Option pricing under deformed Gaussian distributions

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Abstract

In financial literature many have been the attempts to overcome the option pricing drawbacks that affect the Black and Scholes model. Starting from the Tsallis deformation of the usual exponential function, this paper presents, in a complete market setup, a class of deformed geometric Brownian motions flexible enough to reproduce fat tails and to capture the volatility behavior observed in models that consider both stochastic volatility and jumps.

Keywords: Derivative pricing, Stochastic volatility, Deformed exponential, Fat tails, Tsallis exponential, Complete markets

1. Introduction

In this paper, we present a family of complete market option pricing models with stochastic volatility in which the underlying asset exhibits heavier tails than those of a log-normal distribution. More precisely, the rate of return of this asset follows a diffusion process in the usual form of a geometric Brownian motion but whose source of uncertainty is played by a power-tailed continuous Markov process. Further, the distribution of this process is, at each time t, related to the Tsallis [1] deformation of the exponential function and reduces, as a particular case, to the standard Gaussian distribution.

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It is well known that Black and Scholes [2] (B&S thereafter) and Merton [3] proposed a path-breaking model in modern finance when they derived a closed form formula to price European options under the assumption that the dynamics of the underlying asset follows a standard geometric Brownian motion. This pricing expression is a simple pricing tool for both practitioners and researchers.

If the B&S formula is used to determine the underlying asset implied volatility, it turns out that such volatility is not constant (as instead assumed by the model), being dependent on both the moneyness (i.e. the ratio between the price of the underlying asset and the option strike price) as well as the time to maturity of the option.

The fact that the distribution of the random rate of return of the underlying asset is normal is, unfortunately, a major drawback for the B&S model. Different approaches, that exhibit heavier tails than those obtained with the log-normal distribution, provide a better fit for the observed returns for many equities as well as stock indices (see, for instance, Platen and Rendek [4]). To mention just a few, local volatility models (for instance Derman and Kani [5]), stochastic volatility models (see, amongst others, Heston [6]), and stochastic volatility models with jumps either only in the dynamics of the underlying asset (Bates [7]) or in the dynamics of both underlying asset and its stochastic volatility (Eraker et al. [8], D'Ippoliti et al. [9]).

Any model with more than one source of uncertainty leads to an incomplete market (Bjork [10]) in which, unless a sensible function expressing the market price of risk is exogenously introduced, it would be impossible to price derivatives. This is not the case with the present article that follows the approach of Hobson and Rogers [11]; in their work these authors deal with stochastic volatility in a complete market framework. Borland [12] and [13], Borland and Bouchaud [14], and Vellekoop and Nieuwenhuis [15] also follow this vein.

In Borland, [12] and [13], the author proposed a model for stock prices log-returns in the form of

$$d\ln(S_t) = \mu\, dt + \sigma_S d\Omega_t,$$

where μ and σ_S are constants and the source of uncertainty, Ω_t, is no longer a Wiener process as in the B&S model but, rather, a continuous Markov process evolving in time according to the stochastic differential equation (SDE)

$$d\Omega_t = g_\alpha(t, \Omega_t)\, dW_t, \tag{1}$$

where $\alpha \leq 1$ is a given parameter, W_t is a Wiener process, $g_\alpha(t, \Omega_t) = f(t, \Omega_t)^{\alpha/2}$, and the probability distribution $f(t, x)$ satisfies the purely subdiffusive Fokker-Planck (FP) equation

$$\frac{\partial f(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t,x)^{1+\alpha}. \tag{2}$$

It is well known (see e.g. Plastino and Plastino [16], Tsallis and Bukman [17]) that the solution of (2) is in the form of $f(t,x) \propto \exp_\alpha\left(-\frac{1}{2\sigma^2(t)}x^2\right)$, where $\sigma^2(t) \propto t^{2/(2+\alpha)}$ and $\exp_\alpha(x) = (1+\alpha x)^{1/\alpha}$ is the deformation of the usual exponential function introduced by Tsallis [1] in Statistical Mechanics. As in Hobson and Rogers [11], the Borland approach [12] and [13] leads to a pricing model that lies between the Heston model, because of a stochastic volatility behaviour, and the B&S model, because of its completeness due to the use of the same Wiener process that drives both the price of the underlying asset and its volatility processes. Unfortunately, as observed in Vellekoop and Nieuwenhuis [15], this approach admits arbitrage. These authors, instead, proposed to use the following "deformed" Geometric Brownian Motion

$$dS_t = \mu S_t \, dt + \sigma_S S_t d\Omega_t,$$

or equivalently

$$d\ln(S_t) = \left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(t, \Omega_t)\right) dt + \sigma_S d\Omega_t,$$

thereby keeping the original idea to replace the usual Wiener process with a heavier tailed process. However, as all improvements come with a cost, in all those models, unlike B&S and Heston, it is no longer possible to derive closed or semi-closed formulæ for European option prices. Therefore, some numerical methods have to be used.

The above mentioned results is the motivation for this article: we intend to shed some light into how to depart from the standard rate of return process by introducing a broad range of Tsallis deformations. In fact, unlike earlier papers, to derive the "generalized Gaussian" process having, at time t, the Tsallis distribution $f(t,x) \propto \exp_\alpha\left(-\frac{1}{2\sigma^2(t)}x^2\right)$ we do not require $\sigma(t)$ to be an *a priori* specified function. In other words, $f(t,x)$ satisfies a nonlinear Fokker-Planck equation which is time-dependent through the function $\sigma(t)$. A positive consequence of this is that our models are suitable to describe different types of (possibly non-linear) variance changes with time. However, the flexibility in the choice of $\sigma(t)$ requires to guarantee that the process Ω_t is well defined; the proof of this result is one of the main theoretical contribution of the present work.

Another interesting feature we show is that in our framework we can derive an expression that resembles the market price of risk. This formula obviously depends on the choice made for $\sigma(t)$ so that our model reveals to be very flexible in terms of capturing and representing a large class of volatility surfaces. What in incomplete markets is a necessary input, namely some functions describing the market price of risk, becomes in our approach an output that depends on $\sigma(t)$.

Unlike the papers by Borland, [12] and [13], Borland and Bouchaud [14] and Vellekoop and Nieuwenhuis [15], here we also explicitly derive the dynamics of the variance process that shows a mean-reverting pattern with both long-term mean and speed reversion that are function of time. This result clearly indicates that the deformed Gaussian approach leads to a sensible representation of the volatility process. A detailed analysis of the behavior of log return mean and variance is relegated in Appendix A.

Remarkably, once prices of European call options obtained with our approach have been numerically determined, the resulting implied volatilities show a smile that captures both the short and long-term features present in jump-diffusion stochastic volatility models. Further, a comparison between our model and the Heston model is achieved by choosing $\sigma(t)$ so that the mean value of the squared volatility process in both models are equal.

The paper is organized as follows. In Section 2, we recall the definition of φ-deformed exponential and logarithm as given by Naudts [18] and, as an example, we introduce the Tsallis deformed exponential and logarithm. In Section 3, we define the τ-Gaussian distribution, while in Section 4 we define the class of τ-Gaussian processes, whose heavy-tailed distributions solve a suitable nonlinear time-dependent FP equation. These processes can be viewed as a generalization of the Wiener process, since their law is an extension of the Gaussian law based on the Tsallis exponential. In Sections 5 and 6 we discuss the use of τ-Gaussian processes in modeling prices for underlying assets under different choices of $\sigma(t)$, pointing out how these non-Gaussian deformations differ from the standard geometric Brownian motion. A Monte Carlo method is used to price European options from which we finally obtain their implied volatilities. Section 7 concludes.

2. Deformed exponentials

Naudts [18] (Chapter 10) gives a general definition of φ-deformed exponential. The author firstly defines the φ-deformed logarithm as

$$\ln_\varphi(x) = \int_1^x \frac{dt}{\varphi(t)}, \quad x > 0, \tag{3}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly positive, nondecreasing and continuous function. As a consequence, the φ-logarithm

$$\ln_\varphi : \mathbb{R}_+ \to \Big(-\int_0^1 \frac{dt}{\varphi(t)}, \int_1^{+\infty} \frac{dt}{\varphi(t)} \Big) = (-m, +M)$$

is a strictly increasing and concave function, satisfying $\ln_\varphi(1) = 0$ and $(d/dx)\ln_\varphi(x) = 1/\varphi(x)$. In particular, $\ln_\varphi(x)$ is negative on $(0,1)$ and positive on $(1,+\infty)$. The natural logarithm is obtained for $\varphi(t) = t$.

