



Article

 $\omega - k$ MUSIC Algorithm for Subsurface Target Localization

Antonio Cuccaro ^{1,†} , Angela Dell'Aversano ^{2,*} , Maria Antonia Maisto ^{2,†} , Rosa Scapaticci ^{3,†} ,
Adriana Brancaccio ^{2,†} and Raffaele Solimene ^{2,†}

¹ Department of Information, Modeling, Electronic Systems Engineering, University of Calabria, 87036 Rende, Italy; antonio.cuccaro@unical.it

² Department of Engineering, University of Campania, 81031 Aversa, Italy; mariaantoniamaiuto@unicampania.it (M.A.M.); adriana.brancaccio@unicampania.it (A.B.); raffaele.solimene@unicampania.it (R.S.)

³ Institute for the Electromagnetic Sensing of the Environment, CNR, 80124 Naples, Italy; scapaticci.r@irea.cnr.it

* Correspondence: angela.dellaversano@unicampania.it

† These authors contributed equally to this work.

Abstract

This paper addresses the problem of subsurface target localization from single-snapshot multimonostatic and multifrequency radar measurements. In this context, the use of subspace projection methods—known for their super-resolution capabilities—is hindered by the rank deficiency of the data correlation matrix and the lack of a Vandermonde structure, especially in near-field configurations and layered media. To overcome this issue, we propose a novel pre-processing strategy that transforms the measured data into the $\omega - k$ domain, thereby restoring the structural conditions required for subspace-based detection. The resulting algorithm, referred to as $\omega - k$ MUSIC, enables the application of subspace projection techniques in scenarios where traditional smoothing procedures are not viable. Numerical experiments in a 2-D scalar configuration demonstrate the effectiveness of the proposed method in terms of resolution and robustness under various noise conditions. A Monte Carlo simulation study is also included to provide a quantitative assessment of localization accuracy. Comparisons with conventional migration imaging highlight the superior performance of the proposed approach.

Keywords: subsurface localization; near-field; MUSIC algorithm; radar imaging; spatial smoothing



Academic Editor: Stefano Tebaldini

Received: 13 June 2025

Revised: 1 August 2025

Accepted: 7 August 2025

Published: 15 August 2025

Citation: Cuccaro, A.; Dell'Aversano, A.; Maisto, M.A.; Scapaticci, R.; Brancaccio, A.; Solimene, R. $\omega - k$ MUSIC Algorithm for Subsurface Target Localization. *Remote Sens.* **2025**, *17*, 2838. <https://doi.org/10.3390/rs17162838>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Subsurface imaging is a hot topic of research [1,2], with many practical applications ranging from security [3] and archaeological investigations to civil engineering [4–6]. Ground-penetrating radar (GPR) has played a major role in this area, and a variety of data processing strategies have been proposed, based both on back-propagation principles and on inverse scattering methods [7], to provide reliable and accurate reconstructions of the subsurface scene. These various classes of methods come with distinct advantages and limitations.

Migration-based techniques are computationally efficient and require minimal modeling assumptions, but their resolution is inherently limited by the aperture and frequency content [8,9]. Inverse scattering methods can achieve more accurate reconstructions, especially when an accurate physical model is exploited, but are typically computationally much more expensive and sensitive to modeling errors since nonlinear inversion is required [10]. Subspace-based methods [11], such as MUSIC and ESPRIT, are attractive for

their super-resolution capabilities and relatively low computational cost. Indeed, provided that the signal-to-noise ratio is sufficiently high, subspace projection methods can overcome the Rayleigh limit. This has motivated the development of subspace-based algorithms for GPR applications [12–14]. These methods also alleviate the storage burden typical of model-based inverse techniques, as they are gridless and do not require the storage of the forward model [15]. However, their applicability is traditionally limited to far-field or multi-snapshot configurations, where the rank of the correlation matrix can be effectively managed through spatial smoothing. In single-snapshot or near-field scenarios, by contrast, the data matrix is rank-deficient, preventing the direct application of subspace projection methods.

In this paper, we focus on the case of single-snapshot multimonostatic, multifrequency GPR imaging, where data are gathered over a contactless small aperture working in the near-field of the region of interest. This case is of interest in all those applications where measurement time is crucial and when far-field conditions are difficult to achieve. Moreover, the small aperture enables accounting for possible inhomogeneities in the moving direction, opening the way to fast imaging approaches in a *marching on* fashion [16].

In such cases, standard subspace projection methods fail due to the rank deficiency of the data correlation matrix (because of the single-snapshot acquisition) and the lack of structural conditions required for spatial smoothing. Indeed, for near-field configurations and/or the layered background medium, smoothing strategies cannot be exploited since the scattering model matrix lacks the required Vandermonde structure.

In order to extend the possibility to exploit subspace methods to the case at hand, we introduce a suitable pre-processing step that transforms the data into the $\omega - k$ domain, which allows one to restore the required Vandermonde structure, thus enabling the application of a subspace projection method. While we specifically adopt the MUSIC algorithm in this study, leading to the $\omega - k$ MUSIC method, the proposed pre-processing framework is general and may enable the use of other subspace-based techniques. The algorithm is tested on numerically simulated data under different noise levels in a 2-D scalar setting. The results show a significant improvement in resolution and robustness compared to classical back-propagation methods.

The paper is organized as follows. In Section 2, the mathematical formulation of subsurface imaging is recalled. Section 3 is the core of this work; it explains how the proposed approach works, and in Section 4, numerical results are presented and discussed. Finally, conclusions follow in Section 5.

2. Problem Description and Migration Imaging

In this section, we briefly describe the subsurface imaging problem under investigation, along with the necessary mathematical framework and related notation. In addition, we provide a few details on the back-propagation algorithm used as a benchmark.

We consider the 2-D scalar scattering scenario illustrated in Figure 1, assuming invariance along the y -axis. The background is a two-layer medium separated by a flat interface at $z = 0$. The upper half-space ($z > 0$) represents free space, characterized by permittivity ϵ_0 and wavenumber k_0 . The lower half-space ($z < 0$), representing soil, has permittivity $\epsilon_l = \epsilon_r \epsilon_0$ and wavenumber $k_l = \sqrt{\epsilon_r} k_0$. Both media are non-magnetic, with permeability μ_0 .

The targets are buried in the lower half-space, within a rectangular region $SD = [-X_D, X_D] \times [z_{min}, z_{max}]$. The choice of this scattering domain can reflect rough a priori knowledge about target location, or more generally define a partitioning of the subsurface region for the imaging process. In the latter case, multiple imaging steps are required, one per subdomain. Nevertheless, focusing on smaller regions can reduce data re-

quirements and storage burden [16], and is helpful when the lower half-space has spatially varying properties.

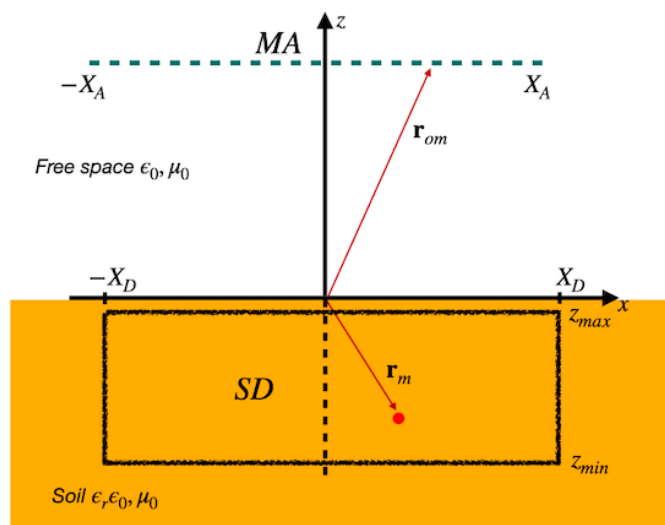


Figure 1. Measurement geometry.

The scattered field is collected along a measurement line in the upper half-space under a monostatic configuration, by scanning the aperture $MA = [-X_A, X_A]$ at height $z_o > 0$ along the x -axis.

The antenna is modeled as a y -polarized filamentary current operating over the frequency band $\Omega = [\omega_{\min}, \omega_{\max}]$, corresponding to a wavenumber band $\Omega_K = [k_{l\min}, k_{l\max}]$ defined with respect to the lower half-space.

Under the Born approximation [17], the y -component of the scattered field is given by

$$E_s(x_o, \omega) = j\omega\mu_0 k_l^2 \int_{SD} G^2(x_o, \mathbf{r}, \omega) \chi(\mathbf{r}) d\mathbf{r} \quad (1)$$

where $(x_o, \omega) \in MA \times \Omega$, $\chi(\cdot)$ is the contrast function describing dielectric deviations from the background, and $G(\cdot)$ is the Green's function of the two-layer medium. The squared Green's function arises from the monostatic configuration. Dependence on z_o is omitted, as the measurement height is constant.

