Skewed coherent electrostatic structures: their electric field and particle distributions

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Abstract

We present a study of the electric field and of the statistical distributions of the electrons and ions of asymmetric solitary structures in a collisionless plasma. We devise a "structure function", related to the electric potential, which obeys a generalised Korteweg de Vries equation: waveforms associated with unsymmetrical electron and ion holes, monotonic and non monotonic double layers and tripolar spikes are found in this way. The electric potential thus found is then introduced into Poisson's equation. We show that the electron and ion distributions which solve this equation, subject to appropriate boundary conditions, are elliptic functions of energy and we analyse in detail their singular structure.

1. Introduction

SKEWED coherent electrostatic waves have been reported in a wide variety of space plasmas. Phenomena falling under this rubric include non monotonic double layers, isolated electrostatic structures, asymmetrical electron and ion holes and, lately, tripolar spikes [1]. At variance with the properties of the well known electrostatic solitary waves, these structures display a distinctive lack of symmetry (or skew) in the spatial distribution of their electric potential.

Since skewed electrostatic waves (**SkEWs** for short) are intermingled with symmetrical solitary waves (sometimes both appear in the same time series) it is natural to analyse them by the collisionless kinetic approach, so successfully used for the latter structures. However, this approach presents a number of challenges, which precisely originate from the mentioned skew of the potential waveform.

One such challenge is the construction and solution of a differential equation governing the potential amplitude: indeed, the solutions of all of the known model equations have, in a way or another, some built-in symmetry. A second difficulty is the fact that the distributions of the particles sustaining the SkEW are in general different on each side of the particles' potential barrier: this implies that Poisson's equation must be solved separately on each side of the barrier.

We tackle the former task in Section 3. Using a new rectification procedure, we work out a generalised Korteweg de Vries equation for the potential related "structure function" as a direct consequence of the potential's morphological and functional properties. We do so without making any reference to the velocity distributions of the electrons and of the ions which sustain the SkEW, nor do we use any reductive, regular perturbation technique.

The second task is tackled in Section 4. Using the properties of the structure function, we reduce Poisson's equation to an integral equation of the Abel type, which has an unique structure over the whole extent of the SkEW. We solve this equation, subject to appropriate boundary conditions, in favour of the electron and ion distributions, which turn out to be elliptic functions of the particle energy and are affected by a number of integrable singularities.

2. Assumptions, notation and boundary conditions

Let the SkEW develop in a steady state, x-dependent, electrostatic, collisionless plasma made of electrons and ions labelled by subscript $\mathrm{e,i}$. Let $\lambda_{\mathrm{De}_\infty} x$ and $T_{\mathrm{e}_\infty}\phi/e$ be the space coordinate and electric potential (Fig. 1) and $v_{\mathrm{Te,i}_\infty} v_{\mathrm{e,i}}$, $-T_{\mathrm{e,i}_\infty} U_{\mathrm{e,i}}$, $T_{\mathrm{e,i}_\infty} W_{\mathrm{e,i}}$, $n_{\mathrm{e,i}_\infty} f_{\mathrm{e,i}}(x, v_{\mathrm{e,i}})$ the particle velocities, potential and total energies and distribution functions, where $\lambda_{\mathrm{De}_\infty}$, $T_{\mathrm{e,i}_\infty}$, n_{i_∞}, $n_{\mathrm{e}_\infty} = Z_\mathrm{i} n_{\mathrm{i}_\infty}$ are the electron Debye length, the particle temperatures and densities as $x \to +\infty$, and $+Z_\mathrm{i}e$ is the ion charge. The bi-directional distribution functions $F_{\mathrm{e,i}}(W_{\mathrm{e,i}}) = f_{\mathrm{e,i}}(x, +\surd\{2[W_{\mathrm{e,i}} + U_{\mathrm{e,i}}]\}) + f_{\mathrm{e,i}}(x, -\surd\{2[W_{\mathrm{e,i}} + U_{\mathrm{e,i}}]\})$ are subject to the **boundary conditions**

$$\text{for } x \to +\infty: \ F_\mathrm{e}(W_\mathrm{e}) = \surd 2 e^{-W_\mathrm{e}},\ F_\mathrm{i}(W_\mathrm{i}) = \surd 2 e^{-W_\mathrm{i}} \text{ only if } W_\mathrm{i} \geq 0. \quad (1)$$

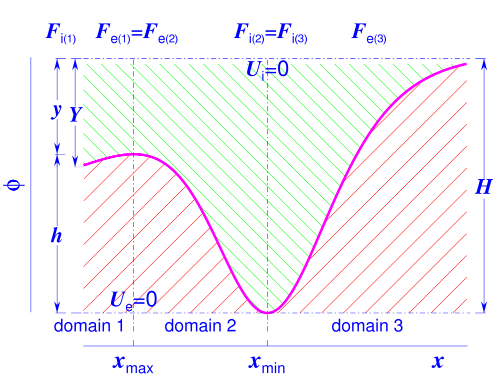

Figure 1: The potential ϕ of a typical SkEW vs. space x. Shown are the potential jumps H, h, Y, y, the positions of the electron (ion) potential barriers at $x = x_{\mathrm{min}}$ ($x = x_{\mathrm{max}}$), the space partition into the three domains $j = 1, 2, 3$, the corresponding equivalence relations between the distributions $F_{\mathrm{e,i}(j)}$ of the particles and the zeroes of their potential energies $U_{\mathrm{e,i}}$. Trapped electrons (ions) have position and energy in the red (green) area.