The φ-exponential is then defined as the inverse function of $\ln_\varphi$:

$$\exp_\varphi = \ln_\varphi^{-1} : (-m, +M) \to \mathbb{R}_+.$$

This is a positive, increasing and convex function, satisfying $\exp_\varphi(0) = 1$ and $(d/dx)\exp_\varphi(x) = \varphi(\exp_\varphi(x))$.

Until now, the φ-exponential has mainly been used in physics literature to explain, for suitable choices of the deformation function φ, a very large class of experimentally observed phenomena which are described by distribution functions that exhibit power law tails. In finance and related fields, deformed exponentials, in the form presented in Tsallis [1] and Kaniadakis [19], were used by, e.g., Trivellato [20], [21], Imparato and Trivellato [22], Popescu et al. [23], and Preda et al. [24] and [25] to define new families of relative φ-entropies which are closely related to the relative Shannon entropy.
We focus our attention on the Tsallis logarithm with parameter $\alpha \leq 1$ that is a deformed logarithm with $\varphi(t) = t^{1-\alpha}$ and

$$m = \begin{cases} \frac{1}{\alpha}, & \text{if} \quad 0 < \alpha \leq 1, \\ +\infty, & \text{if} \quad \alpha \leq 0, \end{cases} \qquad M = \begin{cases} +\infty, & \text{if} \quad 0 \leq \alpha \leq 1, \\ -\frac{1}{\alpha}, & \text{if} \quad \alpha < 0, \end{cases}$$

and is defined by

$$\ln_\alpha(x) = \begin{cases} \frac{x^\alpha - 1}{\alpha}, & \text{if} \quad \alpha \leq 1,\ \alpha \neq 0, \\ \ln x, & \text{if} \quad \alpha = 0. \end{cases} \tag{4}$$

Its inverse is defined by

$$\exp_\alpha(x) = \begin{cases} (1+\alpha x)^{\frac{1}{\alpha}}, & \text{if} \quad (0<\alpha\leq 1,\ x>-1/\alpha) \quad \text{or} \quad (\alpha<0,\ x<-1/\alpha) \\ \exp x, & \text{if} \quad \alpha=0, \quad (x\in\mathbb{R}). \end{cases}$$

An interesting property of the Tsallis exponential function is the asymptotic behavior of its power law:

$$\exp_\alpha(x) \sim |\alpha x|^{\pm 1/|\alpha|} \quad \text{as } |x|\to+\infty. \tag{5}$$

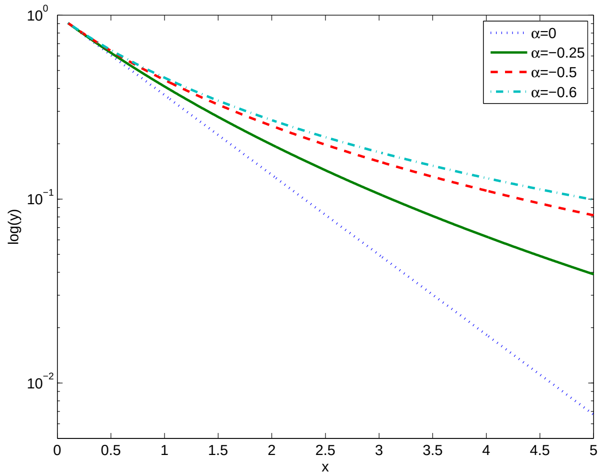

Figure 1: *Plot of the function* $\exp_\alpha(-x)$ *for different values of* α.

In Figure 1, function $\exp_\alpha(-x)$ is plotted for different values of α. We note that, for $x>0$, $\exp_\alpha(-x)\geq\exp(-x)$ and, as $\alpha\to 0$, the Tsallis exponential approaches the ordinary one.
In the next section, we use the Tsallis exponential to generalize the Gaussian distribution.

3. Tsallis distribution

The (centered) τ-Gaussian probability density of parameters[1] $\alpha\in(-2,0)$ and $\sigma>0$ is defined as

[1]For $\alpha\leq -2$ the distribution cannot be normalized, while for $\alpha>0$ it vanishes outside the interval $\left(-\frac{\sqrt{2}}{\alpha}\sigma, \frac{\sqrt{2}}{\alpha}\sigma\right)$.

$$f_\alpha^{\mathrm{T}}(x) = Z_\alpha^{-1} \exp_\alpha\left(-\frac{1}{2\sigma^2}x^2\right), \quad x \in \mathbb{R}, \tag{6}$$

where

$$Z_\alpha^{-1} = \frac{1}{\sqrt{2\pi}\sigma}\sqrt{-\alpha}\,\frac{\Gamma\left(-\frac{1}{\alpha}\right)}{\Gamma\left(-\frac{1}{\alpha}-\frac{1}{2}\right)}. \tag{7}$$

It can be checked that a random variable X with a T-Gaussian distribution can be written as

$$X = \sqrt{-\frac{2}{\alpha\nu}}\,\sigma\, T_\nu = \sqrt{\frac{2}{2+\alpha}}\,\sigma\, T_\nu, \tag{8}$$

where T_ν is a random variable having a Student t-distribution with $\nu = -\frac{2}{\alpha} - 1 > 0$ (real) degrees of freedom. Since

$$\mathbb{E}[T_\nu^n] \;=\; \begin{cases} \nu^{n/2}\prod_{i=1}^{n/2}\frac{2i-1}{\nu-2i}, & \text{if } n \text{ even}, \quad 0 < n < \nu \\ 0, & \text{if } n \text{ odd}, \quad 0 < n < \nu, \end{cases} \tag{9}$$

the T-Gaussian has finite moments only under some suitable choices of the parameter α.

In general, from (8) and (9) it follows that, if $-2/(2n+1) < \alpha < 0$ ($n = 1, 2, \dots$) then $\mathbb{E}[X^{2n}]$ is finite and proportional to σ^{2n}. Specifically, if $-2/3 < \alpha < 0$, the variance of the T-Gaussian distribution is finite and equals

$$\int_{\mathbb{R}} x^2 f_\alpha^{\mathrm{T}}(t,x)\,dx = \Gamma_\alpha\,\sigma^2, \tag{10}$$

where

$$\Gamma_\alpha = \frac{2}{3\alpha+2}. \tag{11}$$

Moreover, if $-2/5 < \alpha < 0$, the fourth moment of the T-Gaussian is also finite and equals

$$\int_{\mathbb{R}} x^4 f_\alpha^{\mathrm{T}}(t,x)\,dx = C_\alpha\,\sigma^4, \tag{12}$$

where

$$C_\alpha = \frac{12}{(5\alpha+2)(3\alpha+2)}. \tag{13}$$

When α goes to zero, the T-Gaussian distribution approaches the usual centered Gaussian distribution with variance σ^2, while for larger $|\alpha|$ it shows fat tails which vanish according to a power law.

The T-Gaussian distribution for $\sigma^2 = 1$ and different values of α is plotted in Figure 2.

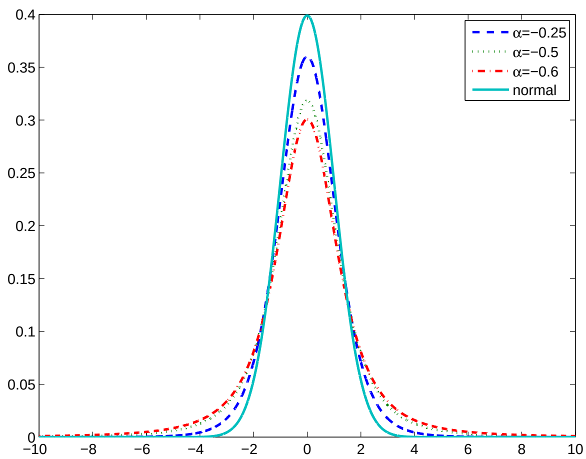

Figure 2: *Comparison between different Tsallis distributions and a standard normal distribution.*

4. Tsallis deformed Gaussian processes

In this section, we define stochastic processes which can be viewed as a generalization of the Wiener process since their law is an extension of the Gaussian law based on the Tsallis exponential.

