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A boundary control problem for a possibly singular phase field system with dynamic boundary conditions

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ABSTRACT

This paper deals with an optimal control problem related to a phase field system of Caginalp type with a dynamic boundary condition for the temperature. The control placed in the dynamic boundary condition acts on a part of the boundary. The analysis carried out in this paper proves the existence of an optimal control for a general class of potentials, possibly singular. The study includes potentials for which the derivatives may not exist, these being replaced by well-defined subdifferentials. Under some stronger assumptions on the structure parameters and on the potentials (namely for the regular and the logarithmic case having single-valued derivatives), the first order necessary optimality conditions are derived and expressed in terms of the boundary trace of the first adjoint variable.

1. Introduction

In this paper we are concerned with an optimal control problem for a nonlinear phase field system of a standard form (cf. the monograph [6]), but with a possibly singular double well potential, like in the logarithmic case (cf. the later (1.4)), and with a dynamic boundary condition for the temperature, in which also the time derivative of the boundary temperature plays a role and where the control variable appears in the external term (see (1.6)). Let us now introduce and discuss the problem in precise terms.

A rather general version of the phase field system of Caginalp type [7] reads as follows

$$\partial_t \vartheta - \Delta\vartheta + \lambda(\varphi)\partial_t\varphi = 0 \qquad \text{in } Q := (0,T)\times\Omega \tag{1.1}$$

$$\partial_t\varphi - \sigma\Delta\varphi + \mathcal{W}'(\varphi) = \vartheta\lambda(\varphi) \quad \text{in } Q \tag{1.2}$$

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where Ω is the domain where the evolution takes place, T is some final time, ϑ denotes the relative temperature around some critical value that is taken to be 0 without loss of generality, and φ is the order parameter. Moreover, λ is a given real function, whose meaning is related to the latent heat, and σ is a positive constant. Finally, $\mathcal{W}'$ represents the derivative of a double-well potential $\mathcal{W}$. Typical examples for $\mathcal{W}$ are the regular potential

$$\mathcal{W}_{reg}(r) = r^2(r-1)^2, \quad r \in \mathbb{R} \tag{1.3}$$

with two absolute minima located in 0 and 1, and the logarithmic potential

$$\mathcal{W}_{log}(r) = ((1+r)\ln(1+r) + (1-r)\ln(1-r)) - a\,r^2, \quad r \in (-1,1) \tag{1.4}$$

where the coefficient $a > 0$ is large enough in order to kill convexity. The potential (1.3) is a shifted version of the usual classical potential given by $r \mapsto \frac{1}{4}(r^2-1)^2$ and precisely satisfies our general assumptions given below, while (1.4) has a derivative which behaves singularly in the neighborhoods of -1 and 1. Generally speaking, the potential $\mathcal{W}$ could be just the sum $\mathcal{W} = \widehat{\beta} + \widehat{\pi}$, where $\widehat{\beta}$ is a convex function that is allowed to take the value $+\infty$, and $\widehat{\pi}$ is a smooth perturbation (not necessarily concave). In such a case, $\widehat{\beta}$ is supposed to be proper and lower semicontinuous so that its subdifferential $\beta := \partial\widehat{\beta}$ is well defined and can replace the derivative which might not exist. Of course, $\mathcal{W}'$ has to be read as $\beta + \pi$, where $\pi := \widehat{\pi}'$, and equation (1.2) becomes a differential inclusion.

Moreover, initial conditions like $\vartheta(0) = \vartheta_0$ and $\varphi(0) = \varphi_0$ and suitable boundary conditions must complement the above equations. As far as the latter are concerned, we take the homogeneous Neumann condition for φ, that is,

$$\partial_n\varphi = 0 \quad \text{on } \Sigma := (0,T) \times \Gamma \tag{1.5}$$

where Γ is the boundary of Ω and ∂_n is the (say, outward) normal derivative. The position (1.5) is mostly the rule in the literature for the order parameter φ. Concerning the temperature ϑ, in order to address the boundary control problem, we choose the following dynamic boundary condition

$$\partial_n\vartheta + \tau\partial_t\vartheta_\Gamma + \alpha(\vartheta_\Gamma - mu) = 0 \quad \text{on } \Sigma \tag{1.6}$$

where $\vartheta_\Gamma := \vartheta_{|\Gamma}$ is the trace of ϑ on the boundary and u is the control, which is supposed to vary in some control box $\mathcal{U}_{ad}$. Moreover, in (1.6), τ is a positive time relaxation parameter, α is a positive constant and m is a nonnegative function defined on Γ. Notice that, in fact, the control u can act just on the subset of Γ where m is positive. Thus, the state system takes the following form

$$\partial_t\vartheta - \Delta\vartheta + \lambda(\varphi)\partial_t\varphi = 0 \qquad \text{in } Q \tag{1.7}$$

$$\partial_t\varphi - \sigma\Delta\varphi + \beta(\varphi) + \pi(\varphi) \ni \vartheta\lambda(\varphi) \qquad \text{in } Q \tag{1.8}$$

$$\partial_n\vartheta + \tau\partial_t\vartheta_\Gamma + \alpha(\vartheta_\Gamma - mu) = 0 \quad \text{and} \quad \partial_n\varphi = 0 \qquad \text{on } \Sigma \tag{1.9}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0 \qquad \text{on } \Omega. \tag{1.10}$$

The cost functional we consider depends on two nonnegative constants κ_1 and κ_2 and two functions ϑ_Q and φ_Ω on Q and Ω, respectively. We want to minimize

$$\mathcal{J}(u) := \frac{\kappa_1}{2}\int_Q |\vartheta - \vartheta_Q|^2 + \frac{\kappa_2}{2}\int_\Omega |\varphi(T) - \varphi_\Omega|^2 \tag{1.11}$$

where u, ϑ and φ vary under the constraint of the state system and $u \in \mathcal{U}_{ad}$, where the control box $\mathcal{U}_{ad}$ is defined by

$$\mathcal{U}_{ad} := \left\{ u \in L^2(\Sigma) : \; u_{\min} \le u \le u_{\max} \text{ a.e. on } \Sigma \right\} \tag{1.12}$$

for some given bounded functions $u_{\min}$ and $u_{\max}$. The analysis carried out in this paper shows the existence of an optimal control for a general class of potentials $\mathcal{W} = \widehat{\beta} + \widehat{\pi}$: indeed, for this purpose $\widehat{\beta}$ is just assumed to be a general convex and lower semicontinuous function with minimum 0 attained at 0, that is, $\widehat{\beta}(0) = 0$, which is physically reasonable. On the other hand, the derivation of the first order necessary optimality conditions can be made only in case of regular (like (1.3)) and singular (like (1.4)) potentials. Linearized and adjoint problems are under our investigation and, subsequently, the optimality conditions can be expressed in terms of the adjoint variables (see the Theorem 2.9 stated in the next section).

Let us mention here some related work. As far as we know, the contributions on optimal control problems for phase field models are quite a few and often restricted to the case of regular potentials, or dealing with approximations of the actual systems when the first order optimality conditions are discussed. In this respect, we point out the papers [11,20,21,29] concerned with distributed control problems; we also refer to [2,4,12,13,16,23,26,28] for different types of phase field models and other kinds of control problems. The main features of our paper are the study of a boundary control problem and the consideration of a dynamic boundary condition, in a very simple form: indeed, (1.6) is an affine condition involving the temperature and its time derivative on the boundary, with an external term carrying out the action of the control. About dynamic boundary conditions, also of nonlinear type and possibly involving the Laplace–Beltrami operator, let us quote the articles [3,8–10,14,15,17–19].

The paper is organized as follows. In the next section, we list our assumptions, state the problem in a precise form and present our results. The well-posedness of the state system, the regularity results and the existence of an optimal control will be shown in Sections 3 and 4, respectively, while the rest of the paper is devoted to the derivation of first order necessary conditions for optimality, which are computed in the case of potentials that generalize (1.3)–(1.4) (some cases of more singular potentials being the subject of a future project of ours). The final result will be proved in Section 6 and it is prepared in Section 5 with the study of the control-to-state mapping. Finally, Appendix A is devoted to the rigorous proof of an estimate that is derived just formally in Section 3.

2. Statement of the problem and results

In this section, we describe the problem under study and present our results. As in the Introduction, Ω is the body where the evolution takes place. We assume $\Omega \subset \mathbb{R}^3$ to be open, bounded, connected, and smooth, and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_n still stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, we set for convenience

$$Q_t := (0,t) \times \Omega \quad \text{and} \quad \Sigma_t := (0,t) \times \Gamma \quad \text{for every } t \in (0,T] \tag{2.1}$$

$$Q := Q_T \,, \quad \text{and} \quad \Sigma := \Sigma_T \,. \tag{2.2}$$

Now, we specify the assumptions on the structure of our system. We assume that

$$\sigma, \tau \in (0,+\infty), \quad m \in L^\infty(\Omega) \quad \text{and} \quad m \ge 0 \quad \text{a.e. in } \Omega \tag{2.3}$$

$$\widehat{\beta} : \mathbb{R} \to [0,+\infty] \quad \text{is convex, proper and l.s.c.} \quad \text{with} \quad \widehat{\beta}(0) = 0 \tag{2.4}$$

$$\widehat{\pi}, \widehat{\lambda} : \mathbb{R} \to \mathbb{R} \quad \text{are } C^1 \text{ functions} \quad \text{and} \quad \widehat{\pi}', \widehat{\lambda}' \quad \text{are Lipschitz continuous} \tag{2.5}$$

We set for convenience

$$\beta := \partial\widehat{\beta}, \quad \pi := \widehat{\pi}\,' \quad \text{and} \quad \lambda := \widehat{\lambda}\,' \tag{2.6}$$

and denote by $D(\beta)$ and $D(\widehat{\beta})$ the effective domains of β and $\widehat{\beta}$, respectively. Moreover, $\beta^\circ(r)$ is the element of $\beta(r)$ having minimum modulus for every $r \in D(\beta)$ (see, e.g., [5, p. 28]). It is well known that β is a maximal monotone operator from $\mathbb{R}$ to $\mathbb{R}$ (see, e.g., [5, Ex. 2.3.4, p. 25]). Next, in order to simplify notations, we set

$$\begin{gathered} V := H^1(\Omega),\ H := L^2(\Omega),\ W := \{v \in H^2(\Omega) : \partial_n v = 0\} \\ \text{as well as} \ \ H_\Gamma := L^2(\Gamma) \quad \text{and} \quad V_\Gamma := H^1(\Gamma) \end{gathered} \tag{2.7}$$

and endow these spaces with their natural norms. The symbol $\|\cdot\|_X$ stands for the norm in the generic Banach space X, while $\|\cdot\|_p$ is the usual norm in anyone of the L^p spaces on Ω, Γ, Q and Σ, for $1 \le p \le \infty$, provided that no confusion can arise. Furthermore, it is understood that H is embedded in V^*, the dual space of V, in the standard way, i.e., in order that $\langle u, v\rangle = \int_\Omega uv$ for every $u \in H$ and $v \in V$, where $\langle\cdot\,,\cdot\rangle$ denotes the duality product between V^* and V. Finally, for $v \in L^2(Q)$ the symbol $1 * v$ is the usual time convolution, i.e.,

$$(1 * v)(t) := \int_0^t v(s)\,ds \quad \text{for } t \in [0, T]. \tag{2.8}$$

At this point, we describe the state system. Given ϑ_0 and φ_0 such that

$$\vartheta_0 \in V, \quad \varphi_0 \in W \quad \text{and} \quad \beta^\circ(\varphi_0) \in H \tag{2.9}$$

we look for a quadruplet $(\vartheta, \vartheta_\Gamma, \varphi, \xi)$ satisfying

$$\vartheta \in H^1(0,T;H) \cap L^\infty(0,T;V) \tag{2.10}$$

$$\vartheta_\Gamma \in H^1(0,T;H_\Gamma) \quad \text{and} \quad \vartheta_\Gamma(t) = \vartheta(t)_{|\Gamma} \quad \text{for a.a. } t \in (0,T) \tag{2.11}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \tag{2.12}$$

$$\xi \in L^\infty(0,T;H) \tag{2.13}$$

and solving the problem

$$\begin{gathered} \int_\Omega \partial_t\vartheta\, v + \int_\Omega \nabla\vartheta\cdot\nabla v + \int_\Omega \lambda(\varphi)\partial_t\varphi\, v + \tau\int_\Gamma \partial_t\vartheta_\Gamma\, v_\Gamma + \alpha\int_\Gamma (\vartheta_\Gamma - mu)\, v_\Gamma = 0 \\ \text{for every } (v, v_\Gamma) \in V \times V_\Gamma \text{ such that } v_\Gamma = v_{|\Gamma} \text{ and a.e. in } (0,T) \end{gathered} \tag{2.14}$$

$$\partial_t\varphi - \sigma\Delta\varphi + \xi + \pi(\varphi) = \vartheta\lambda(\varphi) \quad \text{and} \quad \xi \in \beta(\varphi) \quad \text{a.e. in } Q \tag{2.15}$$

$$\partial_n\varphi = 0 \quad \text{a.e. on } \Sigma \tag{2.16}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega. \tag{2.17}$$

Remark 2.1. The variational equation (2.14) is the weak formulation of equation (1.7) and of the dynamic boundary condition contained in (1.9). Let us notice that we can deduce both (1.7) and the first condition in (1.9) from (2.14). Indeed, by writing (2.14) with an arbitrary $v \in H_0^1(\Omega)$ and $v_\Gamma = 0$, we derive (1.7) in the sense of distributions on Q. From (2.12) we infer that φ is bounded since $W \subset L^\infty(\Omega)$. As λ is continuous, the same holds for $\lambda(\varphi)$, so that $\lambda(\varphi)\partial_t\varphi \in L^2(0,T;H)$. By comparison of terms in (1.7), it turns out that $\Delta\vartheta$ belongs to $L^2(0,T;H)$ and (1.7) holds a.e. in Q. It also follows that the normal derivative $\partial_n\vartheta$ makes sense in a proper Sobolev space of negative order on the boundary and that the integration-by-parts formula holds in a generalized sense. By applying it, one immediately derives the dynamic boundary condition contained in (1.9) in a generalized sense: comparing terms in it yields $\partial_n\vartheta \in L^2(\Sigma)$ and consequently the boundary condition holds a.e. on Σ. Finally, we remark that (2.10) and the trace theorem imply $\vartheta_\Gamma \in L^\infty(0,T;H^{1/2}(\Gamma))$.

Our first result ensures well-posedness with the prescribed regularity, stability and continuous dependence on the control variable in suitable topologies.

Theorem 2.2. *Assume (2.3)–(2.5) and (2.9). Then, for every $u \in L^2(\Sigma)$, problem (2.14)–(2.17) has a unique solution $(\vartheta, \vartheta_\Gamma, \varphi, \xi)$ satisfying (2.10)–(2.13), and the estimate*

$$\begin{aligned}&\|\vartheta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\vartheta_\Gamma\|_{H^1(0,T;H_\Gamma)}\\&\quad + \|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\xi\|_{L^\infty(0,T;H)} \le C_1\end{aligned} \tag{2.18}$$

holds true for some constant C_1 that depends only on Ω, T, the structure (2.3)–(2.5) of the system, the norms of the initial data associated to (2.9) and an upper bound for $\|u\|_2$. Moreover, if $u_i \in L^2(\Sigma)$, $i = 1,2$, are given and $(\vartheta_i, \vartheta_{i,\Gamma}, \varphi_i, \xi_i)$ are the corresponding solutions, then the estimate

$$\begin{aligned}&\|\vartheta_1 - \vartheta_2\|_{L^\infty(0,T;H)} + \|(1*\vartheta_1) - (1*\vartheta_2)\|_{L^\infty(0,T;V)}\\&\quad + \|\vartheta_{1,\Gamma} - \vartheta_{2,\Gamma}\|_{L^2(\Sigma)} + \|\varphi_1 - \varphi_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\\&\le C'\,\|u_1 - u_2\|_{L^2(\Sigma)}\end{aligned} \tag{2.19}$$

holds true where C' depends only on Ω, T and on the structure (2.3)–(2.5) of the system.

Remark 2.3. Since W is compactly embedded in $C^0(\overline{\Omega})$, the space of continuous functions on $\overline{\Omega}$, the regularity (2.12) implies $\varphi \in C^0(\overline{Q}) := C^0([0,T];C^0(\overline{\Omega}))$ and the estimate $\|\varphi\|_\infty \le c\|\varphi\|_{L^\infty(0,T;W)}$, where c depends only on Ω. Therefore, we also have

$$\varphi \in C^0(\overline{Q}) \quad \text{and} \quad \|\varphi\|_\infty \le C_2 \tag{2.20}$$

where C_2 is a multiple of the constant C_1 of (2.18).

Some further regularity of the solution is stated in the next result of ours.

Theorem 2.4. *Under assumptions (2.3)–(2.5) and (2.9), the following properties hold true.*

i) If in addition $\vartheta_0 \in L^\infty(\Omega)$ and $u \in L^\infty(\Sigma)$, we also have

$$\vartheta \in L^\infty(Q) \quad \textit{and} \quad \|\vartheta\|_\infty \le C_3 \tag{2.21}$$

where C_3 is a constant with the same dependencies as C_1 and depending on $\|\vartheta_0\|_\infty$ and on an upper bound for $\|u\|_\infty$, in addition.

ii) By also assuming $\beta^\circ(\varphi_0) \in L^\infty(\Omega)$, we have that $\xi \in L^\infty(Q)$ and

$$\|\xi\|_{L^\infty(Q)} \le C_4 \tag{2.22}$$

with a constant C_4 that depends on the norm $\|\beta^\circ(\varphi_0)\|_\infty$ as well.

