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A transformed gamma process for bounded degradation phenomena

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Abstract

Most of the stochastic models adopted to describe the evolution over time of degradation phenomena of technological units assume that their degradation level can increase indeterminately. However, these degradation phenomena are typically subjected to obvious bounds, if only because technological units have finite size. In fact, very often, this inconsistency does not significantly affect the effectiveness of unbounded degradation models, since degrading units are usually assumed to fail when their degradation level exceeds a failure threshold that is much smaller than the obvious bounds. Nevertheless, in some cases, due to the very nature of the underlying degradation mechanism, less obvious bounds could exist, which are not necessarily far from the failure thresholds. The question that arises is whether the use of a bounded degradation model, in this latter type of experimental situations, could be beneficial. For this purpose, since a bounded degradation process should necessarily have dependent increments, in this paper we investigate the potential of a new bounded transformed gamma (TG) process to adequately describe bounded degradation phenomena and predict their future evolution. Differently from other existing gamma process based bounded degradation models, here the upper bound is treated as an unknown parameter that has to be estimated from the available degradation data. A numerical example is presented where the parameters of the proposed model are estimated from simulated data. Then the model is applied to a set of wear measures of cylinder liners that equip a diesel engine for marine propulsion, which have also stimulated this study. Model parameters are estimated by using the maximum likelihood (ML) method. The fitting ability of the proposed new bounded process is compared to that of an unbounded gamma process, which was previously adopted to analyze the same liner wear data. Obtained results are critically discussed in the paper.

KEYWORDS
bounded degradation phenomena, maximum likelihood estimation, remaining useful life, residual reliability, transformed gamma process

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1 | INTRODUCTION

Stochastic models usually adopted to describe degradation phenomena of technological units, such as the gamma, the inverse Gaussian, and the Wiener ones (e.g., see Kahle et al.[1]), assume that the degradation level can increase indeterminately. However, this assumption is not always realistic. In fact, if only because their physical dimensions are finite, the degradation processes of technological units are certainly bounded from above. For example, it is obvious that an upper bound surely exists when the degradation is quantified in terms of loss of material. Likewise, an upper bound also surely exists when the degradation state of a monotonic increasing process is expressed in terms of percent reduction of the quality of the unit with respect to its initial value. In fact, in this latter case the degradation level is obviously constrained to range between 0 and 100.

Despite this evident incongruity, a vast variety of successful applications to real world degradation data demonstrate that unbounded degradation models are able to provide accurate descriptions of many real world degradation phenomena. In fact, the reason why the existence of these obvious bounds does not inhibit the successful use of unbounded model is that technological units are usually assumed to fail when their degradation level passes a threshold limit that is typically largely far from the obvious bounds. Yet, this is not always the case. In fact, there are phenomena where, due to inherent features of the degradation causing mechanism, it is possible to say that less obvious bounds certainly exist. Indeed, in most of these cases, the exact values of these bounds cannot be determined a priori, because the considered degradation phenomena are not sufficiently well understood. Thus, in particular, it is not possible to say if they are close to the considered conventional threshold limits.

The question that arises is whether, in this latter kind of experimental situation, the use of a degradation model that explicitly accounts for the presence of an upper bound could be beneficial. With this objective in mind, and focusing our interest on increasing degradation phenomena, in this paper we investigate the potential of transformed gamma (TG) process (Giorgio et al.[2]) to tackle this specific modeling task. The idea of using the TG process is mainly motivated by the circumstance that this process has increments that are not independent. Indeed, it is not hard to recognize that a monotonic increasing bounded degradation process should necessarily have this feature, if only because (for example) both the conditional mean and variance of its future degradation increment should necessarily go to zero as the current degradation level approaches the upper bound.

It can be highlighted that the idea of using a transformation of the gamma process to describe bounded degradation phenomena is not entirely new. In fact, just Giorgio et al.[2] used a bounded TG process with a log-based state function to describe the percentage reduction of strength of certain polymers. Contemporarily, Ling et al.[3] used a gamma process to describe the negative logarithm of the percentage reduction of light intensity of some light emitting diode (LED), an approach that is (implicitly) consistent with the assumption that the percentage reduction of light intensity follows a log-transformation of the gamma process. Indeed, although the motivating ideas of these papers were slightly different, the proposed modeling solutions can be deemed quite similar. Likewise, Deng and Pandey[4] used a logit transformation (e.g., see Baum[5]) of the gamma process to describe the wall thickness reduction of certain pipes.

However, all these papers consider upper bounds of the type we have previously labeled as "obvious" and, being relatively simple to determine their values, assume that these bounds are known a priori. In fact, in the case of the former models the bound it is set equal to 100%, while in the case of the latter one it is set equal to the initial wall thickness.

Here, differently than in the mentioned papers, we focus on cases where the upper bound is not obvious, with specific regard to the experimental situations where the available knowledge does not allow to determine its value a priori. In fact, coherently with this assumption, we treat the upper bound as one of the unknown parameters of the TG process and assume that it has to be estimated on the basis of the observed degradation data, together to all the other parameters.

To investigate the potential of the proposed modeling solution, we have formulated and then applied three different characterizations of the bounded TG process to a set of wear measurements of the liners of an 8-cylinder diesel engine equipping a cargo ship of the Grimaldi Lines. In fact, the considered degradation phenomenon has three specific features that have stimulated this study. The first one is that a liner is assumed to fail when its wear level exceeds 4 mm, a value that is far smaller than the thickness of the liner itself, which is equal to 100 mm. The second reason is that the observed wear mechanism cannot lead the liner wear to grow up to its thickness, because the upper bound is expected to be not much larger than the threshold limit. The third reason is that the considered wear phenomenon is not sufficiently well known to define (a priori) the exact value of the upper bound.

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It is worth to underline that, under these circumstances, assuming that the upper bound coincides with the obvious physical bound determined by the thickness of the liner (i.e., 100 mm) would be practically equivalent to assume that the considered wear process is unbounded, because the failure threshold (i.e., 4 mm) is hugely far from it.

The rest of the paper is structured as it follows. The bounded TG process is introduced in Section 2. The reliability function and the conditional distribution of the remaining useful life are formulated in Section 3. The maximum likelihood (ML) estimation of model parameters is addressed in Section 4. A numerical example is presented in Section 5, where the parameters of the proposed model are estimated from simulated data. The example of application, developed on the basis of the real set of wear data of cylinder liners of a diesel engine, which have motivated this work, is presented in Section 6. The results presented in Section 6 are critically discussed in Section 7. Final considerations are given in Section 8.

2 | THE BOUNDED TRANSFORMED GAMMA PROCESS

The bounded transformed gamma (BTG) process $\{W(t);\ t \geq 0\}$ is a bounded monotonic increasing Markovian process that has dependent increments. In fact, once an initial condition is assigned (in this paper we assume that $W(0) = 0$), a BTG process is completely defined by the conditional probability density function (pdf) of its generic increment $\Delta W(t, t + \Delta t) = W(t + \Delta t) - W(t)$, given $W(t) = w_t$, that has the following structure:

$$f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t) = g'(w_t + \delta)\frac{[\Delta g(w_t, w_t + \delta)]^{\Delta\eta(t,t+\Delta t)-1}}{\Gamma(\Delta\eta(t, t + \Delta t))}\exp(-\Delta g(w_t, w_t + \delta)),$$
$$0 < \delta < w_{\text{lim}} - w_t, \tag{1}$$

where w_{lim} denotes the upper bound of the degradation process, $g(w)$ is a non-negative, monotone increasing and differentiable function of the degradation level w ($0 \leq w < w_{\text{lim}}$) with $g(0) = 0$ and

$$\lim_{w\to w_{\text{lim}}} g(w) = \infty, \tag{2}$$

$g'(w_t + \delta)$ is the first derivative of $g(\cdot)$ evaluated at $w_t + \delta$, $\Delta g(w_t, w_t + \delta) = g(w_t + \delta) - g(w_t)$, $\eta(t)$ is a non-negative, monotone increasing function, with $\eta(0) = 0$, $\Delta\eta(t, t + \Delta t) = \eta(t + \Delta t) - \eta(t)$, and $\Gamma(\cdot)$ is the complete gamma function. Hereinafter, the functions $\eta(\cdot)$ and $g(\cdot)$ will be called age function and "bounded" state function, respectively. Indeed, the function $g(\cdot)$ is not bounded above but, for simplicity, we will call it "bounded" state function to intend that it is the state function of the BTG process.

To obtain a fully-operative characterization of the BTG process it is necessary to assign functional forms to $\eta(\cdot)$ and $g(\cdot)$ which under the general model are left unspecified for the sake of flexibility.

Suitable forms for the age function are, for example, those already proposed for the TG process in Giorgio et al.[2,6] such as the classical power-law function $\eta(t) = (t/a)^b$ and the exponential function $\eta(t) = ab[\exp(t/b) - 1]$.