Let $\sigma(t)$ be a C^1 function from $\mathbb{R}_+$ to $\mathbb{R}_+$, possibly depending on α, and assume $\sigma(0) = 0$.
Define

$$f_\alpha^{\mathrm{T}}(t,x) = (Z_\alpha(t))^{-1} \exp_\alpha \left(-\frac{1}{2\sigma^2(t)} x^2 \right), \quad \text{for } t > 0,\ x \in \mathbb{R} \tag{14}$$

where $(Z_\alpha(t))^{-1}$ is given by (7) after replacing σ with $\sigma(t)$.

Proposition 1. *$f_\alpha^{\mathrm{T}}(t,x)$ given by (14) solves the nonlinear time-dependent Fokker-Planck equation:*

$$\begin{aligned} \frac{\partial f(t,x)}{\partial t} &= \sigma(t)\dot{\sigma}(t) \frac{\partial}{\partial x} \left[\Big(Z_\alpha(t) f(t,x) \Big)^\alpha \frac{\partial f(t,x)}{\partial x} \right] \\ &= \sigma(t)\dot{\sigma}(t) \frac{\partial^2}{\partial x^2} \left[\frac{(Z_\alpha(t))^\alpha}{1+\alpha} f(t,x)^{1+\alpha} \right] \end{aligned}$$

For the proof refer to Trivellato [21].

Remark 1. Solving the equation

$$\sigma(t)\dot{\sigma}(t)\frac{(Z_\alpha(t))^\alpha}{1+\alpha}=\frac{1}{2} \tag{15}$$

leads to $\sigma(t) \propto t^{1/(2+\alpha)}$ and therefore to the purely sub-diffusive FP equation (2).

Let us now introduce the process

$$d\Omega_t = g_\alpha(t,\Omega_t)dW_t, \quad t>0 \tag{16}$$

described by the time-dependent nonlinear FP equation of Proposition 1, where W_t is the Wiener process and

$$g_\alpha(t,x) \;=\; \sqrt{2\sigma(t)\dot{\sigma}(t)\frac{(Z_\alpha(t)f_\alpha^{\mathrm{T}}(t,x))^\alpha}{1+\alpha}} = \sqrt{2\sigma(t)\dot{\sigma}(t)\,\frac{\phi_\alpha(t,x)}{1+\alpha}},$$

with $\phi_\alpha(t,x) = 1-\frac{\alpha x^2}{2\sigma^2(t)}$ going to 1 as α goes to zero.

Letting $\Omega_0 = 0$ and $\sigma^2(t) \equiv \sigma_\alpha^2(t) \to t$ as α goes to zero, (16) defines a "generalized Wiener process" with a Tsallis distribution at each time $t>0$. In particular, from (10), we deduce that $\mathbb{E}\left[\Omega_t^2\right] = \Gamma_\alpha\,\sigma^2(t)$.
Let us observe that, if $f_\alpha^{\mathrm{T}}(t,\Omega_t)$ is small, then $g_\alpha(t,\Omega_t)$ is large. Consequently, an unlikely step for Ω_t tends to be followed by a large "jump".

Since $g_\alpha(t,x)$ is not uniformly Lipschitz in x, standard results on the existence of strong solutions to stochastic differential equations do not apply. Therefore, we prove the following proposition which represents our main theoretical result.

Proposition 2. *The stochastic differential equation (16) admits a strong solution on the finite time interval* $[0,T]$.

For the proof see Appendix B.

5. Deformed geometric Brownian motion

Here, Ω_t is used as the driving noise in the dynamics of the asset price process S (under the historical probability measure $\mathbb{P}$)

$$dS_t = \mu S_t\,dt + \sigma_S S_t\,d\Omega_t \tag{17}$$

where μ and σ_S are constant parameters. Since

$$d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(t, \Omega_t)\right) dt + \sigma_S \, d\Omega_t, \tag{18}$$

the solution is given by

$$S_t = S_0 \, e^{\int_0^t \left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(s,\Omega_s)\right) ds + \sigma_S \, \Omega_t}. \tag{19}$$

Let us observe that, by Hölder inequality and by the definition of $g_\alpha^2(t, x)$, we deduce that if $\mathbb{E}[\Omega_t^4] < +\infty$ (and thus $\mathbb{E}[\Omega_t^4] = C_\alpha \sigma^4(t)$), then $\mathrm{Var}(\ln(S_t/S_0)) < +\infty$. In fact, we get that

$$\mathbb{E}[(\ln(S_t/S_0))^2] \leq C(t) \left(1 + \mathbb{E}[\Omega_t^2] + \int_0^t \mathbb{E}[\Omega_s^4] \, ds\right),$$

for a suitable positive function $C(t)$.

In particular, it is

$$\Omega_t = \sqrt{\frac{2}{2+\alpha}} \, \sigma(t) \, T_\nu(t), \tag{20}$$

where $T_\nu(t)$ is, for any t, a random variable having a Student distribution with $\nu = -\frac{2}{\alpha} - 1 > 0$ (real) degrees of freedom. As a consequence,

$$\begin{aligned} S_t &= S_0 \, e^{\mu t - \frac{1}{2}\sigma_S^2 \int_0^t (g_\alpha(s,\Omega_s))^2 \, ds + \sigma_S \, \Omega_t} \\ &= S_0 \, e^{\mu t - \frac{\sigma_S^2}{2(1+\alpha)}\sigma^2(t) + \frac{\alpha \sigma_S^2}{(1+\alpha)(2+\alpha)} \int_0^t \sigma(s)\dot{\sigma}(s) T_\nu^2(s) \, ds + \sigma_S \sqrt{\frac{2}{2+\alpha}} \, \sigma(t) \, T_\nu(t)}. \end{aligned} \tag{21}$$

If $\sigma^2(t) = t$, we deduce in particular that

$$S_t = S_0 \, e^{\left(\mu - \frac{\sigma_S^2}{2(1+\alpha)}\right) t + \frac{\alpha \sigma_S^2}{2(1+\alpha)(2+\alpha)} \int_0^t T_\nu^2(s) \, ds + \sigma_S \sqrt{\frac{2t}{2+\alpha}} \, T_\nu(t)}. \tag{22}$$

This formula immediately shows the difference between our T-deformed pricing and the B&S models: the role of the standard normal distribution is played by the Student distribution with $\nu = -\frac{2}{\alpha} - 1$ degrees of freedom, and the two terms similar to those of the B&S stock price are affected by α. Furthermore, there is an additional (negative) term related to the history of $T_\nu^2(s)$ up to t, which vanishes as α goes to zero.

Figure 3 shows a box-plot in which 100,000 values of the T-deformed geometric Brownian motion, simulated using formula (21), are compared with those obtained from a geometric Brownian motion. It results that the first shows a larger dispersion than the latter. This behavior is further confirmed by fat tails displayed by Tsallis densities (when compared with the B&S ones) as depicted in Figure 4, where their normalized log-returns, for different choices of $\sigma(t)$ and $\alpha = -0.2$, are compared. In Appendix A, the first and second moments of the log-return process, used to achieve normalization, are mathematically derived.