The well-posedness result for problem (2.14)–(2.17) given by Theorem 2.2 allows us to introduce the control-to-state mapping $\mathcal{S}$ and to address the corresponding control problem. We define

$$\mathcal{X} := L^\infty(\Sigma) \quad \text{and} \quad \mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3 \quad \text{where} \tag{2.23}$$

$$\mathcal{Y}_1 := \{v \in L^2(Q) : \ 1 * v \in L^\infty(0,T;V)\}, \quad \mathcal{Y}_2 := L^2(\Sigma) \tag{2.24}$$

$$\text{and} \quad \mathcal{Y}_3 := C^0([0,T];H) \cap L^2(0,T;V) \tag{2.25}$$

$$\mathcal{S} : \mathcal{X} \to \mathcal{Y}, \quad u \mapsto \mathcal{S}(u) =: (\vartheta, \vartheta_\Gamma, \varphi) \quad \text{where}$$

$$(\vartheta, \vartheta_\Gamma, \varphi, \xi) \text{ is the unique solution to (2.10)–(2.17) corresponding to } u. \tag{2.26}$$

Next, in order to introduce the control box and the cost functional, we assume that

$$u_{\min}, u_{\max} \in L^\infty(\Sigma) \quad \text{satisfy} \quad u_{\min} \le u_{\max} \quad \text{a.e. on } \Sigma \tag{2.27}$$

$$\kappa_1\,,\, \kappa_2 \in [0,+\infty), \quad \vartheta_Q \in L^2(Q) \quad \text{and} \quad \varphi_\Omega \in H^1(\Omega) \tag{2.28}$$

and define $\mathcal{U}_{ad}$ and $\mathcal{J}$ according to the Introduction. Namely, we set

$$\mathcal{U}_{ad} := \{u \in \mathcal{X} : \ u_{\min} \le u \le u_{\max} \text{ a.e. on } \Sigma\} \tag{2.29}$$

$$\mathcal{J} := \mathcal{J}_0 \circ \mathcal{S} : \mathcal{X} \to \mathbb{R} \quad \text{where} \quad \mathcal{J}_0 : \mathcal{Y} \to \mathbb{R} \quad \text{is defined by} \tag{2.30}$$

$$\mathcal{J}_0(\vartheta, \vartheta_\Gamma, \varphi) := \frac{\kappa_1}{2} \int_Q |\vartheta - \vartheta_Q|^2 + \frac{\kappa_2}{2} \int_\Omega |\varphi(T) - \varphi_\Omega|^2. \tag{2.31}$$

Here is our first result on the control problem.

Theorem 2.5. *Assume (2.3)–(2.5) and (2.9), and let $\mathcal{U}_{ad}$ and $\mathcal{J}$ be defined in (2.23)–(2.31). Then, there exists $u^* \in \mathcal{U}_{ad}$ such that*

$$\mathcal{J}(u^*) \le \mathcal{J}(u) \quad \textit{for every } u \in \mathcal{U}_{ad}. \tag{2.32}$$

From now on, it is understood that the assumptions (2.3)–(2.5) and (2.9) on the structure and on the initial data are satisfied and that the map $\mathcal{S}$, the cost functionals $\mathcal{J}_0$ and $\mathcal{J}$ and the control box $\mathcal{U}_{ad}$ are defined by (2.23)–(2.31). Thus, we do not remind anything of that in the statements given in the sequel.

Our next aim is to formulate first order necessary optimality conditions. As $\mathcal{U}_{ad}$ is convex, the desired necessary condition for optimality is

$$\langle D\mathcal{J}(u^*), u - u^* \rangle \ge 0 \quad \text{for every } u \in \mathcal{U}_{ad} \tag{2.33}$$

provided that the derivative $D\mathcal{J}(u^*)$ exists in the dual space $\mathcal{X}^*$ at least in the Gâteaux sense. Then, the natural approach consists in proving that $\mathcal{S}$ is Fréchet differentiable at u^* and applying the chain rule to $\mathcal{J} = \mathcal{J}_0 \circ \mathcal{S}$. We can properly tackle this project under some further assumptions that are satisfied for each of the potentials (1.3)–(1.4). We also have to require something more on λ. Namely, we also suppose that

$$D(\beta) \text{ is an open interval and } \beta \text{ is a single-valued on } D(\beta) \tag{2.34}$$

$$\beta, \pi \text{ and } \lambda \text{ are } C^2 \text{ functions.} \tag{2.35}$$

In particular, $\beta^\circ = \beta$. Furthermore, the inclusion in (2.15) reduces to $\xi = \beta(\varphi)$, and it is no longer necessary to split the nonlinear contribution into the equation in the form $\xi + \pi(\varphi)$. Hence, we set for brevity

$$\gamma := \beta + \pi \tag{2.36}$$

and observe that γ is a C^2 function on $D(\beta)$.

As assumptions (2.34)–(2.35) force $\beta(r)$ to tend to $\pm\infty$ as r tends to a finite end-point of $D(\beta)$, if any, we see that combining the further requirement (2.34)–(2.35) with the boundedness of φ and ξ given by Theorems 2.2 and 2.4 immediately yields

Corollary 2.6. *Under all the assumptions of Theorem 2.4, suppose that (2.34)–(2.35) hold, in addition. Then, the component φ of the solution $(\vartheta, \vartheta_\Gamma, \varphi, \xi)$ also satisfies*

$$\varphi_\bullet \le \varphi \le \varphi^\bullet \quad \textit{in } \overline{Q} \tag{2.37}$$

for some constants $\varphi_\bullet, \varphi^\bullet \in D(\beta)$ that depend only on Ω, T, the structure (2.3)–(2.5) and (2.34)–(2.35) of the system, the norms of the initial data associated to (2.9), the norms $\|\vartheta_0\|_\infty$ and $\|\beta(\varphi_0)\|_\infty$, and an upper bound for $\|u\|_\infty$.

As we shall see in Section 5, the computation of the Fréchet derivative of $\mathcal{S}$ leads to the linearized problem that we describe at once and that can be stated starting from a generic element $\overline{u} \in \mathcal{X}$. Let $\overline{u} \in \mathcal{X}$ and $h \in \mathcal{X}$ be given. We set $(\overline{\vartheta}, \overline{\vartheta}_\Gamma, \overline{\varphi}) := \mathcal{S}(\overline{u})$. Then the linearized problem consists in finding $(\Theta, \Theta_\Gamma, \Phi)$ satisfying

$$\Theta \in H^1(0,T;H) \cap L^\infty(0,T;V) \tag{2.38}$$

$$\Theta_\Gamma \in H^1(0,T;H_\Gamma) \quad \text{and} \quad \Theta_\Gamma(t) = \Theta(t)_{|_\Gamma} \quad \text{for a.a. } t \in (0,T) \tag{2.39}$$

$$\Phi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{2.40}$$

and solving the following problem

$$\begin{aligned}
&\int_\Omega \partial_t\Theta\, v + \int_\Omega \nabla\Theta \cdot \nabla v + \int_\Omega \lambda(\overline{\varphi})\partial_t\Phi\, v + \int_\Omega \lambda'(\overline{\varphi})\partial_t\overline{\varphi}\,\Phi\, v \\
&\quad + \tau \int_\Gamma \partial_t\Theta_\Gamma\, v_\Gamma + \alpha \int_\Gamma \Theta_\Gamma\, v_\Gamma = \alpha \int_\Gamma mh\, v_\Gamma \\
&\quad \text{for every } (v, v_\Gamma) \in V \times V_\Gamma \text{ such that } v_\Gamma = v_{|_\Gamma} \text{ and a.e. in } (0,T)
\end{aligned} \tag{2.41}$$

$$\partial_t\Phi - \sigma\Delta\Phi + \gamma'(\overline{\varphi})\,\Phi = \overline{\vartheta}\lambda'(\overline{\varphi})\,\Phi + \lambda(\overline{\varphi})\,\Theta \quad \text{a.e. in } Q \tag{2.42}$$

$$\partial_n\Phi = 0 \quad \text{a.e. on } \Sigma \tag{2.43}$$

$$\Theta(0) = 0 \quad \text{and} \quad \Phi(0) = 0 \quad \text{a.e. in } \Omega. \tag{2.44}$$

Proposition 2.7. *Let $\overline{u} \in \mathcal{X}$ and $(\overline{\vartheta}, \overline{\vartheta}_\Gamma, \overline{\varphi}) = \mathcal{S}(\overline{u})$. Then, for every $h \in \mathcal{X}$, there exists a unique triplet $(\Theta, \Theta_\Gamma, \Phi)$ satisfying (2.38)–(2.40) and solving the linearized problem (2.41)–(2.44). Moreover, the inequality*

$$\|(\Theta, \Theta_\Gamma, \Phi)\|_{\mathcal{Y}} \le C_5 \|h\|_{\mathcal{X}} \tag{2.45}$$

holds true with a constant C_5 that depend only on Ω, T, the structure (2.3)–(2.5) and (2.34)–(2.35) of the system, the norms of the initial data associated to (2.9), and the norms $\|\overline{u}\|_\infty$, $\|\vartheta_0\|_\infty$ and $\|\beta(\varphi_0)\|_\infty$. In particular, the linear map $\mathcal{D} : h \mapsto (\Theta, \Theta_\Gamma, \Phi)$ is continuous from $\mathcal{X}$ to $\mathcal{Y}$.

Namely, we shall prove in Section 5 that the map $\mathcal{D} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ introduced in the last statement exactly provides the Fréchet derivative $D\mathcal{S}(\bar{u})$ of $\mathcal{S}$ at $\bar{u}$. Once this is done, we may use the chain rule with $\bar{u} := u^*$ to prove that the necessary condition (2.33) for optimality takes the form

$$\kappa_1 \int_Q (\vartheta^* - \vartheta_Q)\Theta + \kappa_2 \int_\Omega \big(\varphi^*(T) - \varphi_\Omega\big)\Phi(T) \ge 0 \quad \text{for any } u \in \mathcal{U}_{ad} \tag{2.46}$$

where $(\vartheta^*, \vartheta^*_\Gamma, \varphi^*) = \mathcal{S}(u^*)$ and, for any given $u \in \mathcal{U}_{ad}$, the triplet $(\Theta, \Theta_\Gamma, \Phi)$ is the solution to the linearized problem corresponding to $h = u - u^*$.

The final step then consists in eliminating the pair (Θ, Φ) from (2.46). This will be done by introducing a triplet (p, p_Γ, q) that fulfills the regularity requirements

$$p\,, q \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad q \in L^2(0,T;W) \tag{2.47}$$

$$p_\Gamma \in H^1(0,T;H_\Gamma) \quad \text{and} \quad p_\Gamma(t) = p(t)_{|\Gamma} \quad \text{for a.a. } t \in (0,T) \tag{2.48}$$

and solves the following adjoint system:

$$- \int_\Omega \partial_t p\, v + \int_\Omega \nabla p \cdot \nabla v - \int_\Omega \lambda(\varphi^*) q v - \tau \int_\Gamma \partial_t p_\Gamma\, v_\Gamma + \alpha \int_\Gamma p_\Gamma v_\Gamma$$
$$= \kappa_1 \int_\Omega (\vartheta^* - \vartheta_Q) v$$
$$\text{for every } v \in V,\ v_\Gamma := v_{|\Gamma}, \text{ and a.e. in } (0,T) \tag{2.49}$$

$$- \int_\Omega \partial_t q\, v + \sigma \int_\Omega \nabla q \cdot \nabla v + \int_\Omega \big(\gamma'(\varphi^*) - \vartheta^* \lambda'(\varphi^*)\big) q v = \int_\Omega \lambda(\varphi^*) \partial_t p\, v$$
$$\text{for every } v \in V \text{ and a.e. in } (0,T) \tag{2.50}$$

$$p(T) = 0 \quad \text{and} \quad q(T) = \kappa_2\big(\varphi^*(T) - \varphi_\Omega\big) \quad \text{a.e. in } \Omega. \tag{2.51}$$

Clearly, (2.49)–(2.50) are the variational formulation of a boundary value problem. Namely, p and q solve two backward parabolic equations complemented by a dynamic boundary condition for p and the homogeneous Neumann boundary condition for q. However, it is more convenient to keep the problem in that form.

Theorem 2.8. *Let u^* and $(\vartheta^*, \vartheta^*_\Gamma, \varphi^*) = \mathcal{S}(u^*)$ be an optimal control and the corresponding state. Then the adjoint problem (2.49)–(2.51) has a unique solution (p, p_Γ, q) satisfying the regularity conditions (2.47)–(2.48).*

We recall that, if K is a closed interval and $y_0 \in K$, the normal cone to K at y_0 is the set of $z \in \mathbb{R}$ such that $z(y - y_0) \le 0$ for every $y \in K$. Here is our last result.

Theorem 2.9. *Let u^* be an optimal control and $(\vartheta^*, \vartheta^*_\Gamma, \varphi^*) = \mathcal{S}(u^*)$ denote the associate state. Moreover, let (p, p_Γ, q) be the unique solution to the adjoint problem (2.49)–(2.51) given by Theorem 2.8. Then, for a.a. $(t,x) \in \Sigma$, we have*

$$m(x) p_\Gamma(t,x)\big(y - u^*(t,x)\big) \ge 0 \quad \textit{for every } y \in [u_{\min}(t,x), u_{\max}(t,x)] \tag{2.52}$$

that is, $-m(x)p_\Gamma(t,x)$ *belongs to the normal cone to* $[u_{\min}(t,x), u_{\max}(t,x)]$ *at* $u^*(t,x)$*. In particular, we have*

$$u^* = u_{\max}, \quad u^* = u_{\min} \quad \textit{and} \quad u^* \in [u_{\min}, u_{\max}]$$

a.e. in the subsets of Σ *where* $mp_\Gamma < 0$*,* $mp_\Gamma > 0$*,* $mp_\Gamma = 0$*, respectively.*

In performing our a priori estimates in the remainder of the paper, we often use the Hölder inequality (with the standard notation p' for the conjugate exponent of p), its consequences and the elementary Young inequalities

$$ab \le \omega\, a^{1/\omega} + (1-\omega)\, b^{1/(1-\omega)} \quad \text{and} \quad ab \le \delta a^2 + \frac{1}{4\delta}\, b^2$$
$$\text{for every } a, b \ge 0,\ \omega \in (0,1) \text{ and } \delta > 0 \tag{2.53}$$

as well as the continuous (in fact compact) embedding $V \subset L^4(\Omega)$. Moreover, in order to avoid a boring notation, we follow a general rule to denote constants. The small-case symbol c stands for different constants which depend only on Ω, on the final time T, the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements. A small-case symbol with a subscript like c_δ indicates that the constant might depend on the parameter δ, in addition. Hence, the meaning of c and c_δ might change from line to line and even in the same chain of equalities or inequalities. On the contrary, we mark precise constants which we can refer to by using different symbols, e.g., capital letters.

3. The state system

This section is devoted to the proofs of Theorems 2.2 and 2.4. As for the former, we start proving its second part, i.e., the continuous dependence formula (2.19). From this we derive uniqueness as well.