For the state function we suggest the following three functional forms:

$$g_1(w) = -\beta \ln\left(1 - \frac{w}{w_{\text{lim}}}\right), \tag{3}$$

$$g_2(w) = \beta\frac{w}{w_{\text{lim}} - w}, \tag{4}$$

and

$$g_3(w) = \beta \tan\left(\frac{\pi}{2}\frac{w}{w_{\text{lim}}}\right), \tag{5}$$

which provide a good trade-off between model simplicity and flexibility and are differentiable with respect to w, with derivative:

$$g_1'(w) = \frac{\beta}{w_{\text{lim}} - w}, \tag{6}$$

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$$g_2'(w) = \frac{\beta w_{\lim}}{(w_{\lim} - w)^2}, \tag{7}$$

and

$$g_3'(w) = \frac{\beta}{w_{\lim}} \quad \frac{\pi/2}{\cos^2\left(\frac{\pi}{2}\frac{w}{w_{\lim}}\right)}, \tag{8}$$

respectively.

It is worth to note that, when $\eta(t)$ is a linear function of t, so that $\Delta\eta(t, t + \Delta t) \propto \Delta t$, the BTG process reduces to an age-independent process. In fact, from Equation (1), when $\eta(t)$ is linear in t, we have that the conditional distribution of the degradation increment $\Delta W(t, t + \Delta t)$, given the current state w_t, depend only on Δt, being independent of the current age t. Otherwise, when $\eta(t)$ is non-linear, the BTG process is both age- and state-dependent. In fact, in this latter case, the conditional distribution of $\Delta W(t, t + \Delta t)$, given the current state w_t, depends both on Δt and t.

Similarly, the functional form of the state function $g(w)$ determines how the increment $\Delta W(t, t + \Delta t)$ depends on the current degradation state $W(t) = w_t$, given the current age t. In fact, in particular, if $g(w)$ is convex upward, as the state functions (3)-(5), the conditional mean of the degradation increment in the time interval $(t, t + \Delta t)$ of the TG process decreases monotonically with the current state w_t.

From Equation (1), the conditional cumulative distribution function (Cdf) of the degradation increment $\Delta W(t, t + \Delta t)$, given $W(t) = w_t$, can be expressed as:

$$F_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t) = \begin{cases} \frac{\gamma(\Delta g(w_t, w_t+\delta);\Delta\eta(t,t+\Delta t))}{\Gamma(\Delta\eta(t,t+\Delta t))}, & \text{for}\delta < w_{\lim} - w_t \\ 1, & \text{for}\delta \geq w_{\lim} - w_t \end{cases}, \tag{9}$$

where:

$$\gamma(x; a) = \int_0^x u^{a-1} \exp(-u)\ \mathrm{d}u \tag{10}$$

is the (lower) incomplete gamma function. Accordingly, from Equations (1) and (9), the pdf and the Cdf of the degradation level $W(t)$ of a new unit, being $W(t) = \Delta W(0, t)$, are given by:

$$f_{W(t)}(w) = g'(w) \frac{[g(w)]^{\eta(t)-1}}{\Gamma(\eta(t))} \exp[-g(w)], 0 < w < w_{\lim} \tag{11}$$

and

$$F_{W(t)}(w) = \begin{cases} \dfrac{\gamma(g(w); \eta(t))}{\Gamma(\eta(t))}, & \text{for } w < w_{\lim} \\ 1, & \text{for } w \geq w_{\lim} \end{cases}, \tag{12}$$

respectively.

The conditional mean and variance of the degradation increment, given the state w_t at the time t, do not allow for closed form expressions. Yet, they can be easily computed by numerically evaluating the following integrals:

$$E\{\Delta W(t, t + \Delta t)|W(t) = w_t\} = \int_0^{w_{\lim}-w_t} \delta\ f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)\ \mathrm{d}\delta \tag{13}$$

$$V\{\Delta W(t, t + \Delta t)|W(t) = w_t\} = \int_0^{w_{\lim}-w_t} \delta^2\ f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)\mathrm{d}\delta - E^2\{\Delta W(t, t + \Delta t)|W(t) = w_t\} \tag{14}$$

Accordingly, the mean and variance of the degradation level $W(t)$ of new unit are given by:

$$E\{W(t)\} = \int_0^{w_{\lim}} w\ f_{W(t)}(w)\ \mathrm{d}w \tag{15}$$

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and

$$V\{W(t)\} = \int_0^{w_{\lim}} w^2 \, f_{W(t)}(w) \, \mathrm{d}w - E^2\{W(t)\} \ . \tag{16}$$

It is worth to note that, due to the "bounded" nature of the state function $g(w)$, the BTG process has the following peculiar features:

- When $t \to \infty$ the pdf (11) tends to the Dirac delta distribution with support $w_{\lim}$, the mean $E\{W(t)\}$ of the degradation process tends to $w_{\lim}$, and its variance $V\{W(t)\}$, that initially grows with time, approaches to zero.
- For $\Delta t \to \infty$ (given t) the conditional pdf (1) tends to the Dirac delta distribution with support $w_{\lim} - w_t$, the conditional mean $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ of the degradation increment tends to $w_{\lim} - w_t$, and the conditional variance $V\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ approaches to zero. Of course, for $w_t \to w_{\lim}$, the conditional mean $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ tends to 0.
- When $\eta(t)$ is convex downward, for $t \to \infty$ (given Δt) the conditional pdf (1) tends to the Dirac delta distribution with support $w_{\lim} - w_t$, the conditional mean $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ of the degradation increment tends to $w_{\lim} - w_t$, and the conditional variance $V\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ approaches to zero.

Moreover, from Equation (11), we have that, under the proposed state functions (3)-(5), the quantity $w_{\lim}$ acts as a scale parameter for the pdf of $W(t)$. In fact, given that $f_{W(t)}(w)$ can be rewritten as:

$$f_{W(t)}(w) = \frac{1}{w_{\lim}} \, h\left(\frac{w}{w_{\lim}}; \eta(t), \beta\right), \tag{17}$$

where the function $h(z; \eta(t), \beta)$ depends on $\eta(t)$ and on the functional form of $g(w)$ and β only, the (dimensionless) variable $Z(t) = W(t)/w_{\lim}$ has pdf $f_{Z(t)}(z) = h(z; \eta(t), \beta)$ that does not depend on $w_{\lim}$. Indeed, for example, if the state function is modelled by the functional form $g_1(w)$ given in Equation (3), the function $f_{Z(t)}(z)$ has the following expression:

$$f_{Z(t)}(z) = \frac{\beta^{\eta(t)}[-\ln(1-z)]^{\eta(t)-1}}{\Gamma(\eta(t))}(1-z)^{\beta-1}, 0 < z < 1. \tag{18}$$

As a consequence, from Equations (15) and (16) we (obviously) have that, under the suggested state functions (3)-(5), the mean:

$$E\{W(t)\} = \int_0^{w_{\lim}} w f_{W(t)}(w) \, \mathrm{d}w = w_{\lim} \int_0^1 z f_{Z(t)}(z) \, \mathrm{d}z \tag{19}$$

depends linearly on $w_{\lim}$ and also that, being:

$$E\left\{W^2(t)\right\} = \int_0^{w_{\lim}} w^2 f_{W(t)}(w) \, \mathrm{d}w = w_{\lim}^2 \int_0^1 z^2 f_{Z(t)}(z) \, \mathrm{d}z, \tag{20}$$

the variance of $W(t)$ depends linearly on the square of $w_{\lim}$.

In addition, it is interesting to note that, under the function (3), the quantity $w_{\lim} - w_t$ acts as a scale parameter in the conditional pdf $f_{\Delta W(t,t+\Delta t)}(\delta|w_t)$ of the degradation increment $\Delta W(t, t+\Delta t)$, given $W(t) = w_t$. In fact, the functions $\Delta g_1(w_t, w_t + \delta)$ and $g_1'(w_t + \delta)$ can be rewritten as:

$$\Delta g_1 \ (w_t, w_t + \delta) = -\beta \left[\ln\left(1 - \frac{w_t + \delta}{w_{\lim}}\right) - \ln\left(1 - \frac{w_t}{w_{\lim}}\right)\right] = -\beta \ \ln\left(1 - \frac{\delta}{w_{\lim} - w_t}\right) \tag{21}$$

and

$$g_1'(w_t + \delta) = \frac{\beta}{w_{\lim} - w_t - \delta} = \frac{\beta}{(w_{\lim} - w_t)\left(1 - \frac{\delta}{w_{\lim} - w_t}\right)}. \tag{22}$$