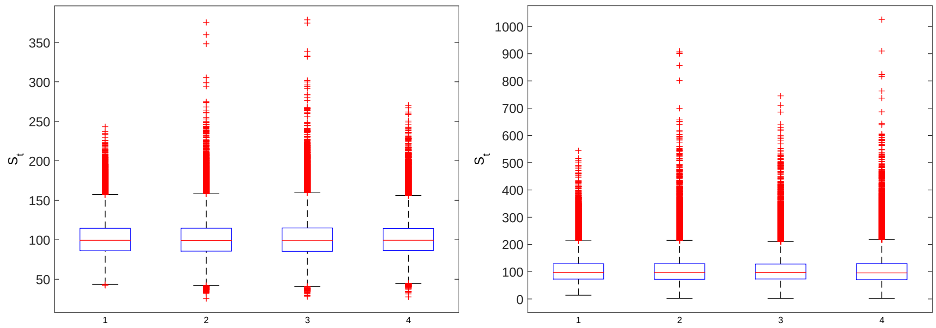

Figure 3: *Boxplot of S_t in the Tsallis model for different choices of $\sigma^2(t)$ with $\alpha = -0.2$, for maturity 0.5 years (left) and 2 years (right). Models are denoted with numbers (1: B&S, 2: $\sigma^2(t) = t$, 3: $\sigma^2(t) = t^{\frac{2}{2+\alpha}}$, 4: $\sigma^2(t) = t^{\frac{2}{2-\alpha}}$).*

The role of the squared volatility in stochastic volatility models is played here by the process $g_\alpha^2(t, \Omega_t)$. Its mean is given by

$$\mathbb{E}\,[g_\alpha^2(t, \Omega_t)\,] = 2\Gamma_\alpha\, \sigma(t)\dot{\sigma}(t).$$

Since $\Omega_t = \int_0^t g_\alpha(s, \Omega_s)\, dW_s$, by assuming the variance of Ω_t to be finite, we in fact have that $\Gamma_\alpha \sigma^2(t) = \mathbb{E}\,[\Omega_t^2\,] = \int_0^t \mathbb{E}\,[g_\alpha^2(s, \Omega_s)\,]\, ds$. Furthermore, the variance of $g_\alpha^2(t, \Omega_t)$ results to be, up to a constant depending on α, proportional to $(\sigma(t)\dot{\sigma}(t))^2$, i.e. $\mathrm{Var}(g_\alpha^2(t, \Omega_t)) \propto (\sigma(t)\dot{\sigma}(t))^2$.

The flexibility in the choice of $\sigma^2(t)$ allows us to reproduce different behaviors for $g_\alpha^2(t, \Omega_t)$ which, in turn, generate different implied volatilities, as we will see later. If $\sigma^2(t) \propto t$ then both the mean and the variance of $g_\alpha^2(t, \Omega_t)$ are constant. If $\sigma^2(t) \propto t^\gamma$, with $\gamma > 1$, then $\sigma(t)\dot{\sigma}(t) \propto t^{\gamma-1}$, and therefore both the mean and the variance of $g_\alpha^2(t, \Omega_t)$ tend to 0 as $t \to 0$ and to $+\infty$ as $t \to +\infty$, while the converse is true when $\sigma^2(t) \propto t^\beta$, with $0 < \beta < 1$.

In the sequel, as examples of possible choices for $\sigma^2(t)$, we let $\sigma^2(t) \equiv \sigma_\alpha^2(t) = t,\, t^{\gamma(\alpha)},\, t^{\beta(\alpha)}$, where $\gamma(\alpha) = 2/(2+\alpha)$ and $\beta(\alpha) = 2/(2-\alpha)$ and α varies

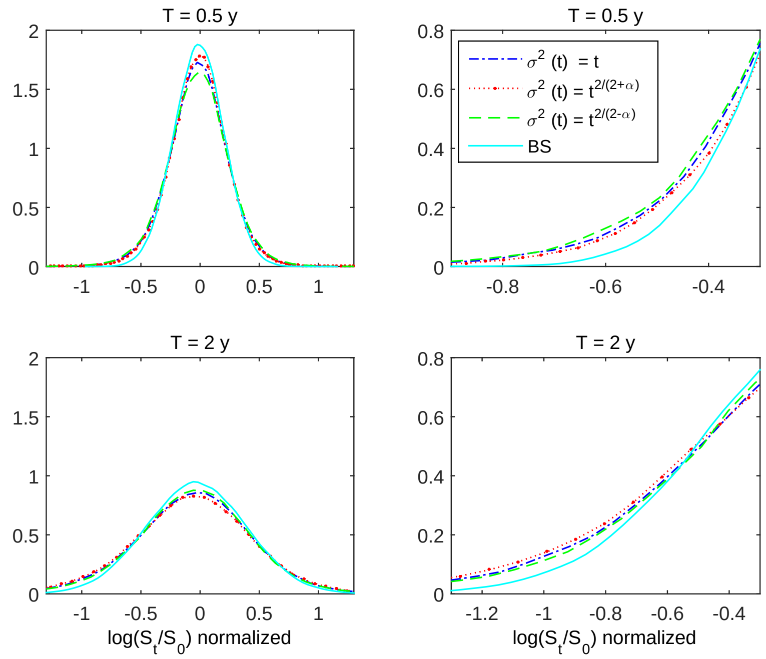

Figure 4: *Tsallis density log-return for maturities* 0.5 *years (top left) and* 2 *years (bottom left) for different choices of* $\sigma(t)$ *and* $\alpha = -0.2$. *In the right graphs, the left tails of the respective densities are magnified. Probability densities have been normalized to have zero mean and unit variance.*

in $(-2/5, 0)$ in order to guarantee the finiteness of the mean and variance of $g_\alpha^2(t, \Omega_t)$ (which, in turn, means the finiteness of the second and forth moments of Ω_t).

We recall that, as noted in Remark 1, $\gamma(\alpha) = 2/(2+\alpha)$ leads to the purely sub-diffusive FP equation (2). To the best of our knowledge, this choice is (up to a constant) the only one used in the existing literature related to this topic.

Applying Ito's formula to $g_\alpha^2(t, \Omega_t)$, we obtain the following dynamics:

$$dg_\alpha^2(t, \Omega_t) = \kappa(t) \left(\theta(t) - g_\alpha^2(t, \Omega_t)\right) dt + 2a(t)b(t)\, \Omega_t g_\alpha(t, \Omega_t)\, dW_t, \tag{23}$$

where

$$a(t) = \frac{2\sigma(t)\dot{\sigma}(t)}{1+\alpha}, \qquad b(t) = -\frac{\alpha}{2\sigma^2(t)}$$

$$\kappa(t) = -\left(\frac{\dot{a}(t)}{a(t)} + a(t)b(t) + \frac{\dot{b}(t)}{b(t)}\right), \qquad \theta(t) = \frac{-\frac{a(t)\dot{b}(t)}{b(t)}}{\kappa(t)}.$$

Since $b(t) > 0$, and since $\dot{b}(t) = \alpha\frac{\dot{\sigma}(t)}{\sigma^3(t)}$ and $a(t)$ have opposite sign, we deduce that $\kappa(t)\theta(t) > 0$. It is worth stressing that Equation (23) is a very general way to express a mean-reverting variance process with time-dependent long-term mean, reversion speed and volatility. This result clearly shows that the deformed Gaussian approach leads to a sensible representation of the volatility process.

As a peculiar case, if $\sigma^2(t) = t$, we get

$$a(t) = \frac{1}{1+\alpha}, \quad b(t) = -\frac{\alpha}{2t}, \quad \kappa(t) = \frac{1}{(1+\alpha)\Gamma_\alpha\, t}, \quad \theta(t) = \Gamma_\alpha,$$

where

$$\Gamma_\alpha = \frac{2}{3\alpha+2} = \mathbb{E}\left[g_\alpha^2(t,\Omega_t)\right],$$

so that equation (23) boils down to a mean-reverting dynamics with constant mean and time-decaying speed reversion.

If, instead, $\sigma^2(t) = t^{\frac{2}{2+\alpha}}$ we obtain

$$a(t) = \frac{2}{(1+\alpha)(2+\alpha)} t^{\frac{-\alpha}{2+\alpha}}, \quad b(t) = -\frac{\alpha}{2} t^{\frac{-2}{2+\alpha}},$$

$$\kappa(t) = \frac{\alpha^2+4\alpha+2}{(1+\alpha)(2+\alpha)\, t}, \quad \theta(t) = \frac{3\alpha+2}{\alpha^2+4\alpha+2}\mathbb{E}\left[g_\alpha^2(t,\Omega_t)\right],$$

where

$$\mathbb{E}\left[g_\alpha^2(t,\Omega_t)\right] = \frac{4}{(2+\alpha)(3\alpha+2)} t^{\frac{-\alpha}{2+\alpha}}.$$

Similarly, if $\sigma^2(t) = t^{\frac{2}{2-\alpha}}$, we obtain

$$a(t) = \frac{2}{(1+\alpha)(2-\alpha)} t^{\frac{\alpha}{2-\alpha}}, \quad b(t) = -\frac{\alpha}{2} t^{\frac{-2}{2-\alpha}},$$

$$\kappa(t) = \frac{-\alpha^2+2\alpha+2}{(1+\alpha)(2-\alpha)\, t}, \quad \theta(t) = \frac{3\alpha+2}{-\alpha^2+2\alpha+2}\mathbb{E}\left[g_\alpha^2(t,\Omega_t)\right],$$

where

$$\mathbb{E}\left[g_\alpha^2(t,\Omega_t)\right] = \frac{4}{(2-\alpha)(3\alpha+2)} t^{\frac{\alpha}{2-\alpha}}.$$

As can be seen, in all cases the speed reversion $\kappa(t)$ decays as $1/t$, while $\theta(t)$ is proportional to $\mathbb{E}\left[g_\alpha^2(t,\Omega_t)\right]$.

