Consiglio Nazionale delle Ricerche

ISTITUTO DI ELABORAZIONE DELLA INFORMAZIONE

PISA

INCOMPLETE FAULT COVERAGE IN MODULAR MULTIPROCESSOR SYSTEMS

L. Simoncini, A.D. Friedman

L78-19

ACM 78, Proceedings 1978 Annual Conference,
Washington, 1978. PP. 210-216

INCOMPLETE FAULT COVERAGE IN MODULAR MULTIPROCESSOR SYSTEMS *

L. Simoncini
IEI-CNR
46, Via S. Maria
Pisa, Italy

A.D. Friedman
Department of Electrical Engineering
and Computer Science
The George Washington University
Washington, D. C.

In an early paper Preparata, Metze, and Chien formulated a model of system level diagnosis in which it is assumed that a fault-free module can detect any fault in a module it is testing. In practice this assumption may not be true. If a fault-free module can only detect p x 100% of all faults (or equivalently detect a fault with probability p) we refer to this as incomplete fault coverage. With incomplete fault coverage it is possibile that a system will fail to detect a faulty module. In this paper we consider the problem of designing systems which minimize the probability of failure to detect for a given fault coverage p. We show that the ability to detect faults does not only depend on the number of modules n and testing links m, but also in general, on the structure of the network (i.e. the exact interconnection pattern of testing links). Systems which are optimal with respect to fault detectability are presented for various n and m, and a correspondence between detectability and the girth of the system testing graph is presented. The effect of system structure on diagnosability is briefly discussed.

INTRODUCTION

An early paper by Preparata, Metze and Chien [1] formulated a model of digital system diagnosis which has been widely researched. In this model a system is represented as a graph with the nodes corresponding to replaceable modules in the system and directed arcs corresponding to the diagnosis of one module by another.

It is assumed that a fault free module can detect <u>any</u> fault in the module it is testing and that, if the testing module is faulty, we cannot rely on its judgement. This assumption may be somewhat optimistic in systems where the modules are complex (e.g. microprocessors) due to the difficulties associated with testing a complex module and the possibility that in the network the diagnosing module has incomplete access to the module being tested. In this paper we consider the effect of incomplete fault detection (i.e. a fault free module may fail to detect some faults in a module it is testing) on system level detection.

We consider the following model The system S is composed of n interconnected units, each able to test and diagnose other units and can be represented by a directed diagnostic graph G(V, E), $|V|=n, |E|=m$, where each node $v_i \varepsilon V$ corresponds to a unit of S and there is a directed arc from v_i to v_j if the unit v_i is able to test and diagnose v_j. Each arc (test) has associated with it a probability p_i where p_i is the coverage of the test t_i for possible faults in unit v_j. (i.e. the percentage of faults in v_j which can be detected by v_i). Each diagnostic test t_i determines a binary outcome: a 0 outcome on t_i means that unit v_i has judged unit v_j as fault-free, whereas a 1 outcome indicates that unit v_i has judged unit v_j as faulty. By the previous assumptions, each test outcome has a probability associated with it as shown in Figure 1.

○ fault free unit ● faulty unit

	v_i	$\longrightarrow v_j$	outcome from t_i	probability of this outcome
a)	○	○	0	1
b)	○	●	1	p_i
c)	○	●	0	$1-p_i$
d)	●	○	0	r_i
e)	●	○	1	$1-r_i$
f)	●	●	1	s_i
g)	●	●	0	$1-s_i$

Figure 1

From figure 1 we see that the probabilistic model we propose is a simplified version of a model proposed by Blount [13], as it doesn't consider the possibility of having a 1 outcome for case a). Note that the Preparata, Metze, and Chien model [1] corresponds to the special case $p_i=1$.

---

*This work was supported in part by the National Science Foundation under Grant MCS 77-21569 and by the National Council of Research of Italy through a NATO fellowship for Dr. Simoncini.

