

Spin-Glass Complexity

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We study the quenched complexity in spin-glass mean-field models satisfying the Becchi-Rouet-Stora-Tyutin supersymmetry. The outcome of such study, consistent with recent numerical results, allows, in principle, to conjecture the absence of *any supersymmetric contribution* to the complexity in the Sherrington-Kirkpatrick model. The same analysis can be applied to any model with a full replica symmetry breaking phase, e.g., the Ising p -spin model below the Gardner temperature. The existence of different solutions, breaking the supersymmetry, is also discussed.

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Mean-field spin-glass models can display different kinds of *frozen phases* according to the choice of the interaction between their constituent elements. Such a phase can be either full replica symmetry breaking (FRSB), i.e., described by means of an order parameter that is a *function* of the replica-group index x [the *overlap* $q(x)$], or one step replica symmetry breaking (1RSB), in which only two groups of replicas are necessary for a proper representation of the properties of the system and the overlap consequently takes only two values. The prototype of the first kind is the Sherrington-Kirkpatrick (SK) model [1], whereas for the second kind we will discuss the Ising p -spin model, displaying discontinuous overlap at the paramagnet/spin-glass transition at temperature T_s and, moreover, a second 1RSB/FRSB transition deep in the frozen phase at the Gardner temperature T_G [2].

By decreasing the temperature from the paramagnetic phase, both the SK and the p -spin models undergo a transition to a phase where a dynamic aging regime sets up. Below the transition temperature, the infinite system never equilibrates; two-time quantities are not time translational invariant and do not satisfy the fluctuation-dissipation theorem. However, the nature of the aging regime is different in the two models. Furthermore, in the SK model such transition coincides with the static transition, taking place at temperature T_s and the large time values of one-time intensive quantities, e.g., the energy, tend to their equilibrium value (at finite temperature). In p -spin models, the temperature T_d at which the aging regime arises is above T_s and the dynamical energy never converges to its equilibrium value, which remains above some threshold. It is largely believed that the dynamical features of these systems are connected to the presence of a high number of metastable states, the logarithm of which, divided by the size of the system, is called configurational entropy or complexity.

In this Letter we present a quenched FRSB computation of the complexity of the SK model based on the Becchi-Rouet-Stora-Tyutin supersymmetry (BRST-SUSY). While in 1RSB models the complexity saddle point is supersymmetric and finite, in the SK model our

computation shows that a SUSY solution would be compatible only with a subextensive number of metastable states. It is reasonable to think that some of the differences between the dynamics of the two classes of models amount to this difference.

We will first concentrate on the complexity of the SK model, consisting of N Ising spins connected to each other by random variables J_{ik} of variance $1/N$. The states are usually identified with a proper subset of solutions of the Thouless-Anderson-Palmer (TAP) [3] equations, and the computation attempts to count these solutions. The TAP equations are mean-field equations for the single-site magnetizations $\{m_i\}$:

$$m_i = \tanh \left[\beta \left(\sum_{k=1}^N J_{ik} m_k - m_i (1 - q) \right) \right], \quad (1)$$

where $q = \sum_i m_i^2 / N$. They can be obtained by extremizing the following TAP free energy function $F(\{m_i\})$ with respect to the $\{m_i\}$:

$$\beta F(\{m\}) = -\beta \sum_{i < j} J_{ij} m_i m_j - \frac{N\beta^2}{4} (1 - q)^2 + \sum_i \left[\frac{1 + m_i}{2} \ln \frac{1 + m_i}{2} + \frac{1 - m_i}{2} \ln \frac{1 - m_i}{2} \right]. \quad (2)$$

Not all TAP solutions can be identified with physical states. An important condition is that they must be the minima of the TAP free energy. Furthermore, in order to yield the correct magnetic susceptibility of a state in zero external field, i.e., $\chi = \beta(1 - q)$, the solutions must satisfy the *Plefka criterion* $x_p \equiv 1 - \beta^2 \sum (1 - m_i^2)^2 / N \geq 0$ [4]. This criterion is also encountered in the replica computation of the equilibrium free energy as the central stability condition of the saddle point with respect to the fluctuation of the order parameter Q_{ab} . Indeed, it corresponds to the condition of positivity of the replicon eigenvalue. Equivalently, it can be recovered in the context of the cavity method as the condition of positivity of the spin-glass susceptibility $\chi_{SG} = \sum_{ij} \chi_{ij}^2 / N$ [5].