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The conditional pdf $f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$ in Equation (1) can be rewritten as:

$$f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$$
$$= \frac{1}{w_{\lim} - w_t} \frac{\beta^{\eta(t,t+\Delta t)}\left[-\ln\left(1 - \frac{\delta}{w_{\lim}-w_t}\right)\right]^{\Delta\eta(t,t+\Delta t)-1}}{\Gamma(\Delta\eta(t,t+\Delta t))} \left(1 - \frac{\delta}{w_{\lim} - w_t}\right)^{\beta-1}, \tag{23}$$

and the (dimensionless) variable $\Delta Z(t, t+\Delta t) = \Delta W(t, t+\Delta t)/(w_{\lim} - w_t)$ has pdf:

$$f_{\Delta Z(t,t+\Delta t)}(\delta_z) = \frac{\beta^{\Delta\eta(t,t+\Delta t)}[-\ln(1-\delta_z)]^{\Delta\eta(t,t+\Delta t)-1}}{\Gamma(\Delta\eta(t,t+\Delta t))}(1-\delta_z)^{\beta-1}, 0 < \delta_z < 1 \tag{24}$$

that does not depend on $w_{\lim}$ and w_t. As a consequence, being:

$$E\{\Delta W(t, t+\Delta t)|W(t) = w_t)\} = \int_0^{w_{\lim}-w_t} \delta f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)\mathrm{d}\delta$$
$$= (w_{\lim} - w_t)\int_0^1 \delta_z f_{\Delta Z(t,t+\Delta t)}(\delta_z)\,\mathrm{d}\delta_z \tag{25}$$

and

$$E\{\Delta W^2(t, t+\Delta t)|W(t) = w_t\} = \int_0^{w_{\lim}-w_t} \delta^2 f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)\mathrm{d}\delta$$
$$= (w_{\lim} - w_t)^2\int_0^1 \delta_z^2 f_{\Delta Z(t,t+\Delta t)}(\delta_z)\,\mathrm{d}\delta_z, \tag{26}$$

it (obviously) results that the conditional mean and variance of the increment $\Delta W(t, t+\Delta t)$, given $W(t) = w_t$, depend linearly on $(w_{\lim} - w_t)$ and $(w_{\lim} - w_t)^2$, respectively. Differently, under the state functions (4) and (5) the conditional pdf of the increment $\Delta W(t, t+\Delta t)$, given $W(t) = w_t$, depends both on $\delta/w_{\lim}$ and on $w_t/w_{\lim}$.

It is also worth to remark that the age function $\eta(t)$, being non negative, continuous, and unbounded, only affects the "time" scale of the process. Thus, for example, this implies that the functional form of $\eta(t)$ does not affect the maximum value of the variance function.

The curves depicted in Figure 1 show the behavior of the mean function $E\{W(t)\}$ of a BTG process with power-law age function $\eta(t) = (t/a)^b$, for $w_{\lim} = 10$ and five different sets of values of the model parameters a, b, and β: $(1, 1, 1)$, $(1, 0.5, 1)$, $(1, 1, 2)$, $(1, 2, 5)$, and $(3, 1, 1)$, respectively. The bounded state function used here is the function $g_1(w)$ given in Equation (3). The figure shows that when the age parameter b is larger than 1 (that is, when the age function is convex) the mean curve has an inflection point. Otherwise, its first derivative with respect to t decreases monotonically.

Figure 2 shows the behavior of the variance $V\{W(t)\}$ of the same degradation model for $w_{\lim} = 10$ and the same values of the parameters a, b, and β used for Figure 1. Numerical investigations confirm the conjecture that the maximum value $V_{\max}\{W(t)\}$ of the variance does not depend on the age function (i.e., on the values of the age parameters a and b), and hence that the ratio $V_{\max}\{W(t)\}/w_{\lim}^2$ depends on the parameter β of the state function only (see Figure 2).

In addition, the time at which the variance reaches its maximum value given a, b, and β does not depend on $w_{\lim}$ and, given b and β, is proportional to a.

Finally, in Figure 3 the variance-to-mean ratio $V\{W(t)\}/E\{W(t)\}$ of the same degradation model is depicted for $w_{\lim} = 10$ and the same selected values of the process parameters a, b, and β used for Figures 1 and 2. We note that the variance-to-mean ratio is always decreasing with time, and tends to zero for $t \to \infty$.

Mean, variance, and variance-to-mean ratio functions of the BTG processes obtained by using the state functions (4) and (5) have analogous properties and behaviors.

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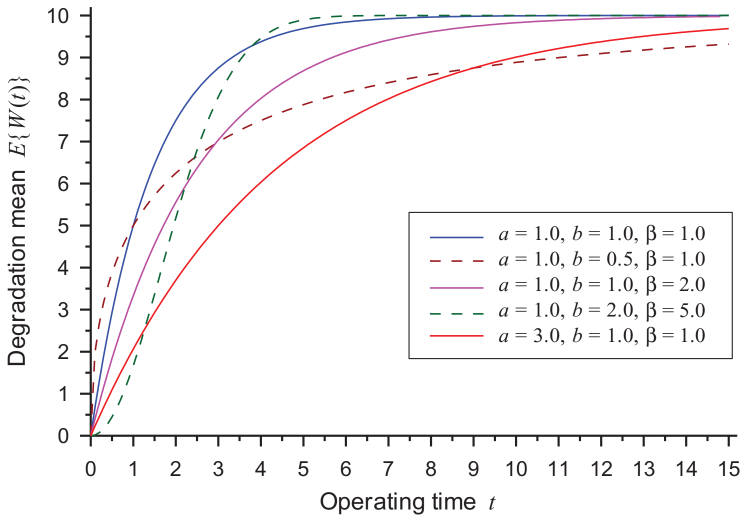

FIGURE 1 Behavior of the degradation mean $E\{W(t)\}$ for $w_{\text{lim}} = 10$ and a, b, and β set to the values indicated in the figure, when $g(w) = -\beta \ln(1 - w/w_{\text{lim}})$

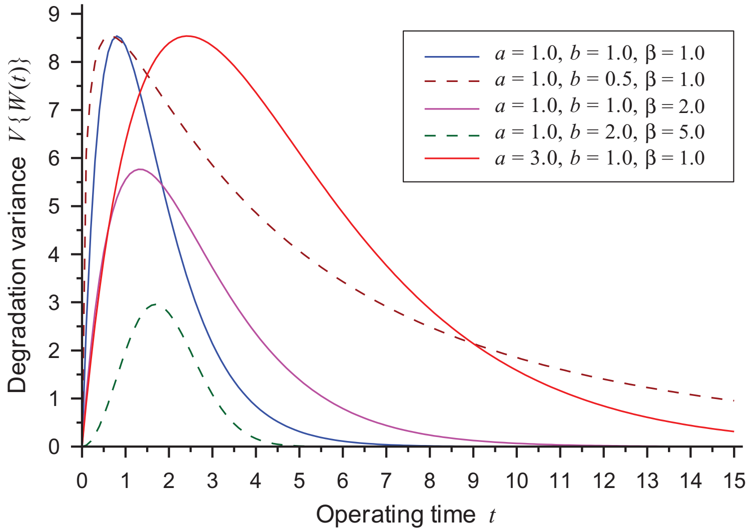

FIGURE 2 Behavior of the degradation variance $V\{W(t)\}$, for $w_{\text{lim}} = 10$ and a, b, and β set to the values indicated in the figure, when $g(w) = -\beta \ln(1 - w/w_{\text{lim}})$

3 | REMAINING USEFUL LIFE AND RELIABILITY FUNCTION

In the context of increasing degradation processes, a unit is assumed to fail when its degradation level W exceeds a threshold limit D ($D < w_{\text{lim}}$). Then, the lifetime of the unit X can be defined as the operating time to first, and sole, passage beyond the limit D; that is:

$$X = inf\{x : W(x) > D\}. \tag{27}$$

Accordingly, the remaining useful life (RUL) X_t of a unit at time t is defined as:

$$X_t = \max\{0, X - t\}, \tag{28}$$

so that X_t is equal to $X - t$ if the unit at t is unfailed, and is assumed to be 0 otherwise.