1. MEASUREMENT OF DETECTABILITY OF FAULTS IN A SYSTEM

As a consequence of incomplete coverage, (i.e. $p_i<1$) it is possible that one or more faulty units are actually present in the system S, even if the test outcome S_0 of all 0's is obtained.*

Hence it is necessary to consider problems associated with detecting faults in this model. We introduce a new parameter associated with S; the <u>detectability</u> of S, D(S) which defines the <u>effectiveness</u> of S in detecting its own faults (i.e. the probability that S can detect a fault in some set of its modules). Correspondingly U(S)=1-D(S) is the <u>undetectability</u> of S.

<u>Definition:</u> The undetectability parameter U(S) of a system S is the probability of obtaining the test outcome S_0, when one or more units are actually faulty in S.

For the moment, we will consider the evaluation of the parameter $U_k(S)$, that is the undetectability of S, with respect to the set F_k of fault patterns of given cardinality k, k=1,2,...,n. We shall see later the relation of $U_k(S)$ with U(S).

There are $\binom{n}{k}$ different fault patterns of cardinality k. Any such pattern F_k^j can be represented in the diagnostic graph G(V,E) as in <u>Figure 2</u>.

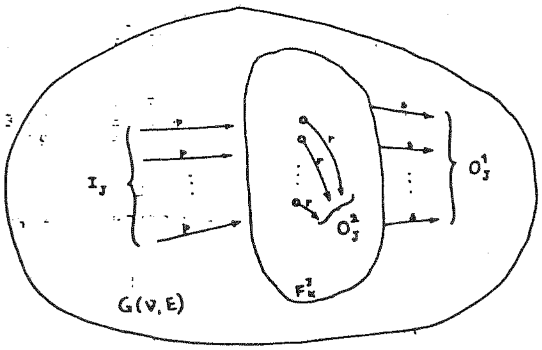

Figure 2

In order to simplify the mathematics we make the following assumptions:

a) all fault patterns of cardinality k are equally probable i.e. $P\left(F_k^j \mid F_k\right)=\frac{1}{\binom{n}{k}}$

b) all p_i are equal to p, all r_i are equal to r and all s_i are equal to s and are independent of each other.

---

*S_0 is the test outcome defined by $t_i=0$ for all i. In the deterministec model of [1], S_0 can occur only when all the units of S are actually fault-free or all are faulty.

For any specific fault pattern F_k^j let the number of arcs to elements in F_k^j from elements outside F_k^j be I_j; the number of arcs from elements in F_k^j to elements outside F_k^j be o_j^1 and the number of arcs among elements in F_k^j be o_j^2. Then the probability of obtaining the test outcome S for the fault pattern F_k^j is given by

$$P\left(S \mid F_k^j\right) = (1-p)^{I_j} \quad (1-s)^{o_j^2} \, r^{o_j^1}$$

$$\text{Then } U_k(S) = \sum_{j=1}^{\binom{n}{k}} P(S_o \mid F_k^j) \cdot P(F_k^j \mid G_k)$$

$$= \frac{1}{\binom{n}{k}} \sum_{i=1}^{\binom{n}{k}} (1-p)^{I_j} (1-s)^{o_j^2} \, r^{o_j^1}$$

If p_k is the probability of the system having any fault of cardinality k then we can express the undetectability U(S) as:

$$U(S) = \sum_{k=1}^{n} p_k U_k(S)$$

In this equation the value of p_k is independent of the particular structure of S. It depends only on the failure rate of each unit in S and on the time between successive applications of the diagnostic tests in S.

We also consider the expression

$$U_t(S) = \sum_{k=1}^{t} p_k U_k(S)$$

which represents the undetectability of S if we disregard the possibility of faults of cardinality k>t.

2. RELATION BETWEEN DETECTABILITY AND STRUCTURE OF S.

We will now consider the effect of the structure of the system on its detectability. We show that for different systems with the same number of arcs and edges, the detectability is not necessarily equal. This difference can be attributed to structure. We characterize some systems with optimal structure.

<u>Definition:</u> A system S with n nodes and m arcs is optimal with respect to detectability if for any S; having n nodes and m arcs $U_k(S)<U_k(S')$ for all k=1,...,n.