Continuous dependence and uniqueness. We first derive an identity that is satisfied by any solution. By integrating (2.14) with respect to time, we obtain

$$\int_\Omega \vartheta v + \int_\Omega \nabla(1 * \vartheta) \cdot \nabla v + \tau \int_\Gamma \vartheta_\Gamma v_\Gamma + \alpha \int_\Gamma \bigl(1 * (\vartheta_\Gamma - mu)\bigr) v_\Gamma$$
$$= \int_\Omega \vartheta_0\, v + \tau \int_\Gamma (\vartheta_0|_\Gamma)\, v_\Gamma - \int_\Omega \bigl(\widehat\lambda(\varphi) - \widehat\lambda(\varphi_0)\bigr) v$$
$$\text{for every } (v, v_\Gamma) \in V \times V_\Gamma \text{ such that } v_\Gamma = v|_\Gamma \text{ and a.e. in } (0,T). \tag{3.1}$$

Now, we pick two elements $u_i \in L^2(\Sigma)$, $i = 1, 2$, and consider two corresponding solutions $(\vartheta_i, \vartheta_{i,\Gamma}, \varphi_i, \xi_i)$. We write (3.1) for both controls and solutions and test the difference by choosing $v = \vartheta := \vartheta_1 - \vartheta_2$ and $v_\Gamma = \vartheta_\Gamma := \vartheta_{1,\Gamma} - \vartheta_{2,\Gamma}$. Then, we integrate over $(0,t)$, where $t \in (0,T)$ is arbitrary. At the same time, we write (2.15) for both solutions, multiply the difference by $\varphi := \varphi_1 - \varphi_2$ and integrate over Q_t. Finally, we add the equalities we have obtained to each other. By also setting $u := u_1 - u_2$ and $\xi := \xi_1 - \xi_2$ for brevity, we infer that

$$\int_{Q_t} |\vartheta|^2 + \frac{1}{2} \int_\Omega |\nabla(1 * \vartheta)(t)|^2 + \frac{\tau}{2} \int_{\Sigma_t} |\vartheta_\Gamma|^2 + \frac{\alpha}{2} \int_\Gamma |(1 * \vartheta_\Gamma)(t)|^2$$
$$+ \frac{1}{2} \int_\Omega |\varphi(t)|^2 + \sigma \int_{Q_t} |\nabla\varphi|^2 + \int_{Q_t} \xi\varphi$$

$$= \alpha \int_{\Sigma_t} m(1 * u)\vartheta_\Gamma - \int_{Q_t} \big(\widehat{\lambda}(\varphi_1) - \widehat{\lambda}(\varphi_2)\big)\vartheta$$

$$- \int_{Q_t} \big(\pi(\varphi_1) - \pi(\varphi_2)\big)\varphi + \int_{Q_t} \big(\vartheta_1 \lambda(\varphi_1) - \vartheta_2 \lambda(\varphi_2)\big)\varphi. \tag{3.2}$$

All the terms on the left-hand side are nonnegative, including the last one since β is monotone. We estimate each term on the right-hand side, separately. In the sequel, δ is a positive parameter. We have

$$\alpha \int_{\Sigma_t} m(1 * u)\vartheta_\Gamma = -\alpha \int_{\Sigma_t} mu(1 * \vartheta_\Gamma) + \alpha \int_\Gamma m(1 * u)(t)\,(1 * \vartheta_\Gamma)(t)$$

$$\le c \int_{\Sigma_t} |1 * \vartheta_\Gamma|^2 + c \int_{\Sigma_t} |u|^2 + \delta \int_\Gamma |(1 * \vartheta_\Gamma)(t)|^2 + c_\delta \int_\Gamma |(1 * u)(t)|^2 \tag{3.3}$$

and the last integral is bounded by $c_\delta \|u\|^2_{L^2(\Sigma)}$ due to the Hölder inequality. Next, owing to the boundedness of φ_1 and φ_2 ensured by Remark 2.3 and to the regularity of $\widehat{\lambda}$ on bounded intervals, we infer that

$$-\int_{Q_t} \big(\widehat{\lambda}(\varphi_1) - \widehat{\lambda}(\varphi_2)\big)\vartheta \le c \int_{Q_t} |\varphi|\,|\vartheta| \le \delta \int_{Q_t} |\vartheta|^2 + c_\delta \int_{Q_t} |\varphi|^2$$

and the third integral on the right-hand side of (3.2) can be treated in a similar way. Finally, we deal with the last term. As λ is Lipschitz continuous (see (2.5)) and φ_2 is bounded, we have

$$\int_{Q_t} \big(\vartheta_1 \lambda(\varphi_1) - \vartheta_2 \lambda(\varphi_2)\big)\varphi = \int_{Q_t} \vartheta_1 \big(\lambda(\varphi_1) - \lambda(\varphi_2)\big)\varphi + \int_{Q_t} \vartheta \lambda(\varphi_2)\varphi$$

$$\le c \int_{Q_t} |\vartheta_1|\,|\varphi|^2 + c \int_{Q_t} |\vartheta|\,|\varphi| \le c \int_{Q_t} |\vartheta_1|\,|\varphi|^2 + \delta \int_{Q_t} |\vartheta|^2 + c_\delta \int_{Q_t} |\varphi|^2.$$

On the other hand, by combining the Hölder inequality, the Sobolev inequality $\|v\|_4 \le c\|v\|_V$ for every $v \in V$, and the regularity $\vartheta_1 \in L^\infty(0, T; V)$, we obtain

$$\int_{Q_t} |\vartheta_1|\,|\varphi|^2 \le \int_0^t \|\vartheta_1(s)\|_4\, \|\varphi(s)\|_4\, \|\varphi(s)\|_2\, ds$$

$$\le c \int_0^t \|\varphi(s)\|_V\, \|\varphi(s)\|_2\, ds \le \delta \int_{Q_t} |\nabla \varphi|^2 + c_\delta \int_{Q_t} |\varphi|^2.$$

At this point, we collect all the estimates we have derived, choose δ small enough and apply the Gronwall lemma. Thus, we obtain (2.19) and uniqueness easily follows. Indeed, taking $u_1 = u_2$ in (2.19) immediately yields $\vartheta_1 = \vartheta_2$, $\vartheta_{1,\Gamma} = \vartheta_{2,\Gamma}$ and $\varphi_1 = \varphi_2$. Using finally (2.15), we also have $\xi_1 = \xi_2$. □

In order to complete the proof of Theorem 2.2, we have to show the existence of a solution and to establish estimate (2.18). To this end, we first replace β, $\widehat{\lambda}$ and λ by the smooth approximation of them β_ε, $\widehat{\lambda}_\varepsilon$ and λ_ε we introduce below, where ε is a positive parameter, say $\varepsilon \in (0, 1)$. By doing that, we obtain the approximating problem of finding a quadruplet $(\vartheta_\varepsilon, \vartheta_{\varepsilon,\Gamma}, \varphi_\varepsilon, \xi_\varepsilon)$ satisfying regularity requirements of type (2.10)–(2.13) and solving

$$\int_\Omega \partial_t \vartheta_\varepsilon \, v + \int_\Omega \nabla \vartheta_\varepsilon \cdot \nabla v + \int_\Omega \partial_t \widehat{\lambda}_\varepsilon(\varphi_\varepsilon)\, v + \tau \int_\Gamma \partial_t \vartheta_{\varepsilon,\Gamma}\, v_\Gamma + \alpha \int_\Gamma (\vartheta_{\varepsilon,\Gamma} - mu)\, v_\Gamma = 0$$

$$\text{for every } (v, v_\Gamma) \in V \times V_\Gamma \text{ such that } v_\Gamma = v_{|\Gamma} \text{ and a.e. in } (0,T) \tag{3.4}$$

$$\partial_t \varphi_\varepsilon - \sigma \Delta \varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) = \vartheta_\varepsilon \lambda_\varepsilon(\varphi_\varepsilon) \quad \text{and} \quad \xi_\varepsilon = \beta_\varepsilon(\varphi_\varepsilon) \quad \text{a.e. in } Q \tag{3.5}$$

$$\partial_n \varphi_\varepsilon = 0 \quad \text{a.e. on } \Sigma \tag{3.6}$$

$$\vartheta_\varepsilon(0) = \vartheta_0 \quad \text{and} \quad \varphi_\varepsilon(0) = \varphi_0 \quad \text{a.e. in } \Omega. \tag{3.7}$$

The regularity we require for the solution is the following

$$\vartheta_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \tag{3.8}$$

$$\vartheta_{\varepsilon,\Gamma} \in H^1(0,T;H_\Gamma) \quad \text{and} \quad \vartheta_{\varepsilon,\Gamma}(t) = \vartheta_\varepsilon(t)_{|\Gamma} \quad \text{for a.a. } t \in (0,T) \tag{3.9}$$

$$\varphi_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W). \tag{3.10}$$

The lower level (3.10) with respect to (2.12) has been chosen for convenience. However, the solution we find also satisfies (2.12), as it will be clear from the proof. In the above equations, β_ε is the Yosida regularization of β at level ε (see, e.g., [5, p. 28]). It is well known that β_ε is maximal monotone, single-valued and Lipschitz continuous. We also introduce the function $\widehat{\beta}_\varepsilon$ defined by

$$\widehat{\beta}_\varepsilon(r) := \int_0^r \beta_\varepsilon(s)\, ds \quad \text{for } r \in \mathbb{R} \tag{3.11}$$

and recall that

$$|\beta_\varepsilon(r)| \le |\beta^\circ(r)| \quad \text{and} \quad \beta_\varepsilon(r) \to \beta^\circ(r) \quad \text{for } r \in D(\beta) \tag{3.12}$$

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r) \quad \text{for every } r \in \mathbb{R}. \tag{3.13}$$

For (3.12) see, e.g., [5, Prop. 2.6, p. 28], while (3.13) follows from (3.12) and $\beta_\varepsilon(0) = 0$ (cf. (2.4)). Furthermore, $\widehat{\lambda}_\varepsilon$ as well as its derivative λ_ε are defined by

$$\widehat{\lambda}_\varepsilon(r) := \widehat{\lambda}(r)\, \zeta(\varepsilon r) \quad \text{and} \quad \lambda_\varepsilon(r) := \frac{d}{dr} \widehat{\lambda}_\varepsilon(r) \quad \text{for } r \in \mathbb{R}, \quad \text{where}$$

$$\zeta \in C^\infty(\mathbb{R}) \ \text{ satisfies } \ \zeta(r) = 1 \text{ for } |r| < 1 \text{ and } \zeta(r) = 0 \text{ for } |r| > 2. \tag{3.14}$$

Notice that both $\widehat{\lambda}_\varepsilon$ and λ_ε are bounded and Lipschitz continuous, and we set

$$\widehat{\Lambda}_\varepsilon := \sup |\widehat{\lambda}_\varepsilon| \quad \text{and} \quad \Lambda_\varepsilon := \sup |\lambda_\varepsilon|. \tag{3.15}$$

Our project is the following: *i*) we prove that problem (3.4)–(3.7) has at least a solution by a fixed point argument; *ii*) using compactness and monotonicity methods we show that its solution tends to a solution of problem (2.14)–(2.17) as $\varepsilon \searrow 0$, at least for a subsequence. We need two lemmas.

Lemma 3.1. *Let $\vartheta_\varepsilon \in L^2(0,T;H)$. Then, there exists a unique φ_ε satisfying (3.10), (3.5)–(3.6) and the second Cauchy condition in (3.7). Moreover, the estimate*

$$\|\varphi_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \le C_\varepsilon \big(1 + \|\vartheta_\varepsilon\|_{L^2(0,T;H)}\big) \tag{3.16}$$

holds true with a constant C_ε that depends on the structure (2.3)–(2.5), the norms involved in (2.9) and ε, but it is independent of ϑ_ε.

Proof. We are dealing with a standard semilinear parabolic problem that has a unique solution with the required regularity. We just derive estimate (3.16) and control the dependence of constants. We multiply equation (3.5) by $\partial_t \varphi_\varepsilon$ and add the same integral to both sides, for convenience. We have

$$\int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + \int_{Q_t} |\varphi_\varepsilon|^2 + \frac{\sigma}{2} \int_\Omega |\nabla \varphi_\varepsilon(t)|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))$$
$$= \frac{\sigma}{2} \int_\Omega |\nabla \varphi_0|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) - \int_{Q_t} \pi(\varphi_\varepsilon)\partial_t \varphi_\varepsilon + \int_{Q_t} \vartheta_\varepsilon\, \lambda_\varepsilon(\varphi_\varepsilon)\, \partial_t \varphi_\varepsilon + \int_{Q_t} |\varphi_\varepsilon|^2 \tag{3.17}$$

where $\widehat{\beta}_\varepsilon$ is given by (3.11). We recall (3.13) and (2.4), which imply that $\widehat{\beta}_\varepsilon(\varphi_0) \le \widehat{\beta}(\varphi_0) \le \beta^\circ(\varphi_0)\varphi_0$ a.e. in Ω. Thus (2.9) yields $\|\widehat{\beta}_\varepsilon(\varphi_0)\|_1 \le c$. Furthermore, notice that $|\lambda_\varepsilon(\varphi_\varepsilon)| \le \Lambda_\varepsilon$ (see (3.15)). On the other hand, π has a linear growth, so that

$$-\int_{Q_t} \pi(\varphi_\varepsilon)\partial_t \varphi_\varepsilon \le \frac{1}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + c \int_{Q_t} |\varphi_\varepsilon|^2 + c$$

and the last integral, which coincides with the last term on the right-hand side of (3.17), can be treated as follows: for every $s \in [0, t]$, we have

$$\varphi_\varepsilon(s) = \varphi_0 + \int_0^s \partial_t \varphi_\varepsilon(s')\, ds', \quad \text{whence}$$
$$|\varphi_\varepsilon(s)|^2 \le 2|\varphi_0|^2 + 2\left|\int_0^s \partial_t \varphi_\varepsilon(s')\, ds'\right|^2 \le c + c\int_0^s |\partial_t \varphi_\varepsilon(s')|^2\, ds' .$$

It turns out that

$$\int_\Omega |\varphi_\varepsilon(t)|^2 \le c + c\int_{Q_t} |\partial_t \varphi_\varepsilon|^2 \quad \text{and} \quad \int_{Q_t} |\varphi_\varepsilon|^2 \le c + c\int_0^t \Big(\int_{Q_s} |\partial_t \varphi_\varepsilon|^2\Big)\, ds . \tag{3.18}$$

The second inequality in (3.18) implies that its left-hand side, i.e., the integral we are dealing with, can be handled by the Gronwall lemma and we conclude that

$$\|\varphi_\varepsilon\|_{H^1(0,T;H)} + \|\nabla \varphi_\varepsilon\|_{L^\infty(0,T;H)} \le c + c\big(1 + \Lambda_\varepsilon\big)\|\vartheta_\varepsilon\|_{L^2(0,T;H)}$$

where c depends only on the structure and the initial datum φ_0. From this estimate and the first inequality in (3.18) it follows that

$$\|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c + c\big(1 + \Lambda_\varepsilon\big)\|\vartheta_\varepsilon\|_{L^2(0,T;H)}$$

with a similar new constant c. Now, for a.a. $t \in (0, T)$, we write (3.5) at time t, multiply by $\xi_\varepsilon(t)$ and integrate over Ω. We obtain a.e. in $(0, T)$

$$\sigma \int_\Omega \beta'_\varepsilon(\varphi_\varepsilon)|\nabla \varphi_\varepsilon|^2 + \int_\Omega |\xi_\varepsilon|^2 = \int_\Omega \big(-\partial_t \varphi_\varepsilon - \pi(\varphi_\varepsilon) + \vartheta_\varepsilon\, \lambda_\varepsilon(\varphi_\varepsilon)\big)\, \xi_\varepsilon$$
$$\le \frac{1}{2} \int_\Omega |\xi_\varepsilon|^2 + c\int_\Omega \big(|\partial_t \varphi_\varepsilon|^2 + 1 + |\varphi_\varepsilon|^2 + |\vartheta_\varepsilon \lambda_\varepsilon(\varphi_\varepsilon)|^2\big)$$

whence immediately for a.a. $t \in (0,T)$

$$\|\xi_\varepsilon(t)\|_H \le c\big(1 + \|\partial_t\varphi_\varepsilon(t)\|_H + \|\varphi_\varepsilon(t)\|_H + \|\vartheta_\varepsilon(t)\lambda_\varepsilon(\varphi_\varepsilon(t))\|_H\big). \tag{3.19}$$

In view of the regularity of φ_ε already achieved and the one of ϑ_ε, recalling (3.15) we deduce that

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \le c + c\,(1+\Lambda_\varepsilon)\,\|\vartheta_\varepsilon\|_{L^2(0,T;H)}$$

where c has the same dependencies as above. As the estimate of the norm of φ_ε in $L^2(0,T;W)$ follows by comparison of terms in (3.5) and elliptic regularity, the proof of (3.16) is complete. □

Lemma 3.2. *Let $\varphi_\varepsilon \in H^1(0,T;H)$. Then, there exists a unique pair $(\vartheta_\varepsilon, \vartheta_{\varepsilon,\Gamma})$ satisfying (3.8)–(3.9) and the first initial condition in (3.7) and solving the variational equation (3.4). Moreover, the estimates*

$$\|\vartheta_\varepsilon\|_{L^2(Q)} \le R_\varepsilon \tag{3.20}$$

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\vartheta_{\varepsilon,\Gamma}\|_{H^1(0,T;H_\Gamma)} \le D_\varepsilon\big(1 + \|\varphi_\varepsilon\|_{H^1(0,T;H)}\big) \tag{3.21}$$

hold true with constants R_ε and D_ε that depend on the structure (2.3)–(2.5), the norms involved in (2.9), the norm $\|u\|_2$ and ε, but they are independent of φ_ε.