Then, by using the conditional Cdf (9), the conditional probability that the RUL X_t exceeds the time x, given the current state $w_t < D$ at the current age t (here also referred to as the residual reliability), is given by:

$$R_t(x|W(t) = w_t) = \Pr\{\Delta W(t, t + x) \le D - w_t | W(t) = w_t\} = \frac{\gamma(\Delta g(w_t, D); \Delta\eta(t, t + x))}{\Gamma(\Delta\eta(t, t + x))}. \tag{29}$$

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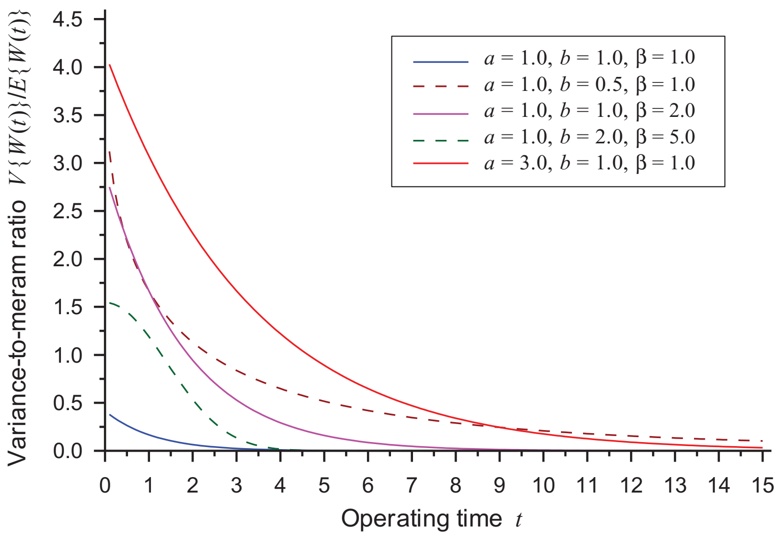
FIGURE 3 Behavior of the variance-to-mean ratio $V\{W(t)\}/E\{W(t)\}$, for $w_{\lim} = 10$ and a, b, and β set to the values indicated in the figure, when $g(w) = -\beta \ln(1 - w/w_{\lim})$



Accordingly, the reliability function of a new unit, which can be readily obtained from Equation (29) $w_t = 0$ and $t = 0$, is given by:

$$R(x) = \Pr\{W(x) \le D\} = \frac{\gamma(g(D); \eta(x))}{\Gamma(\eta(x))}. \tag{30}$$

It is worth to remark that, since the BTG is a Markovian process, the residual reliability function (29) should be intended to incorporate all the information useful for prognostic purposes provided by the past history $H_t = \{W(s), s \le t\}$ of the process up to (and included) the time t.

If the age function $\eta(t)$ is differentiable with respect to t, then the conditional pdf of the RUL X_t can be obtained by deriving the residual reliability (29) with respect to x:

$$f_{X_t|W(t)}(x|w_t) = -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{g(w_t,D)} \frac{u^{\Delta\eta(t,t+x)-1}}{\Gamma(\Delta\eta(t,t+x))}\exp(-u)\,\mathrm{d}u. \tag{31}$$

Then, by using the arguments given in Giorgio et al.[2] it is possible to obtain a series representation of the pdf of X_t, which does not involve any numerical integration:

$$f_{X_t|W(t)}(x|w_t) = \frac{\mathrm{d}\Delta\eta(t,t+x)}{\mathrm{d}x}\quad\frac{1}{\Gamma(\Delta\eta(t,t+x))}\Bigg\{\gamma(\Delta g(w_t,D); \Delta\eta(t,t+x))(\psi(\Delta\eta(t,t+x)) - \ln(\Delta g(w_t,D))) + \sum_{k=0}^{\infty}\frac{(-1)^k[\Delta g(w_t,D)]^{\Delta\eta(t,t+x)+k}}{[\Delta\eta(t,t+x)+k]^2 k!}\Bigg\} \tag{32}$$

where $\psi(z)$ is the digamma function.

Accordingly, from Equation (32), the pdf of the lifetime X of a new unit can be expressed as:

$$f_X(x) = \frac{\mathrm{d}\eta(x)}{\mathrm{d}x}\frac{1}{\Gamma(\eta(x))}\left\{\gamma(g(D); \eta(x))(\psi(\eta(x)) - \ln(g(D))) + \sum_{k=0}^{\infty}\frac{(-1)^k[g(D)]^{\eta(x)+k}}{[\eta(x)+k]^2 k!}\right\}. \tag{33}$$

Finally, the mean RUL $E\{X_t|W(t) = w_t\}$ and the mean lifetime $E\{X\}$ of a new item can be computed as:

$$E\{X_t|W(t) = w_t\} = \int_0^{\infty} R_t(x|W(t) = w_t)\mathrm{d}x \tag{34}$$

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and

$$E\{X\} = \int_0^\infty R(x)\,\mathrm{d}x, \tag{35}$$

respectively.

4 | THE ESTIMATION PROCEDURE

Let us consider m identical units which operate under the same conditions and suppose that their degradation level is periodically measured by performing ad hoc inspections. Moreover, let us denote by n_i the number of measurements performed on the unit i $(i = 1, \dots, m)$, by $t_{i,j}$ $(i = 1, \dots, m;\ j = 1, \dots, n_i)$ the age of the unit i at the epoch of the j-th inspection, and by $w_{i,j}$ the degradation level of the unit at $t_{i,j}$. Finally, let $\delta_{i,j} = w_{i,j} - w_{i,j-1}$, $\Delta g_{i,j} = \Delta g(w_{i,j-1}, w_{i,j-1} + \delta_{i,j})$, $\Delta\eta_{i,j} = \Delta\eta(t_{i,j-1}, t_{i,j})$, $w_{i,0} = t_{i,0} = 0$ for all i, and let $\boldsymbol{\theta}$ denote the vector of parameters which, together to w_{lim}, index the age and state functions. Therefore, for example, if the state function has one of the functional forms given in the Equations (3) to (5) and the age function is the power-law function $\eta(t) = (t/a)^b$, then $\boldsymbol{\theta} = (a, b, \beta)$.

All this stated, under the BTG process, the log-likelihood function relative to the observed data $\boldsymbol{w} = (w_{1,1}, \cdots, w_{1,n_1}, \cdots, w_{m,1}, \dots, w_{m,n_m})$ is:

$$\ell\,(\boldsymbol{w}; \boldsymbol{\theta}, w_{\text{lim}}) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \ln(f_{\Delta W(t_{i,j-1}, t_{i,j})|W(t_{i,j-1})}(\delta_{i,j}|w_{i,j-1})), \tag{36}$$

where, from Equation (1), the conditional pdf of the wear increment $\Delta W(t_{i,j-1}, t_{i,j})$ accumulated by the unit i over the inspection interval $(t_{i,j-1}, t_{i,j})$, given $W(t_{i,j-1}) = w_{i,j-1}$, is:

$$f_{\Delta W(t_{i,j-1}, t_{i,j})|W(t_{i,j-1})}(\delta_{i,j}|w_{i,j-1}) = g'\left(w_{i,j-1} + \delta_{i,j}\right) \frac{\left(\Delta g_{i,j}\right)^{\Delta\eta_{i,j}-1}}{\Gamma(\Delta\eta_{i,j})} \exp\left(-\Delta g_{i,j}\right),$$
$$0 < \delta_{i,j} < w_{\text{lim}} - w_{i,j-1}, \tag{37}$$

The maximum likelihood (ML) estimates $\hat{\boldsymbol{\theta}}$ and $\hat{w}_{\text{lim}}$ of the process parameters are the values of $\boldsymbol{\theta}$ and w_{lim} that maximize the log-likelihood function (23). These estimates are not available in closed form. Yet, they can be readily retrieved by using a numerical procedure.

In several circumstances, previous experiences and/or physical considerations on the degradation phenomenon allow the analyst to fix a lower limit or an interval of plausible values for w_{lim}. An example of application where there is this kind of prior knowledge is discussed in Section 6. In these circumstances, a proper (constrained) maximization procedure of the log-likelihood (36) could be used in order both to satisfy the above constraints and to take advantage of the available knowledge.

Approximate confidence intervals for the model parameters $(\boldsymbol{\theta}, w_{\text{lim}})$ can be obtained by using asymptotic results. For example, by assuming that the ML estimator of the parameter β of the "bounded" state functions (3)-(5) is asymptotically distributed as a normal random variable, the approximate equal-tails $(1-q)$ confidence interval for β is given by:

$$\hat{\beta} \pm z_{q/2}\hat{\sigma}\left(\hat{\beta}\right), \tag{38}$$

where $z_{q/2}$ denotes the $q/2$ quantile of the standard normal distribution, and $\hat{\sigma}(\hat{\beta})$ is the estimated standard deviation of $\hat{\beta}$. This estimated standard deviation is given by the squared root of the element (1,1) of the estimated covariance matrix $[\boldsymbol{J}(\hat{\boldsymbol{\theta}}, \hat{w}_{\text{lim}})]^{-1}$, which is obtained by inverting the observed Fisher information matrix $\boldsymbol{J}(\hat{\boldsymbol{\theta}}, \hat{w}_{\text{lim}})$, whose entries are the negative second derivatives of the log-likelihood function (23) with respect to the model parameters, evaluated at the ML estimate of the parameter vector $(\hat{\boldsymbol{\theta}}, \hat{w}_{\text{lim}})$.