We would like to compute the minima of the TAP free energy functional satisfying the Plefka criterion. Unfortunately, the only expression one can handle is

$$\rho_s(f) = \int_{-1}^1 \prod_{i=1}^N dm_i \delta(\partial_i F) \det(\partial_i \partial_j F) \delta(F - Nf). \quad (3)$$

Strictly speaking, this corresponds to count all the solutions of the TAP equations of a given free energy f times the sign of the determinant of the Hessian. Therefore, the assumption that this expression corresponds to the number of physical states must be justified *a posteriori*. Through standard manipulations we can express ρ_s as an integral over fermionic, $\{\psi_i, \bar{\psi}_i\}$, and bosonic, $\{m_i, x_i\}$, variables of the exponential of the following action [6,7]:

$$S = x_i \partial_{m_i} F(\{m\}) + \bar{\psi}_i \psi_j \partial_i \partial_j F(\{m\}) + u[F(\{m\}) - Nf]. \quad (4)$$

The variable u is given by the δ -function over the free energy. This action posses the BRST-SUSY [6], i.e., it is invariant under the transformation $\delta m_i = \epsilon \psi_i$; $\delta x_i = \epsilon u \psi_i$; $\delta \bar{\psi}_i = -\epsilon x_i$; $\delta \psi_i = 0$. We can perform annealed or quenched averages yielding $\ln \bar{\rho}_s$ and $\overline{\ln \rho_s}$, respectively; the quenched averages are the physical ones, i.e., those describing the properties of a typical system. They can be computed through the replica method, considering n copies of the system and taking the limit $n \rightarrow 0$. Eventually one obtains an integral of the exponential of a ‘‘macroscopic’’ action S_{macro} depending on four bosonic and four fermionic (in case replicated) variables (see, e.g., [7]). If the complexity is extensive, the integral may be evaluated by steepest descent, solving saddle point equations for the macroscopic action. Furthermore, the fermionic part always gives a subextensive contribution that is neglected, yielding the well-known expression obtained more than 20 years ago by Bray and Moore (BM) [6,8–10].

The problem of finding a solution to the saddle point equations obtained in order to extremize the macroscopic action is highly nontrivial. The annealed case [6,8,10] tells us that there are different solutions to these equations and we face the problem of selecting, if any, the correct one. In particular, there is a BRST-SUSY solution and a BRST-SUSY-breaking solution. The selection problem has been recently considered by the authors [10] and it has been shown that all solutions currently known for the annealed case present some problems. In the quenched case it has been shown [11] that the complexity curve $\Sigma(f)$, if it exists, vanishes at the equilibrium free energy. In [10] it has been pointed out that such a point of the complexity curve is described by a BRST-SUSY saddle point.

The fermionic, subextensive part, usually neglected in the computation of the complexity by means of the saddle point approximation, is actually very important; taking it

into account, one sees that the macroscopic action too is invariant under a proper macroscopic BRST-SUSY transformation between its eight (eventually replicated) variables [7] and the prefactor of the exponential of the BM saddle point to be zero at all orders in an expansion in $1/N$.

Considering a BRST-SUSY saddle point simplifies the computation of $\int e^{S_{\text{macro}}}$. Indeed, as already discussed in [6,10], one recovers the computation scheme proposed in [12,13]. We, thus, consider the replicated expression of the variational free energy used to compute the equilibrium free energy, which depends on the $n \times n$ matrix Q_{ab} . In the limit $n \rightarrow 0$, Q_{ab} is parametrized by means of the Parisi Ansatz in terms of the function $q(x)$. We impose by hand a break point of $q(x)$ at $x = m < x_{\text{static}}$ (where x_{static} is the break point of the Parisi solution), and then we extremize the free energy with respect to $q(x)$, obtaining the function $F(\beta, m)$. The complexity $\Sigma(\beta, f)$ at a given free energy f is obtained through the following equations:

$$\Sigma(\beta, f) = \beta m^2 \frac{\partial F}{\partial m}, \quad f = \frac{\partial(mF(\beta, m))}{\partial m} \quad (5)$$

with

$$F(\beta, m) = \text{ext} - \frac{\beta}{4} \left[1 - 2q(1) + \int_0^1 q^2(x) dx \right] - f(0, 0). \quad (6)$$