<u>Defnintion:</u> A system S with n nodes and m arcs is optimally t-detectable if for any S' having n nodes and m arcs, $U_k(S)\leq U_k(S')$ for all k=1,...,t,t<n.

If $U_k(S)$ is minimal for all k=1,...,n (k=1,...,t), the expression for U(S) (and $U^t(S)$) is also minimal since p_k,k=1,...,n (k=1,...,t,t<n), doesn't depend on the structure of S. Therefore

we consider the expression of $U_k(S)$ and minimize it, extending the results to $U(S)$ and $U^t(S)$. For a system S, if we consider the set F_k of all fault patterns of cardinality k, we have

$$U_k(S) = \frac{1}{\binom{n}{k}} \sum_{j=1}^{\binom{n}{k}} (1-p)^{I_j} (1-s)^{o_j^2} r^{o_j^1}$$

However, we also have the following constraints:

1) $\sum_{j=1}^{\binom{n}{k}} I_j = m \binom{n-2}{k-1}$

2) $\sum_{j=1}^{\binom{n}{k}} o_j^1 = m \binom{n-2}{k-1}$

3) $\sum_{j=1}^{\binom{n}{k}} o_j^2 = m \binom{n-2}{k-2}$.

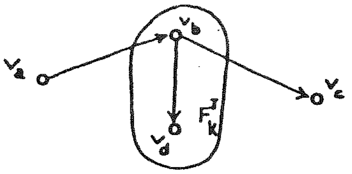

Figure 3

The first constraint is obtained as follows: an arc from v_a to v_b (Figure 3) contributes to the term I_j for a specific fault pattern F_k^j if and only if $v_a \notin F_k^j$ and $v_b \in F_k^j$. There are $\binom{n-2}{k-1}$ such patterns in F_k. As $|E|=m$, the first constraint follows. The second constraint is obtained as follows: an arc from v_b to v_c contributes to the term o_j^1 for a specific fault pattern F_k^j if and only if $v_b \in F_k^j$ and $v_c \notin F_k^j$. There are $\binom{n-2}{k-1}$ such fault patterns in F_k. As $|E|=m$, the second constraint follows.

The third constraint is obtained as follows: an arc from v_b to v_d contributes to the term o_j^2 for a specific fault pattern F_k^j if and only if $v_b \in F_k^j$ and $v_d \in F_j^j$. There are $\binom{n-2}{k-2}$ such fault patterns F_k. As $|E| = m$, the third constraint follows. The minimization of (2) subject to the constraints 1), 2) and 3) has the following solution:

$$I_j = \frac{m(n-k)k}{n(n-1)} = I$$

$$o_j^1 = \frac{mk(n-k)}{n(n-1)} = o^1 = I \qquad j=1,\dots,\binom{n}{k} \qquad (3)$$

$$o_j^2 = \frac{mk(k-1)}{n(n-1)} = o^2$$

and the minimal expression for $U_k(S)$ is:

$$U_k(S)_{min} = (1-p)^I (1-s)^{o^2} r^{o^1} = (1-p)r^I (1-s)^{o^2}$$

when k=1, $o_j^2 = o$ and $I_j = o_j^1 = \frac{m}{n}$ for $j=1,\dots\binom{n}{k}$. Since in any graph I_j and o_j are integers, the lower bound on $U_k(S)$ can be matched only when $\frac{m}{n}=L$ is an integer. (i.e. the number of interconnections is an integer multiple of the number of nodes). Let us consider the class of <u>regular</u> graphs.

<u>Definition</u>: A graph G=G(V,E) is <u>regular</u> of degree 2L if each of its n nodes has exactly L incoming arcs and L outgoing arcs.

<u>Theorem 1:</u> Among all the possible connections for which $\frac{m}{n}=L$ is integer, the systems which are optimally detectable (optimally t-detectable) have a regular diagnostic graph.