Proof. We set

$$\mathcal{V} := \{(v, v_\Gamma) \in V \times H_\Gamma : \ v_\Gamma = v_{|_\Gamma}\}, \quad \mathcal{H} := H \times H_\Gamma \tag{3.22}$$

and endow these spaces with the scalar products defined on $\mathcal{V}^2$ and $\mathcal{H}^2$ by

$$\big((w,w_\Gamma),(v,v_\Gamma)\big)_{\mathcal{V}} := \int_\Omega (\nabla w\cdot\nabla v + wv) + \int_\Gamma w_\Gamma v_\Gamma \tag{3.23}$$

$$\big((w,w_\Gamma),(v,v_\Gamma)\big)_{\mathcal{H}} := \int_\Omega wv + \tau\int_\Gamma w_\Gamma v_\Gamma \tag{3.24}$$

respectively. Then, we obtain two Hilbert spaces and $\mathcal{V}$ is continuously and densely embedded in $\mathcal{H}$, so that we can construct the Hilbert triplet $(\mathcal{V},\mathcal{H},\mathcal{V}^*)$ in the usual way. Moreover, we define the continuous bilinear form a on $\mathcal{V}^2$ by the formula

$$a\big((w,w_\Gamma),(v,v_\Gamma)\big) := \int_\Omega \nabla w\cdot\nabla v + \alpha\int_\Gamma w_\Gamma v_\Gamma \tag{3.25}$$

and consider the operator $\mathcal{A} \in \mathcal{L}(\mathcal{V},\mathcal{V}^*)$ associated to a. As a clearly satisfies

$$a\big((v,v_\Gamma),(v,v_\Gamma)\big) + \big(1+\tfrac{1}{\tau}\big)\|(v,v_\Gamma)\|_{\mathcal{H}}^2 \ge \|(v,v_\Gamma)\|_{\mathcal{V}}^2 \quad \text{for every } (v,v_\Gamma)\in\mathcal{V}$$

the general theory (see, e.g., [24]) ensures that, for every $F \in L^2(0,T;\mathcal{V}^*)$ and $U_0 \in \mathcal{H}$, there exists a unique U satisfying

$$U \in H^1(0,T;\mathcal{V}^*) \cap L^2(0,T;\mathcal{V}) \subset C^0([0,T];\mathcal{H})$$
$$U'(t) + \mathcal{A}\,U(t) = F(t) \quad \text{a.e. in } (0,T) \quad \text{and} \quad U(0) = U_0$$

and that $U \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V})$ whenever $F \in L^2(0,T;\mathcal{H})$ and $U_0 \in \mathcal{V}$. In view of our assumption on φ_ε and (2.9), we choose

$$F = \big(-\partial_t \widehat{\lambda}_\varepsilon(\varphi_\varepsilon), \alpha m u\big) \in L^2(0,T;\mathcal{H}) \quad \text{and} \quad U_0 = \big(\vartheta_0\,,\vartheta_0{}_{|_\Gamma}\big) \in \mathcal{V}$$

and obtain a unique pair $(\vartheta_\varepsilon, \vartheta_{\varepsilon,\Gamma})$ satisfying (3.8)–(3.9) and the first initial condition (3.7) and solving the variational equation (3.4). Let us now prove estimates (3.20)–(3.21). We observe that the analog of (3.1) obtained by replacing $\widehat{\lambda}$ by $\widehat{\lambda}_\varepsilon$ holds for $(\vartheta_\varepsilon, \vartheta_{\varepsilon,\Gamma}, \varphi_\varepsilon)$. So, we take $(\vartheta_\varepsilon, \vartheta_{\varepsilon,\Gamma})$ as test pair and have

$$\int_{Q_t} |\vartheta_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla(1*\vartheta_\varepsilon)(t)|^2 + \tau \int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma}|^2 + \frac{\alpha}{2}\int_\Gamma |(1*\vartheta_{\varepsilon,\Gamma})(t)|^2$$
$$= \alpha \int_{\Sigma_t} m(1*u)\vartheta_{\varepsilon,\Gamma} + \int_{Q_t} \vartheta_0 \vartheta_\varepsilon + \tau \int_{\Sigma_t} (\vartheta_0{}_{|_\Gamma})\vartheta_{\varepsilon,\Gamma} + \int_{Q_t} (\widehat{\lambda}_\varepsilon(\varphi_0) - \widehat{\lambda}_\varepsilon(\varphi_\varepsilon))\vartheta_\varepsilon\,. \tag{3.26}$$

All the integrals on the left-hand side are nonnegative and the first term on the right-hand side can be estimated as we did in (3.3), namely

$$\alpha \int_{\Sigma_t} m(1*u)\vartheta_{\varepsilon,\Gamma} \le c \int_{\Sigma_t} |1*\vartheta_{\varepsilon,\Gamma}|^2 + c\int_{\Sigma_t} |u|^2 + \delta \int_\Gamma |(1*\vartheta_{\varepsilon,\Gamma})(t)|^2 + c_\delta \int_\Gamma |(1*u)(t)|^2$$

and the last integral is bounded by $c_\delta \|u\|^2_{L^2(\Sigma)}$ due to the Hölder inequality. The remaining terms on the right-hand side of (3.26) are treated in the usual way with the help of the Hölder and Young inequalities; just for the last term we point out that

$$\int_{Q_t} (\widehat{\lambda}_\varepsilon(\varphi_0) - \widehat{\lambda}_\varepsilon(\varphi_\varepsilon))\vartheta_\varepsilon \le 2\widehat{\Lambda}_\varepsilon \int_{Q_t} |\vartheta_\varepsilon| \le \delta \int_{Q_t} |\vartheta_\varepsilon|^2 + c_\delta \big|\widehat{\Lambda}_\varepsilon\big|^2$$

thanks to (3.15). Thus, by choosing δ small enough and applying the Gronwall lemma, it is straightforward to obtain the desired estimate (3.20). Let us now prove (3.21). The rigorous argument could rely on testing (3.4) by a $\mathcal{V}$-valued approximation of $(\partial_t\vartheta_\varepsilon, \partial_t\vartheta_{\varepsilon,\Gamma})$. However, we prefer to avoid such a detail and formally test the equation by $(\partial_t\vartheta_\varepsilon, \partial_t\vartheta_{\varepsilon,\Gamma})$, directly. We have

$$\int_{Q_t} |\partial_t\vartheta_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla\vartheta_\varepsilon(t)|^2 + \tau\int_{\Sigma_t} |\partial_t\vartheta_{\varepsilon,\Gamma}|^2 + \frac{\alpha}{2}\int_\Gamma |\vartheta_{\varepsilon,\Gamma}(t)|^2$$
$$= -\int_{Q_t} \lambda_\varepsilon(\varphi_\varepsilon)\partial_t\varphi_\varepsilon\,\partial_t\vartheta_\varepsilon + \alpha\int_{\Sigma_t} mu\,\partial_t\vartheta_{\varepsilon,\Gamma} + \frac{1}{2}\int_\Omega |\nabla\vartheta_0|^2 + \frac{\alpha}{2}\int_\Gamma |\vartheta_0{}_{|_\Gamma}|^2.$$

As $|\lambda_\varepsilon(\varphi_\varepsilon)| \le \Lambda_\varepsilon$ (see (3.15)) and ϑ_0 satisfies (2.9), we immediately derive (3.21) owing to the Hölder and Young inequalities. □

At this point, we are ready to complete the proof of Theorem 2.2 by following our project sketched above.

Existence of the approximating solution. As said before, we are going to use a fixed point argument. We often avoid stressing the dependence on ε, which is fixed. We consider the closed ball of $L^2(Q)$

$$\mathcal{B} := \{v \in L^2(Q) : \ \|v\|_2 \le R_\varepsilon\} \tag{3.27}$$

where R_ε is given by Lemma 3.2 (see (3.20)) and define the map $\mathcal{F} : \mathcal{B} \to L^2(Q)$ by the following steps: *i*) for $\bar\vartheta \in \mathcal{B}$ we apply Lemma 3.1 where ϑ_ε is replaced by $\bar\vartheta$, find the solution φ_ε and term it $\mathcal{E}(\bar\vartheta)$; *ii*) by starting from such a φ_ε, we apply Lemma 3.2, find the solution $(\vartheta_\varepsilon, \vartheta_{\varepsilon,\Gamma})$ and set $\mathcal{F}(\bar\vartheta) := \vartheta_\varepsilon$. By construction, it turns out that $\mathcal{F}(\bar\vartheta) \in \mathcal{B}$: indeed, the constant R_ε in (3.20) is independent of $\|\varphi_\varepsilon\|_{H^1(0,T;H)}$. Moreover, with the above notation, we deduce from Lemmas 3.1 and 3.2 (cf. (3.16) and (3.21))

$$\|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le C_\varepsilon\big(1 + R_\varepsilon\big) \tag{3.28}$$

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\vartheta_{\varepsilon,\Gamma}\|_{H^1(0,T;H_\Gamma)} \le D_\varepsilon\big(1 + C_\varepsilon(1 + R_\varepsilon)\big). \tag{3.29}$$

Therefore, $\mathcal{F}(\mathcal{B})$ is relatively compact by the Aubin–Lions lemma (see, e.g., [25, Thm. 5.1, p. 58]). Now, we verify that $\mathcal{F}$ is continuous. So, we assume that $\bar\vartheta_n \in \mathcal{B}$ and that $\bar\vartheta_n$ converges to $\bar\vartheta$ in $L^2(Q)$, and we prove that $\vartheta_n := \mathcal{F}(\bar\vartheta_n)$ converges to $\mathcal{F}(\bar\vartheta)$ in $L^2(Q)$. We set for convenience $\varphi_n := \mathcal{E}(\vartheta_n)$ and $\vartheta_{n,\Gamma} := \vartheta_{n|_\Gamma}$. As $\bar\vartheta_n \in \mathcal{B}$, estimates (3.28)–(3.29) hold for φ_n, ϑ_n and $\vartheta_{n,\Gamma}$. Hence, for a subsequence, φ_n, ϑ_n and $\vartheta_{n,\Gamma}$ converge to some φ, ϑ and ϑ_Γ in the corresponding weak or weak star topologies. Clearly, $\vartheta_\Gamma = \vartheta_{|_\Gamma}$. Moreover, φ_n and ϑ_n converge to φ and ϑ strongly in $L^2(Q)$ by the Aubin–Lions lemma. This implies that $f(\varphi_n)$ converges to $f(\varphi)$ strongly in $L^2(Q)$ for any Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$, and this is the case if f is either λ_ε, or $\widehat\lambda_\varepsilon$, or β_ε, or π. As a consequence, $\partial_t\widehat\lambda_\varepsilon(\varphi_n) = \lambda_\varepsilon(\varphi_n)\partial_t\varphi_n$ and $\vartheta_n\lambda_\varepsilon(\varphi_n)$ converge to $\lambda_\varepsilon(\varphi)\partial_t\varphi$ and $\vartheta\lambda_\varepsilon(\varphi)$ at least weakly in $L^1(Q)$. Therefore, the quadruplet $(\vartheta, \vartheta_\Gamma, \varphi, \xi)$, where $\xi := \beta_\varepsilon(\varphi)$, solves the integrated version of the variational formulation of the approximating problem (3.4)–(3.7) with smooth test functions, with ϑ_ε replaced by $\bar\vartheta$ in (3.5). This easily implies that $\vartheta = \mathcal{F}(\bar\vartheta)$. As the same argument holds for any subsequence extracted from $\{\vartheta_n\}$, the continuity we have claimed is proved. Therefore, we can apply the Schauder fixed point theorem and conclude that $\mathcal{F}$ has a fixed point ϑ_ε. As ϑ_ε belongs to $\mathcal{F}(\mathcal{B})$, it satisfies (3.8). Moreover, $\vartheta_{\varepsilon,\Gamma} := \vartheta_{\varepsilon|_\Gamma}$ belongs to $H^1(0,T;H_\Gamma)$, the function $\varphi_\varepsilon := \mathcal{E}(\vartheta_\varepsilon)$ satisfies (3.10) and the quadruplet $(\vartheta_\varepsilon, \vartheta_{\varepsilon,\Gamma}, \varphi_\varepsilon, \xi_\varepsilon)$, where $\xi_\varepsilon := \beta_\varepsilon(\varphi_\varepsilon)$, solves (3.4)–(3.7). □

The last step consists in letting ε tend to zero and getting a solution to (2.14)–(2.17). To this aim, we derive a priori estimates on the approximating solution that are uniform with respect to $\varepsilon \in (0, 1)$. As such estimates are conserved in the limit as $\varepsilon \searrow 0$, inequality (2.19) is established as a consequence.

First a priori estimate. We choose $v = \vartheta_\varepsilon$ and $v_\Gamma = \vartheta_{\varepsilon,\Gamma}$ in (3.4) and integrate over $(0, t)$. At the same time, we multiply (3.5) by $\partial_t\varphi_\varepsilon$ and integrate over Q_t. Then, we add the equalities obtained in this way and observe that the terms involving the nonlinear function λ_ε cancel out. Hence, we have

$$\begin{aligned}
&\frac12\int_\Omega |\vartheta_\varepsilon(t)|^2 + \int_{Q_t} |\nabla\vartheta_\varepsilon|^2 + \frac{\tau}{2}\int_\Gamma |\vartheta_{\varepsilon,\Gamma}(t)|^2 + \alpha\int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma}|^2 \\
&\quad + \int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + \frac{\sigma}{2}\int_\Omega |\nabla\varphi_\varepsilon(t)|^2 + \int_\Omega \widehat\beta_\varepsilon(\varphi_\varepsilon(t)) \\
&= \frac12\int_\Omega |\vartheta_0|^2 + \frac{\tau}{2}\int_\Gamma |\vartheta_{0|_\Gamma}|^2 + \frac{\sigma}{2}\int_\Omega |\nabla\varphi_0|^2 + \int_\Omega \widehat\beta_\varepsilon(\varphi_0) \\
&\quad + \alpha\int_{\Sigma_t} mu\vartheta_{\varepsilon,\Gamma} - \int_{Q_t} \pi(\varphi_\varepsilon)\partial_t\varphi_\varepsilon\,.
\end{aligned}$$

As (3.13) holds, π is Lipschitz continuous, m and u are bounded, and the data satisfy (2.9), we easily deduce that

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\vartheta_{\varepsilon,\Gamma}\|_{L^\infty(0,T;H_\Gamma)} + \|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c\,. \tag{3.30}$$

Since $|\lambda_\varepsilon(r)| \le |\lambda(r)| + c(1+|r|) \le c(1+|r|)$ (by (3.14) and (2.5)) and $V \subset L^6(\Omega)$, it follows that

$$\|\lambda_\varepsilon(\varphi_\varepsilon)\|_{L^2(0,T;L^6(\Omega))} \le c \quad \text{and} \quad \|\vartheta_\varepsilon \lambda_\varepsilon(\varphi_\varepsilon)\|_{L^2(0,T;L^3(\Omega))} \le c\,. \tag{3.31}$$

Therefore, by applying (3.19), we deduce that

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \le c\left(1 + \|\varphi_\varepsilon\|_{H^1(0,T;H)} + \|\vartheta_\varepsilon \lambda_\varepsilon(\varphi_\varepsilon)\|_{L^2(0,T;H)}\right) \le c\,. \tag{3.32}$$

Hence, in view of (3.5) we easily derive an estimate of $\Delta\varphi_\varepsilon$ in $L^2(0,T;H)$, whence

$$\|\varphi_\varepsilon\|_{L^2(0,T;W)} \le c \tag{3.33}$$

by (3.6) and elliptic regularity.

Second a priori estimate. The estimate we need next is the following

$$\begin{aligned} &\|\vartheta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\vartheta_{\varepsilon,\Gamma}\|_{H^1(0,T;H_\Gamma)} \\ &\quad + \|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le c\,. \end{aligned} \tag{3.34}$$

A rigorous proof is given in Appendix A. Here, we proceed formally. We take $v = \partial_t\vartheta_\varepsilon$ and $v_\Gamma = \partial_t\vartheta_\varepsilon|_\Gamma$ as test functions in (3.4) and integrate over $(0,t)$. At the same time, we multiply the equation obtained by differentiating (3.5) with respect to time by $\partial_t\varphi_\varepsilon$ and integrate over Q_t. Then, we add the equalities just derived to each other. Since the terms involving the product $\partial_t\vartheta_\varepsilon\,\partial_t\varphi_\varepsilon$ cancel out, we have

$$\begin{aligned} &\int_{Q_t} |\partial_t\vartheta_\varepsilon|^2 + \frac12\int_\Omega |\nabla\vartheta_\varepsilon(t)|^2 + \tau\int_{\Sigma_t} |\partial_t\vartheta_{\varepsilon,\Gamma}|^2 + \frac{\alpha}{2}\int_\Gamma |\vartheta_{\varepsilon,\Gamma}(t)|^2 \\ &\quad + \frac12\int_\Omega |\partial_t\varphi_\varepsilon(t)|^2 + \sigma\int_{Q_t} |\nabla\partial_t\varphi_\varepsilon|^2 + \int_{Q_t} \beta_\varepsilon'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \\ &= \frac12\int_\Omega |\nabla\vartheta_0|^2 + \frac{\alpha}{2}\int_\Gamma |\vartheta_0|_\Gamma|^2 + \frac12\int_\Omega |\partial_t\varphi_\varepsilon(0)|^2 + \int_{Q_t} \vartheta_\varepsilon\lambda_\varepsilon'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \\ &\quad + \alpha\int_{\Sigma_t} mu\,\partial_t\vartheta_{\varepsilon,\Gamma} - \int_{Q_t} \pi'(\varphi_\varepsilon)\,|\partial_t\varphi_\varepsilon|^2. \end{aligned}$$

All the terms on the left-hand side are nonnegative, in particular for the last one we use the monotonicity of β_ε. Moreover, the first two terms on the right-hand side are finite by (2.9) and the last two integrals can be treated in an obvious way by also taking the Lipschitz continuity of π into account. In order to control the term involving λ_ε', we observe that (3.14) easily implies that $|\lambda_\varepsilon'(r)| \le c$ for all $r \in \mathbb{R}$ and $\varepsilon \in (0,1)$. We also recall that $V \subset L^4(\Omega)$ and term C the norm of the embedding. Therefore, owing to the Hölder and Young inequalities, we obtain

$$\begin{aligned} &\int_{Q_t} \vartheta_\varepsilon\lambda_\varepsilon'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2 \le c\int_0^t \|\vartheta_\varepsilon(s)\|_4\,\|\partial_t\varphi_\varepsilon(s)\|_4\,\|\partial_t\varphi_\varepsilon(s)\|_2\,ds \\ &\le \frac{\sigma}{2C^2}\int_0^t \|\partial_t\varphi_\varepsilon(s)\|_4^2\,ds + c\int_0^t \|\vartheta_\varepsilon(s)\|_4^2\,\|\partial_t\varphi_\varepsilon(s)\|_2^2\,ds \end{aligned}$$

$$\leq \frac{\sigma}{2} \int_0^t \|\nabla \partial_t \varphi_\varepsilon(s)\|_H^2 \, ds + \frac{\sigma}{2} \int_0^t \|\partial_t \varphi_\varepsilon(s)\|_H^2 \, ds + c \int_0^t \|\vartheta_\varepsilon(s)\|_V^2 \, \|\partial_t \varphi_\varepsilon(s)\|_2^2 \, ds$$

and we remark at once that $s \mapsto \|\vartheta_\varepsilon(s)\|_V^2$ is bounded in $L^1(0,T)$ by (3.30). Finally, we estimate $\partial_t \varphi_\varepsilon(0)$ in H. We formally have from (3.5)

$$\partial_t \varphi_\varepsilon(0) = \sigma \Delta \varphi_0 - \beta_\varepsilon(\varphi_0) - \pi(\varphi_0) + \vartheta_0 \lambda_\varepsilon(\varphi_0) \quad \text{a.e. in } \Omega$$

and we recall (2.9), (3.12) and the inequality $|\lambda_\varepsilon(r)| \leq |\lambda(r)| + c(1+|r|) \leq c(1+|r|)$ for every $r \in \mathbb{R}$. Then, accounting for the continuous embedding $V \subset L^4(\Omega)$ once more, the bound $\|\partial_t \varphi_\varepsilon(0)\|_H \leq c$ follows and (3.34) is obtained via the Gronwall lemma.