The function $f(x, y)$ satisfies the Parisi differential equation $\partial_x f = \dot{q}/2[\partial_y^2 f + x(\partial_y f)^2]$ with initial condition

$$f(m^-, y) = \frac{1}{\beta m} \ln \int dp_{\Delta q}(z) \cosh^m(\beta y + \beta z) dz + \frac{\ln 2}{\beta}, \quad (7)$$

where $dp_{\Delta q}(z)$ is a Gaussian measure over z with mean zero and variance Δq , which accounts for a possible discontinuity at $x = m$. We have computed both $q(x)$ and the resulting complexity as a function of m by numerical integration and by an exact series expansion in powers of the reduced temperature through the techniques of [14]. We find that when setting m smaller than the static break point x_{static} , the function $q(x)$, which extremizes the free energy, develops a discontinuity at $x = m$. In general, $q(x)$ is a continuous monotonous function for $x < m$; at some $x_0 < m$, it develops a plateau such that $q(x) = q(x_0)$ for $x_0 \leq x \leq m$, in particular, $q(m^-) = q(x_0)$. Eventually, it displays a positive discontinuity Δq at $x = m$. For $x > m$, it is a constant. At some $m = m_{\text{max}}$, corresponding to a threshold free energy f_{th} , $\Sigma(\beta, m)$ takes its maximal value (see Fig. 1). We can identify $\Sigma(f_{\text{th}})$ with the total complexity.

Our most important result is that this solution must be rejected on a physical ground. In Fig. 1 the BRST-SUSY quenched complexity is plotted versus f together with the annealed complexity [6,10]. Convexity implies that $\ln \bar{\rho}_s \geq \overline{\ln \rho_s}$. The total complexities, in the annealed

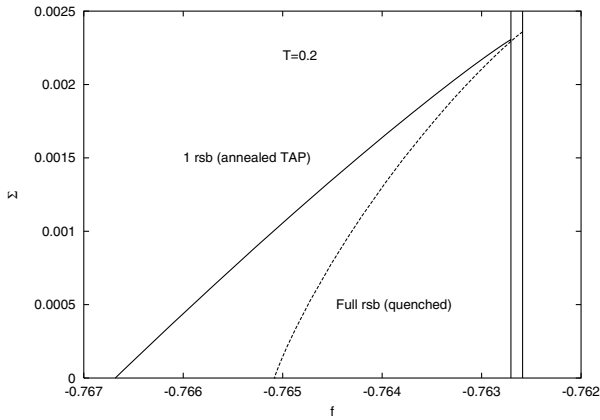


FIG. 1. Annealed and quenched complexity of the BRST solution in SK model.

and quenched case, can be identified with those of the states with higher complexity, i.e., $\Sigma_{\text{tot}} = \Sigma(f_{\text{th}})$ and, thus, the previous convexity condition is violated. Moreover, convexity implies that at any f the annealed complexity must be greater than the quenched one, but f_{th} is greater in the quenched case; therefore, there is a region where the annealed complexity is zero while the quenched complexity is finite.

Most importantly, we have checked (details elsewhere [15]) that the solution does not satisfy the Plefka criterion, i.e., the replicon eigenvalue is negative as soon as $m < x_{\text{static}}$. This means that, provided it actually describes some TAP solutions, they have no physical meaning. As discussed in [10], the violation of the Plefka criterion leads also to a mathematical inconsistency with some assumptions implicit in the solution (i.e., the condition $B = 0$; see [10]). A direct computation shows that this result can be extended to generic mean-field models with a continuous FRSB $q(x)$ [15].

Since the solution is unphysical for any $f > f_{\text{eq}}$, we obtain that BRST-SUSY in the SK model implies zero complexity. In order to account for a finite complexity, one has to look for another solution. In the BRST-SUSY case, the number of states is subextensive and all of them have the same free energy per spin, equal to f_{eq} . Furthermore, all states verify $x_p = 0$. These properties are in complete agreement with both old and recent numerical findings [16,17], not implying any supersymmetry. In [16] the minima of the TAP free energy with $x_p \geq 0$ (i.e., those that can be physically identified with states of the system) are studied by means of some modified TAP equations [16] which allow separation of this set from the non-physical solutions; in [17] the original equations are employed. In both cases it turns out that all these minima satisfy $x_p = 0$, as $N \rightarrow \infty$, which is in disagreement with the prediction of the annealed BRST-SUSY-breaking solution [11,10]. We also mention that a zero complexity was also obtained in the dynamical reformulation of Parisi solution performed in Ref. [18].