<u>Proof:</u> In a regular graph for k=1, $I_j = o_j^1 = \frac{m}{n} = L$; $U_n(S)$ is minimal and therefore from the definition of optimality the theorem follows. This is because detectability is directly related to the worst case number of interconnections from fault-free to faulty modules. If $\frac{m}{n} = L$ and the graph is regular, there exists a node with fewer than L incoming edges, thus decreasing the detectability of such a system.

<u>Lemma 1</u>: All regular graphs are optimally 1-detectable. All regular graphs minimize $U_{n-1}(S)$.

<u>Proof:</u> In regular graphs all faults patterns for k=1 have $I_j = \frac{m}{n} = L$ and $o_j^1 = \frac{m}{n} = L$ matching the required I_j and o_j^1 in (3) and similarly for fault patterns of cardinality n-1, $I_j = \frac{m}{n} = L$, $o_j^1 = \frac{m}{n} = L$ and $o_j^2 = L(n-2)$.

The lower bound for $U_k(S)$ predicted by (4) can only be obtained by a regular graph. However not all regular graphs will satisfy this lower bound and in fact for some values of m, n, and k, there may not exist any regular graph which matches this lower bound.

We will now consider some cases for which the predicted lower vound for $U_k(S)$, derived in Theorem 1, can be matched by a regular graph.

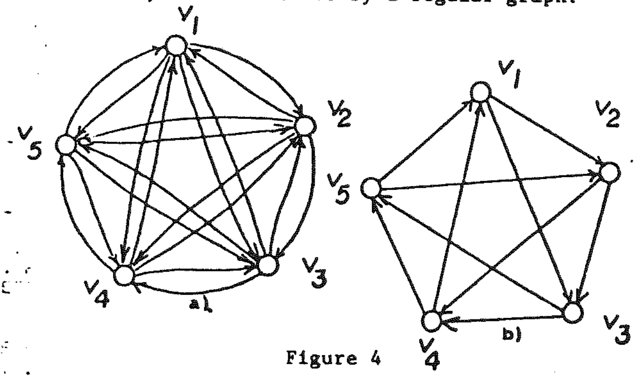

Figure 4

<u>Definition</u>: A system S is a $D_{\delta t}$ system if there exists an arc from v_i to v_j if and only if $j-i = \delta a$ (modulo n), a=1,2,...,t. D_{1L} systems have edges

from a node i to i+1, i+2,...,i+L.

Thus the system of Figure 4A is a D_{1L} system for L=n-1=4 and the system of Figure 4b is a D_{1L} system for $L = \frac{n-1}{2} = 2$. D_{1L} systems with L=(n-1)/2, n odd, can be shown to yield the same values of I and σ_j^2 as (3) and hence are optimal.

Theorem 2: D_{1L} systems for L=n-1 and $L=\frac{n-1}{2}$, n odd, are optimally detectable systems.

Proof: Follows directly from the fact that the actual value of $U_k(S)$, for k=1,...,n, for D_{1L}, L=n-1 or $L=\frac{n-1}{2}$ and n odd, corresponds to the minimal theoretical value obtainable, given by Theorem 1.

The D_{1L} system, with L=n-1, is the only regular graph by which we can connect n nodes with m=n(n-1) arcs. However, the D_{1L} system with $L=\frac{n-1}{2}$ and n odd is not the only regular graph we may have given n nodes and $m=\frac{n(n-1)}{2}$ arcs. The following example compares $U_k(S)$ for two different regular systems with n=5 and m=10 ($L=\frac{m}{n}=2$).

Example: Let us consider the graphs $G(D_{12})$ and G' for n=5, m=10 in Figure 5.