Subsurface radar imaging amounts to solving the ill-posed linear inverse problem associated with (1). Numerous methods have been developed for this task, broadly classified into inverse filtering and migration approaches [18].

In this work, a migration method is employed as the benchmark. Reconstructions are obtained using the adjoint of the scattering operator in (1) [19], a widely adopted approach in radar imaging due to its implicit regularization and reduced computational cost, especially when combined with FFT acceleration [19–22].

The reconstruction is computed as

$$\mathcal{G}^+ : E_s \mapsto \tilde{\chi} = -j\omega\mu_0 k_l^2 \int_{MA \times \Omega} G^{2*}(x_o, \mathbf{r}, \omega) E_s(x_o, \omega) dx_o d\omega \quad (2)$$

where $\mathcal{G}^+$ is the adjoint of the scattering operator introduced in (1), G^{2*} denotes the complex conjugate of the squared Green function, and $\tilde{\chi}$ stands for the estimated contrast profile.

The scattered field is sampled following the approach described in [23]. Although this may result in oversampling relative to non-uniform sampling strategies (e.g., [24]), it is adopted here to allow for implementation via the Fast Fourier Transform (FFT).

Let N_{x_0} and N_ω be the number of spatial and frequency samples, respectively. The discrete reconstruction is given by

$$\mathbf{G}^\dagger : \mathbf{e}_s \mapsto \tilde{\chi} = \mathbf{G}^\dagger \mathbf{e}_s \quad (3)$$

where $\mathbf{e}_s = [\mathbf{e}_{s,1}^T \mathbf{e}_{s,2}^T \dots \mathbf{e}_{s,N_\omega}^T]^T \in \mathbb{C}^{N_{x_0} N_\omega \times 1}$ collects the scattered data at all frequencies, and each $\mathbf{e}_{s,i} \in \mathbb{C}^{N_{x_0} \times 1}$ corresponds to a single frequency.

The discretized scattering operator is

$$\mathbf{G} = [\mathbf{G}_1^T \mathbf{G}_2^T \dots \mathbf{G}_{N_\omega}^T]^T \in \mathbb{C}^{N_{x_0} N_\omega \times N_x N_z}$$

where each $\mathbf{G}_i \in \mathbb{C}^{N_{x_0} \times N_x N_z}$ is defined by $\{\mathbf{G}_i\}^{mn} = j\omega_i \mu_0 k_{li}^2 G^2(x_{0,m}, \mathbf{r}_n, \omega_i) \Delta_x \Delta_z$, with $x_{0,m}$ being the m -th measurement point on MA, and $\mathbf{r}_n = (x_n, z_n)$ denoting the n -th point in the imaging grid. Finally, $\tilde{\chi} \in \mathbb{C}^{N_x N_z \times 1}$ is the vectorized reconstruction of the contrast function.

Although migration techniques are widely used in subsurface imaging due to their simplicity and computational efficiency [18], their resolution is fundamentally constrained by the Rayleigh limit, particularly in scenarios involving limited aperture, near-field propagation, and single-snapshot acquisitions [8]. This limitation constitutes an important bottleneck in GPR imaging and directly motivates the development of the methodology proposed in this work and detailed below.

3. $\omega - k$ MUSIC Algorithm

Subspace projection methods, such as MUSIC, rely on the ability to separate the signal and noise subspaces via eigenstructure analysis of the data correlation matrix. In standard multi-snapshot configurations, this matrix can be estimated directly by averaging over multiple realizations. However, in the case of single-snapshot data, as considered in this work, the correlation matrix is rank-deficient, and its rank must be recovered through suitable smoothing procedures.

A key requirement for applying such procedures is that the data matrix exhibits a Vandermonde structure. In the present scattering configuration, however, this requirement is not fulfilled. Although the data are sampled uniformly along the measurement aperture, the underlying propagator does not exhibit a linear exponential dependence on the observation variable, as it is for the case of far-field homogeneous background medium configuration. As a result, the sampled measurement vector does not satisfy the constant ratio condition between consecutive entries, which is the defining feature of a Vandermonde vector.

This limitation precludes the use of standard smoothing strategies. To address this, we introduce a suitable pre-processing step that enables the application of smoothing-based rank recovery techniques by restoring the necessary structural conditions by basically working in the $\omega - k$ domain.

3.1. $\omega - k$ Domain Representation

In order to pass to the $\omega - k$ domain, the Green function can be conveniently expressed in the spectral wavenumber domain. For the considered two-layered background medium and $z_0 > 0, z < 0$, this yields

$$\mathbf{G}(\mathbf{r}_0, \mathbf{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(k_x, z_0, z, \omega) e^{-jk_x(x_0-x)} dk_x \quad (4)$$

where $\hat{G}$ is the Fourier transform of the Green function (with respect to x , k_x being the corresponding spectral variable) given by

$$\hat{G}(k_x, z_o, z, \omega) = \frac{1}{2} \tau(k_x, \omega) e^{jk_{lz}z} e^{-jk_{0z}z_o}$$

$$\tau(k_x, \omega) = \frac{2}{k_{lz} + k_{0z}}$$

taking into account refraction through the separation interface and being related to the transmission coefficient, and

$$k_{lz} = \begin{cases} \sqrt{k_l^2 - k_x^2} & \text{if } k_x \leq k_l \\ -j\sqrt{k_x^2 - k_l^2} & \text{if } k_x > k_l \end{cases}$$

and

$$k_{0z} = \begin{cases} \sqrt{k_0^2 - k_x^2} & \text{if } k_x \leq k_0 \\ -j\sqrt{k_x^2 - k_0^2} & \text{if } k_x > k_0 \end{cases}$$

Basically, (4) expresses the Green's function as a superposition of plane-waves that propagate across the separation interface. It is worth pointing out that the integral representation in (4) clarifies why the propagator in this configuration does not induce a Vandermonde structure. Although the integrand contains exponential terms in x_o , the Green function is a weighted superposition of such terms over k_x , modulated by the spectral kernel $\hat{G}(k_x, z, z_o, \omega)$, which depends on the transmission coefficient and the depth. As a result, the sampled measurement vector $\{G(x_{om}, \mathbf{r}, \omega)\}_m$ is not composed of samples of a single exponential, and the ratio between consecutive entries is not constant. This violates the key requirement for the Vandermonde structure, namely, linear exponential dependence on the observation coordinate.

The kernel of the scattering operator provided in (1) actually involves the square of the Green function. Therefore, what we really need is the spectral representation of the squared Green function. A common way to avoid dealing with the related convolution in the spectral domain is to employ the so-called exploding source model [1]. Via this model, the two-way propagation is replaced by a one-way path through a medium whose propagation speed is halved. This is equivalent to considering 2ω in place of ω in the Green function expression. Accordingly,

$$G^2(\mathbf{r}_o, \mathbf{r}, \omega) \simeq G(\mathbf{r}_o, \mathbf{r}, 2\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(k_x, z_o, z, 2\omega) e^{-jk_x(x_o - x)} dk_x \quad (5)$$

It must be remarked that G^2 is not even an actual Green function; it is not the solution of the Helmholtz equation describing the propagation when a point-source is considered. Therefore, it is necessary to understand in which way $G(\mathbf{r}_o, \mathbf{r}, 2\omega)$ can be considered an approximation of $G^2(\mathbf{r}_o, \mathbf{r}, \omega)$. It is easy to recognize that the approximation in (5) exactly reproduces the propagation kernel (i.e., the exponential factors) while it fails to accurately approximate the amplitude term. For imaging purposes, this is not considered a serious issue when a migration scheme is adopted, as the phase term is the primary factor of interest. For example, while implementing the so-called $\omega - k$ algorithm [25,26], the amplitude term is often neglected. In this case, inserting the plane-wave representation of $G(\mathbf{r}_o, \mathbf{r}, 2\omega)$ in (1) and by Fourier transforming the scattered field with respect to x_o yields

$$\hat{E}_s(k_x, \omega) = \alpha(k_x, \omega) e^{-jk_{0z}z_o} \hat{\chi}(k_x, k_{lz}) \quad (6)$$

where $\hat{\chi}(\cdot)$ is the Fourier transform of the contrast function and the amplitude term is

$$\alpha(k_x, \omega) = \frac{-\omega\mu_0 k_l^2}{4\pi} \tau(k_x, 2\omega) \quad (7)$$

Note that since the Green function is considered as a function of 2ω , in (6) and in the equations below, $k_{0z} = \sqrt{(2k_0)^2 - k_x^2}$ instead of $k_{0z} = \sqrt{k_0^2 - k_x^2}$ and $k_{lz} = \sqrt{(2k_l)^2 - k_x^2}$. Then, since

$$\hat{\chi}(k_x, k_{lz}) = \hat{E}_s(k_x, \omega) / \alpha(k_x, \omega) e^{jk_{0z}z_0} \quad (8)$$

reconstruction can be obtained by running an FFT, of course once proper filtering is introduced and the Stolt interpolation achieved [25].

