The meaning of the power law distributions in describing complex systems: implications for social sciences.

Abstract

Power laws are a special class of functions, which have been demonstrated to describe a lot of natural, economic and social phenomena. They assume often the form of a probability distribution, but they have been also used to explain other occurrences like the geometrical growing of biological or social entities (i.e. trees, cities, enterprises, etc.). Starting from the description of the physical and mathematical reasons for their emergence, we present some outstanding properties of the power laws (for example the scale invariance) and we review several studies carried out in many research areas. We focus particularly on social studies, in order to demonstrate that the understanding of the mechanism of action of the power laws has impact on the performances in multi-agents contexts and on the prevention of potentially disastrous events. First results on the extension to the Italian context of previous studies are also shown.

Keywords: complex systems, non-equilibrium dynamics, power laws, self-organized criticality, fractal structures, risk assessment.

1. Introduction

Real biological and natural world is characterized by complex phenomena, which are typically have irreversible, non-linear, non-at-equilibrium behavior.

The study of non-equilibrium dynamic systems became quite popular since the eighties, mainly for the work of the Belgian physical chemist Ilya Prigogine, Nobel Prize in Chemistry in 1977 for his contribution to the research on irreversible phenomena and non–equilibrium thermodynamics. He stated that the nature shows both irreversible and reversible processes, but the former are the rule, the latter the exception (Prigogine, 1997a). The irreversible behavior of a thermodynamic system cannot be described by the Newton's law, that is reversible and deterministic. The "arrow of time" - to say with the Prigogine's words - that is the possibility to predict the state of a dynamic system at any instant of time, future or past, only by the knowledge of its initial conditions has not any sense when we deal with irreversibility (Prigogine, 1997b). A system at the equilibrium state, according to the second law of thermodynamics, is characterized by maximum entropy and minimum free energy and it is typically disordered; far from equilibrium, systems are generally open and can import energy from the environment, increasing their negentropy (the negative entropy) and producing self-organized ordered structures (*dissipative structures*) where imported energy is then again dissipated (with a new increase of entropy).

In the literature, a complex system is described as typically dynamic and exhibiting scaling properties (Stanley et al., 1996; Lee et al., 1998; Mandelbrot, 2001). Generally such a behavior is index of a system: (1) composed by a large number of non-linearly interacting agents; (2) openly in relation with the external environment; (3) where some kind of self-organization emerge, as it is far from the equilibrium; (4) where the emergent phenomena can be not explained by the properties of the single component and they happen without any central control, as theorized by the swarm theory (Helbing, 1991).

2. Non-linearity, chaos and irreversibility in the dynamic systems

The challenge of the complexity theory is to prevent catastrophic events potentially generated by the chaotic behavior of some systems (i.e. cancer in biology, thunderbird in meteorology, market crash in economy, etc.). Often concepts like "complexity", "non-linearity" and "chaos" are confused or used as synonyms. There is, nevertheless, a substantial difference among them. Complex systems, as Prigogine demonstrated, are in general described by non-linear equations and the associated phenomena are mainly irreversible. He focused his work on the inadequacy of classical and quantum mechanics to describe natural and biological phenomena and developed a new probabilistic approach to the study of such systems. Almost contemporary, at the Santa Fe Institute in New Mexico, several scientists dealt with non-linearity and chaos theory, using cybernetics and computer simulation, but their approach to the chaotic systems was, actually, totally deterministic as it followed the classical Newtonian equation of trajectories., Indeed, if small errors are made in measuring, at time t, the variables describing two apparently identical states of a dynamic system described by non-linear equations, then even large differences can be observed at time t+1. This can generate an apparently stochastic behavior, even if only deterministic equations have been used to describe the system: it is the so called *deterministic chaos* (Schuster and Just, 2005; Anderson, 1999). Random fluctuations, then, can potentially affect the system and amplify non–linear interactions, since a critical point, called "bifurcation point", where chaos prevails and any symmetry is broken. Below this threshold, the system can continue absorbing energy and different organized configurations can emerge: stationary states (stable attractors); oscillatory states (limit cycle attractors). Non periodic strange attractors, instead, characterize systems with chaotic behavior.

Indeed, given a dynamic system $u(t)$ in an Euclidean space $\mathfrak{R}^N$ where $u(0)$ is the state of the systems at t=0, the system has an attractor if exists a subset A of $\mathfrak{R}^N$ so that, for each $u(0)$ and for large values of t, $u(t)$ is close to A (Mandelbrot, 1983). Strange attractors are those points around which the system continuously and chaotically orbits, without any clear reason. Small perturbations to the initial conditions of a chaotic system can produce amplified systemic effects. This is sometimes referred to as "butterfly effect" as non-linear systems, even starting by the same initial conditions, can evolve differently in the course of the time. This concept is also at the base of the so called "catastrophe theory" (Zeeman, 1977; Thom, 1977), for which small shift to some parameters can generate different final equilibrium states.

