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ON SPACE-TIME ELEMENTS VERSUS THE WEIGHTED RESIDUAL METHOD.

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INTRODUCTION

Some remarks from Prof. O.C. Zienkiewicz have stimulated the authors to make a direct comparison of the space-time finite element method proposed by them [1, 2, 3,] against weighted residuals extended to time (see, for instance [4]). The comparison is focused on:

- the discretized equations;
- the space-time element matrix;
- the system of linear equations to be solved.

The comparison is carried out thoroughly for a simple linear parabolic problem, where parallel formulations exist. The comparisons is sketched for hyperbolic problems; here the computation at each step of the initial velocity becomes critical for numerical accuracy. An euristic two step procedure was successfully tested, and a chart given for its extended use.

SPACE-TIME VERSUS WEIGHTED RESIDUALS: PARABOLIC

The comparison is best exemplified by a one dimensional linear heat transfer problem; starting from the strong differential (see Problem PP [1]) formulation

$$(1.1) \qquad -K u_{xx} + c\dot{u} = f \qquad x \in (0,b),\ t \in [0,T]$$

$$(1.2) \qquad u(x,0) = u_0 \qquad x \in (0,b)$$

$$(1.3) \qquad u(0,t) = u(b,t) = 0 \qquad t \in [0,T]$$

Introducing the notation for inner product in space and time

$$(2) \qquad ((u,v)) = \int_0^T \int_0^b uv \, dx\, dt$$

Lion's weak formulation for problem (1.1.-3) becomes

$$K((u_x, v_x)) - c((u, \dot{v})) = ((f, v)) + c(u_0, v_0) \tag{3}$$

while weighted residuals lead to (see [4] eq.5)

$$K((u_x, v_x)) + c((\dot{u}, v)) = ((f, v)) \tag{4}$$

Clearly the latter formulation falls short of the partial integration on time; that induces a change of sign in the second term of the l.h.s. and the appearence of (u_0, v_0) on the r.h.s. Most important is the unloading of the time derivative from the solution u onto the test function v.

Discretizing into finite elements, linear in space and time, (see [1] pag. 17)

$$u(x,t) = (1, x, t, xt)\, N\, \underline{u} \tag{5}$$

where $\underline{u}$ are the values of the temperature at the nodes of the element, expression (3) becomes for one time step (0,T) (see [1] pag. 6)

$$\underline{u}^T K_x \underline{v} - \underline{u}^T K_t \underline{v} = \underline{f}\, \underline{v} + \underline{u}_0 Q\, \underline{v}_0 \tag{6}$$

Calling $K = K_x - K_t$, partitioning eq. 6 for t=o (index 1) and t=T (index 2) and recalling that $\underline{v}_2 = 0$

$$[\underline{u}_1, \underline{u}_2] \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \underline{v}_1 \\ \underline{0} \end{Bmatrix} = [\underline{f}_1, \underline{f}_2] \begin{Bmatrix} \underline{v}_1 \\ \underline{0} \end{Bmatrix} + \underline{u}_1 Q\, \underline{v}_1 \tag{7}$$

Considering that $\underline{u}_1 = \underline{u}_0$ is given, the unknown is $\underline{u}_2 = \underline{u}(T)$ and eq. 7 becomes

$$(\underline{u}_1 K_{11} + \underline{u}_2 K_{12})\, \underline{v}_1 = \underline{f}_1\, \underline{v}_1 + \underline{u}_1 Q\, \underline{v}_1$$

eliminating $\underline{v}_1$ and using the symmetry of the matrices

$$K_{12}\underline{u}_2 = \underline{f}_1 + Q\underline{u}_1 - K_{11}\underline{u}_1 \tag{8}$$

Eq. 8 gives the set of linear equations to be solved at each time step for $\underline{u}_2$.