Here, however, we need a more precise amplitude term compensation, which asks for a more accurate approximation of the Fourier transform of the squared Green function, and eventually of the scattered field in the $\omega - k$ domain. This can be achieved by resorting to stationary phase arguments obtaining the following:

$$\hat{E}_s(k_x, \omega) = \tilde{\alpha}(k_x, \omega) e^{-jk_{0z}z_0} \hat{\chi}(u, k_{lz}) \quad (9)$$

As can be seen, (9) is formally identical to (6), confirming that the exploding source model works well in approximating the exponential factor. However, the amplitude factor is different. For the sake of readability, the detailed derivation of (9) is reported in Appendix A.

3.2. $\omega - k$ Subspace Based Detection

We have now all the ingredients to proceed to detail the proposed detection algorithm. In particular, we focus on the case where the targets consist of an ensemble of M point-like scatterers. Accordingly, the contrast function is expressed as

$$\chi(\mathbf{r}) = \sum_{m=1}^M \chi_m \delta(\mathbf{r} - \mathbf{r}_m) \quad (10)$$

where $\mathbf{r}_m = (x_m, z_m)$, $m = 1, 2, \dots, M$ are the targets' positions all located within SD .

In particular, the detection scheme can be divided into two phases. The first one addresses a proper pre-processing, which makes the data suitable for a subspace projection method scheme; the second phase consists of the actual detection scheme.

3.3. Pre-Processing Stage

According to the assumed contrast function in (10), the scattered field (9) is particularized as

$$\hat{E}_s(k_x, \omega) = \tilde{\alpha}(k_x, \omega) e^{-j\sqrt{(2k_0)^2 - k_x^2}z_0} \sum_{m=1}^M \chi_m e^{jk_x x_m} e^{j\sqrt{(2k_l)^2 - k_x^2}z_m} \quad (11)$$

Considering now that actual measurements can be collected over a discrete grid of the $MA \times \Omega$ domain, and by a slight change of notation, we obtain the discrete version of (11) as

$$\hat{E}_s(k_{xn}, \omega_h) = \tilde{\alpha}(k_{xn}, \omega_h) e^{-jk_{znh}^0 z_0} \sum_{m=1}^M \chi_m e^{jk_{xn} x_m} e^{jk_{znh}^l z_m} \quad (12)$$

where k_{xn} , $n = 1, 2, \dots, N'_{x_0}$ is the conjugated (of x_0) sampled spectral variable, $k_{znh}^0 = \sqrt{(2k_{0h})^2 - k_{xn}^2}$ and $k_{znh}^l = \sqrt{(2k_{lh})^2 - k_{xn}^2}$, with k_{0h} and k_{lh} being the wavenumbers in the

upper and lower half-paces, respectively, in correspondence of the h -frequency. Note that $N'_{x_0} > N_{x_0}$ if zero-padding is exploited while computing the FFT to obtain $\hat{E}_s$.

Now, let us introduce the $N'_{x_0} \times N_\omega$ matrices $\hat{\mathbf{E}}_s$, $\mathbf{K}_z^0$, $\mathbf{K}_z^1$, and $\tilde{\mathbf{A}}$, whose entries are $\hat{E}_s(k_{xn}, \omega_h)$, k_{zn}^0 , k_{zn}^1 , and $\tilde{a}(k_{xn}, \omega_h)$, respectively. Moreover, let $\mathbf{k}_x = (k_{x1}, k_{x2}, \dots, k_{xN'_{x_0}})$.

Then, (12) can be compactly and conveniently rewritten as

$$\hat{\mathbf{E}}_s = e^{-j\mathbf{K}_z^0 z_0} \odot \tilde{\mathbf{A}} \odot \sum_{m=1}^M \chi_m \text{diag}\{e^{j\mathbf{k}_x x_m}\} e^{j\mathbf{K}_z^1 z_m} \quad (13)$$

where $\odot$ denotes the Hadamard entrywise product and $\text{diag}\{e^{j\mathbf{k}_x x_m}\}$ is a $N'_{x_0} \times N'_{x_0}$ diagonal matrix.

Let us introduce the $N'_{x_0} \times N_\omega$ matrix $\mathcal{I}$, whose n, h entry is $[1 + \text{sign}(4k_{0h}^2 - k_{xn}^2)]/2$. Basically, $\mathcal{I}$ is a discrete indicator function (in fact it is a matrix) whose entries are 1 for the grid points (k_{xn}, ω_h) , for which k_{zn}^0 is real and zero elsewhere. Such a matrix is then exploited to build

$$\bar{\mathbf{E}}_s = \mathcal{I} \odot e^{j\mathbf{K}_z^0 z_0} \oslash \tilde{\mathbf{A}} \odot \hat{\mathbf{E}}_s = \mathcal{I} \odot \sum_{m=1}^M \chi_m \text{diag}\{e^{j\mathbf{k}_x x_m}\} e^{j\mathbf{K}_z^1 z_m} \quad (14)$$

where $\oslash$ denotes the entrywise division.

Basically, (14) is the equivalent of (13), in which data are demodulated along the range as well as filtered in order to remove the evanescent contributions (thanks to the structure of $\mathcal{I}$). Note this last step acts as a regularization procedure and stabilize (with respect to the noise) the reconstruction process.

$\bar{\mathbf{E}}_s$ does not exhibit the necessary structure to run a subspace projection method. This is due to the product by $\mathcal{I}$ and because the spectral points (k_{xn}, k_{zn}^1) lay over circles of radius $2k_{lh}$ and hence form a non-uniform grid in the $k_x - k_z^1$ domain. Indeed, while k_{xn} are already uniformly arranged, $k_{zn}^1 = \sqrt{(2k_{lh})^2 - k_{xn}^2}$ is not. To restore the Vandermonde structure, it is needed that $\bar{\mathbf{E}}_s$ be given over a uniform 2D grid (k_{xm}, k_{zh}^1) , with k_{zh}^1 being independent on k_{xn} . To solve this issue, we employ the classical Stolt interpolation. More specifically, for each k_{xn} (i.e., along the columns) the data matrix in (14) is linearly interpolated and resampled within the interval $[0, 2k_{max}^1]$ over $N'_\omega > N_\omega$ uniform points k_{zh}^1 . In particular, in the following we will consider $N'_\omega = 2N_\omega$, with N_ω set as described in [23].

Accordingly, after the Stolt interpolation (14) becomes

$$\bar{\mathbf{E}}_s = \mathcal{I} \odot \sum_{m=1}^M \chi_m e^{j\mathbf{k}_x^T x_m} e^{j\mathbf{k}_z^1 z_m} \quad (15)$$

with $\mathbf{k}_x = (k_{x1}, k_{x2}, \dots, k_{xN'_{x_0}})$ and $\mathbf{k}_z^1 = (k_{z1}^1, k_{z2}^1, \dots, k_{zN'_\omega}^1)$. It must be remarked that, for the sake of simplicity of notation, in (15) we continued to denote the data and the indicator matrix as in (14). However, in (15), $\bar{\mathbf{E}}_s \in \mathbb{C}^{N'_{x_0} \times N'_\omega}$, and it is given over a uniform two-dimensional grid. Also, $\mathcal{I}$ is still the indicator matrix but defined over the uniform grid arising after the Stolt interpolation.

The resulting matrix in (15) does not have the required structure yet, since some entries are zero due to $\mathcal{I}$. In order to circumvent this drawback, a rectangular sub-matrix, of a size, say, $N_{x_0}^{sub} \times N_\omega^{sub}$, is extracted from $\bar{\mathbf{E}}_s$, with care taken to select the row and the column indexes so that they run within the support of $\mathcal{I}$. This leads to

$$\bar{\mathbf{E}}_{s\text{sub}} = \sum_{m=1}^M \chi_m e^{j\mathbf{k}_x^T x_m} e^{j\mathbf{k}_z^1 z_m} \quad (16)$$

which finally has the requested structure and can be arranged as shown below as a two-dimensional Vandermonde matrix [27], with $\mathbf{k}_x \in R^{1 \times N_{x_0}^{sub}}$ and $\mathbf{k}_z^l \in R^{1 \times N_{\omega}^{sub}}$ being the corresponding sub-arrays along k_x and k_z , respectively.