When the parameter is constrained to be positive, as it occurs for example for the parameters of the proposed state functions, the normal approximation is sometimes unsatisfactory, because the distribution of the estimator of the parameter

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TABLE 1 Degradation level $w_{i,j} = W(t_j)$ accumulated by the unit i up to time t_j

i	$w_{i,1}$	$w_{i,2}$	$w_{i,3}$	$w_{i,4}$	$w_{i,5}$	$w_{i,6}$	$w_{i,7}$	$w_{i,8}$	$w_{i,9}$	$w_{i,10}$
1	1.97	3.16	3.40	3.83	4.02	4.19	4.31	4.37	4.40	4.49
2	0.51	1.95	2.48	3.51	3.72	3.99	4.10	4.22	4.32	4.43
3	0.42	0.89	1.89	3.05	3.71	3.90	4.11	4.21	4.30	4.37
4	0.77	2.94	3.63	4.00	4.10	4.28	4.38	4.47	4.52	4.55
5	0.64	1.60	2.70	3.64	3.87	3.94	4.03	4.14	4.20	4.33
6	2.75	3.48	3.63	3.84	3.89	4.01	4.11	4.30	4.33	4.40
7	1.37	2.29	2.91	3.31	3.85	3.94	4.00	4.13	4.20	4.27
8	0.14	1.33	2.70	3.32	3.38	3.61	3.77	3.98	4.20	4.34
9	1.12	2.04	3.22	3.57	3.81	3.94	4.03	4.30	4.38	4.43
10	1.45	2.56	3.44	3.52	3.76	3.90	4.09	4.20	4.23	4.34

can be highly skewed when the sample size is small or moderate. In this case, for example, the use of a normal approximation for $\ln(\hat{\beta})$, rather than for $\hat{\beta}$, can be more suitable, even because it prevents the lower bound of the resulting confidence interval from being negative. Thus, if $\ln(\hat{\beta})$ is asymptotically normal, then:

$$\frac{\ln\left(\hat{\beta}\right) - \ln(\beta)}{\hat{\sigma}\left(\hat{\beta}\right)/\hat{\beta}} \tag{39}$$

is asymptotically standard normal (Bishop et al.[7]) and the $(1-q)$ approximate confidence interval for β is given by:

$$\hat{\beta}\exp\left(\pm z_{q/2}\hat{\sigma}\left(\hat{\beta}\right)/\hat{\beta}\right). \tag{40}$$

Of course, in the cases where, as mentioned before, further constrains can be imposed to w_{lim}, the confidence interval on w_{lim} must account for it. In particular, when a lower limit, say w^*_{lim}, is imposed then an approximate lower confidence limit for w_{lim}, say w^L_{lim}, when the asymptotic normal approximation is assumed for $\ln(\hat{w}_{\text{lim}})$, can be computed as:

$$w^L_{\text{lim}} = \begin{cases} \hat{w}_{\text{lim}}\exp\left(\pm z_{q/2}\hat{\sigma}\left(\hat{w}_{\text{lim}}\right)/\hat{w}_{\text{lim}}\right), & \text{if}\, w^*_{\text{lim}} < \hat{w}_{\text{lim}}\exp\left(\pm z_{q/2}\hat{\sigma}\left(\hat{w}_{\text{lim}}\right)/\hat{w}_{\text{lim}}\right) \\ w^*_{\text{lim}}, & \text{if}\, w^*_{\text{lim}} \geq \hat{w}_{\text{lim}}\exp\left(\pm z_{q/2}\hat{\sigma}\left(\hat{w}_{\text{lim}}\right)/\hat{w}_{\text{lim}}\right) \end{cases}. \tag{41}$$

Once the vector of unknown parameters has been estimated, the ML estimate of any function thereof, say $\phi(\theta, w_{\text{lim}})$, can be easily obtained by substituting $\hat{\theta}$ and $\hat{w}_{\text{lim}}$ to θ and w_{lim} in $\phi(\theta, w_{\text{lim}})$. In particular, by using this approach, one can easily obtain the ML estimates of the conditional distribution of the RUL, the mean of the RUL, the residual reliability, and the conditional distribution of the degradation growth over a future time interval.

5 | NUMERICAL EXAMPLE

In order to give evidence of the utility of the proposed BTG processes and to demonstrate the affordability of the estimation procedure suggested in Section 4, we have developed a preliminary numerical example by using the simulated data reported in Table 1. The same data are also displayed in Figure 4. These data have been generated under a BTG2 process with parameters $w_{\text{lim}} = 5$, $\beta = 5$ $a = 1$, $b = 1.2$. There are $m = 10$units and $n_i = 10$ degradation measures for each unit (i.e., $n_i = 10\ \forall i$). It is assumed that measurements are performed at 10 equally-spaced inspection times that are the same for all the measurements (i.e., $t_{i,j} = t_j \quad \forall i$): $t_1 = 2, t_2 = 4, \ldots, t_{10} = 20$.

The ML estimates of the parameters of the proposed BTG models obtained from these data are reported in the Table 2. For comparative purposes, this table also reports the ML estimates of the parameters of the unbounded TG process with the same age function and the power-law state function $g(w) = (w/\alpha)^{\beta}$.

The obtained results show that, according to the Akaike information criterion (AIC) (see, Akaike[8]), the BTG processes fit the considered data better than the unbounded TG process and that, among them, the one that provides the better fit is just the BTG2 process we have used to generate the data.

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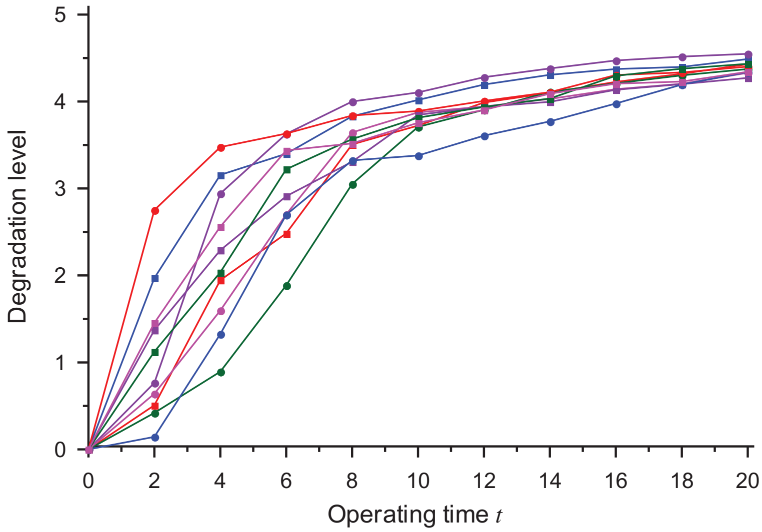

FIGURE 4 Degradation data. Lines connect points pertaining to the same degradation path

TABLE 2 Estimation results under the BTG and the TG processes

Process	$\hat{a}$	$\hat{b}$	$\hat{w}_{\text{lim}}$ or $\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC
BTG1	0.657	1.004	4.597	9.554	71.08	−134.2
BTG2	1.450	1.349	4.921	4.011	81.12	−154.2
BTG3	0.984	1.186	4.913	5.795	80.20	−152.4
TG	6.173	2.591	12.40	3.444	63.01	−118.0

TABLE 3 Approximate 0.90 confidence intervals on the parameters of the BTG2 process

Parameter	0.90 confidence interval	True value
a	(0.619, 1.564)	1
b	(1.037, 1.356)	1.2
w_{lim}	(4.796, 5.032)	5
β	(3.983, 8.431)	5

Although estimates obtained from a single dataset do not allow to evaluate the performances of the ML estimators, it is still useful to observe that all the estimates obtained under the BTG processes are rather close to the true values of the corresponding true parameters. Specifically, it is to note that, the estimate of the bound w_{lim} obtained under the (true) BTG2 process model is very close to its true value. In fact, it is also interesting to remark that all the 0.90 confidence intervals of the parameters reported in Table 3, obtained under the BTG2 process, by using the asymptotic normal approximation for the logarithm of their estimators, include the corresponding true values.

Figures 5 and 6 display the ML estimates of the mean and variance functions obtained under the considered competing models. The same figures also show the "true" mean and variance function (i.e., those computed under the BTG2 process by setting the parameters to their true values) and the empirical estimates of the mean and variance of the degradation process computed at the inspection times from the simulated data.

Figures 5 and 6 confirm the conclusions driven by analyzing the results reported in Tables 1, 2, and 3. In fact, they show that the BTG2 model fits the mean and variance function of the true process adequately. They also show that the BTG3 process provides estimates that are very similar to those obtained under the BTG2 one, and that the estimates obtained under the BTG1 process are slightly different from those obtained under the other two bounded processes. Finally, Figures 5 and 6 give evidence that unbounded TG process is not able to fully capture the behaviors of the true mean and variance functions.