For the weighted residuals method, calling $K = K_{sc} + K_t$, eliminating $\underline{v}$, one obtains

$$K\underline{u} = \underline{f} \tag{9}$$

partitioning eq. 9 and carrying $\underline{u}_1$ onto the r.h.s.

$$K_{22}\underline{u}_2 = \underline{f}_2 - K_{21}\underline{u}_1 \tag{10}$$

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A comparison of eq.s 8 and 10 shows that space-time elements use the offdiagonal "evolution" matrix K_{12} to solve for $\underline{u}_2$, while weighted residuals use the diagonal "distribution" matrix K_{22} . The right end sides are totally different.

Notice that K is a space-time matrix, and that isdifferent in the two cases.

The equivalence of Galerkin and other time stepping formulas to the procedure of weighting residuals on time was proved by Zienkiewicz [5] .

Clearly, if the space-time approximating function in eq. 5 can be split into the product of a function approximating in space by a function approximating on time, the weighted residuals method extended to time can be rendered computationally equivalent to some time stepping method.

The space-time procedure, however, is inherently different from

weighted residuals and, therefore, from time stepping Galerkin methods.

The numerical results obtained prove the point.

SPACE-TIME VERSUS WEIGHTED RESIDUALS: HYPERBLIC

Starting from the one dimensional problem in strong (differential) formulation (see problem PI [1])

$$(11.1) \quad -Ku_{xx} + \ddot{u} = f \qquad x \in (0,b),\ t \in [0,T]$$

$$(11.2) \quad u(x,0) = u_0 \qquad x \in (0,b)$$

$$(11.3) \quad \dot{u}(x,0) = \dot{u}_0 \qquad x \in (0,b)$$

$$(11.4) \quad u(0,t) = u(b,t) = 0 \qquad t \in [0,T]$$

Lion's weak formulation becomes

$$(12) \quad K((u_x, v_x)) - \rho((\dot{u}, \dot{v})) = ((f, v)) + (\dot{u}_0, v_0)$$

whereas weighted residuals gives

$$(13) \quad K((u_x, v_x)) + \rho((\ddot{u}, v)) = ((f, v))$$

Notice that, in eq. 12, the time derivatives are lowered and balanced between the solution and the test function. The starting velocity $\dot{u}_0$ must however be computed at all steps, except for the first one. The procedure is the following:

- 1 The displacements are computed (at each step $(0,T)$) for two time values $t = T$, $t = \psi T$, $0 < \psi < 2$, $\psi \neq 1$

- 2 The time derivatives, i.e. the initial velocity for the next step, are, from the incremental ratio,

$$\dot{u}_0 \simeq (u(T) - u(\psi T)) / |T - \psi T|$$

The choice of ψ is based on minimizing the error in wave amplitude for the first half wave. Numerical tests on the one dimensional wave propaga-

tion problem, where the exact solution is known, have shown that the dependence of the error in wave amplitude is linear with ψ , regardless of wave number and wave lenght. A chart has been drawn where, for a given grid lengh α , step of integration τ and value of ψ , the percentage error in wave amplitude for the integration of the half wave is computed (see fig. 18). A correct choice of ψ has proved to lead the space-time procedure to extreme accuracy over many wave lengths of numerical integration. That can be explained by the fact that the numerical accuracy of the velocity is not subjected to control either in the weighted residuals or in the Galerkin methods.

The authors believe that values of ψ chosen from the chart, based on time step and wave lenght, can be successfully applied to space-time integration over multidimensional domains.

CONCLUSIONS

The space-time finite element method is affected in its (good) numerical performance by a few basic factors:

- the partial integration on time unloads the time derivative from the solution to the test function, that disappears from the solving procedure. Mathematical analysist have proved that this leads to higher ragularity; numerical analysts believe that this leads to better accuracy.
- the condition $\sigma(\tau)=0$ induces a shift, in the assembled discrete equations, from the diagonal to the offdiagonal matrix as the basic set of equations to be solved at each step. The matrix is a space-time matrix: the offdiagonal terms represent "evolution", in time, while the diagonal terms represent "distribution" in space.
- for hyperbolic problems, in addition, a proper guess of the velocity keeps the numerical solution well in phase, with great improvements in accuracy.

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