3.4. Imaging Stage

After completing the pre-processing described above, it is now possible to run a subspace projection method to achieve target detection and localization. A number of subspace projection methods have been developed in the literature [28]. Here, among the others, we exploit the well known MUSIC algorithm (Algorithm 1).

The problem at hand involves a 2-D localization. This requires one to consider the vectorized form of the data matrix to obtain the column vector $\mathbf{x} = \text{vect}[\tilde{\mathbf{E}}_{\text{Sub}}] \in C^{N_{x_0}^{sub} N_{\omega}^{sub} \times 1}$, from which the correlation matrix $\mathbf{R} = \mathbf{x}\mathbf{x}^H$ is obtained, with $(\cdot)^H$ denoting transposition and conjugation. The eigenspectrum of the correlation matrix is then exploited to build the so-called pseudospectrum indicator that peaks at the target locations.

A key question to be addressed concerns the rank of the correlation matrix. Indeed, the very definition of the correlation matrix entails that for single snapshot data it is rank-deficient with a rank equal to 1. Therefore, usually multi snapshots are employed to obtain a maximum likelihood estimation of the correlation matrix. However, in the single snapshot case, the rank of the correlation matrix can be recovered by exploiting a spatial smoothing procedure [29] or by simply rearranging the data to form a Toeplitz or Hankel matrix. This basically is because $\mathbf{x}$, after a simple recasting of (16), enjoys the representation

$$\mathbf{x} = \mathbf{A} \cdot \chi \quad (17)$$

where $\mathbf{A} = [e^{jk_x x_1} \otimes e^{jk_z^l z_1} \ e^{jk_x x_2} \otimes e^{jk_z^l z_2} \ \dots \ e^{jk_x x_M} \otimes e^{jk_z^l z_M}]$, with $\otimes$ denoting the Kronecker product and χ the column vector $[\chi_1, \chi_2, \dots, \chi_M]^T$. The point is that $\mathbf{A}$ exhibits a (2-D) Vandermonde structure, which is a mandatory whatever rank recovering procedure one may want to employ. Note that, for the considered configuration, this is solely possible thanks to the pre-processing stage described above.

Accordingly, we can run a rank recovering procedure. Here, we employ the forward-backward spatial smoothing (FBSS) algorithm [29]. To this end, fixed $\bar{N}_{x_0}^{sub}, \bar{N}_{\omega}^{sub} > p_1, p_2 \geq M + 1$, then $Q = (\bar{N}_{x_0}^{sub} + 1 - p_1)(\bar{N}_{\omega}^{sub} + 1 - p_2)$ sub-matrices of size $p_1 \times p_2$ can be extracted from $\tilde{\mathbf{E}}_{\text{Sub}}$ by sliding a $p_1 \times p_2$ window. Each such sub-matrix is stacked to form the vector $\mathbf{x}_q \in C^{p_1 p_2 \times 1}$, and the corresponding correlation matrix, $\mathbf{R}_q = \mathbf{x}_q \mathbf{x}_q^H$, is obtained. Then the correlation matrix is estimated as [29]

$$\tilde{\mathbf{R}} = \frac{1}{Q} \sum_{q=1}^Q (\mathbf{R}_q + \mathbf{J} \mathbf{R}_q^H) \quad (18)$$

where $\mathbf{J}$ is the exchange matrix of appropriate dimension. The choice of p_1 and p_2 is a delicate question since it impacts on the number of targets that can be detected, the resolution, and the degree of decorrelation [30]. According to [29], we select $p_1 = \frac{2}{3} \bar{N}_{x_0}^{sub}$ and $p_2 = \frac{2}{3} \bar{N}_{\omega}^{sub}$, respectively. Now, the eigenspectrum of $\tilde{\mathbf{R}}$ is computed to obtain the so-called *signal subspace* (i.e., the range of $\tilde{\mathbf{R}}$) and the *noise subspace* (i.e., the kernel of $\tilde{\mathbf{R}}$). Finally, the localization of the targets is achieved by taking the peaks of the MUSIC pseudospectrum

$$\phi(\mathbf{r}) = \frac{1}{\|\mathcal{P}_{\mathcal{N}} \psi(\mathbf{r})\|^2} \quad (19)$$

where $\psi(\mathbf{r})$ is the the *steering matrix* corresponding to the trial position $\mathbf{r} = (x, y) \in SD$ and $\mathcal{P}_{\mathcal{N}}$ denotes the projector operator onto the noise subspace.

The indicator peaks when $\mathbf{r}$ coincides with one of the actual target positions. The dimension of the signal subspace is determined by the AIC (Akaike Information Crite-

rion) [31] applied to only the first $\min(p_1, p_2) - 1$ singular values, which is the maximum number of detectable scatterers according to the FBSS (forward-backward spatial smoothing) theory.

The overall procedure is summarized in pseudocode (Algorithm 1).

Algorithm 1 $\omega - k$ MUSIC Algorithm

Require: Scattered field data $\mathbf{E}_S(x_o, \omega)$

Ensure: Target locations via $\omega - k$ MUSIC algorithm

1: **Fourier transform:** Apply Fourier transform with respect to x_o to obtain:

$$\hat{\mathbf{E}}_S = e^{-j\mathbf{K}_z^0 z_o} \odot \tilde{\mathbf{A}} \odot \sum_{m=1}^M \chi_m e^{j\mathbf{k}_x x_m} e^{j\mathbf{K}_z^1 z_m}$$

2: **Regularization:** Apply regularization and stabilization to handle noise:

$$\bar{\mathbf{E}}_S = \mathcal{I} \odot e^{j\mathbf{K}_z^0 z_o} \odot \tilde{\mathbf{A}} \odot \hat{\mathbf{E}}_S = \mathcal{I} \odot \sum_{m=1}^M \chi_m \text{diag}\{e^{j\mathbf{k}_x x_m}\} e^{j\mathbf{K}_z^1 z_m}$$

3: **Stolt interpolation:** Resample $\bar{\mathbf{E}}_S$ onto a uniform (k_x, k_z) grid

4: **Sub-matrix extraction:** Extract rectangular sub-matrix $\bar{\mathbf{E}}_{Sub}$

5: **Vectorization:** Stack sub-matrix to form vector $\mathbf{x}_q \in \mathbb{C}^{p_1 p_2 \times 1}$

6: **Correlation matrix:** Compute $\mathbf{R}_q = \mathbf{x}_q \mathbf{x}_q^H$

7: **Rank recovering:** Apply forward-backward spatial smoothing:

$$\tilde{\mathbf{R}} = \frac{1}{Q} \sum_{q=1}^Q (\mathbf{R}_q + \mathbf{J} \mathbf{R}_q^H)$$

8: **MUSIC pseudospectrum:** Compute:

$$\phi(\mathbf{r}) = \frac{1}{\|\mathcal{P}_{\mathcal{N}} \psi(\mathbf{r})\|^2}$$

9: **Target localization:** Identify peaks in $\phi(\mathbf{r})$

4. Numerical Validation

In this section, a numerical analysis is presented to evaluate the performance of the proposed procedure. Specifically, the two-point resolution in both range and cross-range directions is assessed for varying signal-to-noise ratio (SNR) levels and different values of the dielectric permittivity of the lower half-space (ϵ_r) [32]. The results are compared with those obtained using a migration algorithm that approximates the inverse of the scattering operator with its adjoint, as described in (3). In addition, the analysis is extended to scenarios involving distributed (extended) targets.

The first numerical example presented in this section aims to illustrate the role of the amplitude factor $\alpha(k_x, \omega)$ in (6) and to assess the effect of its approximation $\tilde{\alpha}(k_x, \omega)$ introduced in (9).

To this end, let us consider a frequency bandwidth of [0.5–1.1] GHz. The scattered field is collected at $z_0 = 8\lambda$, where λ denotes the free-space wavelength at the center frequency, with respect to the separation interface located at $z = 0$. Measurements are taken along the observation line $MA = [-5\lambda, 5\lambda]$, using uniform spatial sampling as suggested in [23].

The investigation domain is defined as $SD = [-4\lambda, 4\lambda] \times [-4\lambda, 0]$ and is fully embedded in the lower half-space, which is characterized by a relative dielectric permittivity of $\epsilon_r = 9$.

For cross-range resolution assessment, we consider two point scatterers placed at the same depth $z = -1\lambda$, separated along the x -axis by 0.4λ . Conversely, to evaluate range resolution, two targets are located at $x = 0$ and separated along the z -axis by 0.1λ .