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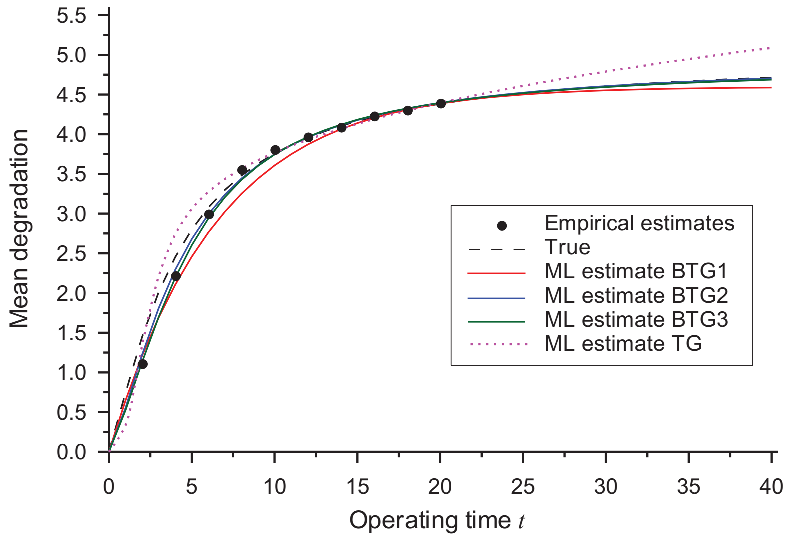

FIGURE 5 True values and empirical estimates of $E\{W(t)\}$ together with the corresponding ML estimates obtained under the TG, BTG1, BTG2, and BTG3 processes

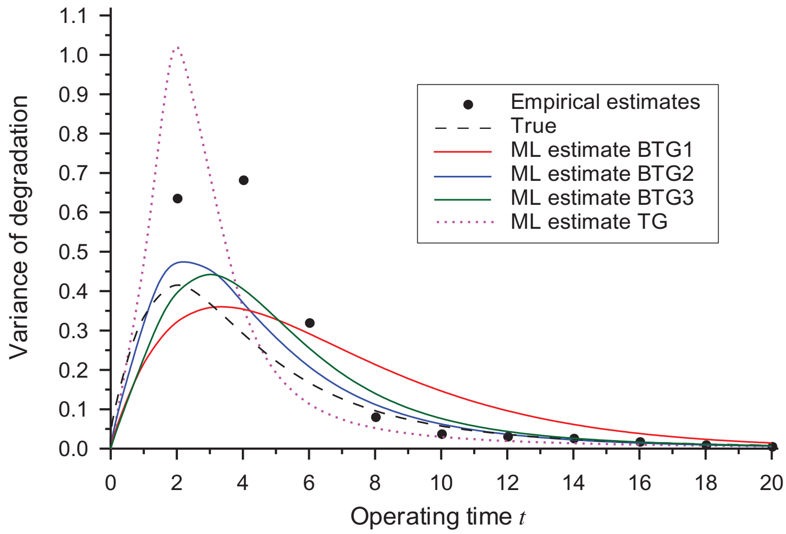

FIGURE 6 True values and empirical estimates of $V\{W(t)\}$ together with the corresponding ML estimates obtained under the TG, BTG1, BTG2, and BTG3 processes

6 | APPLICATION OF THE MODEL TO WEAR MEASURES OF CYLINDER LINERS

This section illustrates the results we have obtained by applying the BTG processes with state functions (3)-(5) to the real set of wear data that has motivated this paper. The dataset consists of 23 wear measures of the liners of a 8-cylinder SULZER marine propulsion diesel engine which equips a cargo ship of the Grimaldi Lines. Data have been gathered via ad hoc inspections between 1994 and 2004. During this period the cargo ship has operated on the same routes under homogeneous operating conditions. At each inspection, the wear of the liner is measured by positioning a caliper inside a predetermined hole situated at the top dead center of the engine, which is the point where (for physical reasons) the liner wear usually reaches its maximum level. Each datum consists of the age of the liner at the inspection epoch and of the associated wear measure. Both number of measures and inspection times changes from liner to liner. The wear data are reported in Table 4 and depicted in Figure 7. In compliance with technical specifications and contractual agreements, a liner is conventionally considered failed when its wear level passes the threshold limit $D = 4$ mm.

This wear process was previously analyzed in Giorgio et al.[6] under a TG process with (unbounded) power-law function $g(w) = (w/\alpha)^{\beta}$, and two different age functions, namely the power-law function $\eta(t) = (t/a)^{b}$ and the exponential function $\eta(t) = ab[\exp(t/b) - 1]$.

The best fit for these data, according to the Akaike information criterion, was obtained by adopting a TG process with power-law age function. Nonetheless, although this TG process was found to fit quite well the empirical mean of the wear process (see the blue curve of Figure 8), it was found unable to fit equally well the empirical variance (see the blue curve

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TABLE 4 Wear $w_{i,j} = W(t_{i,j})$ [mm] accumulated by liner i up to the inspection time $t_{i,j}$ [hours]

i	$w_{i,1}$	$t_{i,1}$	$w_{i,2}$	$t_{i,2}$	$w_{i,3}$	$t_{i,3}$	$w_{i,4}$	$t_{i,4}$
1	0.90	11,300	1.30	14,680	2.85	31,270		
2	1.50	11,300	2.00	21,970				
3	1.00	12,300	1.35	16,300				
4	1.90	14,810	2.25	18,700	2.75	28,000		
5	1.20	10,000	2.75	30,450	3.05	37,310		
6	0.50	6,860	1.45	17,200	2.15	24,710		
7	0.40	2,040	2.00	12,580	2.35	16,620		
8	0.50	7,540	1.10	8,840	1.15	9,770	2.10	16,300

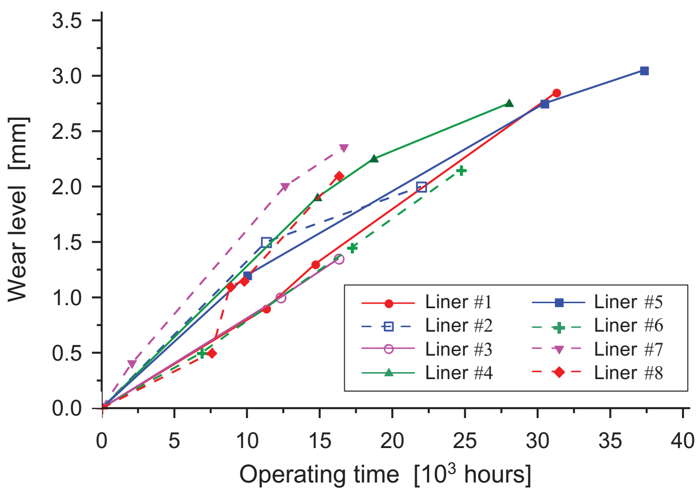

FIGURE 7 Liner wear data

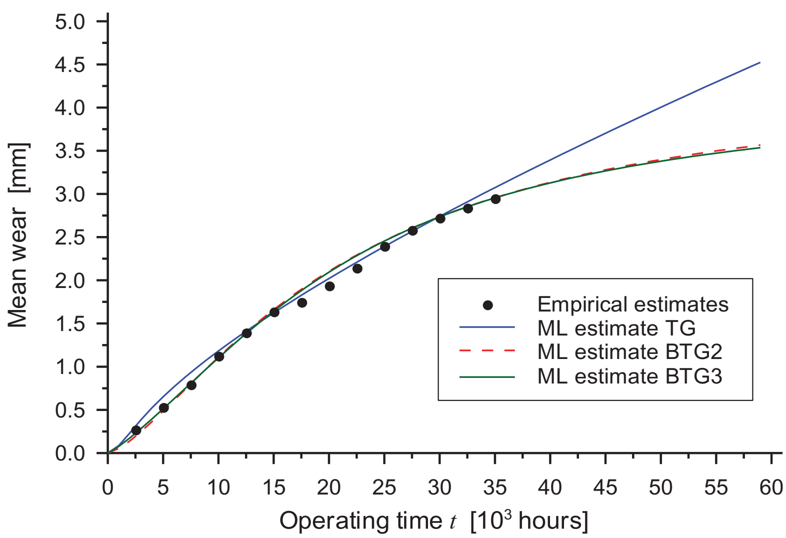

FIGURE 8 Empirical and ML estimates of the mean wear $E\{W(t)\}$ under the TG, BTG2, and BTG3 processes

of Figure 9), in particular when the empirical variance decreases quickly.

Thus, considered that the proposed BTG process can describe the behavior of the empirical variance given in Figure 9, which the TG process fails to do, and that previous experiences suggest that a physical bound to the wear process of cylinder liners exists, we have decided to analyze the liner data under the proposed BTG process.