3. The power laws in the self-organized criticality

In nature, many phenomena have characteristic parameters subjected to recurrent fluctuations (i.e. turbulence in liquids, neuronal activity, movement of the earth crust, etc.), which can generate the emergence of new stationary states not tuned by any external agent (such as a magnetic field, a thermodynamic variable, etc.).
For this reason, great popularity gained in the last thirty years the study of the so called “self-organized criticality”, so defined by Per Bak (Bak, 1990). He studied the capability of some dynamic systems to experiment critical states, during which a self-organized configuration can occasionally evolve, through an “avalanche” effect, also for minimal modifications of the external conditions. A sand pile is the classical example. When some grains of sand are trickled onto the top of a pile, the slope of the heap growths until the pile reached a fixed conic surface and the angle of repose is reached (self-organized stationary state) Any addition of single grains of sand onto the pile can generate small or big avalanches, that bring again the pile at the stationary point (critical state). It is not possible to foreseen if any added grain will generate small or catastrophic avalanche events.
In the same way, earthquakes are generated by the sudden movement of the tectonic plates inside the planet (as a fluctuation of a critical state) and they propagate seismic waves up to the earth’s surface, such as avalanches of big or small intensity.
The Gutenberg-Richter law:

$$N(E) \approx E^{-b}$$

gives the number of earthquakes that release at least a certain energy E, being b a constant value between 1,2 and 1.5 (Gutenberg and Richter, 1956). Computer simulations revealed the chaotic behavior of the earthquakes: small changes in the initial conditions, in fact, can give rise to a large amount of effects, so that it is not possible to formulate predictions about the evolution of the system, without knowing all about it (that is unrealistic). The Gutenberg-Richter law has the typical mathematical expression of many complex phenomena. It is a power law and give us information about the frequency of earthquakes of low or high intensity (respectively with small or large value of E). So it will be much more probable to have many “insignificant” earthquakes than to experiment a “big one”.
Compared with a symmetric Gaussian distribution of independent events, a power law distribution (also called Pareto distribution)

$$Pr[X \geq x] = cx^{-\alpha} \text{ (with c>0 and } \alpha > 0)$$

has a long tail. Usually α falls in the range $0 < \alpha \leq 2$, so both variance (for $\alpha \leq 2$) and mean (for $\alpha \leq 1$) can have infinite value, differently by the normal distribution, where mean and variance are stable and finite. In a log-log plot $Pr[X \geq x]$ is a straight line (Mitzenmacher, 2003), because

$$ln(Pr[X \geq x]) = -\alpha ln\, x + ln\, c$$

Actually, the extreme events (more probable for a long tail distribution than in a Gaussian one), alter the mean of the distribution, so that the potentially infinite variance and the large confidence intervals make findings less significant.

In the case of earthquakes, this conclusion has serious implications: on a hand, there is no way to predict an earthquake; on the other hand, even if extremely improbable, potentially devastating events are those who must be taken particularly into account, as they represent a risk for the people.
Self-organized criticality has been also found in completely different sectors, such as the economy. With the divergent interests of suppliers and customers, the irrationality of people, the political changes affecting a globalized market, it can be easily understood how the economy act in a continuous critical edge, where small fluctuations of one or more parameters can generate a financial bubble or a market crash.

4. The power law in the fractal mathematics

In 1961, Benoit Mandelbrot was invited for a talk to Harvard University, where discovered that diagrams of cotton market price fluctuations on a century base, which Professor Houthakker was at that time working on, fitted like his own graphs on income distribution through the society (Hudson and Mandelbrot, 2004). So he realized that price variations followed a power law distribution, such as the frequency of words in the Zipf' studies (Zipf, 1949) or the wealth distribution in Pareto's ones (Pareto, 1847). The most exiting evidence, anyway, was the fact that the fluctuations on daily, monthly or yearly rate fitted the same way.
This behavior does not depend on data or on the time series analyzed, but on a mathematical property of the power law functions that describes this kind of phenomena, called self-similarity or scalability.
Actually, multiplying the argument of a power law *f(x)*~ $x^{-\alpha}$ for a constant k, the function itself is multiplied for the same value

$$f(kx) = (kx)^{-\alpha} = k^{-\alpha} f(x).$$

Generally, power laws and their probability densities describe phenomena where variables are not independent and an elevated degree of complexity characterizes the system. Unfortunately, as, at different scales, a system remains self-similar, in order to studies its general properties, it is not possible to reduce the complexity looking at the basic components, such as chemists and physicists are used to do (i.e. studying atoms and molecules to derive characteristics of the whole structure).
The power laws, thus, are good clues of a not linear and scale-free dynamic in a system. Mandelbrot, with his studies on cotton market, was one of the first scientists to consider the potential effects of the scale invariance of the power laws on the economic research.
However, in nature many objects or phenomena appear self-similar changing the scale: the trees, the lightning in a thunder, the clouds, the distribution of galaxies and clusters in the universe (Pietronero, 1987) and many others.

4.1. The fractal dimension

Mandelbrot introduced the concept of fractal dimension to describe the way in which such an object fills in the space, using the Kolmogorov's concept of "capacity of a geometrical figure" (Kolmogorov, 1958). According to Mandelbrot (Mandelbrot, 1983), a fractal set is, by definition, an ensemble for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension ($D_{HB}>D_T$). To understand this concept is quite easy. The topological dimension is always an integer natural number lower than 3; more easily, it can be considered like the Euclidean dimension, so that a point has $D_T=0$, a line $D_T=1$, a flat surface $D_T=2$ and a solid figure in the space $D_T=3$. Dividing iteratively an irregular shape by a divisional factor $k = \frac{L}{\varepsilon}$ (where L is the total length and ε is the length of a single part), we obtain for each iteration $N(\varepsilon)$ equal parts with

$$N(\varepsilon) = c\left(\frac{1}{\varepsilon}\right)^D = c\,\varepsilon^{-D}$$

where c is a constant value depending on the measurement unit. This is a particular power function, as in logarithmic scale it defines the Hausdorff-Besicovitch dimension (from now on “fractal dimension”)