6. Option pricing and numerical results

In this section, we start by finding conditions on $\sigma(t)$ which ensure the existence of an equivalent martingale measure. We use the standard tool of Girsanov theorem which we can apply due to the particular shape of $g_\alpha(t,\Omega_t)$. Let

$$u_t = \frac{\mu - r}{\sigma_S g_\alpha(t,\Omega_t)}, \tag{24}$$

where r is the risk free rate of return, with $0 < r < \mu$. Process u_t can be seen as the 'price-of-risk' process. It explicitly shows the impact of the choice of $\sigma(t)$ in the pricing model. If $\alpha \to 0$ then the B&S case is recovered.
Define the measure $\mathbb{Q}$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T u_t dW_t - \frac{1}{2}\int_0^T u_t^2 dt\right). \tag{25}$$

The Girsanov Theorem states that $W_t^{\mathbb{Q}} = W_t + \int_0^t u_s dW_s$ is a $\mathbb{Q}$-Wiener process if $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$. A sufficient condition which guarantees the validity of this equality is the Novikov condition:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T u_t^2 dt}\right] < \infty. \tag{26}$$

Proposition 3. *If*

$$\int_0^T \frac{1}{\sigma(t)\dot{\sigma}(t)}\, dt < \infty, \tag{27}$$

the Novikov condition (26) is satisfied for the model introduced by (16), (17).

For the proof see Appendix B.

Let us note that condition (27) is rather mild and allows a vast range of functions $\sigma(t)$ to fulfil it.

Under the martingale measure $\mathbb{Q}$, we get

$$dS_t = rS_t dt + \sigma_S S_t g_\alpha(t,\Omega_t) dW_t^{\mathbb{Q}},$$

$$d\Omega_t = -\lambda\, dt + g_\alpha(t,\Omega_t) dW_t^{\mathbb{Q}},$$

where $\lambda = (\mu - r)/\sigma_S$.
The following proposition gives two conditions under which the discounted asset price is a (true) $\mathbb{Q}$-martingale. One of these conditions ensures the finiteness of the variance of the log-return process.

Proposition 4. *If condition (27) holds true and if* $\mathbb{E}\left[\Omega_t^4\right]$ *is finite, the stochastic integral process*

$$\int_0^t g_\alpha(s,\Omega_s)dW_s^{\mathbb{Q}} \tag{28}$$

is a (true) $\mathbb{Q}$*-martingale.*

For the proof see Appendix B.

The aim of the rest of this section is to price a European call option with exercise date T and strike price K. Since, by Proposition 4, the price process $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale, and since the couple (S_t,Ω_t) is a strong Markov process, adapted to the filtration $\mathbb{F}_t$ of $W_t^{\mathbb{Q}}$, we can apply the martingale pricing theory to conclude that the call option price can be written as

$$\begin{aligned} C(t,S_t,\Omega_t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T-K)^+ \mid \mathbb{F}_t\right] = e^{-r(T-t)} \qquad (29)\\ \mathbb{E}^{\mathbb{Q}} &\left[\left(S_t\, e^{r(T-t)+\sigma_S\lambda(T-t)+\sigma_S(\Omega_T-\Omega_t)-\frac{1}{2}\sigma_S^2\int_t^T g_\alpha^2(s,\Omega_s)\,ds} - K\right)^+ \mid S_t,\,\Omega_t\right]. \end{aligned}$$

Because of the presence of the quadratic variation integral in the exponent, there is no closed formula for (29). However, to compute the call price we can use a Monte Carlo simulation method or a finite difference method to discretize the nonlinear, time-inhomogeneous partial differential equation associated to (29),

$$\begin{aligned} &\frac{1}{2}\sigma_S^2 s^2 g_\alpha^2(t,w)\frac{\partial^2 C}{\partial s^2} + \sigma_S s g_\alpha^2(t,w)\frac{\partial^2 C}{\partial s\partial w} + \frac{1}{2}g_\alpha^2(t,w)\frac{\partial^2 C}{\partial w^2} + rs\frac{\partial C}{\partial s}\\ &-\lambda\frac{\partial C}{\partial w} - rC + \frac{\partial C}{\partial t} = 0, \end{aligned}$$

with boundary condition $C(T,s,w) = (s-K)^+$.

In Figures 5-7 some volatility surfaces are plotted for $\sigma^2(t) = t$, $\sigma^2(t) = t^{2/(2-\alpha)}$ and $\sigma^2(t) = t^{2/(2+\alpha)}$ and different values of α. Volatilities are obtained from option prices using the standard computation of the implied volatility in the B&S model through the MATLAB function `calcBSImpVol.m`. Prices for European call options are obtained using Monte Carlo simulations with $S_0 = 100$, $\sigma_s = 0.3$, $r = 0.03$ and $\lambda = 0$. To reduce variance, a control variate method has been applied choosing as control variate the underlying asset. We have obtained 100,000 option prices in 106 seconds running a FORTRAN program on a AMD Opteron Processor 6328, 3.2 Ghz.

We can notice that the surfaces exhibit a clear volatility smile, which is more pronounced for shorter maturities and greater values of $|\alpha|$. This important feature of our models is not performed, at short maturities, by diffusion stochastic volatility models, unless they either encompass jumps or perform a change of time driven by a Lèvy process.

The fact that we get different curves for different choices of $\sigma^2(t)$ provides an interesting opportunity to use a rich class of volatility surfaces. As mentioned earlier, the different choices of $\sigma^2(t)$ determine the different behaviors of the volatility surfaces for short and long maturities. We recall that $\sigma^2(t) = t$ (Figure 5) implies that both the mean and the variance of $(g_\alpha(t, \Omega_t))^2$ are constant; $\sigma^2(t) = t^{2/(2+\alpha)}$ (Figure 6) implies that both the mean and the variance of $(g_\alpha(t, \Omega_t))^2$ tend to 0 as $t \to 0$ and tend to $+\infty$ as $t \to +\infty$, while the converse is true when $\sigma^2(t) = t^{2/(2-\alpha)}$ (Figure 7). This justifies why in Figure 6 the surfaces tend to increase in time, mainly for larger values of $|\alpha|$, while the converse is true in Figure 7. In Figure 8 the volatility surfaces are obtained combining $\sigma^2(t) = t^{\frac{2}{2-\alpha}}$, for t less than 1, with $\sigma^2(t) = t^{\frac{2}{2+\alpha}}$, for t greater than 1. As we can see, the resultant surfaces present a smile with respect to both time to maturity and strike variables.

In order to compare our model with the Heston model, another interesting choice for $\sigma^2(t)$ could be the one that matches $\mathbb{E}\left[g^2_\alpha(t, \Omega_t)\right]$ with $\mathbb{E}\left[v_t\right]$, where v_t is the squared volatility in the Heston model. With this choice, we obtain

$$\sigma^2(t) = \frac{3\alpha + 2}{2}\theta^* \left(t + \frac{e^{-k^* t}}{k^*} - \frac{1}{k^*}\right) \tag{30}$$

where k^* and θ^* are, respectively, the speed reversion and the long-run mean of the squared volatility in the Heston model.
In Figure 9, the volatility surface obtained for $\sigma^2(t)$ defined in equation (30) is compared with the Heston volatility surface. The parameter values considered for the comparison are the same used in Heston: $S_0 = 100$, $r = 0$, $\rho = 0$, $\sigma = 0.1$, and $v_0 = 0$. Moreover, for the Tsallis model we consider $\sigma_S = 1$ and $\alpha = -0.3$.

The two volatility surfaces share the same shape in the sense that the smiles have a very closed behavior. The only minor difference shows up for very short times to maturity; such gap vanishes for maturities just below one year.