Third a priori estimate. The Hölder inequality, the continuous embedding $V \subset L^4(\Omega)$ and (3.34) imply

$$\|\vartheta_\varepsilon \lambda_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;H)} \leq c \, \|\vartheta_\varepsilon\|_{L^\infty(0,T;L^4(\Omega))} \big(1 + \|\varphi_\varepsilon\|_{L^\infty(0,T;L^4(\Omega))}\big) \leq c. \tag{3.35}$$

On the other hand, (3.19) holds for the approximating solution. Thus, ξ_ε is bounded in $L^\infty(0,T;H)$ and a bound for $\Delta\varphi_\varepsilon$ in $L^\infty(0,T;H)$ follows by a comparison of terms in (3.5). Hence, by also using (3.6) and the elliptic regularity theory, we have

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;W)} + \|\xi_\varepsilon\|_{L^\infty(0,T;H)} \leq c \, . \tag{3.36}$$

Conclusion of the proof. By standard weak and weak star compactness results, we have for a subsequence

$$\vartheta_\varepsilon \to \vartheta \qquad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \tag{3.37}$$

$$\vartheta_{\varepsilon,\Gamma} \to \vartheta_\Gamma \qquad \text{weakly in } H^1(0,T;H_\Gamma) \tag{3.38}$$

$$\varphi_\varepsilon \to \varphi \qquad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \tag{3.39}$$

$$\xi_\varepsilon \to \xi \qquad \text{weakly in } L^\infty(0,T;H) \tag{3.40}$$

and $\vartheta_\Gamma(t) = \vartheta(t)|_\Gamma$ for a.a. $t \in (0,T)$. Owing to the compact embedding $W \subset C^0(\overline{\Omega})$ and to [27, Sect. 8, Cor. 4], we can also assume that the selected subsequence satisfies

$$\varphi_\varepsilon \to \varphi \quad \text{uniformly in } Q \tag{3.41}$$

so that $\lambda_\varepsilon(\varphi_\varepsilon)$ converges to $\lambda(\varphi)$ uniformly in Q since $\lambda_\varepsilon(r)$ converges to $\lambda(r)$ uniformly on every bounded interval. We deduce that $\lambda_\varepsilon(\varphi_\varepsilon)\partial_t\varphi_\varepsilon$ and $\vartheta_\varepsilon\lambda_\varepsilon(\varphi_\varepsilon)$ converge to $\lambda(\varphi)\partial_t\varphi$ and to $\vartheta\lambda(\varphi)$ at least weakly in $L^2(Q)$. Finally, we have $\xi \in \beta(\varphi)$ a.e. in Q by applying, e.g., [1, Prop. 2.2, p. 38]. Therefore, we can pass to the limit in the integrated version of problem (3.4)–(3.7) written with time dependent test functions and easily conclude that $(\vartheta, \vartheta_\Gamma, \varphi, \xi)$ is a solution to problem (2.14)–(2.17). □

Now, we prove Theorem 2.4. For the claim i), we first consider the Cauchy problem for the linear variational equation

$$\int_\Omega \partial_t \vartheta \, v + \int_\Omega \nabla \vartheta \cdot \nabla v + \tau \int_\Gamma \partial_t \vartheta_\Gamma \, v_\Gamma + \alpha \int_\Gamma \vartheta_\Gamma \, v_\Gamma = \int_\Omega \psi v + \int_\Gamma \psi_\Gamma v_\Gamma$$
$$\text{for every } (v, v_\Gamma) \in V \times V_\Gamma \text{ such that } v_\Gamma = v|_\Gamma \text{ and a.e. in } (0,T) \tag{3.42}$$

where ψ and ψ_Γ are given, and prove the following

Proposition 3.3. *Assume that* $\psi \in L^\infty(0,T;H)$, $\psi_\Gamma \in L^\infty(\Sigma)$ *and* $\vartheta_0 \in V \cap L^\infty(\Omega)$ *and that the corresponding norms are estimated by some constant* C. *Moreover, assume that* $(\vartheta, \vartheta_\Gamma)$ *enjoys the regularity* (2.10)–(2.11) *and satisfies the variational equation* (3.42) *and the initial condition* $\vartheta(0) = \vartheta_0$. *Then,* ϑ *is bounded and the estimate*

$$\|\vartheta\|_\infty \leq \widehat{C}$$

holds true with a constant $\widehat{C}$ *that depends only on* Ω, T, τ, α *and* C.

Proof. This is a regularity result for a linear problem. Hence, it can be established by considering the following cases:

$$a) \quad \psi = 0 \quad \text{and} \quad \vartheta_0 = 0\,; \qquad b) \quad \psi_\Gamma = 0\,.$$

Let us consider the first one. For any integer $n > 0$ and $p \in (2, +\infty)$ we define

$$T_n(r) := n \tanh \frac{r}{n} \quad \text{and} \quad \tilde{T}_{np}(r) := \int_0^{|r|} \bigl(T_n(s)\bigr)^{p-1} ds \quad \text{for } r \in \mathbb{R}.$$

Now, we set $\vartheta_n := T_n(\vartheta)$ and $\vartheta_{n,\Gamma} := T_n(\vartheta_\Gamma)$, and we test (3.42) by $v = |\vartheta_n|^{p-1} \operatorname{sign} \vartheta$ and $v_\Gamma = |\vartheta_{n,\Gamma}|^{p-1} \operatorname{sign} \vartheta_\Gamma$, where the sign function is extended by $\operatorname{sign}(0) = 0$. Then, we integrate over $(0,T)$. As $\operatorname{sign} \vartheta_{n,\Gamma} = \operatorname{sign} \vartheta_\Gamma$, we obtain

$$\int_\Omega \tilde{T}_{np}(\vartheta(T)) + \tau \int_\Gamma \tilde{T}_{np}(\vartheta_\Gamma(T)) + (p-1) \int_Q |\vartheta_n|^{p-2}\, T_n'(\vartheta_n) |\nabla \vartheta|^2 + \alpha \int_\Sigma |\vartheta_\Gamma| |\vartheta_{n,\Gamma}|^{p-1}$$
$$= \int_\Sigma \psi_\Gamma\, |\vartheta_{n,\Gamma}|^{p-1} \operatorname{sign} \vartheta_\Gamma\,.$$

All the terms on the left-hand side are nonnegative. Hence, by observing that $|r| \geq |T_n(r)|$ for every r, and applying the Young inequality, we have

$$\alpha \int_\Sigma |\vartheta_{n,\Gamma}|^p \leq \alpha \int_\Sigma |\vartheta_\Gamma| |\vartheta_{n,\Gamma}|^{p-1} \leq \int_\Sigma |\psi_\Gamma|\, |\vartheta_{n,\Gamma}|^{p-1}$$
$$= \int_\Sigma \alpha^{-1/p'} |\psi_\Gamma|\, \alpha^{1/p'} |\vartheta_{n,\Gamma}|^{p-1} \leq \int_\Sigma \Bigl(\frac{1}{p}\, \alpha^{-p/p'} |\psi_\Gamma|^p + \frac{1}{p'}\, \alpha |\vartheta_{n,\Gamma}|^p \Bigr)$$

whence immediately

$$\frac{\alpha}{p} \int_\Sigma |\vartheta_{n,\Gamma}|^p \leq \frac{1}{p}\, \alpha^{-p/p'} \int_\Sigma |\psi_\Gamma|^p \qquad \text{or} \qquad \alpha^{1/p} \|\vartheta_{n,\Gamma}\|_p \leq \alpha^{-1/p'} \|\psi_\Gamma\|_p\,.$$

By noting that $|T_n(r)| \nearrow |r|$ for every $r \in \mathbb{R}$ as $n \nearrow \infty$, we can let first n and then p tend to infinity and deduce that ϑ_Γ is bounded. Namely

$$\|\vartheta_\Gamma\|_\infty \leq \alpha^{-1} \|\psi_\Gamma\|_\infty \leq C/\alpha\,. \tag{3.43}$$

Hence, we can apply [22, Thm. 7.1] with $q = 2$ and $r = \infty$ and conclude.

In case b) we adapt the proof of [22, Thm. 7.1] (still with $q = 2$ and $r = \infty$) to our situation. For $k \geq \max\{1, C\}$ we set $\vartheta^k := (\vartheta - k)^+$ and $\vartheta^k_\Gamma := (\vartheta_\Gamma - k)^+$ and take $v = \vartheta^k$ in (3.42). By simply writing $\vartheta_\Gamma = (\vartheta_\Gamma - k) + k$ and observing that $\vartheta^k(0) = 0$ since $\vartheta_0 \leq C$, we obtain

$$\frac{1}{2}\int_\Omega |\vartheta^k(t)|^2 + \frac{\tau}{2}\int_\Gamma |\vartheta^k_\Gamma(t)|^2 + \int_{Q_t} |\nabla \vartheta^k|^2 + \alpha \int_{\Sigma_t} |\vartheta^k_\Gamma|^2 + k\alpha \int_{\Sigma_t} \vartheta^k_\Gamma = \int_{Q_t} \psi \vartheta^k$$

whence also

$$\frac{1}{2}\int_\Omega |\vartheta^k(t)|^2 + \int_{Q_t} |\nabla \vartheta^k|^2 \leq \int_{Q_t} \psi \vartheta^k.$$

This corresponds to formula [22, (7.6), p. 183] and the whole argument can be performed in the same way till formula [22, (7.14), p. 186]. Then, just one modification is needed. Namely, we account for [22, Rem. 6.2, p. 103] in applying [22, Thm. 6.1, p. 102] since no upper bound for ϑ_Γ is known now. This leads to the desired estimate from above $\vartheta \leq \widehat{C}$. The corresponding estimate from below is obtained by applying the former to $-\vartheta$. $\square$

Now, we apply the above result by observing that the pair $(\vartheta, \vartheta_\Gamma)$ we are interested in satisfies (3.42) with $\psi := \lambda(\varphi)\partial_t\varphi$ and $\psi_\Gamma := mu$, and notice that these functions belong to $L^\infty(0, T; H)$ and to $L^\infty(\Sigma)$, respectively. Moreover, the corresponding norm of ψ has been already estimated by (2.18), while an upper bound of the norm of ψ_Γ is supposed to be given in the statement. This yields the claim i) of Theorem 2.4.

The next step should be the proof of ii) of Theorem 2.4. To this aim, we just refer to [11, Thm. 2.2, iii)]. In fact, the proof given there shows that the component ξ of a pair (φ, ξ) satisfying (2.12)–(2.13) and solving the homogeneous Neumann problem for the equations

$$\partial_t\varphi - \sigma\Delta\varphi + \xi + \pi(\varphi) = f \quad \text{and} \quad \xi \in \beta(\varphi) \quad \text{a.e. in } Q$$

is bounded whenever $f \in L^\infty(Q)$. Such a statement is proved just with $f = \vartheta$ in the quoted paper by knowing that ϑ is bounded. However, the same proof is valid with any bounded f. Here, we have $f = \vartheta\lambda(\varphi)$, and both ϑ and $\lambda(\varphi)$ are bounded.

4. Existence of an optimal control

We prove Theorem 2.5 by the direct method. Since $\mathcal{U}_{ad}$ is nonempty, we can take a minimizing sequence $\{u_n\}$ for the optimization problem and, for any n, we can consider the corresponding solution $(\vartheta_n, \vartheta_{n,\Gamma}, \varphi_n, \xi_n)$ to problem (2.14)–(2.17). Then, $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and estimate (2.18) holds for $(\vartheta_n, \vartheta_{n,\Gamma}, \varphi_n, \xi_n)$. Therefore, we have

$$\begin{aligned} u_n &\to u^* && \text{weakly star in } L^\infty(\Omega) \\ \vartheta_n &\to \vartheta^* && \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \\ \vartheta_{n,\Gamma} &\to \vartheta^*_\Gamma && \text{weakly star in } H^1(0,T;H_\Gamma) \\ \varphi_n &\to \varphi^* && \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \\ \xi_n &\to \xi^* && \text{weakly star in } L^\infty(0,T;H) \end{aligned}$$

at least for a subsequence. Then, $u^* \in \mathcal{U}_{ad}$ since $\mathcal{U}_{ad}$ is closed in $\mathcal{X}$. Moreover, the initial conditions for ϑ^* and φ^* are satisfied and $\vartheta^*_\Gamma = \vartheta^*|_\Gamma$ a.e. in $(0, T)$. Thus, we can easily conclude by standard argument. Indeed,

$\{\varphi_n\}$ converges to φ^* uniformly in Q due to the compact embedding $W \subset C^0(\overline{\Omega})$ (see, e.g., [27, Sect. 8, Cor. 4]), whence $\pi(\varphi_n)$ and $\lambda(\varphi_n)$ converge to $\pi(\varphi^*)$ and $\lambda(\varphi^*)$ in the same topology. We also deduce that $\lambda(\vartheta_n)\partial_t\varphi_n$ and $\vartheta_n\lambda(\varphi_n)$ converge to $\lambda(\vartheta^*)\partial_t\varphi^*$ and $\vartheta^*\lambda(\varphi^*)$ at least weakly in $L^2(Q)$, and that $\xi^* \in \beta(\varphi^*)$ (note that $\varphi_n \to \varphi^*$ strongly in $L^2(Q)$ and see, e.g., [1, Prop. 2.1, p. 29]). Hence, $(\vartheta^*, \vartheta^*_\Gamma, \varphi^*, \xi^*)$ satisfies the variational formulation in the integral form of problem (2.14)–(2.17) corresponding to u^*. Therefore

$$\mathcal{J}(u^*) = \mathcal{J}_0(\vartheta^*, \vartheta^*_\Gamma, \varphi^*) \leq \liminf_{n\to\infty} \mathcal{J}_0(\vartheta_n, \vartheta_{n,\Gamma}, \varphi_n) = \lim_{n\to\infty} \mathcal{J}(u_n) = \inf_{u\in\mathcal{U}_{ad}} \mathcal{J}(u).$$

5. The control-to-state mapping

As sketched in Section 2, the main point is the Fréchet differentiability of the control-to-state mapping $\mathcal{S}$. This involves the linearized problem (2.41)–(2.44) and we first prove Proposition 2.7, i.e., well-posedness for the linearized problem and the continuous dependence of its solution on the parameter h. It is understood that all the assumptions of Theorem 2.4 as well as (2.34)–(2.36) are in force.