However, this is not the only reason that motivate the use of a BTG model. In fact, the idea that the liner wear process is bounded is also directly suggested by the inherent features of its causing mechanisms. Indeed, the liner wear is jointly caused by the abrasive action of small hard particles, in jargon referred to as soot, and by the corrosive actions of chemical

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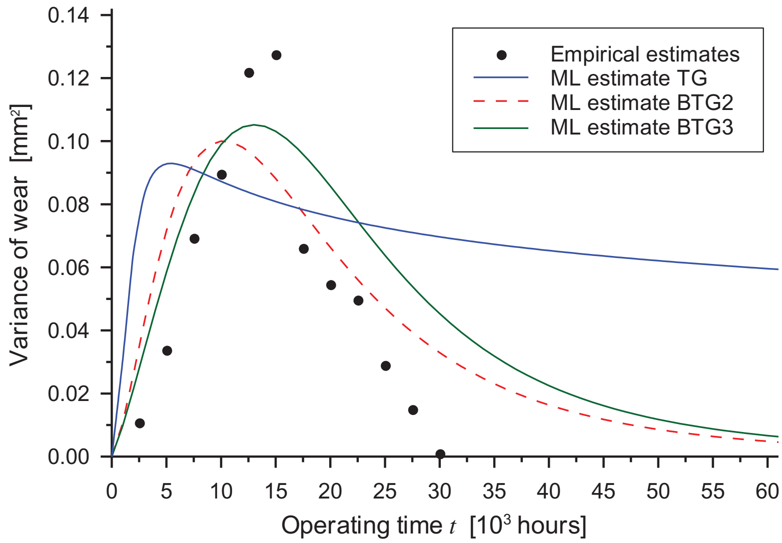

FIGURE 9 Empirical and ML estimates of the variance $V\{W(t)\}$ under the TG, BTG2, and the BTG3 processes

TABLE 5 Estimation results under the BTG and the TG processes

Process	$\hat{a}$ [h]	$\hat{b}$	$\hat{w}_{\mathrm{lim}}$ or $\hat{\alpha}$ [mm]	$\hat{\beta}$	$\hat{\ell}$	AIC
BTG1	1295	1.106	4.3	31.36	2.657	2.686
BTG2	2682	1.434	4.363	18.62	3.843	0.314
BTG3	1504	1.182	4.3	21.67	3.837	0.327
TG	5107	1.701	2.321	0.750	0.590	6.820

agents, identified as sulphuric acid, nitrous/nitric acids, and water. Therefore, considered that the thickness of the liner is 100 mm, it is surely realistic to presume that these wear mechanisms cannot lead the liner wear to grow up to its thickness.

Another interesting feature of the liner wear process is that the aforementioned physical considerations alone do not allow to determine (a priori) the exact value of the upper bound. Indeed, based on previous experiences and wear data of similar liners collected over a time span of about 20 years, it is only possible to say that the liner wear can surely grow up to the value of 4.3 mm. Thus, based on these considerations, under the BTG process the upper bound is treated as an unknown parameter and is estimated by considering the constraint $w_{\mathrm{lim}} \geq 4.3$ mm.

It is worth to note that, in this experimental situation, assuming that the upper bound of the BTG process coincides with the thickness of the liner (that is equal to 100 mm) is practically equivalent to use unbounded TG process.

Table 5 reports the ML estimates of model parameters, the corresponding estimated log-likelihood $\hat{\ell}$ and AIC value obtained under the BTG1, BTG2, and BTG3 processes, with the power-law age function $\eta(t) = (t/a)^b$ and the "bounded" state functions (3)-(5), where the acronyms BTGl indicates the BTG process that uses the state function $g_l(w)$ $(l = 1, 2, 3)$. For comparative purposes, the last row of the same table reports the ML estimates of model parameters, the estimated log-likelihood $\hat{\ell}$, and the AIC value obtained under the TG process adopted in Giorgio et al.[6]

Table 5 shows that all the suggested BTG processes fit the liner wear data better than the (unbounded) TG process. Indeed, all the AIC values obtained under the BTG processes are smaller than the AIC value obtained under the TG process. More in particular, it also results that, according to the Akaike information criterion, the models that provide the best fit for the considered data are the BTG process with state functions $g_2(w)$ given in Equation (4) and the BTG process with state function $g_3(w)$ given in Equation (5), which show a very similar fitting ability.

In Table 6 the approximate 0.90 confidence intervals on the parameters of the BTG2 and BTG3 processes, based on the log-normal approximation, are provided. Clearly, because the lower limit $w^*_{\mathrm{lim}} = 4.3$ mm is here imposed, the approximate lower confidence limits for w_{lim} are obtained from Equation (41).

Figures 8 and 9 show the ML estimates of the mean $E\{W(t)\}$ and variance $V\{W(t)\}$ of the wear process under the TG, the BTG2, and the BTG3 processes. For the sake of providing a graphical demonstration of the fitting ability of the BTG models, the same figures also show the empirical estimates of mean and variance obtained by using the available wear data.

TABLE 6 Approximate 0.90 confidence intervals on the parameters of the BTG2 and BTG3 processes

Parameter	BTG2 process	BTG3 process
a [h]	(1353, 5314)	(665, 3401)
b	(1.119, 1.838)	(0.931, 1.5003)
w_{lim} [mm]	(4.3, 5.506)	(4.3, 5.387)
β	(8.03, 43.13)	(10.55, 44.49)

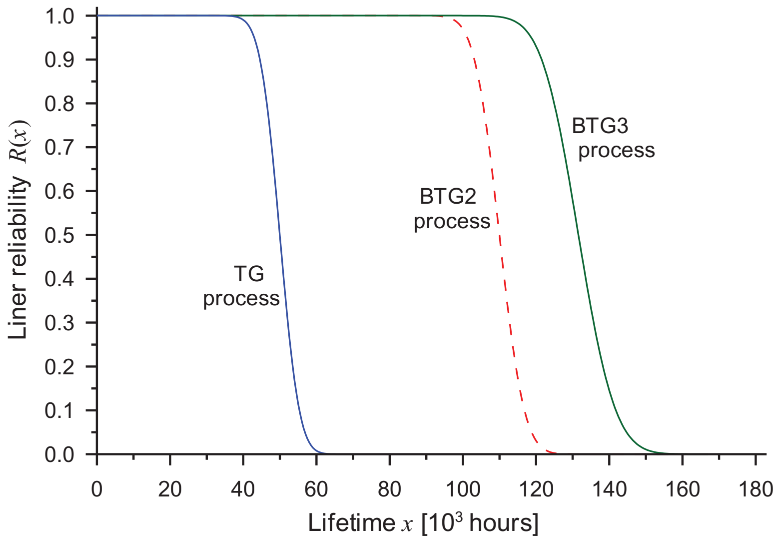

FIGURE 10 ML estimates of the reliability $R(x)$ of a new liner obtained under the TG, BTG2, and BTG3 processes

Note that, since the inspection times differ from liner to liner, and hence the wear measures generally refer to different operating times of the liners, the empirical estimates of mean and variance were obtained by using the linear interpolation procedure of data already used in Giorgio et al.[6]

In particular, the empirical estimates $\tilde{E}\{W(t_k)\}$ and $\tilde{V}\{W(t_k)\}$ of $E\{W(t_k)\}$ and $V\{W(t_k)\}$, are obtained by using the formulas:

$$\tilde{E}\{W(t_k)\} = \frac{\sum_{i:t_{i,n_i} \geq \tau_k} w_i(t_k)}{m_k}, \tag{42}$$

$$\tilde{V}\{W(t_k)\} = \frac{\sum_{i:t_{i,n_i} \geq \tau_k} [w_i(t_k) - \tilde{E}\{W(t_k)\}]^2}{m_k - 1}, \tag{43}$$

where the times $\tau_k = k \cdot 2.5$ (hours 10^3), $k = 0, 1, 2 \ldots, 12$, are arbitrarily selected equally-spaced times, and the corresponding degradation levels $w_i(t_k)$ are values determined by linear interpolation from the available data (i.e., the value used for τ_k and $w_i(t_k)$ are coordinates of points that are on the lines represented in Figure 7). In Equations (42) and (43), m_k denotes the number of liners whose last observation time is larger than or equal to t_k. The variance is computed until $m_k \geq 2$.

Figure 8 shows that the ML estimates of the mean function $E\{W(t)\}$ obtained under the BTG2 and BTG3 processes practically overlap one another, and fit the empirical mean a little better than the TG process, in particular at large time t where the estimate obtained under the (unbounded) TG process seems to increase more rapidly than the last observed data suggest. On the other hand, Figure 9 shows that, differently than the TG process, the BTG2 and BTG3 processes are able to fit adequately both the initial growth and the subsequent rapid reduction of the empirical variance. Indeed, the TG process is not able to provide an adequate fit for the empirical variance, neither for small nor for large values of t.