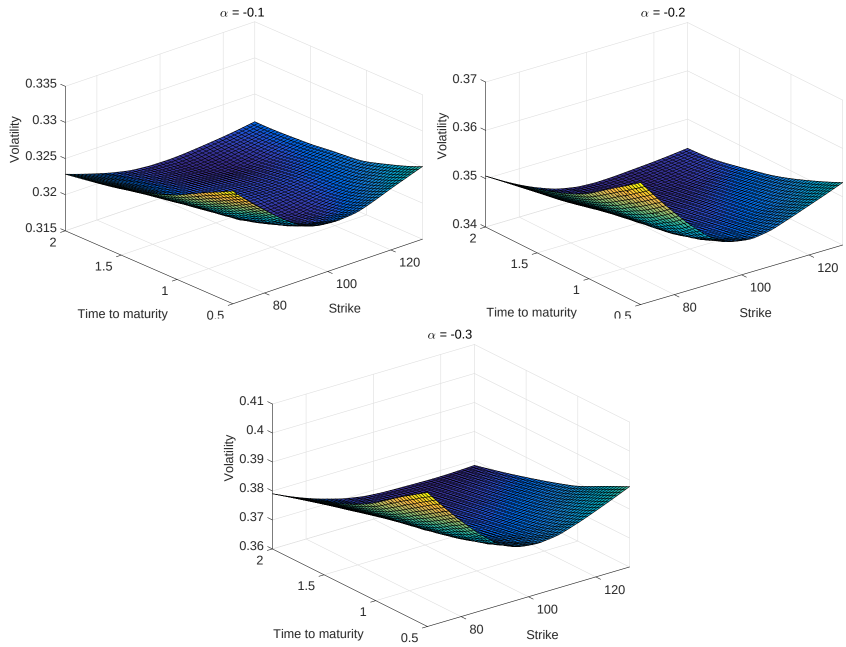

Figure 5: *Tsallis with $\sigma^2(t) = t$ and different values of α.*

7. Concluding remarks

The main aim of this paper is to present a pricing methodology, based on the Tsallis deformation of the Gaussian distribution, that is able to describe different types of variance changes of the underlying asset with respect to time.

Adopting some notions from physics literature into a financial context has allowed us to obtain, in a Black and Scholes complete market setting, results comparable with more complex models that encompass various sources of uncertainty such as stochastic volatility, jump processes or time-changed Lèvy processes.

The examples we have developed have primarily an illustrative purpose: finding the appropriate volatility shapes for different financial markets (FX, commodities and term structure, just to mention a few) is a challenging task that is left for further research.

A final remark for future research is due: to reproduce skew effects in

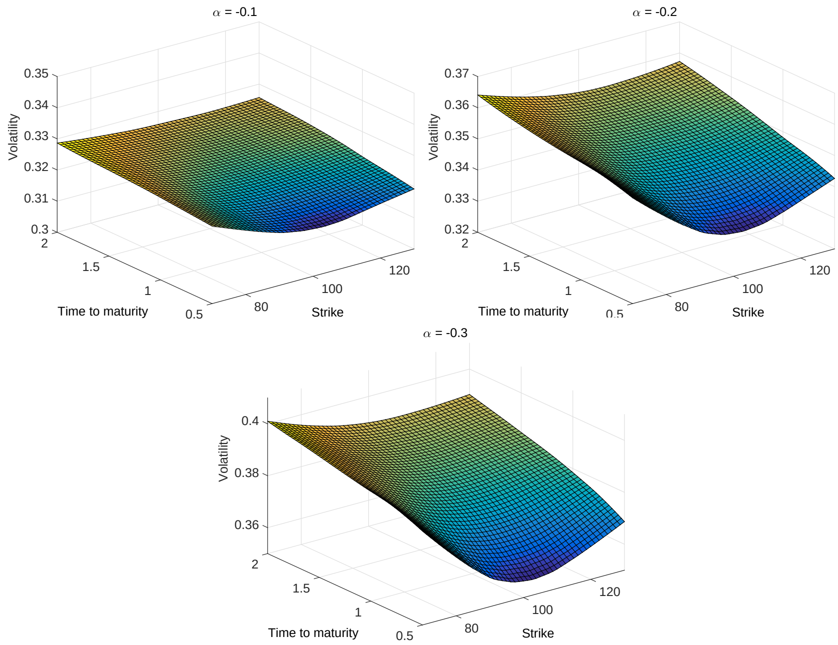

Figure 6: *Tsallis with* $\sigma^2(t) = t^{2/(2+\alpha)}$ *and different values of* α*.*

the implied volatility, Borland and Bouchaud [14] proposed the following deformed constant elasticity of variance (CEV) model:

$$dS_t = \mu S_t \, dt + \sigma S_t^{\gamma} \, d\Omega_t,$$

with γ a parameter that introduces skew into the distribution of log stock returns. Alternatively, as in Hobson and Rogers [11] and Vellekoop and Nieuwenhuis [15], skew effects in the implied volatility could be also obtained if we consider the initial value Ω_0 as a parameter which can range from negative to positive values. Taking $\Omega_0 = 0$, as we have done here, gives symmetric smile curves. Left or right asymmetries in the volatility surfaces could be reproduced by taking Ω_0 negative or positive. This fact could be roughly explained by observing that the sign of the instantaneous correlation between stock log returns and the variance process $g_\alpha^2(t, \Omega_t)$, whose dynamics is given by (23), essentially depends on the sign of Ω_t.

For sake of brevity we postpone this analysis to future research.

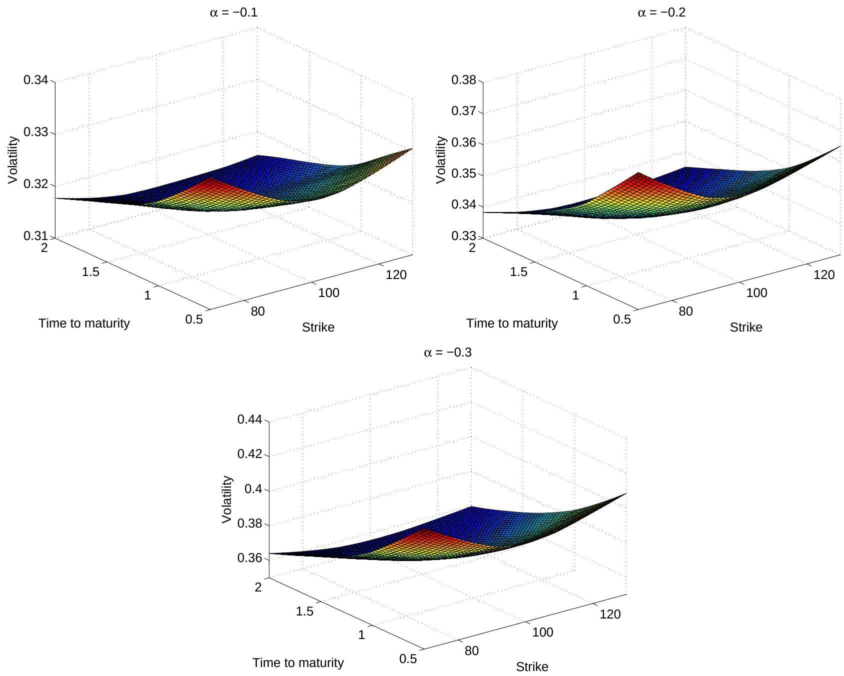

Figure 7: *Tsallis with $\sigma^2(t) = t^{2/(2-\alpha)}$ and different values of α.*

Appendix A. First and second moments of the Tsallis log-return process $\ln(S_t/S_0)$

In this appendix, we show that if $-2/3 < \alpha < 0$, the mean of the log-return process $Y_t = \ln(S_t/S_0)$ is finite and equals

$$m(t) = \mathbb{E}[Y_t] = \mu t - \frac{\sigma_S^2}{2+3\alpha}\sigma^2(t). \tag{A.1}$$