Well-posedness. We aim to apply a contraction argument. To this end, we observe that all the coefficients that enter the equations but $\bar{c} := \lambda'(\overline{\varphi})\partial_t\overline{\varphi}$ are bounded thanks to Corollary 2.6 and that the possibly unbounded coefficient $\bar{c}$ belongs to $L^\infty(0,T;H)$. We define the maps $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}$ in a proper functional framework as follows. For $\overline{\Theta} \in L^2(Q)$, we consider the problem for Φ given by (2.42)–(2.43) and the second condition in (2.44), where Θ in (2.42) is replaced by $\overline{\Theta}$. We obtain a linear parabolic problem which has a unique solution Φ satisfying (2.40). We set $\mathcal{F}_1(\overline{\Theta}) := \Phi$. By doing that, we obtain a map $\mathcal{F}_1 : L^2(Q) \to H^1(0,T;H)$. Now, we fix $\Phi \in H^1(0,T;H)$ and consider the problem for (Θ, Θ_Γ) given by the variational equation (2.41) and the initial condition $\Theta(0) = 0$. Such a problem has a unique solution (Θ, Θ_Γ) satisfying (2.38)–(2.39), as one can see by arguing as in the proof of Lemma 3.2, and we set $\mathcal{F}_2(\Phi) := \Theta$. In this way, we obtain a map $\mathcal{F}_2 : H^1(0,T;H) \to L^2(Q)$. We set $\mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1$ and prove that some iterated $\mathcal{F}^m$ of $\mathcal{F}$ is a contraction in $L^2(Q)$ by deriving some estimates involving $\mathcal{F}_1$ and $\mathcal{F}_2$, separately. For $\overline{\Theta} \in L^2(Q)$ we write (2.42) with $\overline{\Theta}$ in place of Θ and multiply it by Φ. Then, we integrate over Q_t and obtain

$$\frac{1}{2}\int_\Omega |\Phi(t)|^2 + \sigma\int_{Q_t} |\nabla\Phi|^2 = \int_{Q_t} \big(\bar{\vartheta}\lambda'(\overline{\varphi}) - \gamma'(\overline{\varphi})\big)|\Phi|^2 + \int_{Q_t} |\lambda(\overline{\varphi})|\,|\overline{\Theta}|\,|\Phi|.$$

Hence, with the help of the Young inequality and the Gronwall lemma, we deduce the a priori estimate

$$\|\Phi\|_{L^\infty(0,t;H)\cap L^2(0,t;V)} \leq C\,\|\overline{\Theta}\|_{L^2(Q_t)} \quad \text{for every } t \in [0,T]$$

where the constant C we have marked with a capital letter for a future reference does depend neither on t nor on $\overline{\Theta}$. As $\mathcal{F}_1$ is linear, this means that

$$\|\mathcal{F}_1(\overline{\Theta}_1) - \mathcal{F}_1(\overline{\Theta}_2)\|_{L^\infty(0,t;H)\cap L^2(0,t;V)} \leq C\,\|\overline{\Theta}_1 - \overline{\Theta}_2\|_{L^2(Q_t)} \quad \text{for } t \in [0,T] \tag{5.1}$$

for every $\overline{\Theta}_1, \overline{\Theta}_2 \in L^2(Q)$. Now, for given $\Phi_i \in H^1(0,T;H)$, $i = 1, 2$, we consider the problems corresponding to the definition of $\Theta_i = \mathcal{F}_2(\Phi_i)$ and take the difference. By setting $\Phi := \Phi_1 - \Phi_2$ and analogously defining Θ and Θ_Γ for brevity, we see that (2.41) holds with h replaced by 0. We integrate such an equality with respect to time. By observing that $\lambda(\overline{\varphi})\partial_t\Phi + \lambda'(\overline{\varphi})\partial_t\overline{\varphi}\,\Phi = \partial_t(\lambda(\overline{\varphi})\,\Phi)$, we simply obtain

$$\int_\Omega \Theta\, v + \int_\Omega \nabla(1 * \Theta)\cdot\nabla v + \tau\int_\Gamma \Theta_\Gamma\, v_\Gamma + \alpha\int_\Gamma (1 * \Theta_\Gamma)\, v_\Gamma = -\int_\Omega \lambda(\overline{\varphi})\,\Phi\, v \tag{5.2}$$

for every $(v, v_\Gamma) \in V \times V_\Gamma$ such that $v_\Gamma = v_{|_\Gamma}$ and a.e. in $(0,T)$. By choosing $v = \Theta$ and $v_\Gamma = \Theta_\Gamma$, integrating with respect to time and forgetting some nonnegative terms on the left-hand side, we easily infer that

$$\|\Theta_1 - \Theta_2\|_{L^2(0,t;H)} \le D\,\|\Phi_1 - \Phi_2\|_{L^2(0,t;H)} \quad \text{for } t \in [0,T] \tag{5.3}$$

where we have marked the constant D for convenience. If we combine (5.3) written for $\Phi_i = \mathcal{F}_1(\overline{\Theta}_i)$ with (5.1), we deduce the estimate

$$\|\mathcal{F}(\overline{\Theta}_1) - \mathcal{F}(\overline{\Theta}_2)\|^2_{L^2(0,t;H)} = \int_0^t \|(\Theta_1 - \Theta_2)(s)\|_H^2\,ds$$

$$\le D^2 \int_0^t \|(\Phi_1 - \Phi_2)(s)\|_H^2\,ds \le D^2 \int_0^t \|\Phi_1 - \Phi_2\|^2_{L^\infty(0,s;H)}\,ds$$

$$\le D^2 C^2 \int_0^t \|\overline{\Theta}_1 - \overline{\Theta}_2\|^2_{L^2(0,s;H)}\,ds \quad \text{for every } t \in [0,T]$$

and this can be iterated. By doing that, we obtain the inequality

$$\|\mathcal{F}^m(\overline{\Theta}_1) - \mathcal{F}^m(\overline{\Theta}_2)\|^2_{L^2(0,t;H)} \le \frac{(C^2 D^2\,t)^m}{m!}\,\|\overline{\Theta}_1 - \overline{\Theta}_2\|^2_{L^2(0,t;H)}$$

for every $\Theta_i \in L^2(Q)$, $t \in [0,T]$ and $m \ge 1$. By choosing m such that $(C^2D^2\,T)^m < m!$ and $t = T$, we see that $\mathcal{F}^m$ is a contraction in $L^2(Q)$, whence $\mathcal{F}$ has a unique fixed point Θ. Then, Θ and the associated functions Θ_Γ and Φ that enter the construction of $\mathcal{F}_1$ and $\mathcal{F}_2$ provide a unique solution to the linearized problem (2.41)–(2.44) with the regularity (2.38)–(2.40).

Continuous dependence. This is given by the estimate (2.45) we prove at once. By integrating (2.41) with respect to time and proceeding as we did for (5.2), we have

$$\int_\Omega \Theta\,v + \int_\Omega \nabla(1*\Theta)\cdot\nabla v + \tau\int_\Gamma \Theta_\Gamma\,v_\Gamma + \alpha\int_\Gamma (1*\Theta_\Gamma)\,v_\Gamma = -\int_\Omega \lambda(\overline{\varphi})\,\Phi\,v + \alpha\int_\Gamma m(1*h)\,v_\Gamma$$

for every $(v, v_\Gamma) \in V \times V_\Gamma$ such that $v_\Gamma = v_{|_\Gamma}$ and a.e. in $(0,T)$. Now, we take $v = \Theta$, $v_\Gamma = \Theta_\Gamma$ and integrate over $(0,t)$. Besides, we multiply (2.42) by $\delta\partial_t\Phi$, where δ is a positive parameter, and integrate over Q_t. Then, we add the equalities we obtain to each other. Furthermore, we add the same term $(\delta/(4T))\int_\Omega |\Phi(t)|^2$ to both sides for convenience. Also, owing to the boundedness of the coefficients and to the Young inequality, and setting $C := \|\lambda(\overline{\varphi})\|^2_\infty$, we have

$$\int_{Q_t} |\Theta|^2 + \frac{1}{2}\int_\Omega |\nabla(1*\Theta)(t)|^2 + \tau\int_{\Sigma_t} |\Theta_\Gamma|^2 + \frac{\alpha}{2}\int_\Gamma |(1*\Theta_\Gamma)(t)|^2$$

$$+ \delta\int_{Q_t} |\partial_t\Phi|^2 + \frac{\delta\sigma}{2}\int_\Omega |\nabla\Phi(t)|^2 + \frac{\delta}{4T}\int_\Omega |\Phi(t)|^2$$

$$= -\int_{Q_t} \lambda(\overline{\varphi})\Phi\Theta + \alpha\int_{\Sigma_t} m(1*h)\,\Theta_\Gamma$$

$$+ \delta\int_{Q_t} \big(-\gamma'(\overline{\varphi}) + \overline{\vartheta}\lambda'(\overline{\varphi})\big)\Phi\,\partial_t\Phi + \delta\int_{Q_t} \lambda(\overline{\varphi})\,\Theta\,\partial_t\Phi + \frac{\delta}{4T}\int_\Omega |\Phi(t)|^2$$

$$\leq \delta\, C \int_{Q_t} |\Theta|^2 + \frac{1}{4\delta} \int_{Q_t} |\Phi|^2 + \frac{\tau}{2} \int_{\Sigma_t} |\Theta_\Gamma|^2 + c \int_{\Sigma_t} |1 * h|^2$$
$$+ \frac{\delta}{2} \int_{Q_t} |\partial_t \Phi|^2 + \delta\, c \int_{Q_t} |\Phi|^2 + \delta\, C \int_{Q_t} |\Theta|^2 + \frac{\delta}{4T} \int_\Omega |\Phi(t)|^2.$$

Moreover, by arguing as we did to derive the first inequality in (3.18), we see that

$$\frac{\delta}{4T} \int_\Omega |\Phi(t)|^2 \leq \frac{\delta}{4} \int_{Q_t} |\partial_t \Phi|^2.$$

Therefore, by choosing δ such that $2\delta C < 1$ and applying the Gronwall lemma, we obtain

$$\|\Theta\|_{L^2(Q)} + \|1 * \Theta\|_{L^\infty(0,T;V)} + \|\Theta_\Gamma\|_{L^2(\Sigma)} + \|\Phi\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c \|1 * h\|_{L^2(\Sigma)}\,.$$

At this point, it is easy to see that the above estimate implies (2.45). □

Here is the main result of this section.

Theorem 5.1. *Let $\bar{u} \in \mathcal{X}$. Then, $\mathcal{S}$ is Fréchet differentiable at $\bar{u}$ and the Fréchet derivative $[D\mathcal{S}](\bar{u})$ is precisely the map $\mathcal{D} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ defined in the statement of Proposition 2.7.*

Proof. We fix $\bar{u} \in \mathcal{X}$ and the corresponding state $(\bar{\vartheta}, \bar{\vartheta}_\Gamma, \bar{\varphi}) := \mathcal{S}(\bar{u})$ and, for $h \in \mathcal{X}$, we set

$$(\vartheta^h, \vartheta^h_\Gamma, \varphi^h) := \mathcal{S}(\bar{u} + h) \quad \text{and} \quad (\zeta^h, \zeta^h_\Gamma, \eta^h) := (\vartheta^h - \bar{\vartheta} - \Theta, \vartheta^h_\Gamma - \bar{\vartheta}_\Gamma - \Theta_\Gamma, \varphi^h - \bar{\varphi} - \Phi)$$

where $(\Theta, \Theta_\Gamma, \Phi)$ is the solution to the linearized problem corresponding to h. We have to prove that $\|(\zeta^h, \zeta^h_\Gamma, \eta^h)\|_{\mathcal{Y}} / \|h\|_{\mathcal{X}}$ tends to zero as $\|h\|_{\mathcal{X}}$ tends to zero. More precisely, we show that some constant c exists such that

$$\|(\zeta^h, \zeta^h_\Gamma, \eta^h)\|_{\mathcal{Y}} \leq c \|h\|^2_{L^2(\Sigma)} \quad \text{provided that } \|h\|_{\mathcal{X}} \leq 1 \tag{5.4}$$

and this is even stronger than necessary. So, we assume $\|h\|_\infty \leq 1$ and make a preliminary observation. As $\bar{u}$ is fixed and $\|\bar{u} + h\|_\infty \leq \|\bar{u}\|_\infty + 1$ for every h under consideration, we can apply Theorem 2.4 and Corollary 2.6 and find constants $\vartheta_\bullet, \vartheta^\bullet \in \mathbb{R}$ and $\varphi_\bullet, \varphi^\bullet \in D(\beta)$ independent of h such that

$$\vartheta_\bullet \leq \bar{\vartheta} \leq \vartheta^\bullet \quad \text{and} \quad \vartheta_\bullet \leq \vartheta^h \leq \vartheta^\bullet \quad \text{a.e. in } Q \tag{5.5}$$
$$\varphi_\bullet \leq \bar{\varphi} \leq \varphi^\bullet \quad \text{and} \quad \varphi_\bullet \leq \varphi^h \leq \varphi^\bullet \quad \text{a.e. in } Q. \tag{5.6}$$

Moreover, we can also exploit the second part of Theorem 2.2 with $u_1 = \bar{u} + h$ and $u_2 = \bar{u}$. By doing that and also owing to the continuous embedding $V \subset L^4(\Omega)$, we derive from (2.19)

$$\|\vartheta^h - \bar{\vartheta}\|_{L^\infty(0,T;H)} \leq C' \|h\|_{L^2(\Sigma)}\,, \quad \|\varphi^h - \bar{\varphi}\|_{L^4(Q)} \leq M C' \|h\|_{L^2(\Sigma)} \tag{5.7}$$
$$\|\varphi^h - \bar{\varphi}\|_{L^\infty(0,T;L^4(\Omega))} \leq M C' \|h\|_{L^2(\Sigma)} \tag{5.8}$$

where the constants M and C' do not depend on h. Now, the problem solved by $(\zeta^h, \zeta^h_\Gamma, \eta^h)$ is the following

$$\int_\Omega \partial_t\zeta^h\, v + \int_\Omega \nabla\zeta^h \cdot \nabla v + \tau \int_\Gamma \partial_t\zeta_\Gamma^h\, v_\Gamma + \alpha \int_\Gamma \zeta_\Gamma^h v_\Gamma$$

$$= -\int_\Omega \bigl\{\lambda(\varphi^h)\partial_t\varphi^h - \lambda(\overline{\varphi})\partial_t\overline{\varphi} - \lambda(\overline{\varphi})\partial_t\Phi - \lambda'(\overline{\varphi})\partial_t\overline{\varphi}\,\Phi\bigr\}v$$

$$\text{for every } (v, v_\Gamma) \in V \times V_\Gamma \text{ such that } v_\Gamma = v_{|_\Gamma} \text{ and a.e. in } (0,T) \tag{5.9}$$

$$\partial_t\eta^h - \sigma\Delta\eta^h = -E_1^h + E_2^h \qquad \text{a.e. in } Q, \quad \text{where} \tag{5.10}$$

$$E_1^h := \gamma(\varphi^h) - \gamma(\overline{\varphi}) - \gamma'(\overline{\varphi})\,\Phi \tag{5.11}$$

$$E_2^h := \vartheta^h\lambda(\varphi^h) - \overline{\vartheta}\lambda(\overline{\varphi}) - \lambda(\overline{\varphi})\Theta - \overline{\vartheta}\lambda'(\overline{\varphi})\,\Phi \tag{5.12}$$

$$\partial_n\eta^h = 0 \quad \text{a.e. on } \Sigma \quad \text{and} \quad \zeta^h(0) = \eta^h(0) = 0 \quad \text{a.e. in } \Omega. \tag{5.13}$$

By integrating (5.9) with respect to time, we obtain

$$\int_\Omega \zeta^h\, v + \int_\Omega \nabla(1*\zeta^h)\cdot\nabla v + \tau\int_\Gamma \zeta_\Gamma^h\, v_\Gamma + \alpha\int_\Gamma (1*\zeta_\Gamma^h)v_\Gamma = -\int_\Omega Z^h v$$

$$\text{for every } (v, v_\Gamma) \in V \times V_\Gamma \text{ such that } v_\Gamma = v_{|_\Gamma} \text{ and a.e. in } (0,T) \tag{5.14}$$

$$\text{where} \quad Z^h := \widehat{\lambda}(\varphi^h) - \widehat{\lambda}(\overline{\varphi}) - \lambda(\overline{\varphi})\Phi\,. \tag{5.15}$$

At this point, we take $v = \zeta^h$ and $v_\Gamma = \zeta_\Gamma^h$ in (5.14) and integrate over $(0,t)$. Besides, we multiply (5.10) by η^h and integrate over Q_t. Finally, we add the resulting equalities to each other. We obtain

$$\int_{Q_t} |\zeta^h|^2 + \frac{1}{2}\int_\Omega |\nabla(1*\zeta^h)(t)|^2 + \tau\int_{\Sigma_t} |\zeta_\Gamma^h|^2 + \frac{\alpha}{2}\int_\Gamma |(1*\zeta_\Gamma^h)(t)|^2$$

$$+ \frac{1}{2}\int_\Omega |\eta^h(t)|^2 + \sigma\int_{Q_t} |\nabla\eta^h|^2$$

$$= -\int_{Q_t} Z^h\zeta^h - \int_{Q_t} E_1^h\eta^h + \int_{Q_t} E_2^h\eta^h \tag{5.16}$$

and we now estimate each term on the right-hand side. To this end, we represent the functions Z^h and E_i^h, $i = 1, 2$, in different forms. By applying the Taylor expansion to $\widehat{\lambda}$ and γ, we find functions $\widehat{\varphi}$ and $\tilde{\varphi}$ taking values between the ones of $\overline{\varphi}$ and φ^h such that

$$\widehat{\lambda}(\varphi^h) - \widehat{\lambda}(\overline{\varphi}) = \lambda(\overline{\varphi})\,(\varphi^h - \overline{\varphi}) + \frac{1}{2}\,\lambda'(\widehat{\varphi})\,(\varphi^h - \overline{\varphi})^2$$

$$\gamma(\varphi^h) - \gamma(\overline{\varphi}) = \gamma'(\overline{\varphi})\,(\varphi^h - \overline{\varphi}) + \frac{1}{2}\,\gamma''(\tilde{\varphi})\,(\varphi^h - \overline{\varphi})^2$$

whence immediately

$$Z^h = \lambda(\overline{\varphi})\eta^h + \frac{1}{2}\,\lambda'(\widehat{\varphi})\,(\varphi^h - \overline{\varphi})^2 \quad \text{and} \quad E_1^h = \gamma'(\overline{\varphi})\eta^h + \frac{1}{2}\,\gamma''(\tilde{\varphi})\,(\varphi^h - \overline{\varphi})^2\,.$$

In order to rewrite E_2^h, we observe that, if $G : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, we have for every (y, z) and (y', z') in $\mathbb{R}^2$ and for a suitable $(\breve{y}, \breve{z})$ in between

$$G(y', z') - G(y, z) - \nabla_{yz}G(y, z)\cdot(y' - y, z' - z) = \frac{1}{2}\,[y' - y, z' - z]\,D_{yz}^2(\breve{y}, \breve{z})\,{}^t[y' - y, z' - z]\,.$$

