_{2}$ such that $B\left(w_{1}, w_{2}\right)$, the following holds: for each $\downarrow$-path $\pi_{1}:\left[0 ; k_{1}\right] \rightarrow W$ from $w_{1}$ there is $a \downarrow$-path $\pi_{2}:\left[0 ; k_{2}\right] \rightarrow W$ from $w_{2}$ such that $B\left(\pi_{1}\left(k_{1}\right), \pi_{2}\left(k_{2}\right)\right)$ and for each $j \in\left[0 ; k_{2}\right)$ there is $i \in\left[0 ; k_{1}\right)$ such that $B\left(\pi_{1}(i), \pi_{2}(j)\right)$.

Proof. Let $\pi_{1}:\left[0 ; k_{1}\right] \rightarrow W$ be a $\downarrow$-path from $w_{1}$. By Lemma 12 on page 21 we know that there is an $\uparrow \downarrow$-path $\hat{\pi}_{1}:[0 ; 2 h] \rightarrow W$ and total, monotonic nondecreasing surjection $f:[0 ; 2 h] \rightarrow\left[0 ; k_{1}\right]$ such that $\hat{\pi}_{1}(j)=\pi_{1}(f(j))$ for all $j \in[0 ; 2 h]$. Furthermore, by Lemma 13 on page 22 , we know that there is a $\downarrow$-path $\pi_{2}:\left[0 ; k_{2}\right] \rightarrow W$ from $w_{2}$ such that $B\left(\hat{\pi}_{1}(2 h), \pi_{2}\left(k_{2}\right)\right)$ and for each $j \in\left[0 ; k_{2}\right)$ there is $i \in[0 ; 2 h)$ such that $B\left(\hat{\pi}_{1}(i), \pi_{2}(j)\right)$. In addition, $\hat{\pi}_{1}(0)=\pi_{1}(0)=w_{1}$, $B\left(\pi_{1}\left(k_{1}\right), \pi_{2}\left(k_{2}\right)\right)$ since $B\left(\hat{\pi}_{1}(2 h), \pi_{2}\left(k_{2}\right)\right)$ and $\hat{\pi}_{1}(2 h)=\pi_{1}\left(k_{1}\right)$. Finally, for each $j \in\left[0 ; k_{2}\right)$ there is $i \in\left[0 ; k_{1}\right)$ such that $B\left(\pi_{1}(i), \pi_{2}(j)\right)$, since there is $n \in[0 ; 2 h)$ such that $B\left(\hat{\pi}_{1}(n), \pi_{2}(j)\right)$ and $f(n)=i$ for some $i \in\left[0 ; k_{1}\right)$.

## B. 10 Proof of Lemma 7

Lemma 7. Given a polyhedral model $\mathcal{P}=(|K|, V)$, and associated cell poset model $\mathbb{F}(\overparen{\mathcal{P}})=(W, \preceq, \mathcal{V})$, for any $\downarrow$-path $\pi:[0 ; \ell] \rightarrow W$, there is a topological path $\pi^{\prime}:[0,1] \rightarrow|K|$ such that: (i) $\mathbb{F}\left(\pi^{\prime}(0)\right)=\pi(0)$, (ii) $\mathbb{F}\left(\pi^{\prime}(1)\right)=\pi(\ell)$, and (iii) for all $r \in(0,1)$ there is $i<\ell$ such that $\mathbb{F}\left(\pi^{\prime}(r)\right)=\pi(i)$.