Figure 2 presents the corresponding two-point resolution results. The first column shows the reconstructions obtained using the amplitude factor $\alpha(k_x, \omega)$, while the second column refers to its approximation $\tilde{\alpha}(k_x, \omega)$.

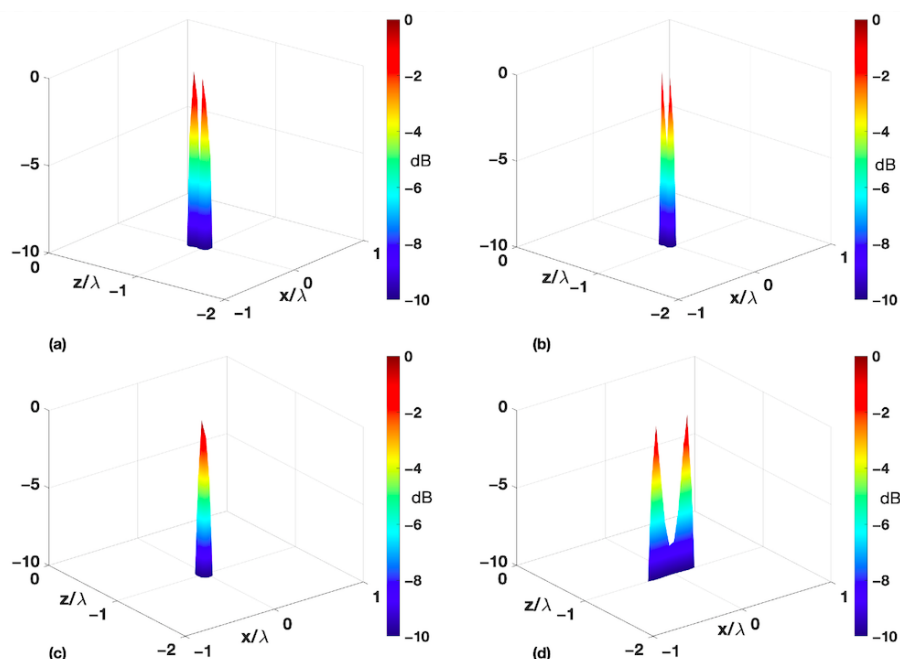


Figure 2. Comparison between reconstructions obtained using $\alpha(k_x, \omega)$ (first-column panels) and $\tilde{\alpha}(k_x, \omega)$ (second-column panels). Cross-range resolution is shown in panels (a,b), while range resolution is illustrated in panels (c,d).

As observed, in the cross-range case, both versions allow for successful discrimination between the two scatterers, although the pseudospectrum obtained with $\tilde{\alpha}(k_x, \omega)$ appears sharper. In contrast, in the range case, the use of $\alpha(k_x, \omega)$ fails to resolve the two targets. This analysis confirms that $\tilde{\alpha}(k_x, \omega)$ is the most appropriate choice for accurate resolution in both dimensions.

4.1. Two-Point Resolution as a Function of SNR

As mentioned above, this analysis is carried out by comparing the obtained results with those of the migration algorithm while varying the SNR. The measurement configuration remains the same as previously described, with the lower half-space having $\epsilon_r = 9$. In particular, the results in Figure 3 illustrate, in the first row, the reconstructions for two scatterers positioned 0.5λ apart in the cross-range and at the same depth of 1λ under the interface. The actual scatterer positions are indicated with blue circles in figure. In panel (a), the migration algorithm, even in absence of noise, fails to distinguish between the two targets, while the proposed method successfully resolves them in the noiseless case and with SNR equal to 10, 5 dB, as shown in panels (b), (c), and (d), respectively. A similar analysis is conducted in the second row of Figure 3, where the two scatterers are placed 0.1λ in the range at $x = 0$. In this case, although the migration algorithm appears to detect the presence of the targets, their positions are not correctly localized (see panel (e)). Conversely, with the proposed method, the scatterers are perfectly localized, even at SNR values of 10 and 5 dB, as shown in panels (g) and (h), respectively.

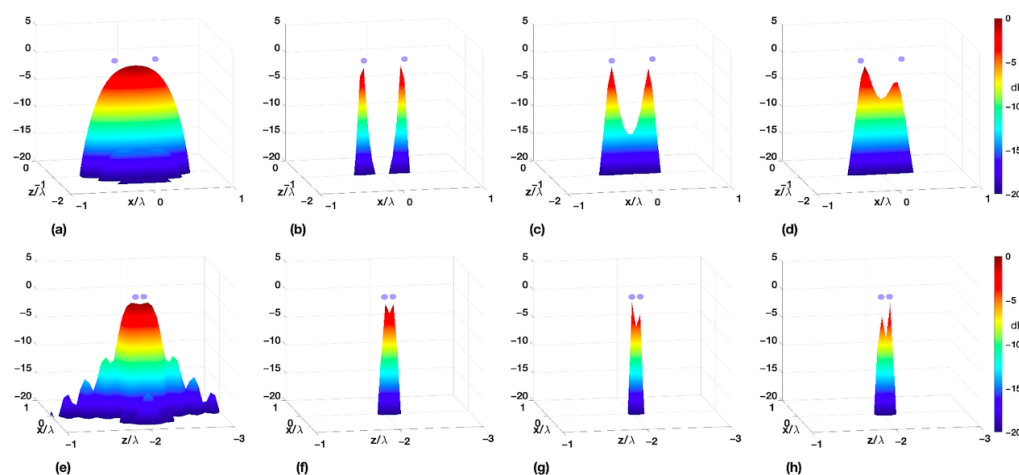


Figure 3. Two-point resolution with $\epsilon_r = 9$ in the lower half-space. Reconstructions in cross-range are shown in the first row, and in range in the second row. Panels (a,e): migration results in the absence of noise. Panels (b,f): $\omega - k$ MUSIC in the noiseless case. Panels (c,g): $\omega - k$ MUSIC with SNR = 10 dB. Panels (d,h): $\omega - k$ MUSIC with SNR = 5 dB.

Next, let us increase the value of ϵ_r in lower half-space to 15: the results are shown in Figure 4. The results concerning the cross-range resolution reported in the first row of Figure 4 are comparable to those of Figure 3. However, the range analysis in the second row of Figure 4 reveals an improvement in the performance of the $\omega - k$ MUSIC (panels (f), (g), (h)). Specifically, the enhanced resolution allows for target distinction even at lower SNR levels. Conversely, the performance of the migration method deteriorates, as it fails to distinguish the two targets (panel (e)).

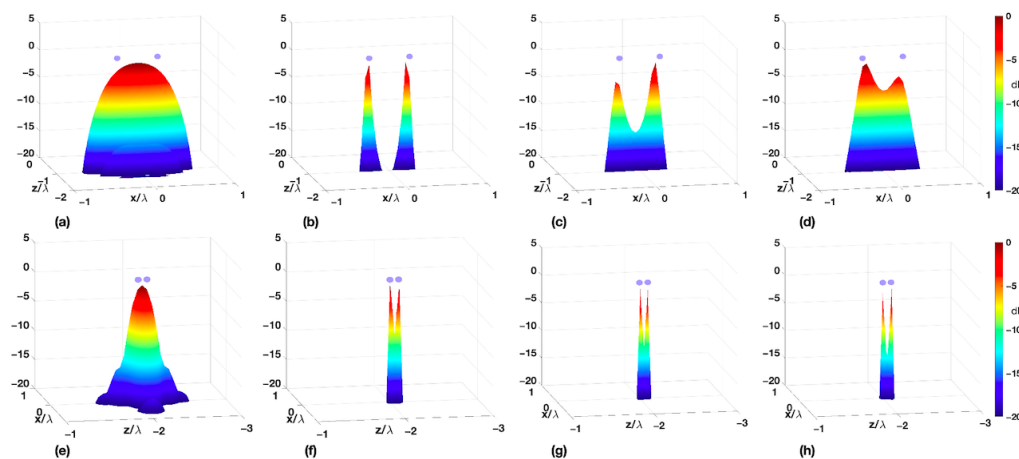


Figure 4. Two-point resolution with $\epsilon_r = 15$ in lower half-space. Reconstructions in cross-range are shown in the first row, and in range in the second row. Panels (a,e): migration results in absence of noise. Panels (b,f): $\omega - k$ MUSIC in the noiseless case. Panels (c,g): $\omega - k$ MUSIC with SNR = 10 dB. Panels (d,h): $\omega - k$ MUSIC with SNR = 5 dB.