In the particular case $G(y,z) = y\lambda(z)$, the above formula becomes

$$y'\lambda(z') - y\lambda(z) - \lambda(z)(y'-y) - y\lambda'(z)(z'-z) \\ = \frac{1}{2}\,\breve{y}\lambda''(\breve{z})(z'-z)^2 + \lambda'(\breve{z})(y'-y)(z'-z).$$

Therefore, there exist functions $\breve{\vartheta}$ and $\breve{\varphi}$ taking values between the ones of $\bar{\vartheta}$ and ϑ^h and the ones of $\overline{\varphi}$ and φ^h, respectively, such that

$$\vartheta^h\lambda(\varphi^h) - \bar{\vartheta}\lambda(\overline{\varphi}) \\ = \lambda(\overline{\varphi})(\vartheta^h - \bar{\vartheta}) + \bar{\vartheta}\lambda'(\overline{\varphi})(\varphi^h - \overline{\varphi}) + \frac{1}{2}\,\breve{\vartheta}\lambda''(\breve{\varphi})(\varphi^h - \overline{\varphi})^2 + \lambda'(\breve{\varphi})(\vartheta^h - \bar{\vartheta})(\varphi^h - \overline{\varphi})$$

so that

$$E_2^h = \lambda(\overline{\varphi})\,\zeta^h + \bar{\vartheta}\lambda'(\overline{\varphi})\,\eta^h + \frac{1}{2}\,\breve{\vartheta}\lambda''(\breve{\varphi})(\varphi^h - \overline{\varphi})^2 + \lambda'(\breve{\varphi})(\vartheta^h - \bar{\vartheta})(\varphi^h - \overline{\varphi}).$$

Notice that $\vartheta_\bullet \le \breve{\vartheta} \le \vartheta^\bullet$ and that $\varphi_\bullet \le \widehat{\varphi}, \tilde{\varphi}, \breve{\varphi} \le \varphi^\bullet$. Therefore, recalling (2.35), the right-hand side of (5.16) can be estimated by the Young inequality as follows

$$- \int_{Q_t} Z^h\zeta^h - \int_{Q_t} E_1^h\eta^h + \int_{Q_t} E_2^h\eta^h \le \frac{1}{2}\int_{Q_t} |\zeta^h|^2 + c\int_{Q_t} |\eta^h|^2 \\ + c\int_{Q_t} |\varphi^h - \overline{\varphi}|^2|\zeta^h| + c\int_{Q_t} |\varphi^h - \overline{\varphi}|^2|\eta^h| + c\int_{Q_t} |\vartheta^h - \bar{\vartheta}|\,|\varphi^h - \overline{\varphi}|\,|\eta^h| \\ \le \frac{3}{4}\int_{Q_t} |\zeta^h|^2 + c\int_{Q_t} |\eta^h|^2 + c\int_{Q} |\varphi^h - \overline{\varphi}|^4 + c\int_{Q_t} |\vartheta^h - \bar{\vartheta}|\,|\varphi^h - \overline{\varphi}|\,|\eta^h|\,.$$

On the other hand, thanks to the Hölder inequality and to the continuous embedding $V \subset L^4(\Omega)$, we have for every $\delta > 0$

$$\int_{Q_t} |\vartheta^h - \bar{\vartheta}|\,|\varphi^h - \overline{\varphi}|\,|\eta^h| \le \int_0^t \|(\vartheta^h - \bar{\vartheta})(s)\|_2\,\|(\varphi^h - \overline{\varphi})(s)\|_4\,\|\eta^h(s)\|_4\,ds \\ \le \delta\int_0^t \|\eta^h(s)\|_V^2\,ds + c_\delta\,\|\varphi^h - \overline{\varphi}\|^2_{L^\infty(0,T;L^4(\Omega))}\,\|\vartheta^h - \bar{\vartheta}\|^2_{L^2(0,T;H)}\,.$$

At this point, we choose δ small enough, apply the Gronwall lemma and account for (5.7)–(5.8) in order to estimate the norms of $\vartheta^h - \bar{\vartheta}$ and of $\varphi^h - \overline{\varphi}$. This yields (5.4) and the proof is complete. □

6. Necessary optimality conditions

In this section, we derive the optimality condition (2.52) stated in Theorem 2.9. We start from (2.33) and first prove (2.46).

Proposition 6.1. *Let u^* be an optimal control and $(\vartheta^*, \vartheta^*_\Gamma, \varphi^*) := \mathcal{S}(u^*)$. Then, condition (2.46) holds.*

Proof. As already said in Section 2, we just have to apply the chain rule for Fréchet derivatives. Clearly, the Fréchet derivative $[D\mathcal{J}_0](\overline{\vartheta}, \overline{\vartheta}_\Gamma, \overline{\varphi})$ of the functional $\mathcal{J}_0$ exists at every point of $\mathcal{Y}$ and it is given by

$$[D\mathcal{J}_0](\overline{\vartheta}, \overline{\vartheta}_\Gamma, \overline{\varphi}) : (h_1, h_2, h_3) \in \mathcal{Y} \mapsto \kappa_1 \int_Q (\overline{\vartheta} - \vartheta_Q)\, h_1 + \kappa_2 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\, h_3(T)\,.$$

In particular, this holds if $(\overline{\vartheta}, \overline{\vartheta}_\Gamma, \overline{\varphi}) = (\vartheta^*, \vartheta^*_\Gamma, \varphi^*) = \mathcal{S}(u^*)$. Therefore, Theorem 5.1 and the chain rule ensure that $\mathcal{J}$ is Fréchet differentiable at u^* and that its Fréchet derivative $[D\mathcal{J}](u^*)$ at any optimal control u^* acts as follows

$$[D\mathcal{J}](u^*) : h \in \mathcal{X} \mapsto \kappa_1 \int_Q (\vartheta^* - \vartheta_Q)\, \Theta + \kappa_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\, \Phi(T)$$

where $(\Theta, \Theta_\Gamma, \Phi)$ is the solution to the linearized problem corresponding to h. Therefore, (2.46) immediately follows from (2.33). □

The next step is the proof of Theorem 2.8. For convenience, we consider the equivalent forward problem in the unknown $(\tilde{p}, \tilde{p}_\Gamma, \tilde{q})$ given by $(\tilde{p}, \tilde{p}_\Gamma, \tilde{q})(t) := (p, p_\Gamma, q)(T - t)$ and corresponding to the new coefficient and given terms defined accordingly. However, to simplify notations, we write p, p_Γ and q instead of $\tilde{p}$, $\tilde{p}_\Gamma$ and $\tilde{q}$ in the sequel. The new problem is to find (p, p_Γ, q) satisfying (2.47)–(2.48) and solving

$$\int_\Omega \partial_t p\, v + \int_\Omega \nabla p \cdot \nabla v + \int_\Omega aqv + \tau \int_\Gamma \partial_t p_\Gamma\, v_\Gamma + \alpha \int_\Gamma p_\Gamma v_\Gamma = \int_\Omega fv$$
$$\text{for every } v \in V,\ v_\Gamma := v_{|\Gamma}, \text{ and a.e. in } (0, T) \tag{6.1}$$

$$\int_\Omega \partial_t q\, v + \sigma \int_\Omega \nabla q \cdot \nabla v + \int_\Omega bqv = \int_\Omega g \partial_t p\, v$$
$$\text{for every } v \in V \text{ and a.e. in } (0, T) \tag{6.2}$$

$$p(0) = 0 \quad \text{and} \quad q(0) = q_0 \quad \text{a.e. in } \Omega \tag{6.3}$$

where a, f, b, g and q_0 are deduced from (2.49)–(2.51). Thus, we have a, b, $g \in L^\infty(Q)$, $f \in L^2(Q)$ and $q_0 \in V$. We aim to use a contraction argument in a weaker functional framework. However, it will be clear from the proof that the unique solution we find satisfies (2.47)–(2.48).

The equivalent fixed point problem. For a given $\bar{q} \in L^2(Q)$, we consider the problem obtained by writing (6.1) with q replaced by $\bar{q}$ and the initial condition $p(0) = 0$. If we introduce the spaces $\mathcal{V}$ and $\mathcal{H}$ and the bilinear form a given by (3.22)–(3.25), and argue as in the proof of Lemma 3.2, we see that the problem under consideration has a unique solution (p, p_Γ) satisfying $p \in H^1(0, T; H) \cap L^\infty(0, T; V)$ and (2.48). However, p is smoother since one can argue as in Remark 2.1 to derive $\Delta p \in L^2(Q)$ and $\partial_n p \in L^2(\Sigma)$ from the regularity already achieved. We set $\mathcal{F}_1(\bar{q}) := p$ and $\tilde{\mathcal{F}}_1(\bar{q}) := (p, p_\Gamma)$. By doing that, we find a map $\mathcal{F}_1 : L^2(Q) \to H^1(0, T; H)$ and an associated map $\tilde{\mathcal{F}}_1$ that we use in the rest of the proof. Now, for $p \in H^1(0, T; H)$, we consider (6.2) complemented by the second initial condition in (6.3). As $b \in L^\infty(Q)$, $g\partial_t p \in L^2(Q)$ and $q_0 \in V$, such a problem has a unique solution q satisfying (2.47), and we set $q := \mathcal{F}_2(p)$. We thus obtain a map $\mathcal{F}_2 : H^1(0, T; H) \to L^2(Q)$ and consider $\mathcal{F} : L^2(Q) \to L^2(Q)$ defined by $\mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1$. Let us point out that, for a given $\bar{q} \in L^2(Q)$, $\mathcal{F}(\bar{q})$ actually takes values in the space $H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)$. The problem under consideration is equivalent to the existence of a unique fixed point for $\mathcal{F}$. Indeed, if q is such a fixed point, q and the corresponding p and p_Γ provide a solution satisfying (2.47)–(2.48) by construction. Conversely, if (p, p_Γ, q) solves the problem, then q is a fixed point of $\mathcal{F}$.

The contraction argument. It suffices to find a constant C such that

$$\|\big(\mathcal{F}(\bar{q}_1) - \mathcal{F}(\bar{q}_2)\big)(t)\|_H \le C\, \|\bar{q}_1 - \bar{q}_2\|_{L^2(0,t;H)} \tag{6.4}$$

for every $\bar{q}_1$, $\bar{q}_2 \in L^2(Q)$ and every $t \in [0,T]$. Indeed, this implies that some iterated $\mathcal{F}^m$ of $\mathcal{F}$ is a contraction. In order to prove (6.4), we take any pair of functions $\bar{q}_i \in L^2(Q)$, $i = 1,2$, consider the pairs $(p_i, p_{i,\Gamma}) := \tilde{\mathcal{F}}_1(\bar{q}_i)$ and the functions $q_i := \mathcal{F}(\bar{q}_i)$ and write their definitions, i.e., (6.1) written with $\bar{q}_i$ in place of q and (6.2) with p_i in place of p. Then we take the two differences. We set for convenience $\bar{q} := \bar{q}_1 - \bar{q}_2$ and similarly introduce p, p_Γ and q. At this point, we formally test the first difference by $v = \partial_t p$ and $v_\Gamma = \partial_t p_\Gamma$ (by avoiding the technical approximation of such test functions by V- and V_Γ-valued functions) and integrate over $(0,t)$. At the same time, we multiply the second one by q and integrate over Q_t. We have

$$\int_{Q_t} |\partial_t p|^2 + \frac{1}{2}\int_\Omega |\nabla p(t)|^2 + \tau \int_{\Sigma_t} |\partial_t p_\Gamma|^2 + \frac{\alpha}{2}\int_\Gamma |p_\Gamma(t)|^2 = -\int_{Q_t} a\,\bar{q}\,\partial_t p$$

as well as

$$\frac{1}{2}\int_\Omega |q(t)|^2 + \sigma \int_{Q_t} |\nabla q|^2 = -\int_{Q_t} b\,|q|^2 + \int_{Q_t} g\,\partial_t p\, q\,.$$

As a, b and g are bounded functions, it is straightforward to deduce that

$$\|p\|_{H^1(0,t;H)\cap L^\infty(0,t;V)} + \|p_\Gamma\|_{L^2(0,t;H_\Gamma)} \le C_1\, \|\bar{q}\|_{L^2(0,t;H)}$$
$$\|q\|_{L^\infty(0,t;H)\cap L^2(0,t;V)} \le C_2\, \|p\|_{H^1(0,t;H)}$$

for every $t \in [0,T]$ and some constants C_1 and C_2 independent of $\bar{q}_i$ and t. By combining such inequalities, we deduce that

$$\|q\|_{L^\infty(0,t;H)} \le C_1 C_2\, \|\bar{q}\|_{L^2(0,t;H)}$$

whence (6.4) follows with $C := C_1C_2$. Thus, Theorem 2.8 is completely proved. □

At this point, we are ready to prove Theorem 2.9 on optimality, i.e., the necessary condition (2.52) for u^* to be an optimal control in terms of the solution (p, p_Γ, q) of the adjoint problem (2.49)–(2.51). So, let u^* be an optimal control and fix an arbitrary $u \in \mathcal{U}_{ad}$. We write both the variational formulations of the linearized problem at the optimal state $(\vartheta^*, \vartheta^*_\Gamma, \varphi^*) := \mathcal{S}(u^*)$ corresponding to $h = u - u^*$ and the adjoint system. We have a.e. in $(0,T)$

$$\int_\Omega \partial_t\Theta\, v + \int_\Omega \nabla\Theta\cdot\nabla v + \int_\Omega \lambda(\varphi^*)\partial_t\Phi\, v + \int_\Omega \lambda'(\varphi^*)\partial_t\varphi^*\,\Phi\, v$$
$$+ \tau\int_\Gamma \partial_t\Theta_\Gamma\, v_\Gamma + \alpha\int_\Gamma \Theta_\Gamma\, v_\Gamma = \alpha\int_\Gamma m(u-u^*)\, v_\Gamma \tag{6.5}$$

$$\int_\Omega \partial_t\Phi\, v + \sigma\int_\Omega \nabla\Phi\cdot\nabla v + \int_\Omega \gamma'(\varphi^*)\,\Phi v = \int_\Omega \vartheta^*\lambda'(\varphi^*)\,\Phi v + \int_\Omega \lambda(\varphi^*)\,\Theta v \tag{6.6}$$

$$-\int_\Omega \partial_t p\, v + \int_\Omega \nabla p\cdot\nabla v - \int_\Omega \lambda(\varphi^*) q v - \tau\int_\Gamma \partial_t p_\Gamma\, v_\Gamma + \alpha\int_\Gamma p_\Gamma v_\Gamma$$

$$= \kappa_1 \int_\Omega (\vartheta^* - \vartheta_Q) v \tag{6.7}$$

$$- \int_\Omega \partial_t q \, v + \sigma \int_\Omega \nabla q \cdot \nabla v + \int_\Omega \big(\gamma'(\varphi^*) - \vartheta^* \lambda'(\varphi^*)\big) q v = \int_\Omega \lambda(\varphi^*) \partial_t p \, v \tag{6.8}$$

where (6.5) and (6.7) hold for every $(v, v_\Gamma) \in V \times V_\Gamma$ such that $v_\Gamma = v_{|\Gamma}$, while (6.6) and (6.8) are required for every $v \in V$. Moreover, the functions at hand satisfy the homogeneous initial or final conditions as specified in (2.44) and (2.51). We choose $(v, v_\Gamma) = (p, p_\Gamma)$ in (6.5), $v = q$ in (6.6), $(v, v_\Gamma) = (-\Theta, -\Theta_\Gamma)$ in (6.7) and $v = -\Phi$ in (6.8). Then, we sum all the resulting equalities and integrate over $(0, T)$. Several terms cancel out and we obtain

$$\int_Q \big\{ \partial_t \Theta \, p + \lambda(\varphi^*) \partial_t \Phi \, p + \lambda'(\varphi^*) \partial_t \varphi^* \, \Phi p + \partial_t \Phi \, q + \partial_t p \, \Theta + \partial_t q \, \Phi + \lambda(\varphi^*) \partial_t p \, \Phi \big\}$$
$$+ \tau \int_\Sigma \big(\partial_t \Theta_\Gamma \, p_\Gamma + \partial_t p_\Gamma \, \Theta_\Gamma \big)$$
$$= \alpha \int_\Sigma m(u - u^*) p_\Gamma - \kappa_1 \int_Q (\vartheta^* - \vartheta_Q) \Theta \, .$$

As the expression in braces is equal to $\partial_t \big(\Theta p + \lambda(\varphi^*) \Phi p + \Phi q \big)$, the above equality becomes

$$\int_\Omega \big(\Theta p + \lambda(\varphi^*) \Phi p + \Phi q \big)(T) - \int_\Omega \big(\Theta p + \lambda(\varphi^*) \Phi p + \Phi q \big)(0)$$
$$+ \tau \int_\Sigma (\Theta_\Gamma p_\Gamma)(T) - \tau \int_\Sigma (\Theta_\Gamma p_\Gamma)(0)$$
$$= \alpha \int_\Sigma m(u - u^*) p_\Gamma - \kappa_1 \int_Q (\vartheta^* - \vartheta_Q) \Theta \, .$$

Owing to the relations $\Theta_\Gamma(0) = \Theta(0)_{|\Gamma}$ and $p_\Gamma(T) = p(T)_{|\Gamma}$, and accounting for the initial conditions (2.44) and the final conditions (2.51), the above equation reduces to

$$\int_\Omega \Phi(T) \, \kappa_2 \big(\varphi^*(T) - \varphi_\Omega \big)$$
$$= \alpha \int_\Sigma m(u - u^*) p_\Gamma - \kappa_1 \int_Q (\vartheta^* - \vartheta_Q) \Theta \, .$$

Therefore, the inequality (2.46) we have already established in Proposition 6.1 implies

$$\alpha \int_\Sigma m(u - u^*) p_\Gamma \geq 0 \quad \text{for every } u \in \mathfrak{U}_{ad} \, .$$

Moreover, the positive coefficient α can be removed. At this point, a standard argument leads to the pointwise relation (2.52) and to its consequences listed in the statement, and the proof of Theorem 2.9 is complete.