The Ising p -spin model [2] is useful to understand the different behavior of the complexity in 1RSB and FRSB models. This system undergoes a first phase transition from a paramagnetic to a 1RSB phase at a temperature T_s and a second, continuous phase transition from the 1RSB phase to a FRSB phase at $T_G < T_s$. The replicon eigenvalue of the 1RSB solution is positive but goes to zero at T_G . At $T < T_G$ the replicon is negative on the 1RSB solution while it is strictly zero on the FRSB solution. At temperatures $T_G < T < T_s$ we can compute the quenched complexity using the standard recipe of [12] complexity, which is zero at the equilibrium free energy and reaches a maximum value at some f_{max} . However, as noted in [19], the replicon eigenvalue, which is positive at $f = f_0$, goes to zero at some $f_G < f_{\text{max}}$ and is negative at free energies $f > f_G$ where the solution must be rejected; see Fig. 2. Thus we identify f_G as the free energy threshold at that temperature. It turns out that $\lim_{T \rightarrow T_G} f_G(T) = f_0(T_G)$. Therefore approaching T_G from above, the range $[f_0, f_G]$, where the complexity is finite, shrinks to zero. Equivalently, the total complexity goes to zero at T_G : $\lim_{T \rightarrow T_G} \Sigma(T) = \lim_{T \rightarrow T_G} \Sigma(f_G(T)) = \Sigma(f_{\text{eq}}(T_G)) = 0$. Therefore at the transition from a 1RSB equilibrium phase to a FRSB equilibrium phase, the complexity vanishes and remains zero in the whole FRSB phase.

We now discuss the possibility of adopting a BRST-SUSY-breaking solution, computed at the annealed level but not (yet) at the quenched one. As discussed in Refs. [7,10] it needs some justification in order to be used. Taking into account corrections to the saddle point, the expansion in powers of $1/N$ of the prefactor of the leading exponential contribution is zero at all orders. This result has been obtained by Kurchan [7] for the total complexity of the naive TAP equations exploiting the BRST-SUSY of the macroscopic action referred above;

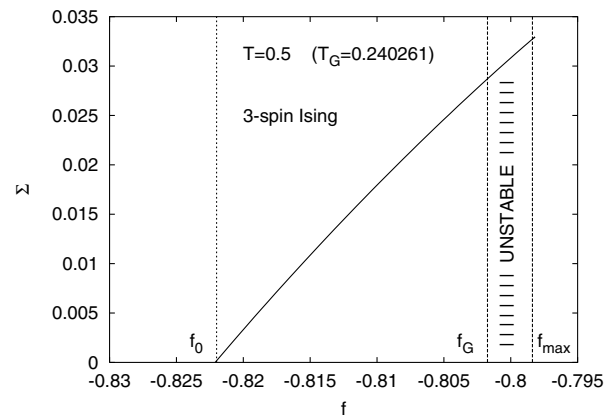


FIG. 2. Complexity of the Ising p -spin model for $p = 3$. It counts (meta) stable states up to a value f_G , lower than f_{max} (it would be $f_G = f_{\text{max}}$ in the spherical p -spin case where no 1RSB/FRSB transition occurs). Beyond such point, the stability condition (Plefka's criterion) is violated. In particular, $f_G \rightarrow f_0$ as $T \rightarrow T_G$.

we have extended this result to the SK model at a generic value of u (the variable conjugated to the free energy), i.e., at any value of the free energy. Clearly a zero or exponentially small prefactor will dramatically change the saddle point prediction. The vanishing prefactor is connected to the isolated zero eigenvalue recently identified in Ref. [20], for $N \rightarrow \infty$, solving the apparent violation of the Morse theorem [6]. In general, this result shows a substantial difference between the BRST-SUSY (counting TAP solutions with a strictly positive spectrum) and BRST-SUSY-breaking solution: while in 1RSB systems a BRST-SUSY complexity counts an extensive number of states with strictly positive eigenvalues, in FRSB systems the BRST-SUSY solution is unphysical and the BRST-SUSY-breaking solution counts, as $N \rightarrow \infty$, an extensive number of configurations with a flat direction out. This could be the cause, e.g., of the difference in dynamical behavior.

Besides this, the quenched solution should coincide with the annealed one at free energies greater than some f_c [8], and it has been shown [10] that this does not compare well with the numerical data of [16,17]. In particular, it predicts a finite value of x_p instead of zero. Even the support of the complexity is unclear at the numerical level, with controversial numerical results [16,21]. However, some, or all, of these latter problems might be clarified by building the quenched solution and checking whether $f_0 = f_{eq}$ and $x_p(f_{eq}) = 0$ [22].