Figure 10 displays the ML estimates of the reliability of a new liner obtained under the BTG2 and BTG3 processes, together with the ML estimate of the same reliability function obtained under the (unbounded) TG process. The figure gives evidence that the (unbounded) TG process provide reliability estimates that are much more pessimistic than those obtained under the BTG processes. Consequently, also the ML estimate of the mean lifetime $E\{X\}$ of a new liner provided

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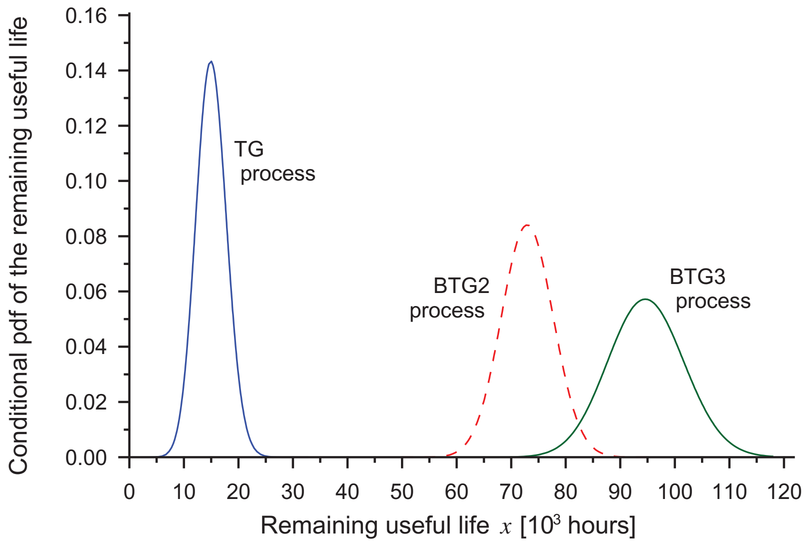

FIGURE 11 ML estimates of the conditional pdf $f_{X_t|W(t)}(x|w_t)$ of the remaining useful life X_t of liner #5, given $w_t = 3.05$ mm, at $t = 37,310$ h, under the TG, BTG2, and BTG3 processes

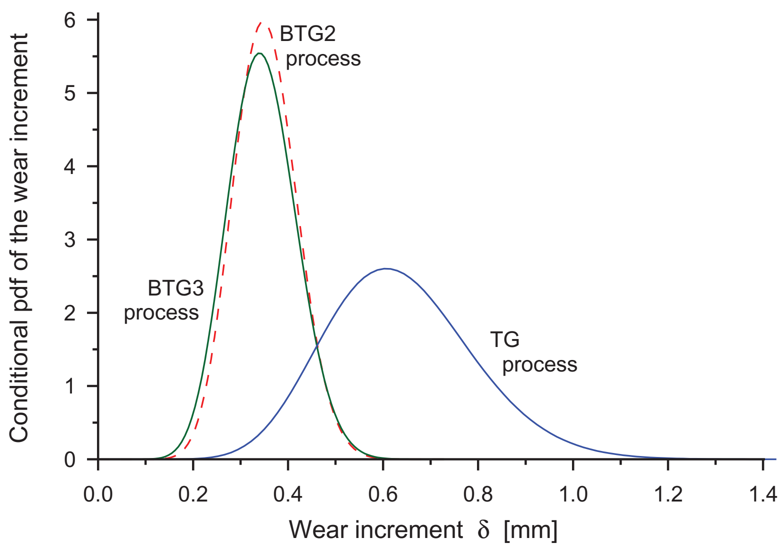

FIGURE 12 ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$ of the wear increment of liner #1 during the interval (31,270 h, 31,270+10,000 h), given $w_t = 2.85$ mm, under the TG, BTG2, and BTG3 processes

by the TG process is very pessimistic with respect to those provided by the BTG2 and BTG3 processes. Indeed, the ML estimate of $E\{X\}$ obtained under TG process is equal to 49,954 h, whereas those provided by the BTG2 and BTG3 processes are equal to 109,994 h and 131,629 h, respectively.

Figure 11 shows the ML estimates of the conditional pdf $f_{X_t|W(t)}(x|w_t)$ of the RUL X_t of liner #5, evaluated at $t = 37,310\ h$, given the current degradation level $w_t = 3.05$ mm, obtained under the BTG2, BTG3, and TG processes. Also in this case, the figure gives clear evidence that the ML estimate of the pdf $f_{X_t|W(t)}(x|w_t)$ obtained under the TG process is much more pessimistic than those obtained under the BTG processes. In fact, for example, while the MLE of the mean of the RUL $E\{X_t|W(t) = w_t\}$ computed by using the TG process is equal 14,965 h, those provide by the BTG2 and BTG3 processes are equal to 72,836 h and 94,537 h, respectively. Similar results have been obtained for the other liners.

Finally, Figure 12 shows the ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t)$ of the wear increment $\Delta W(t, t+\Delta t)$ of liner #1 provided by the TG, BTG2, and BTG3 processes and evaluated at $t = 31,270$ h, by considering an interval width Δt of $10,000$ h, given the current wear level $w_t = 2.85$ mm.

Once again, this latter figure shows that the TG process provides estimates of the future wear growth that are hugely pessimistic with respect to those provided by the BTG processes, that are also very close one to the other. Indeed, the estimated mean wear increment $E\{\Delta W(t, t+\Delta t)|W(t) = w_t\}$ evaluated under the TG process is equal to 0.637 mm, while

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those computed under the BGT2 and BTG3 processes are equal to 0.353 mm and 0.346 mm, respectively. Even in this case, similar results have been obtained for the other liners.

7 | CRITICAL DISCUSSION OF THE RESULTS OBTAINED BY APPLYING THE PROPOSED MODEL TO THE LINER DATA

The example of application presented in Section 6 shows that the BTG processes provide results that are hugely different from those obtained under the unbounded TG process. Indeed, it is apparent that the presence of an upper bound w_{lim} influences in a significant manner the prognostic/predictive ability of the considered models. Thus, while it is clear that an unbounded model risks to produce estimates that are overly conservative, it is also clear that a bounded process could provide estimates that are too optimistic. This potential detrimental effect could be obviously caused by an erratic calibration of the upper bound. Unfortunately, the circumstance that the BTG models fit the available data better than the (unbounded) TG model does not prove that the estimates of the upper bound provided by the former models are close to the true value. In fact, the possibility that the upper bound is underestimated is not negligible, because the estimate is extrapolated from the observed wear data that are (all) smaller and relatively far from it. Indeed, mainly because the threshold limit of 4 mm is imposed by contract clauses, whose failure to comply determines the forfeiture of the guarantee by the manufacturer of the engine, the liners are routinely replaced when their wear reaches a level that is rather smaller than the failure threshold. This early replacement is often also due to the fact that only in certain ports it is possible to replace the liners.

In addition, given that all the observed wear paths are convex upward, it seems realistic to presume that the wear process of the liners up to 4 mm is still in its early phase (e.g., see Gertsbakh and Kordonskiy[9]). Thus, in particular, given that the available experimental data do not allow to investigate how the process behaves for larger values of the wear, it is not possible to exclude that after passing the threshold limit the process could evolve in a different manner.

Another important point of discussion concerns the interaction of the considered degradation induced failure with other failure modes of the liners. In fact, given that the liners are also subjected to other kinds of failure modes (for example thermal crack, see Bocchetti et al.[10]), in order to study the potential detrimental effect that could be caused by using a model that underestimates the upper bound, it would be also important to study whether and how the excess of wear could influence the occurrence and/or the progression of other failure modes (e.g., think about crack initiation). Unfortunately, for the very same reasons mentioned above, the available data do not allow to perform these kinds of analyses.

Therefore, taking into account all this, our idea is that in this kind of experimental situations, only the understanding of the physics of the considered phenomenon could allow the analyst to decide whether to use a bounded or an unbounded degradation process.

Besides these technological aspects, it is clear that the discussed decision of using a bounded or an unbounded degradation process should also be made by taking into careful consideration the replacement cost of a liner and the consequences determined by a failure. In this sense, it is worth to observe that, in the examined case, the consequences of the considered wear induced soft (not catastrophic) failures consist in loss of power and extra costs that are very high but not too different from the cost of a new liner (i.e., thousands of dollars).

Hence, it makes sense investigating the possibility of delaying its replacement as much as possible. In this regard, the use of a predictive/prognostic model that accounts for the presence of an upper bound would certainly represent an element that could allow revising the maintenance strategy.

8 | CONCLUSIONS

This paper suggests a new degradation process, called bounded TG, that has been specifically conceived to describe increasing degradation processes that cannot grow indeterminately. More specifically, the paper focuses on experimental situations where, although it is possible to say that the degradation process of interest is surely bounded above, the available information/knowledge alone does not allow to determine a priori the exact value of the upper bound. In fact, under the suggested bounded TG process, the upper bound is treated as an unknown parameter that, coherently with this setting, should be estimated from the available degradation data.

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The main characteristics of the proposed process have been illustrated. Hence, three possible functional forms have been suggested for its state function, which in the general model is left unspecified for the sake of flexibility.

In order to investigate the potential of the proposed approach, all the suggested characterizations of the bounded TG process have been applied both to a simulated data set and to a set of real data, the last one consisting of wear measures of cylinder liners equipping a diesel engine for marine propulsion, which has also stimulated this study. Model parameters have been estimated by using the ML method.

Obtained results relative to the real wear data have been compared to those obtained under an unbounded TG process previously adopted to analyze the same wear data. The comparison is performed in terms of estimates of the reliability function, of the distribution of the RUL, and of the (conditional) pdf of the future wear increment. The comparison has shown that the suggested bounded TG processes provide reliability estimates and life predictions that are hugely different from those obtained under the considered unbounded gamma process.

Potential pros and cons of using the suggested bounded gamma model have been finally discussed.

DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

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