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Appendix A

This section is devoted to a rigorous proof of (3.34). With respect to the formal procedure of Section 3, we replace derivatives with difference quotients, essentially. For $h \in (0,T)$ and $v \in L^2(Q)$ or $v \in L^2(\Sigma)$, we define v^h on $(0,T-h)$ by setting $v^h(t) := v(t+h)$. We integrate (3.4) with respect to time over $(s,s+h)$ and test the equality we obtain by $v = (\vartheta_\varepsilon^h - \vartheta_\varepsilon)(s)$ and $v_\Gamma = (\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma})(s)$. At the same time, we write (3.5) at times $s+h$ and s, multiply the difference by $(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)$ and integrate over Ω with respect to space. Finally, we add the equalities obtained this way to each other and integrate over $(0,t)$ with respect to s. We have

$$\begin{aligned}
&\int_{Q_t} |\vartheta_\varepsilon^h - \vartheta_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla\big((1*\vartheta_\varepsilon)^h - (1*\vartheta_\varepsilon)\big)(t)|^2 \\
&\quad + \tau \int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma}|^2 + \frac{\alpha}{2}\int_\Gamma |\big((1*\vartheta_{\varepsilon,\Gamma})^h - (1*\vartheta_{\varepsilon,\Gamma})\big)(t)|^2 \\
&\quad + \frac{1}{2}\int_\Omega |(\varphi_\varepsilon^h - \varphi_\varepsilon)(t)|^2 + \sigma\int_{Q_t} |\nabla(\varphi_\varepsilon^h - \varphi_\varepsilon)|^2 + \int_{Q_t} (\xi_\varepsilon^h - \xi_\varepsilon)(\vartheta_\varepsilon^h - \vartheta_\varepsilon) \\
&= -\int_{Q_t} \big(\pi(\varphi_\varepsilon^h) - \pi(\varphi_\varepsilon)\big)(\varphi_\varepsilon^h - \varphi_\varepsilon) + \int_{\Sigma_t} m\big((1*u)^h - (1*u)\big)(\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma}) \\
&\quad + \int_{Q_t} \Big\{ \big(\vartheta_\varepsilon^h \lambda_\varepsilon(\varphi_\varepsilon^h) - \vartheta_\varepsilon\lambda_\varepsilon(\varphi_\varepsilon)\big)(\varphi_\varepsilon^h - \varphi_\varepsilon) - \big(\widehat{\lambda}_\varepsilon(\varphi_\varepsilon^h) - \widehat{\lambda}_\varepsilon(\varphi_\varepsilon)\big)(\vartheta_\varepsilon^h - \vartheta_\varepsilon) \Big\} \\
&\quad + \frac{1}{2}\int_\Omega |\nabla(1*\vartheta_\varepsilon)(h)|^2 + \frac{\alpha}{2}\int_\Gamma |(1*\vartheta_{\varepsilon,\Gamma})(h)|^2 + \frac{1}{2}\int_\Omega |\varphi_\varepsilon(h) - \varphi_0|^2.
\end{aligned} \tag{A.1}$$

All the terms on the left-hand side are nonnegative, the last one by monotonicity, and the first integral on the right-hand side can be estimated in an obvious way by using the Lipschitz continuity of π. Moreover, we have

$$\begin{aligned}
&\int_{\Sigma_t} m\big((1*u)^h - (1*u)\big)(\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma}) \\
&\le \frac{\tau}{2}\int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma}|^2 + c\int_{\Sigma_t} |(1*u)^h - (1*u)|^2 \\
&\le \frac{\tau}{2}\int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma}|^2 + c\,h^2\,\|u\|_{L^2(\Sigma)}^2 = \frac{\tau}{2}\int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma}|^2 + c\,h^2\,.
\end{aligned}$$

The critical term is the next one, since the cancellation of the formal procedure does not occur here. We observe at once that the functions λ_ε have a common Lipschitz constant since λ is Lipschitz continuous. We apply the mean value theorem to $\widehat{\lambda}_\varepsilon$ and obtain a function $\tilde{\varphi}_\varepsilon$, taking values between the ones of φ_ε^h and φ_ε, in order that

$$\int_{Q_t} \left\{ \left(\vartheta_\varepsilon^h \lambda_\varepsilon(\varphi_\varepsilon^h) - \vartheta_\varepsilon \lambda_\varepsilon(\varphi_\varepsilon) \right)(\varphi_\varepsilon^h - \varphi_\varepsilon) - \left(\widehat{\lambda}_\varepsilon(\varphi_\varepsilon^h) - \widehat{\lambda}_\varepsilon(\varphi_\varepsilon) \right)(\vartheta_\varepsilon^h - \vartheta_\varepsilon) \right\}$$

$$= \int_{Q_t} \left\{ \left(\lambda_\varepsilon(\varphi_\varepsilon^h) - \lambda_\varepsilon(\tilde{\varphi}_\varepsilon) \right)(\vartheta_\varepsilon^h - \vartheta_\varepsilon)(\varphi_\varepsilon^h - \varphi_\varepsilon) - \vartheta_\varepsilon \left(\lambda_\varepsilon(\varphi_\varepsilon^h) - \lambda_\varepsilon(\varphi_\varepsilon) \right)(\varphi_\varepsilon^h - \varphi_\varepsilon) \right\}$$

$$\le c \int_{Q_t} |\vartheta_\varepsilon^h - \vartheta_\varepsilon| \, |\varphi_\varepsilon^h - \varphi_\varepsilon|^2 + c \int_{Q_t} |\vartheta_\varepsilon| \, |\varphi_\varepsilon^h - \varphi_\varepsilon|^2 .$$

We treat the last two integrals, separately, owing to the Hölder and Young inequalities and to the continuous embedding $V \subset L^4(\Omega)$. As such embedding is also compact, (3.30) and [27, Sect. 8, Cor. 4] imply that the functions φ_ε are equicontinuous $L^4(\Omega)$-valued functions. Hence, for every $\delta > 0$, there exists $h_\delta > 0$ such that $\|\varphi_\varepsilon^h(s) - \varphi_\varepsilon(s)\|_4 \le \delta$ for every $\varepsilon \in (0,1)$ and $s \in [0, T-h]$ whenever $h \le h_\delta$. In the sequel, δ is a positive parameter, say $\delta \in (0,1)$, whose value is chosen later on, and it is understood that h does not exceed the corresponding h_δ. Therefore, we have

$$\int_{Q_t} |\vartheta_\varepsilon^h - \vartheta_\varepsilon| \, |\varphi_\varepsilon^h - \varphi_\varepsilon|^2 \le \int_0^t \|(\vartheta_\varepsilon^h - \vartheta_\varepsilon)(s)\|_2 \, \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_4^2 \, ds$$

$$\le C\delta \int_0^t \|(\vartheta_\varepsilon^h - \vartheta_\varepsilon)(s)\|_2 \, \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_V \, ds$$

$$\le \delta \int_0^t \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_V^2 \, ds + c\,\delta \int_0^t \|(\vartheta_\varepsilon^h - \vartheta_\varepsilon)(s)\|_H^2 \, ds$$

$$\le \delta \int_0^t \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_V^2 \, ds + c\, h^2 \|\partial_t \vartheta_\varepsilon\|_{L^2(0,T;H)}^2 \le \delta \int_0^t \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_V^2 \, ds + c\, h^2$$

where the marked constant C satisfies $\|v\|_4 \le C\|v\|_V$ for every $v \in V$ and the last inequality is justified by (3.30). Next, we have

$$\int_{Q_t} |\vartheta_\varepsilon| \, |\varphi_\varepsilon^h - \varphi_\varepsilon|^2 \le c \int_0^t \|\vartheta_\varepsilon(s)\|_4 \, \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_2 \, \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_4 \, ds$$

$$\le \delta \int_0^t \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_V^2 \, ds + c_\delta \int_0^t \|\vartheta_\varepsilon(s)\|_V^2 \, \|(\varphi_\varepsilon^h - \varphi_\varepsilon)(s)\|_2^2 \, ds$$

and we observe that the function $s \mapsto \|\vartheta_\varepsilon(s)\|_V^2$ is estimated in $L^1(0,T)$ by (3.30). At this point, we choose δ small enough and apply the Gronwall lemma. We obtain

$$\int_{Q_t} |\vartheta_\varepsilon^h - \vartheta_\varepsilon|^2 + \int_\Omega |\nabla(1 * \vartheta_\varepsilon^h - 1 * \vartheta_\varepsilon)(t)|^2 + \int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma}^h - \vartheta_{\varepsilon,\Gamma}|^2$$

$$+ \int_\Omega |(\varphi_\varepsilon^h - \varphi_\varepsilon)(t)|^2 + \int_{Q_t} |\nabla(\varphi_\varepsilon^h - \varphi_\varepsilon)|^2$$

$$\leq c\,h^2 + c\int_\Omega \big(|\nabla(1*\vartheta_\varepsilon)(h)|^2 + |\varphi_\varepsilon(h) - \varphi_0|^2\big) + c\int_\Gamma |(1*\vartheta_{\varepsilon,\Gamma})(h)|^2 \tag{A.2}$$

for $h > 0$ small enough and for every $t \in [0, T-h]$ and $\varepsilon \in (0,1)$. Now, we estimate the last integrals of (A.2) and show that they have order h^2. To this end, we argue rather similarly as we did in deriving (A.1), but we use $\vartheta_\varepsilon - \vartheta_0$ and $\varphi_\varepsilon - \varphi_0$ as test functions. Namely, we integrate (3.4) with respect to time over $(0,s)$, test the equality we obtain by $v = \vartheta_\varepsilon(s) - \vartheta_0$ and $v_\Gamma = \vartheta_{\varepsilon,\Gamma}(s) - \vartheta_{0|_\Gamma}$, and integrate over $(0,t)$ with respect to s. Besides, we multiply (3.5) by $\varphi_\varepsilon - \varphi_0$ and integrate over Q_t. Finally, we add the resulting equalities to each other and suitably rearrange. We have

$$\int_\Omega |\vartheta_\varepsilon(t) - \vartheta_0|^2 + \frac{1}{2}\int_\Omega |\nabla(1*\vartheta_\varepsilon - 1*\vartheta_0)(t)|^2 + \tau\int_\Gamma |\vartheta_{\varepsilon,\Gamma}(t) - \vartheta_{0|_\Gamma}|^2$$

$$+ \frac{1}{2}\int_\Omega |\varphi_\varepsilon(t) - \varphi_0|^2 + \sigma\int_{Q_t} |\nabla(\varphi_\varepsilon - \varphi_0)|^2 + \int_{Q_t} \big(\beta_\varepsilon(\varphi_\varepsilon) - \beta_\varepsilon(\varphi_0)\big)(\varphi_\varepsilon - \varphi_0)$$

$$= -\int_{Q_t} \big(\pi(\varphi_\varepsilon) - \pi(\varphi_0)\big)(\varphi_\varepsilon - \varphi_0)$$

$$+ \int_{Q_t} \big\{\big(\vartheta_\varepsilon\lambda_\varepsilon(\varphi_\varepsilon) - \vartheta_0\lambda_\varepsilon(\varphi_0)\big)(\varphi_\varepsilon - \varphi_0) - \big(\widehat{\lambda}_\varepsilon(\varphi_\varepsilon) - \widehat{\lambda}_\varepsilon(\varphi_0)\big)(\vartheta_\varepsilon - \vartheta_0)\big\}$$

$$+ \int_{\Sigma_t} m(1*u)(\vartheta_{\varepsilon,\Gamma} - \vartheta_{0|_\Gamma}) + \int_{Q_t} \nabla(1*\vartheta_0)\cdot\nabla(\vartheta_\varepsilon - \vartheta_0)$$

$$+ \int_{Q_t} \big\{\sigma\nabla\varphi_0\cdot\nabla(\varphi_\varepsilon - \varphi_0) + \big(-\beta_\varepsilon(\varphi_0) - \pi(\varphi_0) - \vartheta_0\lambda_\varepsilon(\varphi_0)\big)(\varphi_\varepsilon - \varphi_0)\big\}. \tag{A.3}$$

Even though we are interested in taking $t = h$, it is more convenient to let t vary in order to apply some Gronwall-type lemma. So, we assume $t \in [0,h]$. Also in this case, all the terms on the left-hand side are nonnegative and the first integral on the right-hand side can be easily dealt with. Moreover, the next term can be treated as in the above argument. Furthermore, we have

$$\int_{\Sigma_t} m(1*u)(\vartheta_{\varepsilon,\Gamma} - \vartheta_{0|_\Gamma}) \leq \int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma} - \vartheta_{0|_\Gamma}|^2 + c\int_{\Sigma_t} |1*u|^2$$

$$\leq \int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma} - \vartheta_{0|_\Gamma}|^2 + c\,h^2\|u\|_2^2 = \int_{\Sigma_t} |\vartheta_{\varepsilon,\Gamma} - \vartheta_{0|_\Gamma}|^2 + c\,h^2\,.$$

Next, we point out that

$$\int_{Q_t} \nabla(1*\vartheta_0)\cdot\nabla(\vartheta_\varepsilon - \vartheta_0)$$

$$= \int_\Omega \nabla(1*\vartheta_0)(t)\cdot\nabla(1*\vartheta_\varepsilon - 1*\vartheta_0)(t) - \int_{Q_t} \nabla\vartheta_0\cdot\nabla(1*\vartheta_\varepsilon - 1*\vartheta_0)$$

$$\le \frac{1}{4}\int_\Omega |\nabla(1*\vartheta_\varepsilon - 1*\vartheta_0)(t)|^2 + h^2\int_\Omega |\nabla\vartheta_0|^2 + \int_0^t \|\nabla\vartheta_0\|_H \, \|\nabla(1*\vartheta_\varepsilon - 1*\vartheta_0)(t)\|_H \, dt\,.$$

Finally, the last integral of (A.3) can be written as

$$\int_{Q_t} f_\varepsilon(\varphi_\varepsilon - \varphi_0) \quad \text{where} \quad f_\varepsilon := -\sigma\Delta\varphi_0 - \beta_\varepsilon(\varphi_0) - \pi(\varphi_0) - \vartheta_0\lambda_\varepsilon(\varphi_0)$$

whence it is bounded by

$$\int_0^t \|f_\varepsilon\|_H \, \|(\varphi_\varepsilon - \varphi_0)(t)\|_H \, dt\,.$$

On the other hand, we observe that

$$\int_0^h \|\nabla\vartheta_0\|_H \, dt = h\,\|\nabla\vartheta_0\|_H = c\,h \quad \text{and} \quad \int_0^h \|f_\varepsilon\|_H \, dt = h\,\|f_\varepsilon\|_H \le c\,h$$

by virtue of (2.9) and (3.12). Hence, we can apply well-known Gronwall-type inequalities. Namely we combine, e.g., [5, Lemma A.4, p. 156] and [5, Lemma A.5, p. 157]. By ignoring some nonnegative terms on the left-hand side, we conclude that

$$\int_\Omega \bigl(|\nabla(1*\vartheta_\varepsilon - 1*\vartheta_0)(t)|^2 + |\varphi_\varepsilon(t) - \varphi_0|^2\bigr) + \int_\Gamma |\vartheta_{\varepsilon,\Gamma}(t) - \vartheta_0{}_{|_\Gamma}|^2 \le c\,h^2$$

for $0 \le t \le h$. In particular, we have

$$\int_\Omega \bigl(|\nabla(1*\vartheta_\varepsilon)(h)|^2 + |\varphi_\varepsilon(h) - \varphi_0|^2\bigr)$$
$$\le c\,h^2 + c\int_\Omega |\nabla(1*\vartheta_0)(h)|^2 \le c\,h^2\bigl(1 + \|\nabla\vartheta_0\|_H^2\bigr) = c\,h^2$$

as well as

$$\int_\Gamma |(1*\vartheta_{\varepsilon,\Gamma})(h)|^2 = \bigl\|\textstyle\int_0^h \vartheta_{\varepsilon,\Gamma}(s)\,ds\bigr\|_{H_\Gamma}^2$$
$$\le 2\bigl(\textstyle\int_0^h \|\vartheta_{\varepsilon,\Gamma}(s) - \vartheta_0{}_{|_\Gamma}\|_{H_\Gamma}\,ds\bigr)^2 + 2h^2\,\|\vartheta_0{}_{|_\Gamma}\|_{H_\Gamma}^2 \le c\,h^2.$$

Therefore, the right-hand side of (A.2) has order h^2 and the difference quotients associated to the terms of the left-hand side of (A.1) are bounded in the proper norms. Hence, (3.34) follows, the bound in $L^\infty(0,T;V)$ for ϑ_ε being due to $\nabla\vartheta_\varepsilon = \partial_t(1*\nabla\vartheta_\varepsilon)$.

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