$$D = \frac{logN}{logk}$$

The fractal dimension coincides with the topological dimension for regular geometric figures, like lines, surfaces or cubes because, for any divisional factor k, the ratio between the log of number of resulting parts and the log of k is always 1, 2 or 3 respectively. Vice versa, the Sierpinski gasket is an example of a not regular geometrical figure. Given an equilateral triangle, the median points of each side can be considered the vertexes of a new inscribed overturned triangle (fig. 1). If this is extracted, three new smaller triangles are generated, similar to the first one. The process can be iteratively repeated, each time dividing every side of a triangle in two equal parts (k=2) and giving rise to N=3 new triangles. For this reason, the Sierpinski gasket’s fractal dimension is

$$D = \frac{log3}{log2} \approx 1{,}58$$

This value respect the Mandelbrot definition of a fractal, as the Hausdorff-Besicovitch dimension of a Sierpinski gasket is bigger than its topological dimension (that is 1 as, iterating the triangle extraction at infinitum, it remains only a tangle of straight lines).

Fig.1. The Sierpinski gasket

The topological dimension of the Koch curve (Koch snowflake) is equal to 1 for the same reason, while its fractal dimension is

$$D = \frac{log4}{log3} \approx 1{,}26$$

as, in each iteration, the side of a triangle is divided in three equal parts (k=3) and the middle segment is replaced by two segments having the same length to generate an equilateral triangle (fig. 2), so that 4 identical segments are generated on each side (N=4).

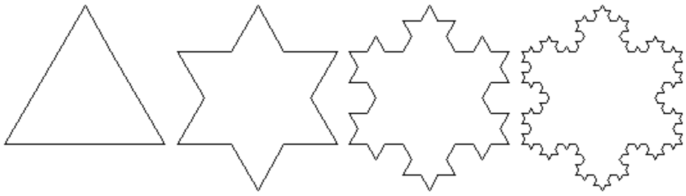

Fig.2. The Koch snowflake

Usually the fractal dimension is a fractional, irrational number, but it can be also an integer. It is the case of a particle, that moves with a Brownian motion on a flat surface. Its trace is considered a fractal, with fractal dimension equal to 2 as, at infinitum, the particle touch all the points of the surface, such as the Peano curve.

These structures are mathematically challenging to represent, as the traditional analytical instruments prove to be useless. In fact, in the most cases, a curve can be expressed by an analytic function so that it is possible to define univocally a tangent curve point by point; for very small scales, the tangent can be also used for approximating the curve itself. In the scale-free world, instead, every part of a system has the same complexity of the whole and it is not possible to describe its contour by a tangent (Pietronero, 2011).

4.2. Fractal evidence of the allometric growth

Scale-free behavior of the fractal structures is the direct consequence of the characteristics of the power laws, which fractal geometry is based on. Fractal evidence has been found in a lot of research fields. In nature, for instance, power laws describe the scaling behavior of several biological parameters. Actually, the so called "allometric equation"

$$Y=Y_0M^b$$

where Y_0 is a normalization constant, M is the body mass and Y is a variable such as the metabolic rate, the blood circulation time, the life span, etc., has a fascinating property: the scale exponent b, whatever parameter Y represents, takes on a value multiple of ¼ for a lot of biological phenomena connected with the metabolic system (West et al., 1997; West et al., 1999). So, the diameter of the tree trunks is proportional to $M^{3/8}$, the blood circulation time and the life span to $M^{1/4}$, the heart rate to $M^{-1/4}$, the same metabolic rate to $M^{3/4}$ (Schmidt-Nielsen, 1984). These relationships explain, for instance, why, for larger organisms, the heart beats slower ($Y\sim M^{-1/4}$); why, for unit mass, they consume less energy ($Y\sim M^{3/4}/M= M^{-1/4}$); why smaller organisms die before $Y\sim M^{1/4}$). Only considering mammals, it is as if nature had replied, at different scales, the same biological mechanism and functionalities for different animals, from mouse to elephant. The characteristic of self-similarity has been explained by the fractal nature of the dynamic constrains inside the organisms. By evolution, living beings have been selected responding to efficiency principles. It implies a balance between the maximization of metabolic performance (capability to produce energy necessary to the survival) and the minimization of distances to be covered to transport materials internally to the organisms (e.g. blood, nourishment, etc.). While the first property is directly connected to the surface area exchanging resources with the external environment (so it should change with mass to the power 2/3), the latter, instead, depends on the volume of the organism (we would expect changing proportionally to the mass). An explanation of the fractional exponent reflects, actually, the existence of fractal-like distribution networks, such as the vascular system, common to different organisms at any scale and emerging to optimize energy transport.

The parallelism between allometric growth and fractal structural development seems to be confirmed also by the studies on the shape and evolution of urban areas. Recently Bettencourt (Bettencourt et al., 2007), on the base of the biological model for metabolic networks, have found that all the cities of a certain country, in the same year, scale according to an allometric power law such as

$$Y=Y_0N(t)^{\beta}$$

where $N(t)$ is the population at time t and the exponent β assumes around the same value for classes of structural or social indicators. For example, they have shown that physical infrastructures like roads or electrical supply networks growth more slowly than the city; vice versa, processes driven by innovation and wealth creation (patents, GDPs, bank deposits, etc.) increase faster than the urban population (Batty, 2008; Bettencourt et al., 2007). Fractal nature of urban agglomerates is far-back known (Batty and Longley, 1994), but it is still to demonstrate even the self-similar urbanization has a responsibility or, on the contrary, it depends on the social and economic dynamics described by the authors. Certainly, growth is governed by availability of resources (including infrastructural assets) and innovation and social wealth could be drivers for the development.