Moreover, if $-2/9 < \alpha < 0$, the second moment of Y_t is also finite and equals

$$\begin{aligned}\mathbb{E}(Y_t^2) &= 2\mu \int_0^t m(s)\, ds + \frac{2\,\sigma_S^2}{2+3\alpha}\,\sigma^2(t) \\ &- \frac{\sigma_S^2}{1+\alpha}\left(\int_0^t 2\sigma(s)\dot{\sigma}(s)m(s)\, ds - \alpha \int_0^t \frac{\dot{\sigma}(s)}{\sigma(s)} v(s)\, ds\right),\end{aligned} \tag{A.2}$$

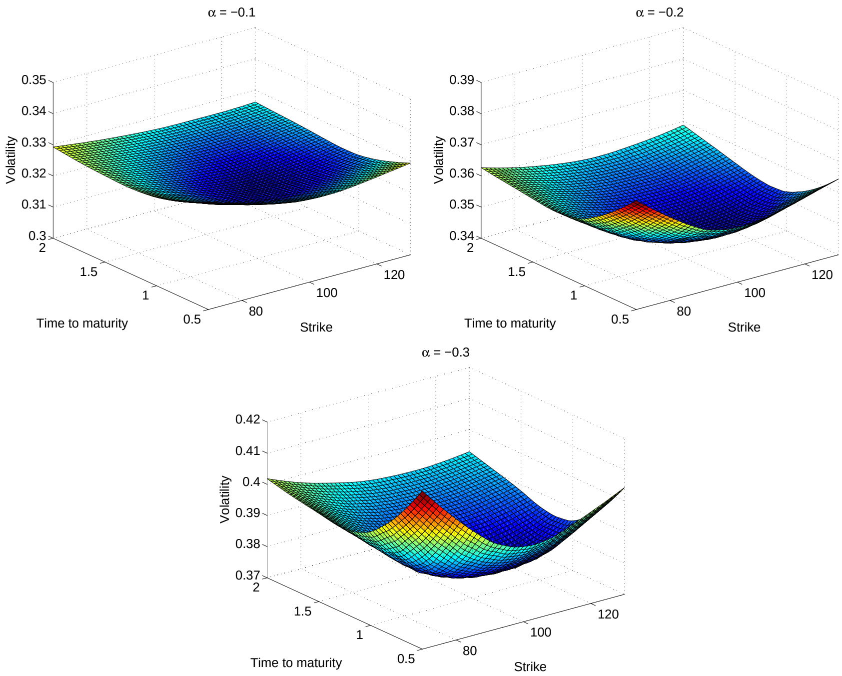

Figure 8: *Tsallis with $\sigma^2(t) = t^{2/(2+\alpha)}$ (for $t \geq 1$) combined with $\sigma^2(t) = t^{2/(2-\alpha)}$ (for $t < 1$) and different values of α.*

where $v(t) = \mathbb{E}[Y_t \Omega_t^2]$, $v(0) = 0$, satisfies for $t > 0$

$$\begin{aligned}\dot{v}(t) &= \mu \Gamma_\alpha \sigma^2(t) - \frac{\sigma_S^2}{1+\alpha}\left(\Gamma_\alpha - \frac{\alpha}{2} C_\alpha\right) \sigma^3(t)\dot{\sigma}(t) \\ &\quad + \frac{2}{1+\alpha}\sigma(t)\dot{\sigma}(t) m(t) - \frac{\alpha}{1+\alpha}\,\frac{\dot{\sigma}(t)}{\sigma(t)} v(t), \qquad \text{(A.3)}\end{aligned}$$

with Γ_α and C_α given by (11) and (13), respectively.

Let us assume that $-2/9 < \alpha < 0$, which implies that, for $n \leq 4$, it is $\mathbb{E}(\Omega_t^{2n}) < +\infty$ and proportional to $\sigma^{2n}(t)$.
Consider the process

$$Y_t = \int_0^t \left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(s, \Omega_s)\right) ds + \sigma_S \Omega_t,$$

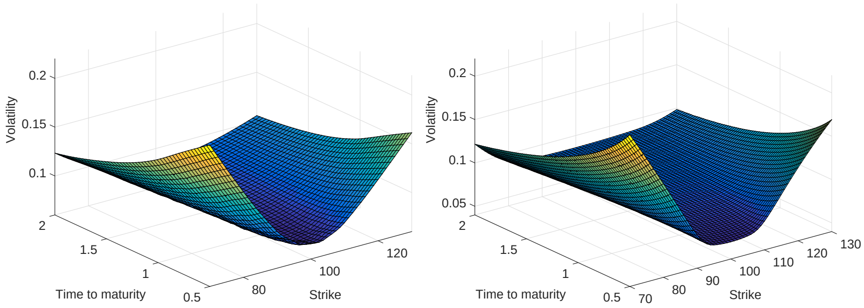

Figure 9: *Volatility surfaces for the model presented in Section 5 with $\sigma^2(t)$ defined in (30) and $\alpha = -0.3$ (left), and for the Heston model (right).*

where

$$g_\alpha^2\,(t, \Omega_t)) = \frac{2\sigma(t)\dot{\sigma}(t)}{1+\alpha}\left(1 - \alpha\frac{\Omega_t^2}{2\sigma^2(t)}\right). \tag{A.4}$$

Since Ω_t is a square-integrable martingale with a Tsallis distribution at each time t such that $\mathbb{E}(\Omega_t^2) = \Gamma_\alpha \sigma^2(t)$, with $\Gamma_\alpha = 2/(2+3\alpha)$, we deduce that

$$\mathbb{E}[g_\alpha^2\,(t, \Omega_t))] = \frac{2\sigma(t)\dot{\sigma}(t)}{1+\alpha}\left(1 - \frac{\alpha}{2}\Gamma_\alpha\right) = \frac{4\sigma(t)\dot{\sigma}(t)}{2+3\alpha} \tag{A.5}$$

and therefore

$$\begin{aligned} m(t) = \mathbb{E}[Y_t] &= \mu t - \frac{\sigma_S^2}{2(1+\alpha)}\left(1 - \frac{\alpha}{2}\Gamma_\alpha\right)\sigma^2(t) \\ &= \mu t - \frac{\sigma_S^2}{2+3\alpha}\sigma^2(t). \end{aligned} \tag{A.6}$$

We now compute the second moment of Y_t. Applying the Ito formula to Y_t^2, we obtain

$$\begin{aligned} dY_t^2 = 2Y_t\, dY_t + d < Y >_t = \left(2Y_t\left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(t, \Omega_t)\right) + \sigma_S^2 g_\alpha^2(t, \Omega_t)\right) dt \\ +2\sigma_S Y_t\, d\Omega_t, \end{aligned} \tag{A.7}$$

where the stochastic integral in (A.7) is a martingale since

$$\mathbb{E}\int_0^t Y_s^2 g_\alpha^2(s, \Omega_s)\, ds < +\infty. \tag{A.8}$$

Using the definition of Y_t and $g_\alpha^2(t,\Omega_t)$, we in fact get that

$$
\begin{aligned}
&Y_s^2 g_\alpha^2(s,\Omega_s) \leq \\
&C(s)\left(1+\Omega_s^2+\Omega_s^4+\Omega_s^6+\left(\int_0^s \Omega_\tau^2\,d\tau\right)^2+\Omega_s^2\left(\int_0^s \Omega_\tau^2\,d\tau\right)^2\right)
\end{aligned} \tag{A.9}
$$

where, from now on, $C(t)$ denotes different integrable functions whose specific values are irrelevant. By Hölder and Schwarz inequalities, we deduce that

$$
\begin{aligned}
&\mathbb{E}[Y_s^2 g_\alpha^2(s,\Omega_s)] \leq C(s) \\
&\left(1+\mathbb{E}[\Omega_s^2]+\mathbb{E}[\Omega_s^4]+\mathbb{E}[\Omega_s^6]+\int_0^s \mathbb{E}[\Omega_\tau^4]\,d\tau+\left(\mathbb{E}[\Omega_s^4]\right)^{1/2}\left(\int_0^s \mathbb{E}[\Omega_\tau^8]\,d\tau\right)^{1/2}\right).
\end{aligned}
$$