The BRST-SUSY is related to the stability of TAP solutions under local perturbation (as shown in Ref. [10]). Breaking it means to allow for bifurcation, birth, or death of solutions when a small local magnetic field is added. This also has deep consequences on the approach by cavity method to the spin-glass problem. Indeed, the method assumes stability adding a new spin, i.e., slightly modifying magnetic local fields. The cavity analogue of the two-group FRSB replica computation [13] should then take into account the onset of new solutions as a new spin is added to the system.

We also mention that in the Ising p -spin model at $T_G < T < T_s$, a BRST-SUSY-breaking solution similar to the annealed one of the SK model can be found [23]. This solution, however, yields a complexity vanishing at a free energy different from the equilibrium one at a given temperature and violating Plefka's criterion for low free energies (see [15] for details). On the contrary, the BRST-SUSY solution of the Ising p -spin model vanishes at exactly the equilibrium 1RSB free energy (Fig. 2).

Regarding the possible existence of quenched BRST-SUSY solutions other than the one we considered, they would violate Plefka criterion too, yielding a negative replicon eigenvalue. Indeed, a direct computation (details elsewhere [15]) shows that in FRSB models the violation of the criterion is caused by the discontinuity at $x = m$,

while a continuous function marginally satisfies it. On the other hand, if $q(x)$ is continuous at $x = m$, we are not able to apply the recipe of Eq. (5). The discontinuity at $x = m$ is therefore unavoidable in order to have a nontrivial complexity but, at the same time, it causes instability.

Looking at the Ising p -spin model, we recall that in [19] the existence of states with free energy higher than f_G has been hypothesized; since the 1RSB solution is unstable, the complexity of these states should be described by a FRSB solution, much as the equilibrium free energy at $T < T_G$. It is not known whether such a solution actually exists, but it can be proved that violation of marginal stability will occur in the p -spin model as well; $q(x)$ should be discontinuous at $x = m$, but at the same time the presence of a FRSB region on the left of the discontinuity will force the replicon to be negative. The problem of the existence of a complexity of the clusters [19] in the p -spin model remains, instead, open.

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- [1] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
 - [2] E. Gardner, Nucl. Phys. B **257**, 747 (1985).
 - [3] D. J. Thouless, P.W. Anderson, and R. G. Palmer, Philos. Mag. **35**, 593 (1977).
 - [4] T. Plefka, J. Phys. A **15**, 1971 (1982); Europhys. Lett. **58**, 892 (2002).
 - [5] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
 - [6] A. Cavagna, I. Giardina, G. Parisi, and M. Mézard, J. Phys. A **36**, 1175 (2003).
 - [7] J. Kurchan, J. Phys. A **24**, 4969 (1991).
 - [8] A. J. Bray and M. A. Moore, J. Phys. C **13**, L469 (1980).
 - [9] A. J. Bray and M. A. Moore, J. Phys. A **14**, L377 (1981).
 - [10] A. Crisanti, L. Leuzzi, G. Parisi, and T. Rizzo, Phys. Rev. B **68**, 174401 (2003).
 - [11] A. J. Bray, M. A. Moore, and A. P. Young, J. Phys. C **17**, L155 (1984).
 - [12] R. Monasson, Phys. Rev. Lett. **75**, 2847 (1995).
 - [13] G. Parisi and M. Potters, Europhys. Lett. **32**, 13 (1995).
 - [14] A. Crisanti and T. Rizzo, Phys. Rev. E **65**, 46137 (2002).
 - [15] A. Crisanti, L. Leuzzi, G. Parisi, and T. Rizzo, cond-mat/0309256.
 - [16] T. Plefka, Phys. Rev. B **65**, 224206 (2002).
 - [17] A. J. Bray and M. A. Moore, J. Phys. C **12**, L441 (1979).
 - [18] J. van Mourik and A. C. C. Coolen, J. Phys. A **34**, L111 (2001).
 - [19] A. Montanari and F. Ricci-Tersenghi, Eur. Phys. J. B **33**, 339 (2003).
 - [20] T. Aspelmeier, A. J. Bray, and M. A. Moore, Phys. Rev. Lett. **92**, 087203 (2004).
 - [21] A. Cavagna, I. Giardina, and G. Parisi, cond-mat/0312534 [Phys. Rev. Lett. (to be published)].
 - [22] A. Crisanti, L. Leuzzi, G. Parisi, and T. Rizzo (to be published).
 - [23] H. Rieger, Phys. Rev. B **46**, 14655 (1992).