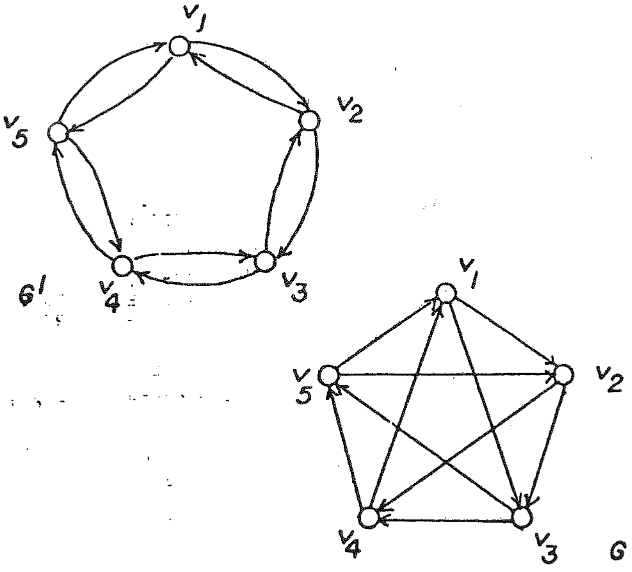

Figure 5

a) As both graphs are regular, the value for $D_1(G)=D_1(G')$ and is from (2):

$$U_1(G) = U_1(G') = (1-p)^2 r^2$$

b) For k=2 we have:

$$U_2(G) = (1-p)^3 (1-s) r_3$$

While for G', as we have two subsets of fault patterns of cardinality 2 with different I_j and σ_j we have

$$U_2(G') = \frac{1}{10} \left(5(1-p)^2 (1-s)^2 r^2 + 5(1-p)^4 r^4\right)$$

and $U_2(G)<U_2(G')$; $U_2(G)=U_2(G')$ only where 1-s=(1-p)r.

For K=3, 4, 5 we have the following values

c) k=3: $U_3(G) =(1-p)^3(1-s)^3r^3$;

$$U_3(G')=\frac{1}{10}\left[5(1-p)^2(1-s)^4r^2+5(1-p)^4 (1-s)^2r^4\right];$$

$U_3(G)=U_3(G')$ only when 1-s=(1-p)r, otherwise $U_3(G)<U_3(G')$.

d) k=4: since k=n-1 we have:

$$U_4(G)=U_4(G')= (1-p)^2(1-s)^6r^2$$

e) k=5: trivial case $U_5(G) =U_5(G')=(1-s)^{10}$

We thus see that for the system $G, U_k(G)<U_k(G')$ for k=1,...,n and therefore has better detectability Since both systems have the same value of n and m, this superiority can be attributed to structure.

The D_{1L} graphs can be shown to be optimally k-detectable for certain values of L,n,k as outlined in Figure 6.

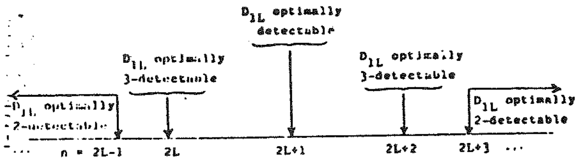

Figure 6

n	
≥2L-1	D_{1L} optimally 2-detectable
2L	D_{1L} optimally 3-detectable
2L+1	D_{1L} optimally detectable for all k
2L+2	D_{1L} optimally 3-detectable
≥2L+3	D_{1L} optimally 2-detectable

2.2. Detectability of regular graphs

All the results obtained so far have been based on finding actual graphs for which the value of $U_k(S)$, k=1,...,n was equal to the theoretical minimal value obtained earlier by (4) and (6). Unfortunately for those cases not represented in Figure 6, the expression for $U_k(S)$, for regular graphs, contains more than two terms. It is thus not possible to prove optimality by a simple comparison with the theoretical lower bound of $U_k(S)$ given by (4) or (6). For general values of n and L and k>2, the graph which has optimal detectability may depend on the relative value of the probabilities p,r and s of detecting a fault in each unit. We consider the case when it is possible to give a value to p through simulation of the units in the system, but we cannot evaluate the values of r and s. In this case we assume that r=s=1/2. For this model let us introduce another parameter for a regular graph G which is strongly related to the detectability of G.

Definition: The girth of a graph G denoted by g(G), is the length of a shortest undirected cycle in G.

The graph girth is related to detectability in the following way. If two graphs G and G' have the same number of edges and nodes and G has girth g and G' has girth g'>g, then there exists a lower bound p' of p such that if $p \geq p'$ G' has a better detectability than G.