3. Waveform rectification and gKdV equation

Any waveform whose domain is partitioned as in Fig. 1 may be "rectified" by the mapping [2]

$$s = \tau\surd\{\surd h + \sigma\surd[\phi - \phi(x_{\mathrm{min}})]\}, \quad (2)$$

where $[\sigma, \tau] = [-1, -1], [-1, 1], [1, 1]$ respectively in domains $1, 2, 3$: s is a monotonic function of x obeying a generalised Korteweg de Vries equation (gKdV) solved by [3]

$$s = [(Z_+ + Z_-)r - (Z_+ - Z_-)]/2, \qquad r = \frac{\surd(1-a)\tanh(kx/2)}{\surd[1 - a\tanh^2(kx/2)]}. \quad (3)$$

Here, k and $Z_\pm$ are the decay rate of ϕ and the values of $\pm s$ as $x \to \pm\infty$.

Varying the constant a, gives a wide class of SkEWs (Figs. 1 and 2).

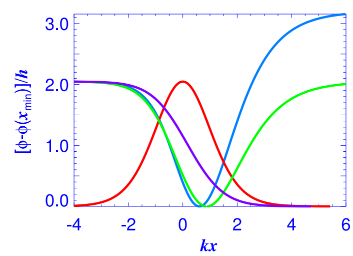

Figure 2: The electric potential ϕ vs. space x for several values of the constant a (Eq. (3)). Symmetric electron hole (—); asymmetric ion hole (—); monotonic (—) and non monotonic (—) double layer. k is the decay rate of $\phi(x)$ as $x \to \pm\infty$ and h is the potential jump (Fig. 1).

4. Singular, elliptic particle distributions

Unlike in symmetric solitary waves, in SkEWs, the particle distributions on each side of the respective potential barrier are not symmetric: in general, they differ in domains $1, 2, 3$ (Fig. 1). Thus, beside the BGK formula [4], which, for $W_\mathrm{i} < 0$, relates $F_{\mathrm{i}(3)}(W_\mathrm{i})$ to the boundary conditions of Eq. (1), we write $F_{\mathrm{i}(1)} = F_{\mathrm{i}(2,3)} + \Delta F_\mathrm{i}$ and $F_{\mathrm{e}(1,2)} = F_{\mathrm{e}(3)} - \Delta F_\mathrm{e}$ where, letting $W_\mathrm{e} = W$, $W_\mathrm{i} = -Z_\mathrm{i} T_{\mathrm{e}\infty}[W + H]/T_{\mathrm{i}\infty}$, the **distribution anomalies**

$$\Delta F_{\mathrm{e,i}}(W_{\mathrm{e,i}}) = (Z_+ - Z_-)\int_{A_{\mathrm{e,i}}(W)}^{B_{\mathrm{e,i}}(W)} \mathrm{d}s \frac{Q(s)}{\surd T(W,s)} \quad (4)$$

are known in terms of 1st and 2nd kind **elliptic integrals**. Here $T(W,s) = \pm[s^4 - \surd h s^2 + h + W]$ ($+$ for ions and $-$ for electrons), $Q(s)$ is a polynomial and $A_{\mathrm{e,i}}$, $B_{\mathrm{e,i}}$ are *algebraic* functions of W. Notice that $\Delta F_{\mathrm{e,i}} = 0$ for symmetric solitary waves ($Z_+ = Z_-$). The resulting distributions $F_{\mathrm{e,i}}$ are shown in Fig. 3.

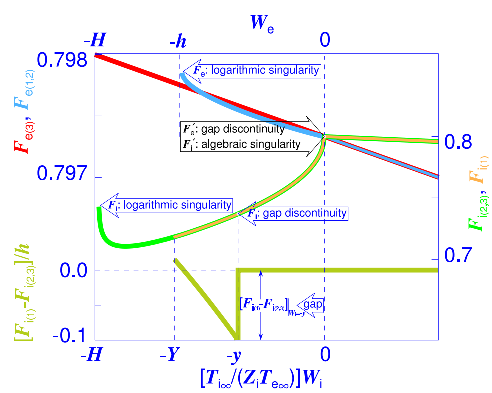

Figure 3: The distributions $F_{\mathrm{e,i}(j)}$ vs. energy $W_{\mathrm{e,i}}$ and their singular structure. $Z_\mathrm{i}e$ is the ion charge, $T_{\mathrm{e,i}\infty}$ are the temperatures as $x \to +\infty$ and $j = 1, 2, 3$ and H, h, Y, y are the space domains and the potential jumps (Fig. 1).

5. Conclusions

An elementary electron and ion plasma configuration in a steady, collision-less state, distributed along one space coordinate, sustains an electric potential which displays a skewed coherent electrostatic waveform (SkEW). Our analysis of this plasma configuration reveals that: (*a*) the differential equation for the SkEW may be mapped into a generalised Korteweg de Vries equation; (*b*) the distributions of the particles in the SkEW are not Maxwellian, rather elliptic functions of energy, which are affected by a number of integrable singularities.

According to our interpretation, the observation of SkEWs in collisionless plasmas provides direct evidence for a new class of nonlinear singular solutions of the Vlasov-Poisson equations.

References

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