In the last case examined in this section, the observation line (MA) remains unchanged, but a smaller number of observation points are selected compared to the previous case. Specifically, k_{lmax} is replaced by $(k_{lmax} - k_{lmin})/2$ in the sampling step suggested in [23]. Also in this case, the performance of the $\omega - k$ MUSIC method, in terms of resolution, proves to be better compared to migration, as shown in Figure 5.

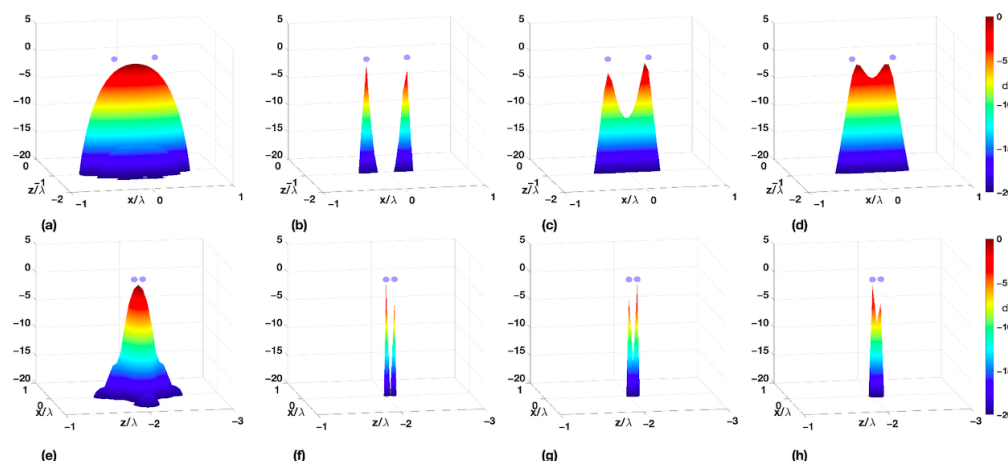


Figure 5. Two-point resolution with a smaller number of observation points compared to the previous case. Reconstructions in cross range in the first row and in range in the second row: migration in absence of noise (panels (a,e)), $\omega - k$ MUSIC in noiseless case (panels (b,f)) and with SNR equal to 10 dB (panels (c,g)) and 5 dB (panels (d,h)).

These results clearly show that, as expected, the MUSIC algorithm provides improved resolution capabilities. This, in turn, indirectly confirms the effectiveness of the proposed pre-processing step in recovering the necessary rank conditions for subspace projection.

4.2. Validation with Respect to More Complex Scenarios

In this section, the effectiveness of the proposed algorithm is demonstrated in a more complex scenario involving multiple targets or extended objects within the investigation domain. Specifically, using the initial measurement configuration as in the previous section, six targets are placed in the lower half-space with $\epsilon_r = 15$. The results are presented in Figure 6. To facilitate the interpretation of the results, the images are visualized in 2D. The proposed algorithm successfully localizes all six scatterers in the noiseless case and at SNR values of 10 and 5 dB (see panels (b), (c), and (d)). Conversely, the migration algorithm fails to distinguish between all the targets, even in the absence of noise (panel (a)).

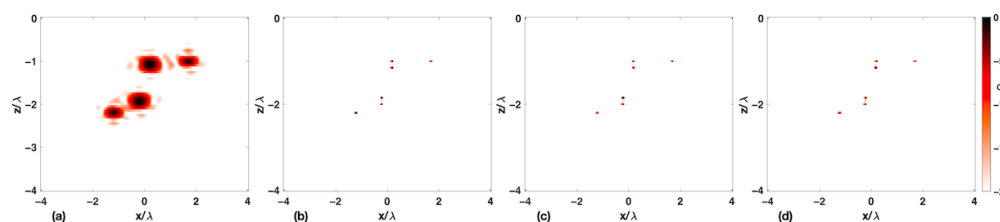


Figure 6. Populated scenario with six targets located within the investigation domain. Reconstructions obtained using migration in the absence of noise are shown in panel (a). Panels (b–d) display the results obtained with $\omega - k$ MUSIC in the noiseless case, and for SNR levels of 10 dB and 5 dB, respectively.

Finally, the case of extended targets is considered. Specifically, in the first case examined, a flat plate is placed inclined with respect to the measurement line, as indicated by the blue line in Figure 7. In the second case, two parallel flat plates are considered.

The first column shows the results obtained with the migration algorithm, which yields very similar reconstructions for both scenarios. In contrast, the proposed method enables the identification of both the position and orientation of the targets across different SNR levels: panels (b), (c), and (d) refer to the first scenario, while panels (f), (g), and (h) correspond to the second one. For clarity, only a portion of the full investigation domain is displayed in the figure.

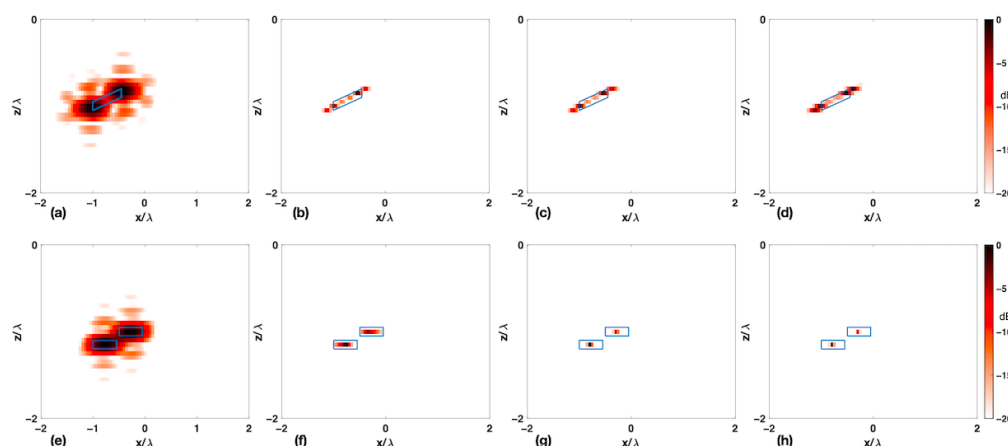


Figure 7. Reconstruction of one oblique flat plate, in first row, and two parallel flat plates, in second row (indicated by blue lines): migration in absence of noise (panels (a,e)), $\omega - k$ MUSIC in noiseless case (panels (b,f)), and with SNR equal to 10 dB (panels (c,g)) and 5 dB (panels (d,h)).

Finally, to gain a clearer understanding of the achievable performance, a Monte Carlo simulation study is carried out by rerunning the cases previously analyzed in Figures 4 and 6 with multiple noise realizations. Table 1 summarizes the performance of the proposed method in terms of the mean and variance of the estimated target positions evaluated over 100 independent noise realizations at SNR levels of 10 dB and 5 dB. Overall, the observed performance trends are consistent with the qualitative results discussed earlier. In particular, the variance of the estimated target positions is found to be very low, indicating that the proposed procedure is robust against noise. Moreover, although in some cases the mean estimated position of a target shows a noticeable deviation from the true location, all targets are consistently detected and correctly distinguished in every realization.

Table 1. Mean values of the estimated target positions, μ_{x_i} and μ_{z_i} , and the corresponding variances, $\sigma_{x_i}^2$ and $\sigma_{z_i}^2$, computed over 100 noise realizations for SNR levels of 10 dB and 5 dB, considering the scenarios shown in Figures 4 and 6.