This is because fault patterns with high probability of non-detection have fewer incoming edges from fault-free modules. If the faulty modules form a connected cycle the number of incoming edges from fault-free modules would be decreased. The girth of a graph defines the minimum number of faulty modules which could define such a cycle.

Unfortunately very little is known in graph theory on how to construct regular graphs of some minimal girth. The only previous results are as follows:

a) a regular graph with n nodes and m arcs of girth $g>3$ can be constructed only if $m<|\frac{n^2}{4}|$, [14]

b) a regular graph with n nodes and $m=nL$ arcs of girth $g>4$ can be constructed only if $n>4L^2+1$.[15]

Graphs which match a) above are bipartite, (i.e. graphs whose set of nodes V can be partitioned into two subsets V_1 and V_2 such that every arc of G connects an element of V_1 to an element of V_2).
From the preceeding we obtain the following result.

Theorem 13: The complete bipartite graph G_B for n even and for $p>p$, has better detectability than the graph D_{1L} with the same number of nodes and arcs.
Proof: As both graphs are regular, they have the same expressions for $U_1(S)$. They also have the same expression for $U_2(S)$ since both have nL fault patterns for which $I_j=2L-1$ and $\frac{n(n-2L-1)}{2}$ fault patterns for which $I_j=2L$. Since $U_3(D_{1L})$ is expressed as (12) and $U_3(G_B)$ is expressed as (11), the Theorem follows.

Example: Let us consider for n=8, L=2, D_{12} and G_B shown in Figure 7.

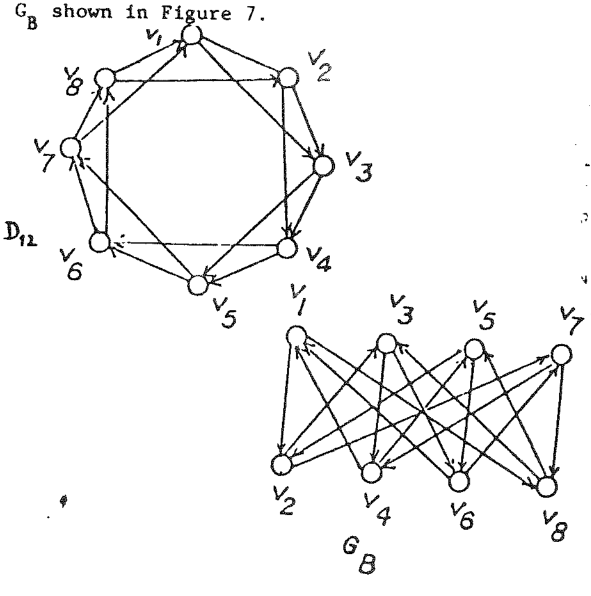

Figure 7

The expressions for $U_k(D_{12})$ and $U_k(G_B)$ for k<3 are:

$$U_1(D_{12}) = U_1(G_B) = \frac{1}{4}(1-p)^2$$

$$U_2(D_{12}) = U_2(G_B) = \frac{1}{448}\{16(1-p)^3 + 12(1-p)^4\}$$

$$U_3(D_{12}) = \frac{1}{3584}\{8(1-p)^3 + 24(1-p)^4 + 24(1-p)^5\}$$

while

$$U_3(G_B) = \frac{1}{3584}\{48(1-p)^4 + 8(1-p)^6\}$$ and

$U_3(G_B)<U_3(D_{12})$ for any value of p.

G_B for this example can be redrawn as two loops of four nodes connected to each other as shown in Figure 8.