5. Complex networks

One of the most interesting methodology for investigating complex systems are the complex networks. As, by definition, a complex system is made by many different interacting components, a lot of scientists have interpreted these interactions as the edges of an intricate network, where the nodes are the single agents. In table 1 we report some illustrative examples of the scientific literature and the correspondent application domains.

Table 1. Examples of complex networks reported in scientific literature

Authors	Described network	Application domain
Jeong, et al., 2000	Metabolic network	Biology-chemistry
Steyvers and Tenenbaum, 2005	Semantic network	Linguistics

Reijneveld et al., 2005	Neuronal brain physiology	Medicine
Onody and de Castro, 2004	Brazilian soccer players' network	Sport
Sznajd-Weron and Sznajd, 2000	Opinion formation	Sociology
Abe and Suzuki, 2006	Earthquakes networks	Seismology
Albert et al., 1999	Internet diameter	ICT
Watts and Strogatz, 1998	Power grid of the Western United States	Energy
Fass et al., 1996	Bacon number, actors's minimal working distance from Kevin Bacon	Cinema
Saito et al., 2007	Interfirm relationships	Economy

The investigated real networks have been demonstrated to show structural properties very different from the random graphs of the Erdős–Rényi (ER)'s probabilistic model, in which N nodes are randomly connected each other with probability p (Erdős and Rényi, 1959). A random network, in fact, is rather homogeneous, due to the Poisson distribution of the nodal degree (the probability that a node has k edges), while, in the real networks, the probability that a vertex is connected with k other nodes follows the power law

$$P(k) \sim k^{-\gamma}$$

This mean that also a complex network behaves as a scale-free system and that there are few nodes with many edges. In the table 2 the main differences between random and real networks (Braha and Bar-Yam, 2007) are presented.

Table 2. Comparison between parameters of random and real complex networks

	Random networks	**Real networks**
Degree distribution	$p(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ with $\lambda = \binom{N-1}{k} p^k(1-p)^{N-1-k}$	$P(k) \sim k^{-\gamma}$
Average degree	$<k> \cong pN$	$<k> = \frac{\sum_{i=1}^{N} k_i}{N}$
Average distance between two nodes (path length)	$l \sim \frac{\ln(N)}{ln<k>}$	$l = \frac{1}{N(N-1)}\sum_{i \neq j} d_{ij}$ with dij number of edges along the shortest path between node i and node j
Clustering coefficient	$C = p \cong \frac{<k>}{N}$	$C = \frac{1}{N}\sum_{i=1}^{N} c_i$ with $c_i = \frac{2n_i}{k_i(k_i-1)}$ k_i = neighbors of vertex i n_i = edges between the k_i nodes

As observed by Barabasi and Albert (1999) random networks model assume a fixed number of nodes, whereas the real networks are often characterized by a continuous addition of new vertices (new web pages in the world wide web, new electric stations in a power grid, etc.). Furthermore, while in the ER model the connection between nodes happens randomly and uniformly, real networks exhibit a *preferential attachment*. In fact, observations demonstrated that the new attaching nodes prefer to connect to nodes with a higher degree. It is trivial to show it in the WWW, where a new web page preferably links to the most popular web sites like Google or Twitter, but it is easy to demonstrate it also for scientific publications, which cite mostly other already very cited publications. The main consequence of this behavior in the real networks is the presence of hubs, highly connected nodes, such as in an airline map, where few airports have connections with a large amount of destinations in the same country or in the world. For this reason real networks are very resilient, that means that they are error tolerant and robust against random failures, as the inhomogeneity of their node distribution. In fact, a random "switching off" of some nodes will affect more likely less-connected vertices (as only a few of nodes are hubs), with no significant malfunctioning of the total network. On the other hand, the same property makes real networks very vulnerable to targeted attacks, with potential disastrous consequences for the network security.
Another property of the real networks is the *small-world* phenomenon (Watts and Strogatz, 1998). Although the real networks are more clustered than the random ones, they show a small path length like the random ER model networks, due to the long-range connections generated both in the real and in the random process. This is also known as the *six degree of separation* phenomenon, by the work of the American sociologist Stanley Milgram (Travers and Milgram, 1969), who empirically demonstrated that two perfectly strangers are, actually, separated only by six handshakes (each person can be connected to any other by a knowledge chain made only of five intermediates). The theory has been often criticized, but also other authors (Dodds et al., 2003) obtained a similar result in a social research experiment where an e-mail message was forwarded to target persons through acquaintances. According to Barabasi, also the WWW is a small world, but with a grade of separation of about 19 (Barabási, 2007).

6. Power laws in the organization sciences

Mandelbrot's studies on price fluctuations of cotton market already put in evidence the action evidence of power laws also in economic phenomena. From Prigogine's studies on, power laws have been associated to systems out of equilibrium, at a critical phase (also called "bifurcation" or "tipping" point), where non-linear interactions take place, with the potential emergence of new properties or behaviors. The challenge is to understand how, when and why an organization show similar conditions. In the social and organization fields, for instance, many scientists have demonstrated that a lot of processes are described by a Pareto, scale-free function. Andriani and McKelvey (2009) have classified them, according to their causes, linking them to causes such as the allometric growth, the self-organized criticality (SOC) or the preferential attachment. Other researches have discussed social and economic issues within the complexity theory (Allen, 2001; Gardini et al., 2008; Schneider and Somers, 2006; Zang, 2009; Plate, 2010). Some of them have introduced mathematical models (most of all concern measurable economic parameters), others, mainly those regarding people behavior, instead, are based on empirical and metaphoric considerations. This is certainly due to the difficulties to model social systems. As observed by Helbing (2010), the study of systems composed by people is drastically different from the study of a physical or chemical phenomenon, typified by atoms and molecules. Humans are influenced by ungovernable parameters like morale, emotions, individual preferences, so that, some "emergent" properties are hardly ascribable to any cause or evolution of the environment. Herein we present some examples of organization and economic researches on systems exhibiting a power law trend, where a scientific approach has been adopted.