Actual Target Positions		SNR [dB]	Means	Variances
$x_1 = -0.25\lambda$ $x_2 = 0.25\lambda$	$z_{1,2} = -1\lambda$	10	$\mu_{x_1} = -0.2520\lambda, \mu_{x_2} = 0.2025\lambda$	$\sigma_{x_1}^2 = 0.0002\lambda, \sigma_{x_2}^2 = 0.0007\lambda$
		5	$\mu_{x_1} = -0.1625\lambda, \mu_{x_2} = 0.1418\lambda$	$\sigma_{x_1}^2 = 0.0079\lambda, \sigma_{x_2}^2 = 0.0039\lambda$
$x_{1,2} = 0$	$z_1 = -2\lambda$ $z_2 = -2.1\lambda$	10	$\mu_{z_1} = -2.0000\lambda, \mu_{z_2} = -2.1000\lambda$	$\sigma_{z_1}^2 = 0.0000\lambda, \sigma_{z_2}^2 = 0.0000\lambda$
		5	$\mu_{z_1} = -2.0000\lambda, \mu_{z_2} = -2.0975\lambda$	$\sigma_{z_1}^2 = 0.0001\lambda, \sigma_{z_2}^2 = 0.0000\lambda$
$x_1 = -1.2\lambda$ $x_2 = -0.2\lambda$ $x_3 = -0.2\lambda$ $x_4 = 0.2\lambda$ $x_5 = 0.2\lambda$ $x_6 = 1.7\lambda$	$z_1 = -2.2\lambda$ $z_2 = -2\lambda$ $z_3 = -1.85\lambda$ $z_4 = -1.15\lambda$ $z_5 = -1\lambda$ $z_6 = -1\lambda$	10	$\mu_{x_1} = -1.2225\lambda, \mu_{z_1} = -2.2000\lambda$	$\sigma_{x_1}^2 = 0.0006\lambda, \sigma_{z_1}^2 = 0.0000\lambda$
			$\mu_{x_2} = -0.2425\lambda, \mu_{z_2} = -2.0000\lambda$	$\sigma_{x_2}^2 = 0.0003\lambda, \sigma_{z_2}^2 = 0.0000\lambda$
			$\mu_{x_3} = -0.2200\lambda, \mu_{z_3} = -1.8500\lambda$	$\sigma_{x_3}^2 = 0.0006\lambda, \sigma_{z_3}^2 = 0.0000\lambda$
			$\mu_{x_4} = 0.1900\lambda, \mu_{z_4} = -1.1500\lambda$	$\sigma_{x_4}^2 = 0.0004\lambda, \sigma_{z_4}^2 = 0.0000\lambda$
			$\mu_{x_5} = 0.2000\lambda, \mu_{z_5} = -1.0000\lambda$	$\sigma_{x_5}^2 = 0.0000\lambda, \sigma_{z_5}^2 = 0.0000\lambda$
			$\mu_{x_6} = 1.7000\lambda, \mu_{z_6} = -1.0000\lambda$	$\sigma_{x_6}^2 = 0.0000\lambda, \sigma_{z_6}^2 = 0.0000\lambda$
		5	$\mu_{x_1} = -1.2100\lambda, \mu_{z_1} = -2.2000\lambda$	$\sigma_{x_1}^2 = 0.0004\lambda, \sigma_{z_1}^2 = 0.0000\lambda$
			$\mu_{x_2} = -0.2425\lambda, \mu_{z_2} = -2.0000\lambda$	$\sigma_{x_2}^2 = 0.0003\lambda, \sigma_{z_2}^2 = 0.0000\lambda$
			$\mu_{x_3} = -0.2275\lambda, \mu_{z_3} = -1.8500\lambda$	$\sigma_{x_3}^2 = 0.0006\lambda, \sigma_{z_3}^2 = 0.0000\lambda$
			$\mu_{x_4} = 0.1850\lambda, \mu_{z_4} = -1.1500\lambda$	$\sigma_{x_4}^2 = 0.0005\lambda, \sigma_{z_4}^2 = 0.0000\lambda$
			$\mu_{x_5} = 0.1900\lambda, \mu_{z_5} = -1.0000\lambda$	$\sigma_{x_5}^2 = 0.0004\lambda, \sigma_{z_5}^2 = 0.0000\lambda$
			$\mu_{x_6} = 1.7000\lambda, \mu_{z_6} = -1.0000\lambda$	$\sigma_{x_6}^2 = 0.0000\lambda, \sigma_{z_6}^2 = 0.0000\lambda$

5. Conclusions

This paper addressed the subsurface localization problem in the presence of single snapshot measurements in a multimonostatic/multifrequency configuration. In particular, ad hoc pre-processing of the data matrix has been introduced to make simple subspace projection methods, such as MUSIC, applicable in practice. The proposed approach consists of restoring the full rank of the data matrix by representing the kernel of the imaging problem in the spectral domain and then applying proper filtering procedures. The developed algorithm has been validated with respect to synthetic data in the 2D scalar case. The obtained results show better performance in terms of resolution and robustness against noise compared to results achieved through the standard migration algorithm, which has been considered as a benchmark herein.

Author Contributions: Conceptualization, A.C., A.D., M.A.M., R.S. (Rosa Scapaticci), A.B., and R.S. (Raffaele Solimene); methodology, A.C., A.D., M.A.M., R.S. (Rosa Scapaticci), A.B., and R.S. (Raffaele Solimene); software, A.C., A.D., M.A.M., R.S. (Rosa Scapaticci), A.B., and R.S. (Raffaele Solimene); validation, A.C., A.D., M.A.M., R.S. (Rosa Scapaticci), A.B., and R.S. (Raffaele Solimene); formal analysis, A.C., A.D., M.A.M., R.S. (Rosa Scapaticci), A.B., and R.S. (Raffaele Solimene); writing—original draft preparation, A.C., A.D., M.A.M., R.S. (Rosa Scapaticci), A.B., and R.S. (Raffaele Solimene); writing—review and editing, A.C., A.D., M.A.M., R.S. (Rosa Scapaticci), A.B., and R.S. (Raffaele Solimene); and funding acquisition, R.S. (Raffaele Solimene). All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the European Union—Next Generation EU under the Italian National Recovery and Resilience Plan (NRRP), Mission 4, Component 2, Investment 1.3, CUP B53C22003970001, partnership on “Telecommunications of the Future” (PE00000001—program “RESTART”).

Data Availability Statement: The raw data supporting the conclusions of this article will be made available by the authors on request.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

In this appendix, we provide some details concerning the $\omega - k$ representation that have been introduced in Section 3.1.

For convenience, we rewrite the scattering model reported in (1)

$$E_s(x_o, \omega) = j\omega\mu_0 k_l^2 \int_{SD} G^2(x_o, \mathbf{r}, \omega) \chi(\mathbf{r}) d\mathbf{r} \quad (\text{A1})$$

which, upon exploiting the Green function representation (4), can be rewritten as

$$E_s(x_o, \omega) = j\omega\mu_0 k_l^2 \cdot \int_{SD} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{G}(k_x, z_o, z, \omega) \hat{G}(k'_x, z_o, z, \omega) e^{-j(k_x + k'_x)x_o} e^{j(k_x + k'_x)x} dk_x dk'_x \right\} \chi(\mathbf{r}) d\mathbf{r} \quad (\text{A2})$$

Fourier transforming with respect to x_o and using the expression of $\hat{G}$ yields

$$\hat{E}_s(u, \omega) = j2\pi\omega\mu_0 k_l^2 \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\chi}(k_x + k'_x, k_{lz} + k'_{lz}) \frac{4e^{-j(k_{0z} + k'_{0z})z_o}}{(k_{0z} + k_{lz})(k'_{0z} + k'_{lz})} \delta[u - (k_x + k'_x)] dk_x dk'_x \quad (\text{A3})$$

or equivalently making explicit the involved convolution

$$\hat{E}_s(u, \omega) = j2\pi\omega\mu_0k_l^2 \int_{-\infty}^{\infty} \hat{\chi}(u, \sqrt{k_l^2 - k_x^2} + \sqrt{k_l^2 - (u - k_x)^2}) \times \frac{4e^{-j(\sqrt{k_0^2 - k_x^2} + \sqrt{k_0^2 - (u - k_x)^2})z_0}}{(\sqrt{k_0^2 - k_x^2} + \sqrt{k_l^2 - k_x^2})(\sqrt{k_0^2 - (u - k_x)^2} + \sqrt{k_l^2 - (u - k_x)^2})} dk_x \quad (A4)$$

Equation (A9) is basically an integration transformation with a rapidly oscillating kernel, that is,

$$I = \int_{-\infty}^{\infty} e^{j\phi(k_x)} f(k_x) dk_x \quad (A5)$$

with the phase term in particular given by $\phi(k_x) = -(\sqrt{k_0^2 - k_x^2} + \sqrt{k_0^2 - (u - k_x)^2})z_0$. Accordingly, it can be evaluated by invoking the stationary phase method by considering the dominant contribution coming from the oscillatory. It can be easily shown that

$$\phi'(k_x) = 0 \quad (A6)$$

where the prime denotes the derivative with respect to the argument admits only one solution (stationary point) for $k_{xs} = u/2$.

Accordingly,

$$I \simeq e^{j\phi(k_{xs})} \sqrt{\frac{2\pi}{|\phi''(k_{xs})|}} e^{j\pi/4 \text{sign}[\phi''(k_{xs})]} f(k_{xs}) \quad (A7)$$

with the second derivative evaluated at the stationary phase point and given by

$$\phi''(k_{xs}) = \frac{2z_0k_0^2}{[k_0^2 - (u/2)^2]^{3/2}} \quad (A8)$$

Accordingly, (A9) can be approximated as

$$\hat{E}_s(u, \omega) = \sqrt{\frac{2\pi}{|\phi''(k_{xs})|}} \times \frac{j32\pi\omega\mu_0k_l^2 e^{j\pi/4 \text{sign}[\phi''(k_{xs})]}}{(\sqrt{(2k_0)^2 - u^2} + \sqrt{(2k_l)^2 - u^2})^2} \times e^{-j\sqrt{(2k_0)^2 - u^2}z_0} \hat{\chi}\left(u, \sqrt{(2k_l)^2 - u^2}\right) = \tilde{a}(k_x, \omega) e^{-j\sqrt{(2k_0)^2 - u^2}z_0} \hat{\chi}\left(u, \sqrt{(2k_l)^2 - u^2}\right) \quad (A9)$$

Finally, by renaming u as k_x , (9) is obtained.