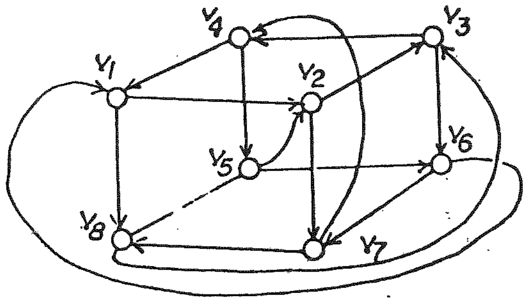

Figure 8

With the same arguments used to develop Theorem 13 we can prove the following:

Theorem 14: Among all regular graphs for which $n>4L^2+1$, those graphs for which the girth $g>4$, have a value of $U_4(S)$ which is less than that of graphs with the same number of nodes and arcs with girth $g<4$ for $p>p$.

Example: Let us consider the case n=20 and L=2. A graph with girth g=5 is shown in Figure 9. From Figure 9 we can see that this graph is drawn as four loops of 5 nodes each, connected together.

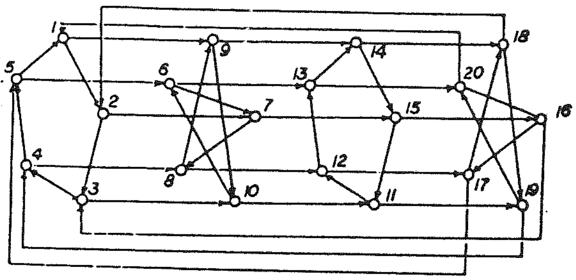

Figure 9

When a graph of some minimum girth g'>g cannot be constructed, it is necessary to construct a graph which has the fewest number of distinct loops of girth g, as illustrated in the following example.

Example: Let us consider the case $m=7$ and $L=2$ and compare $U_k(S)$ for D_{12} and the graph G of Figure 10:

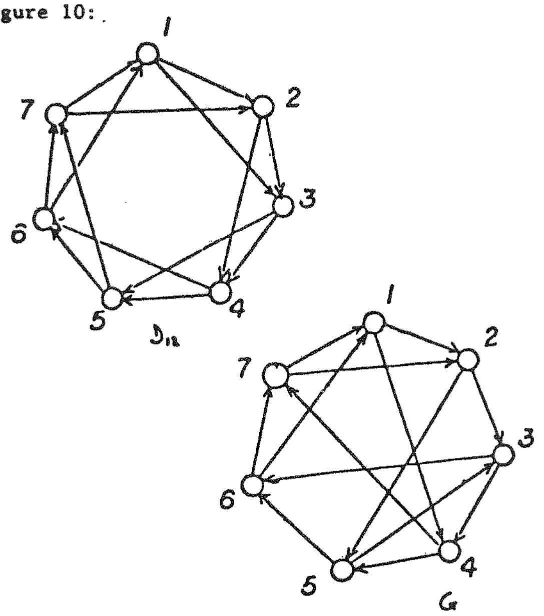

Figure 10

The expressions for $U_k(D_{12})$ and $U_k(G)$ for $k<3$ are:

$$U_2(D_{12}) = U_1(G) = \frac{1}{4}(1-p)^2$$

$$U_2(D_{12}) = U_2(G) = \frac{1}{336}\{14(1-p)^3 + 7(1-p)^4\}$$

$$U_3(D_{12}) = \frac{1}{2240}\{7(1-p)^3 + 21(1-p)^4 + 7(1-p)^5\}$$

$$U_3(G) = \frac{1}{2240}\{6(1-p)^3 + 24(1-p)^4 + 4(1-p)^6\} \text{ and}$$

$$U_3(G) < U_3(D_{12}) \qquad \text{for any value of } p.$$

We have been unable to develop a general proof or a counter-example to the conjectrued relationship between detectability and graph girth. We hypothesize that the detectability of a graph G_1 with girth g_1 will be better than that of a graph G_2 with girth $g_2<g_1$.

Specifically we hypothesize that:

$$U^t(G_1) = U^t(G_2) \qquad \text{for} \qquad t<g_2$$

$$U^t(G_1) < U^t(G_2) \qquad \text{for} \qquad g_2<t<g_1$$

We further conjecture that if $g_1=g_2$ the graph with the fewest number of distinct loops of degree g_1 will have better detectability for $t=g_1$ and the graphs will have the same detectability for $t<g_1$.