6.1. *Growth of companies*

Some authors have analyzed the size dimension of firms and their mechanism of growth, observing entrepreneurial data of a large amount of firms in several years. In particular, Axtell (2001), have considered companies in a large range of employees number (between 1 and 10^6), verifying that they were Zipf-distributed (with an exponent about 1.06) as showed in Fig. 3. Stanley (1996) and Amaral (1997), instead, demonstrated that the conditional (firms aggregate for amount of sales) distribution of the logarithm of the growth rate is not Gaussian, as expected from the Gibrat approach (Gibrat, 1931), but rather exponential (with a tent-shaped form):

$$\rho(r|S0) = \frac{1}{\sqrt{2}}\frac{1}{\sigma(S0)}\exp\left(-\frac{|r-\overline{r(S0)}|}{\sigma(S0)}\right)$$

where $r = \log(S_1/S_0)$ and S_1 and S_0 are the sales at two consecutive years (Fig. 4).
According to Gibrat, in fact, the growth rate of an enterprise doesn't depend on its size, it is not correlated in time and the companies don't interact. Mathematically Gibrat's model is expressed by a stochastic multiplication process (Gibrat's law of proportionate effect) that foresees a lognormal distribution of the growth rate and, consequently, also of the firms' dimension. An in-depth analysis of the statistical modeling of firm growth would bring us far from our aims, but it is here enough to say that other authors have instead demonstrated that the same Gibrat stochastic model can explain also the Zipf fit of the growth rate (Gabaix, 1999), postulating no effect of size on percentage growth rates.
Moreover, Stanley (1996) and Amaral (1997) showed that the fluctuations in the growth rates (the width of the above mentioned distribution) scale, with firm size, as the power law

$$\sigma \propto S_0^{-\beta}$$

where S_0 is the initial value for the observed parameter (sales, employees, etc.) and considering sales, employees number, assets, cost of goods sold, etc. as measure of size (β=0,20±0,03 for sales amount and β=0,18±0,03 for employees number),. This means that largest companies have smaller fluctuations in their growth rate, so they don't experiment big change in a period of time. Results seem universal across markets (same for different SIC codes). A potential explanation of such a behavior could be the analogy between the statistical mechanics of physical systems, composed by many interacting components, and the hierarchical composition of companies, made up of many interrelated subunits.

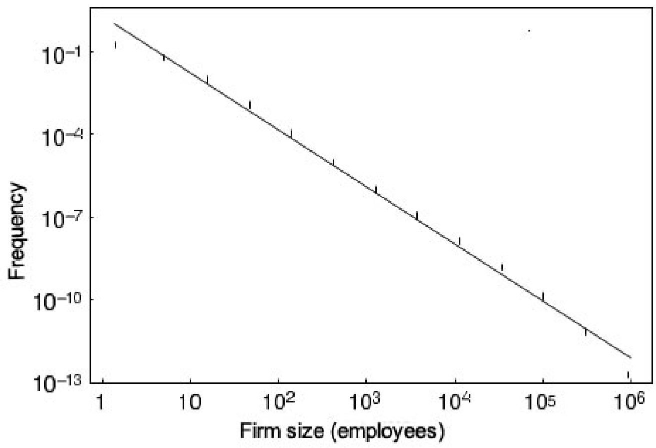

Fig. 3. Power law evidence in the size distribution of US firms by employees (Source: Axtell, 2001)

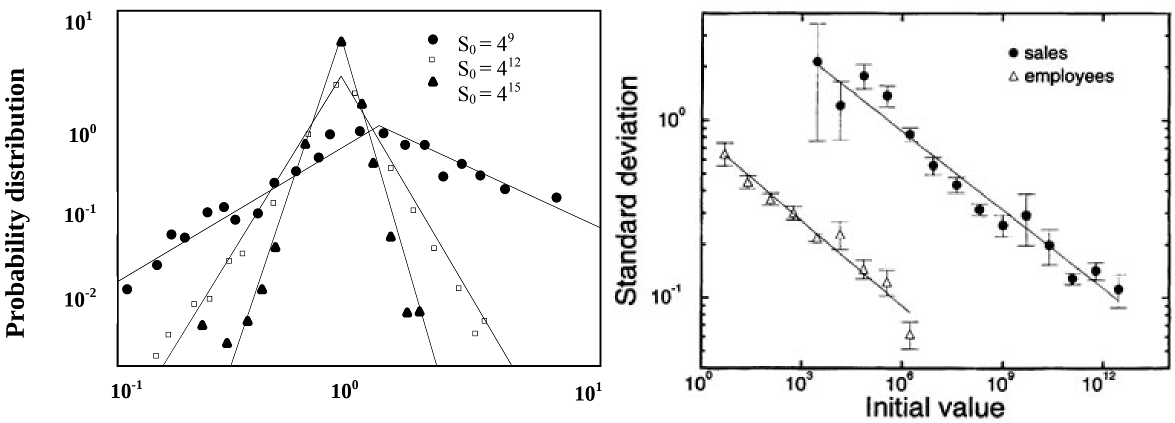

Fig.4. Conditional growth rate distribution of firms and its standard deviation (Source: Stanley, 1996). In the paper has shown that all the data, by a scale factor, collapse in a single curve.