Proof. Since $\pi$ is a $\downarrow$-path, we have that either $\mathcal{C}_{T}\left(\mathbb{F}^{-1}(\pi(k-1))\right) \sqsubseteq \mathcal{C}_{T}\left(\mathbb{F}^{-1}(\pi(k))\right)$ or $\mathcal{C}_{T}\left(\mathbb{F}^{-1}(\pi(k))\right) \sqsubseteq \mathcal{C}_{T}\left(\mathbb{F}^{-1}(\pi(k-1))\right.$ ), for each $k \in\left(0 ; \ell{ }^{9}\right.$. It follows that there is a continuous map $\pi_{k}^{\prime}:\left[\frac{k-1}{\ell}, \frac{k}{\ell}\right] \rightarrow|K|$ such that, in the first case, $\mathbb{F}\left(\pi_{k}^{\prime}\left(\frac{k-1}{\ell}\right)\right)=\pi(k-1)$ and $\pi_{k}^{\prime}\left(\left(\frac{k-1}{\ell}, \frac{k}{\ell}\right]\right) \subseteq \mathcal{C}_{T}\left(\mathbb{F}^{-1}(\pi(k))\right)$, while in the second case, $\pi_{k}^{\prime}\left(\left[\frac{k-1}{\ell}, \frac{k}{\ell}\right)\right) \subseteq \mathcal{C}_{T}\left(\mathbb{F}^{-1}(\pi(k-1))\right)$ and $\mathbb{F}\left(\pi_{k}^{\prime}\left(\frac{k}{\ell}\right)\right)=\pi(k)$. In fact $\pi_{k}^{\prime}$ can be realised as a linear bijection to the line segment connecting the barycenters in the corresponding cell, either in $\mathbb{F}^{-1}(\pi(k))$ or in $\mathbb{F}^{-1}(\pi(k-1))$, respectively.

For each $k \in(0 ; \ell)$, both $\pi_{k}^{\prime}\left(\frac{k}{\ell}\right)$ and $\pi_{k+1}^{\prime}\left(\frac{k}{\ell}\right)$ coincide with the barycenter of $\mathbb{F}^{-1}(\pi(k))$, so that defining $\pi^{\prime}(r)=\pi_{k}^{\prime}(r)$ for $r \in\left[\frac{k-1}{\ell}, \frac{k}{\ell}\right]$ correctly defines a topological path (actually a piece-wise linear path), satisfying (i) and (ii). Finally since $\pi$ is a $\downarrow$-path, $\pi(\ell) \preceq \pi(\ell-1)$, so that $\pi^{\prime}\left(\left[\frac{\ell-1}{\ell}, 1\right)\right) \subseteq \mathbb{F}^{-1}(\pi(\ell-1))$. This implies (iii) above.

[^4]
[^0]:    * This is an extended version of the paper "Weak Simplicial Bisimilarity for Polyhedral Models and $\operatorname{SLCS}_{\eta}{ }^{\prime \prime}$, accepted for publication in the proceedings of the 44th International Conference on Formal Techniques for Distributed Objects, Components, and Systems (FORTE 2024) published as LNCS by Springer. It contains all detailed proofs that are not present in the FORTE 2024 paper due to lack of space.
    The authors are listed in alphabetical order, as they equally contributed to the work presented in this paper.

[^1]:    ${ }^{5}$ But note that the relative interior of a simplex composed of just a single point is the point itself and not the empty set.

[^2]:    ${ }^{6}$ We are interested in model-checking structures resulting from the minimisation, via bisimilarity, of cell poset models, and such structures are often just (reflexive) Kripke models rather than poset models.
    ${ }^{7}$ For undirected path $\pi$ of length $\ell$ we often use the sequence notation $\left(x_{i}\right)_{i=0}^{\ell}$ where $x_{i}=\pi(i)$ for $i \in[0 ; \ell]$.

[^3]:    ${ }^{8}$ It is worth noting that for model-checking purposes we can safely interpret SLCS $_{\eta}$ over (reflexive) Kripke models. The satisfaction relation is defined as in Def. 4 where $\mathcal{F}$ is a Kripke model instead of a poset model (recall that $\pm$-paths are defined on Kripke frames).

[^4]:    ${ }^{9}$ We recall here that $\sigma_{1} \sqsubseteq \sigma_{2}$ iff $\widetilde{\sigma_{1}} \preceq \widetilde{\sigma_{2}}$ and that $\sigma=\mathcal{C}_{T}(\widetilde{\sigma})$.