References

1. Soldovieri, F.; Solimene, R. Ground penetrating radar subsurface imaging of buried objects. In *Radar Technology*; Kouemou, G., Ed.; IntechOpen: London, UK, 2010. [\[CrossRef\]](#)
2. Ivashov, S.I.; Capineri, L.; Bechtel, T.D.; Razevig, V.V.; Inagaki, M.; Gueorguiev, N.L.; Kizilay, A. Design and applications of multi-frequency holographic subsurface radar: Review and case histories. *Remote Sens.* **2021**, *13*, 3487. [\[CrossRef\]](#)
3. Soldovieri, F.; Solimene, R.; Prisco, G. A Multiarrray tomographic approach for through-wall imaging. *IEEE Trans. Geosci. Remote Sens.* **2008**, *46*, 1192–1199.
4. Catapano, I.; Gennarelli, G.; Ludeno, G.; Soldovieri, F. Applying Ground-Penetrating radar and microwave tomography data processing in cultural heritage: State of the art and future trends. *IEEE Signal Process. Mag.* **2019**, *36*, 53–61. [\[CrossRef\]](#)
5. Conyers, L.B. *Ground-Penetrating Radar for Archaeology*, 3rd ed.; AltaMira Press: Lanham, MD, USA, 2013; p. 258.
6. Claerbout J.F. *Imaging the Earth's Interior*; Blackwell Scientific Publications: Oxford, UK, 1985.
7. Catapano, I.; Randazzo, A.; Slob, E.; Solimene, R. GPR Imaging Via Qualitative and Quantitative Approaches. In *Civil Engineering Applications of Ground Penetrating Radar*; Benedetto, A., Pajewski, L., Eds.; Springer Transactions in Civil and Environmental Engineering; Springer: Cham, Switzerland, 2015. [\[CrossRef\]](#)

8. Maisto, M.A.; Solimene, R.; Pierri, R. Transverse resolution in microwave imaging for strip objects Buried in a Half-Space Medium. *Prog. Electromagn. Res. M* **2020**, *88*, 145–157. [\[CrossRef\]](#)
9. Maisto, M.A.; Pierri, R.; Solimene, R. Near-Field Transverse Resolution in Planar Source Reconstructions. *IEEE Trans. Antennas Propag.* **2021**, *69*, 4836–4845. [\[CrossRef\]](#)
10. Fedeli, A.; Schenone, V.; Randazzo, A. Quantitative Inversion of Multiantenna Ground-Penetrating Radar Data with Modeling Error Correction Based on Long Short-Term Memory Cells. *Remote Sens.* **2024**, *16*, 2050. [\[CrossRef\]](#)
11. Schmidt, R. Multiple emitter location and signal parameter estimation. *IEEE Trans. Antennas Propag.* **1986**, *34*, 276–280. [\[CrossRef\]](#)
12. Ammari, H.E.; Iakovleva, D.; Lesselier, D. A MUSIC Algorithm for Locating Small Inclusions Buried in a Half-Space from the Scattering Amplitude at a Fixed Frequency. *Multiscale Model. Simul.* **2005**, *3*, 597–628. [\[CrossRef\]](#)
13. Roy, R.; Kailath, T. ESPRIT-estimation of Signal Parameters via Rotational Invariance Techniques. *IEEE Trans. Acoust. Speech Signal Process.* **1989**, *37*, 984–995. [\[CrossRef\]](#)
14. Yuen, N.; Friedlander, B. Performance Analysis of Higher Order ESPRIT for Localization of Near-Field Sources. *IEEE Trans. Signal Process.* **1998**, *46*, 709–719. [\[CrossRef\]](#)
15. Soldovieri, F.; Hugenschmidt, J.; Persico, R.; Leone, G. A Linear Inverse Scattering Algorithm for Realistic GPR Applications. *Near Surf. Geophys.* **2007**, *5*, 29–41. [\[CrossRef\]](#)
16. Persico, R.; Ludeno, G.; Soldovieri, F.; De Coster, A.; Lambot, S. Two-Dimensional Linear Inversion of GPR Data with a Shifting Zoom along the Observation Line. *Remote Sens.* **2017**, *9*, 980. [\[CrossRef\]](#)
17. Marks, D.L. A Family of Approximations Spanning the Born and Rytov. *Opt. Express* **2006**, *14*, 8837–8848. [\[CrossRef\]](#)
18. Solimene, R.; Catapano, I.; Gennarelli, G.; Cuccaro, A.; Dell’Aversano A.; Soldovieri, F. SAR imaging algorithms and some unconventional applications: A unified mathematical overview. *IEEE Signal Process. Mag.* **2014**, *31*, 90–98. [\[CrossRef\]](#)
19. Özdemir, C.; Demirci, Ş.; Yiğit, E.; Yilmaz, B. A Review on Migration Methods in B-Scan Ground Penetrating Radar Imaging. *Math. Probl. Eng.* **2014**, *2014*, 280738. [\[CrossRef\]](#)
20. Cassereau, D.; Fink, M. Time-reversal of ultrasonic fields. III. Theory of the closed time-reversal cavity. *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **1992**, *39*, 579–592. [\[CrossRef\]](#) [\[PubMed\]](#)
21. Gilmore, C.; Jeffrey, I.; LoVetri, J. Derivation and comparison of SAR and frequency-wavenumber migration within a common inverse scalar wave problem formulation. *IEEE Trans. Geosci. Remote Sens.* **2006**, *44*, 1454–1461. [\[CrossRef\]](#)
22. Zhuge, X.; Yarovoy, A.; Savelyev, T.G.; Ligthart, L.P. Modified kirchhoff migration for uwb mimo array-based radar imaging. *IEEE Trans. Geosci. Remote Sens.* **2010**, *48*, 2692–2703. [\[CrossRef\]](#)
23. Lopez-Sanchez, J.M.; Fortuny-Guasch, J. 3-D Radar Imaging Using Range Migration Techniques. *IEEE Trans. Antennas Propag.* **2000**, *48*, 728–77. [\[CrossRef\]](#)
24. Maisto, M.A.; Bhat, C.; Solimene, R. An Insight into the Warping Spatial Sampling Method in Subsurface Radar Imaging and Its Experimental Validation. *Remote Sens.* **2023**, *15*, 3012. [\[CrossRef\]](#)
25. Stolt, R.H. Migration by Fourier Transform. *Geophysics* **1978**, *43*, 23–48. [\[CrossRef\]](#)
26. Gazdag, J. Wave equation migration with the phase-shift method. *Geophysics* **1978**, *43*, 1342–1351. [\[CrossRef\]](#)
27. Odendaal, J.W.; Barnard, E.; Pistorius, C.W.I. Two-dimensional superresolution radar imaging using the MUSIC algorithm. *IEEE Trans. Antennas Propag.* **1994**, *42*, 1386–1391. [\[CrossRef\]](#)
28. Krim, H.; Viberg, M. Two decades of array signal processing research: The parametric approach *IEEE Signal Process. Mag.* **1996**, *13*, 67–94. [\[CrossRef\]](#)
29. Pillai, S.U.; Kwon, B.H. Forward/backward spatial smoothing techniques for coherent signal identification. *IEEE Trans. Acoust. Speech Signal Process.* **1989**, *37*, 8–15. [\[CrossRef\]](#)
30. Jianxiong, Z.; Zhiguang, S.; Yongfeng, Z. Superresolution performance of different spatial smoothing methods for short uniform linear array. *IEEE Antennas Wirel. Propag. Lett.* **2022**, *21*, 1144–1147. [\[CrossRef\]](#)
31. Akaike, H. Information theory and an extension of the maximum likelihood principle. In *Proceedings of the 2nd International Symposium on Information Theory '73*; Akademiai Kiado: Budapest, Hungary, 1973; pp. 267–281.
32. Daniels, D.J. Properties of materials. In *Ground Penetrating Radar*, 2nd ed.; Institute of Engineering and Technology: London, UK, 2004; pp. 73–130.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.