6.2. *Growth of GDP*

A formally identical result obtained Lee et al. (1998), considering the gross domestic product (GDP) of 152 countries in the period 1950–1992. While data on different GDPs are consistent with a lognormal distribution, the distribution of the logarithm of the annual growth rate r=log[GDP(t+1)/GDP(t)] for countries with about the same GDP has, in a given range, an exponential form

$$\rho(r|GDP)=\frac{1}{\sqrt{2}}\frac{1}{\sigma(GDP)}\exp\left(-\frac{|r-\bar{r}|}{\sigma(GDP)}\right)$$

(Fig. 5a) and it fluctuations follow the power law $\sigma \propto GDP^{-\beta}$ with $\beta \approx 0,15$ (Fig. 5b).
It can be supposed that a common mechanism might describe both processes for firms and countries GDP.

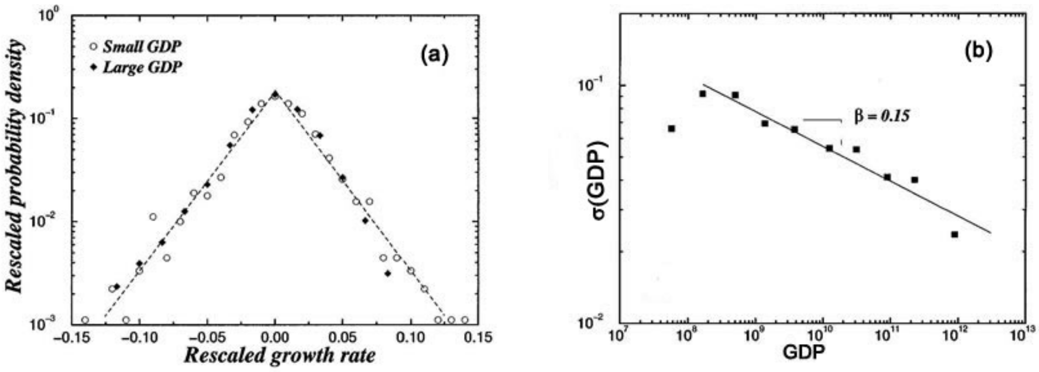

Fig.5. Scaled growth rate distribution of GDP and its standard deviation (Source: Lee 1998). The data, by a scale factor, collapse in a single curve.

6.3. *Growth of universities*

Plerou et al. (1999) analyzed the growth rate of hundreds American, English and Canadian universities during 17 years, in terms of R&D expenditures, number of papers published each year and number of patents granted. The results concerning the dimension are again lognormally distributed and the distribution of the logarithm of the growth rate has again a tent-shaped form (Fig. 6a), with a standard deviation that has a power law fitting with exponent $\beta = 0,25 \pm 0,05$ (Fig. 6b), with the same functional trend than in the studies on firms and GDP dynamics, even if with a larger exponent.
The different value for the exponent is explained again through the random Gibrat multiplicative model: universities, companies and countries are made up of many components (i.e. departments, divisions, cities, respectively), assumed as growing independently one from each other.

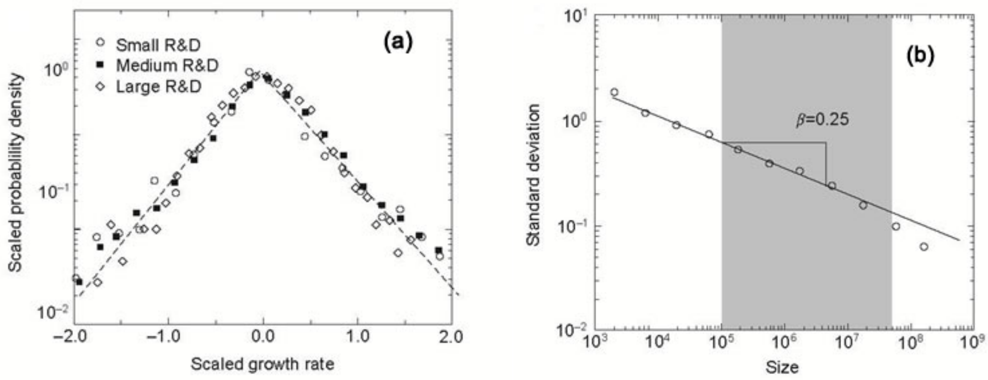

Fig.6. Scaled growth rate distribution of R&D expenditure and its standard deviation (Source: Plerou, 1999). The data, by a scale factor, collapse in a single curve and have the same shape also for papers number and for patents granted.

In this model the exponent β is inversely proportional to the width of the distribution of the minimum sizes of the component units. So, while companies and countries can have units, whose dimension can be very different case by case (with a consequent larger distribution of the minimum sizes), usually there is not such a big difference between universities (with a narrower distribution).

6.4. *Growth of Italian universities*

On the base of the Plerou's work, we analyzed the growth rate of Italian public universities between 1997 and 2013 in terms of published paper according to the database Web of Science (WoS) of the Institute for Scientific Information (ISI). The yearly numbers of papers published by scientists affiliated with Italian universities in ISI journals are not lognormally distributed (Fig. 7), but a it is a power law of the size. As explained by Liang et al. (2006) in the case of Chinese universities, such a difference can be due to the much smaller numbers of papers than in the US universities. Gibrat's model, actually, generates power laws if there is a lower bound for size. The minimum number of papers (1 in our case, excluded by the Fig. 7) acts effectively like a lower bound. This bound is obviously less important for US universities with their huge numbers of papers.