3. CONCLUSIONS

In this paper a new parameter, the detectability of a given system has been introduced. The relation between this parameter and the structure of the system has been demonstrated. Optimal graphs for some cases have been presented, and a conjectured general relationship between graph girth and diagnosability has been presented.

The graphs described in Section 2.2 seem to have a well characterized structure which needs further investigation. We have not examined the relation between diagnosability and the structure of the connections in a probabilistic model. This problem is very difficult since the number of fault patterns which can produce a given syndrome is much greater in the probabilistic model than the corresponding number in a deterministic model, and also each fault pattern has associated with it a non zero probability of producing a given syndrome. These problems will be considered in future papers.

REFERENCES

[1] F.P. Preparata, G. Metze, R.T. Chien: "On the Connection Assignment Problem of Diagnosable Systems", IEEE Trans. Elect. Comp., Vol. EC-16, pp. 848-854, December 1967.

[2] F.P. Preparata: "Some Results on Sequentially Diagnosable Systems", Proc. Hawaii Int. Conf. Systems Sciences, Univ. of Hawaii Press, pp. 623-626, Jan. 1968.

[3] S. L. Hakimi, A.T. Amin: "Characterization of Connection Assignment of Diagnosable Systems", IEEE Trans. Comput., Vol. C-23, pp. 86-88, Jan. 1968.

[4] P. Ciompi, L. Simoncini: "The Boundary Graphs: an Approach to the Diagnosability with Repair of Digital Systems", Proc. Third Texas Conf. Systems, Austin, pp. 9.3.1-9.3.9, Nov. 1974, IEEE CSPO n. n. 74-CH0895-3C.

[5] P. Ciompi, L. Simoncini, "On the Diagnosability with Repair of Digital Systems", Tech. Report n. 75001/E Pisa, Convensione Selehia-CNR, Jan. 1975.

[6] A.D. Friedman, "A New Measure of Digital System Diagnosis", Int. Symp. on Fault Tol. Comp., TEC-5, Paris pp. 167-171, June 1975.

[7] F. Barsi, F. Grandoni, P. Maestrini: "Diagnosability of System Partitioned into Complex Units", Int. Symp. on Fault Tol. Comp., FTC-5, Paris, pp. 171-176, June 1975.

[8] F. Barsi, F. Grandoni, P. Maestrini: "A Study of Self-Diagnosis of Digital Systems", Tech. Rept. n. 75003/E, Pisa, Convenzione Selenia-CNR, Feb. 1975.

[9] J.D. Russell, C.R. Kime, "System Fault Diagnosis: Closure and Diagnosability with Repair", IEEE Trans. Comput., Vol. C-24, pp. 1078-1089, Nov. 1975.

[10] J.D. Russell, C.R. Kime: "System Fault Diagnosis: Exposure and Diagnosability without Repair", IEEE Trans. Comput., Vol. C-24, pp. 1155-1161, Dec. 1975.

[11] S.N. Mahhwari, S. L. Hakimi: "On Models for Diagnosable Systems and Probabilistic Fault Diagnosis", IEEE Trans. Comput., Vol. C-24, pp. 228-236, Jan. 1976.

[12] S. Karunanithi, A.D. Friedman: "System Diagnosis with t/s Diagnosability", Int. Conf. on Fault Tolerance Comp., FTCS-7, Los Angeles, pp. 65-71, June 1977.

[13] M.L. Blount, "Probablistic Treatment of Diagnosis in Digital Systems", Int. Conf. on Fault Tol. Comps., FTCS-7, Los Angeles, pp. 72-80, June 1977.

[14] F. Harary, Graph Theory, Addison-Wesley, 1969.

[15] F. Harary, R.Z. Norman, D. Cartwright, Structural Models: An Introduction to the Theory of Directed Graphs, John Wiley & Sons Inc., 1965.

[16] A.J. Hoffman, R.R. Singleton, "On Moore Graphs with Diameters 2 and 3", IBM J. Res. Develop., pp. 497-504, Nov. 1960.