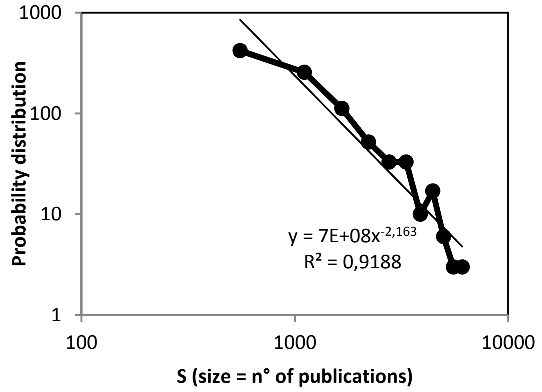

Fig.7. Distribution of the yearly number of papers published by Italian public universities in ISI WoS

The distribution of the logarithm of the growth rate, instead, has the same tent-shaped form already shown by Plerou for US universities (Fig. 8).

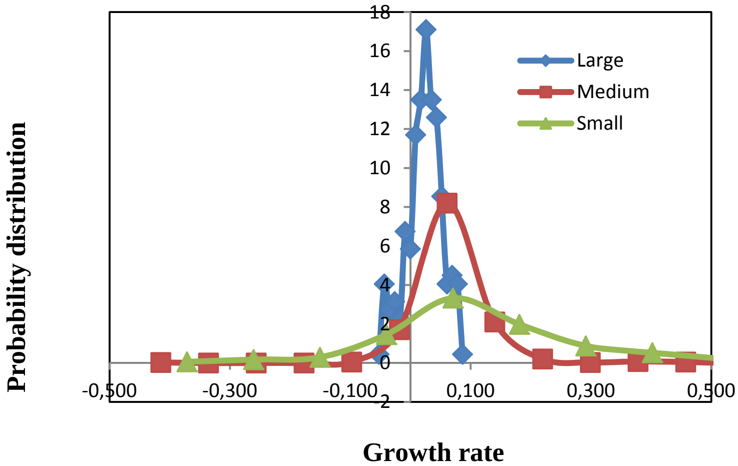

Fig.7. Probability distribution of the annual growth rate for three different dimensional classes of universities (S<200; 201<S<1000; 1001<S<4300 with S is the average number of publication in the period).

The results confirm that, also for Italian universities, the width of the distribution increases while decreasing the size. To confirm this evidence, we plot the conditional standard deviation of the distribution as a function of size. Fig. 8 shows that standard deviation follows the law:

$$\sigma = aS_0^{-\beta}$$

with a = 0,55 and $\beta \approx 0,25$.

The value of β is equal to that of Plerou's paper, which confirm the similar growth model for the university system.

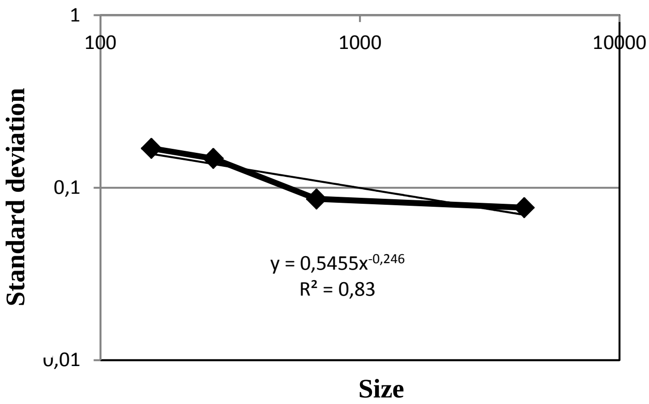

Fig. 8. Standard deviation of the annual growth rate as a function of the size of the universities. The solid line is least-square fit to the data in logarithmic scale with slope $\beta \approx 0,25$.

We plot, then, the conditional probability distribution $p(r|S)$ (12) versus the calculated scaled average vote for the three groups of size. $\bar{r}$ and σ are respectively the average growth rate calculated for each group of areas (large, medium and small) and the standard deviation previously calculated.

$$r_{scal} = \sqrt{2} * \frac{(r - C\bar{r})}{\sigma(S)} \quad \text{(with C constant)}$$

$$p(r|S) = \frac{1}{\sqrt{2}} \frac{1}{\sigma(S)} \exp(-|r_{scal}|)$$

Fig. 9 shows as the three conditional curves collapse upon a same curve.

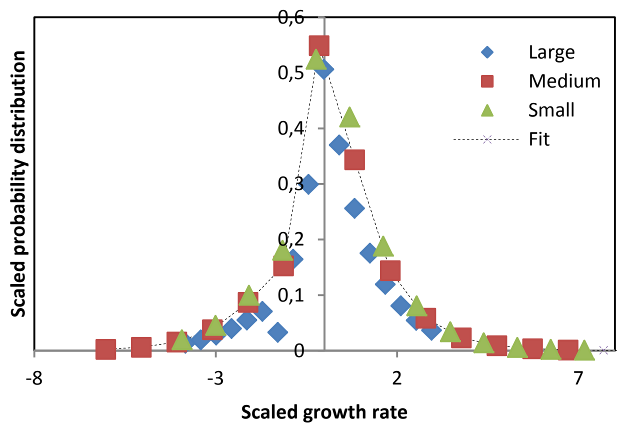

Fig. 9. Scaled probability distribution as function of the calculated value of r_{scal}

7. Discussion and outlook

In this contribution we reviewed several examples of power laws described in literature in many research areas. We have discussed the evidence of power laws in describing complex systems. They are usually so defined by researchers because they: are composed by many interacting elements; are typically in a critical non-equilibrium state; are characterized by non-linear responses to the solicitations. We have also analyzed how, in these systems, self-organized ordered structures (the dissipative structures by Prigogine) can emerge - a phenomenon which Peter Bak has called self-organized criticality. Power laws are also scale invariant, with the fascinating consequence that it is possible sometimes to represent them through the fractal geometry, which give us a powerful instrument for understanding and reproducing some natural or economic events like the structure of the pulmonary alveolus or the price trends. If this characteristic is a notable difficulty from a theoretical and mathematical point of view (the properties of these systems cannot be studied starting from those of their elementary components, using the traditional analytical instruments), under a practical aspect, instead, it is a remarkable property. Let's think to the earthquakes, which distribution of intensity, in terms of energy released, scale with a power law (Gutenberg-Richter law). Even if the tectonic movements of our planet are not yet predictable, this principle tell us that the study of the small more frequent earthquakes give us useful information also on the bigger ones. Actually, it has been well proved that a lot of natural, biological, economic and social phenomena don't answer proportionally to changes in their environment (non-linearity of the response) and that the cumulative distribution of such events is not normal, but rather more heavy-tailed. This means also that rare events are more frequent than expected by a Gaussian distribution (see Fig. 7). For example, in the financial sector, the Black-Scholes formula supposes prices distributed on a bell curve; a constant volatility during the life of the options; prices not experimenting jumps; no taxes and commissions to be paid. According to this model, the 19th of October 1987 (the Black Monday, when world stock markets crashed) would be practically impossible (it is at 35 sigma from the mean, that is like to say zero probability) (Helbing, 2009; Hudson and Mandelbrot, 2004).

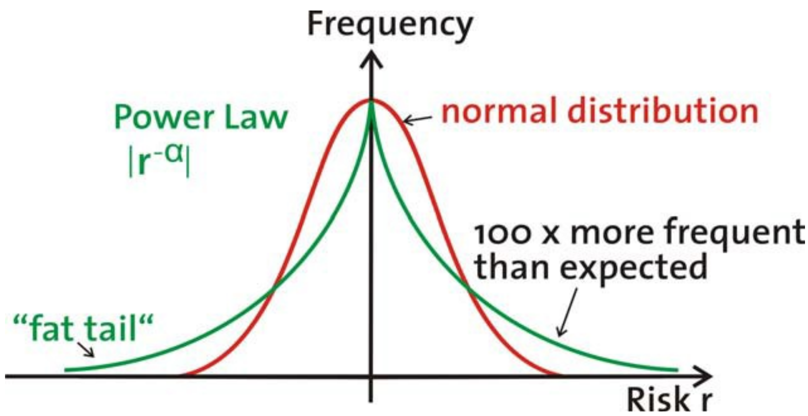

Fig. 7. Comparison between bell and power law distributions (Source: Helbing, 2010).

Consequently, heavy-tailed distributions are now carefully taken into account for the risk assessment, from natural sciences to the business. As we have seen, a power law is the signature for a critical phase, when an abrupt discontinuity can bring to a rapid change in the general environment, such as we have seen in the financial market. For enterprises, mainly for the smallest ones - it is, in fact, demonstrated that the inter-firm relationships vary with the company dimension by a power law (Saito et al., 2007) - a new type of risk is today represented by their dependency by suppliers, financial institutions, technological partners. Due to this interdependence, failure of a firm can have extreme consequences, as its default can propagate its effect through a network of creditors and debtors, potentially causing a cascading of failure events (well illustrated in the complex networks theory) like the bankruptcy (Fujiwara, 2004). The same happens for the spreading of traffic jam or for the blackout induced by a single node of a continental power grid. Anyway, if it is true that there is not a method to forecast such extreme events, a lot of techniques have been developed to reduce their impact or to model complex systems, in order to simulate particular behaviours responding to changes in the characteristic parameters: the agent-base modelling, the network analysis, the cybernetics or the cellular automata are only some examples. Recently, scientists are orienting their efforts not in predicting the behaviour of complex systems, but rather in taking advantage of the emergent properties like the self-organization. The study of the collective intelligences (also called "swarm theory") is becoming an important branch of the social sciences. In fact, in a swarm (let's think at the ants or the bees or the groups of birds flying all together in the sky) the interactions among the single agents, acting with local rules, without a central control neither any knowledge of the global pattern, generate the emergence of a systemic behaviour that is efficient for the group and often very robust against the external attack (i.e. a fish schools able to baffle their predators). This type of property is, for example, currently used in the WWW for marketing scopes (a great number of "I like" under a Facebook tag is both the indicator of a collective preference and a way to catch the attention of other people) or it has been applied to model the collective dynamics of walking people, which self-organize in ordered lanes within a crowd (Helbing, 1991). A lot of qualitative studies have been carried out on the complex behaviour of social and economic organizations. In fact, they have all the characteristics to be considered complex systems, as they experiment continuous critical states (due to the fact that they are open to the external environment, forcing their evolution), are composed by many different agents (i.e. the employees), who interact and influence each other by reciprocal feedbacks and, often, some kind of self-organization emerge (i.e. informal networks, innovation cells, etc.). An increased formalization of this type of studies is the real challenge for the future.

Finally, we have started to present original data on the growth rate of Italian universities. The first results seem confirm the same trend of previous analogous studies carried out on US universities. More developments of the research will be soon available.

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