Since we have assumed that, for $n=1,2,3,4$, $\mathbb{E}[\Omega_t^{2n}]$ is finite, and thus proportional to $\sigma^{2n}(t)$, (A.8) follows.
Taking the expectation of (A.7), from (A.5) we deduce that

$$
\begin{aligned}
\mathbb{E}(Y_t^2) &= 2\mu\int_0^t m(s)\,ds+\sigma_S^2\int_0^t \mathbb{E}(g_\alpha^2(s,\Omega_s))\,ds-\sigma_S^2\int_0^t \mathbb{E}\left(Y_s g_\alpha^2(s,\Omega_s)\right)\,ds \\
&= 2\mu\int_0^t m(s)\,ds+\frac{2\,\sigma_S^2}{2+3\alpha}\sigma^2(t)-\sigma_S^2\int_0^t \mathbb{E}\left(g_\alpha^2(s,\Omega_s)Y_s\right)\,ds.
\end{aligned} \tag{A.10}
$$

Since

$$
g_\alpha^2(t,\Omega_t)Y_t=\frac{2\sigma(t)\dot\sigma(t)}{1+\alpha}Y_t-\frac{\alpha}{1+\alpha}\frac{\dot\sigma(t)}{\sigma(t)}\Omega_t^2Y_t,
$$

we get that

$$
\mathbb{E}\left(g_\alpha^2(t,\Omega_t)Y_t\right)=\frac{2\sigma(t)\dot\sigma(t)}{1+\alpha}m(t)-\frac{\alpha}{1+\alpha}\frac{\dot\sigma(t)}{\sigma(t)}\mathbb{E}\left(\Omega_t^2Y_t\right). \tag{A.11}
$$

Define $v(t)=\mathbb{E}\left(\Omega_t^2Y_t\right)$. From (A.10) and (A.11), we deduce that

$$
\begin{aligned}
\mathbb{E}(Y_t^2) &= 2\mu\int_0^t m(s)\,ds+\frac{2\,\sigma_S^2}{2+3\alpha}\sigma^2(t) \\
&-\frac{\sigma_S^2}{1+\alpha}\left(\int_0^t 2\sigma(s)\dot\sigma(s)m(s)\,ds-\alpha\int_0^t \frac{\dot\sigma(s)}{\sigma(s)}v(s)\,ds\right)
\end{aligned} \tag{A.12}
$$

We are left with the task of computing $v(t)$. Using the Ito formula, we get

$$\begin{aligned}
d\left(\Omega_t^2 Y_t\right) &= \Omega_t^2\, dY_t + Y_t\, d\left(2\Omega_t\, d\Omega_t + d < \Omega >_t\right) + d < Y, \Omega^2 >_t \\
&= \left(\left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(t,\Omega_t)\right)\Omega_t^2 + Y_t g_\alpha^2(t,\Omega_t) + 2\sigma_S \Omega_t g_\alpha^2(t,\Omega_t)\right) dt \\
&\quad + \sigma_S \Omega_t^2\, d\Omega_t + 2Y_t\Omega_t\, d\Omega_t
\end{aligned} \tag{A.13}$$

where the stochastic integrals in (A.13) are martingales since, similarly to (A.9), one can check that

$$\mathbb{E}[g_\alpha^2(s,\Omega_s)\Omega_s^4] \le C(s)\left(\mathbb{E}[\Omega_s^4] + \mathbb{E}[\Omega_s^6]\right)$$

and

$$\begin{aligned}
\mathbb{E}[Y_s^2 g_\alpha^2(s,\Omega_s)\Omega_s^2] \le C(s)\Big(&\mathbb{E}[\Omega_s^2] + \mathbb{E}[\Omega_s^4] + \mathbb{E}[\Omega_s^6] + \mathbb{E}[\Omega_s^8] \\
&+ \left(\mathbb{E}[\Omega_s^4]\right)^{1/2}\left(\int_0^s \mathbb{E}[\Omega_\tau^8]\, d\tau\right)^{1/2} + \left(\mathbb{E}[\Omega_s^8]\right)^{1/2}\left(\int_0^s \mathbb{E}[\Omega_\tau^8]\, d\tau\right)^{1/2}\Big),
\end{aligned}$$

which imply

$$\mathbb{E}\int_0^t g_\alpha^2(s,\Omega_s)\Omega_s^2\, ds < +\infty \quad \text{and} \quad \mathbb{E}\int_0^t Y_s^2 g_\alpha^2(s,\Omega_s)\Omega_s^2\, ds < +\infty. \tag{A.14}$$

Taking the expectation of (A.13), we deduce that

$$\begin{aligned}
\mathbb{E}\left(\Omega_t^2 Y_t\right) &= \int_0^t \mathbb{E}\left(\Omega_s^2\left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(s,\Omega_s)\right)\right) ds + \int_0^t \mathbb{E}\left(g_\alpha^2(s,\Omega_s)Y_s\right) ds \\
&\quad + 2\sigma_S \int_0^t \mathbb{E}\left(g_\alpha^2(s,\Omega_s)\Omega_s\right) ds.
\end{aligned} \tag{A.15}$$

Let us now observe that $\mathbb{E}\left(g_\alpha^2(s,\Omega_s)\Omega_s\right) = 0$ and

$$\begin{aligned}
\mathbb{E}\left(g_\alpha^2(t,\Omega_t)\Omega_t^2\right) &= \frac{2\sigma(t)\dot\sigma(t)}{1+\alpha}\mathbb{E}\left(\Omega_t^2 - \frac{\alpha}{2\sigma^2(t)}\Omega_t^4\right) = \frac{2\sigma^3(t)\dot\sigma(t)}{1+\alpha}\left(\Gamma_\alpha - \frac{\alpha}{2}C_\alpha\right) \\
&= \frac{8}{(2+3\alpha)(2+5\alpha)}\sigma^3(t)\dot\sigma(t).
\end{aligned} \tag{A.16}$$

From (A.11)-(A.16), we deduce that $v(t)$ solves, for $t > 0$, the differential equation

$$\begin{aligned}
\dot v(t) &= \frac{2\mu}{2+3\alpha}\sigma^2(t) - \frac{4\sigma_S^2}{(2+3\alpha)(2+5\alpha)}\sigma^3(t)\dot\sigma(t) \\
&+ \frac{2}{1+\alpha}\sigma(t)\dot\sigma(t)m(t) - \frac{\alpha}{1+\alpha}\frac{\dot\sigma(t)}{\sigma(t)}v(t).
\end{aligned}$$

From (A.1) and (A.2), when $\sigma^2(t) = t$ we obtain

$$m(t) = C_1\, t, \qquad v(t) = C_2\, t^2, \qquad \mathbb{E}[Y_t]^2 = C_3\, t + C_4\, t^2,$$

where

$$\begin{aligned}
C_1 &= \mu - \frac{\sigma_S^2}{2(1+\alpha)}\left(1 - \frac{\alpha}{2}\Gamma_\alpha\right) \\
C_2 &= \frac{1}{5\alpha+4}\left(2(1+\alpha)\mu\Gamma_\alpha - \sigma_S^2\left(\Gamma_\alpha - \frac{\alpha}{2}C_\alpha\right) + 2\mu - \frac{\sigma_S^2}{(1+\alpha)}\left(1 - \frac{\alpha}{2}\Gamma_\alpha\right)\right) \\
C_3 &= \frac{\sigma_S^2}{1+\alpha}\left(1 - \frac{\alpha}{2}\Gamma_\alpha\right) \\
C_4 &= \left(\mu - \frac{\sigma_S^2}{2(1+\alpha)}\right) C_1 + \frac{\alpha}{4(1+\alpha)}\sigma_S^2 C_2
\end{aligned}$$

Recall that, letting $\alpha = 0$ we obtain the first two moments of the log-return process in the Black & Scholes model:

$$m(t) = \left(\mu - \tfrac{\sigma_S^2}{2}\right) t, \qquad v(t) = \left(\mu - \tfrac{\sigma_S^2}{2}\right) t^2, \qquad \mathbb{E}[Y_t]^2 = \sigma_S^2\, t + \left(\mu - \tfrac{\sigma_S^2}{2}\right)^2 t^2.$$

When $\sigma^2(t) = \Sigma_\alpha^2\, t^{\frac{2}{2+\alpha}}$, with $\Sigma_\alpha > 0$ constant, it results that

$$\begin{aligned}
m(t) &= \mu\, t - \frac{\sigma_S^2}{2(1+\alpha)}\left(1 - \frac{\alpha}{2}\Gamma_\alpha\right)\Sigma_\alpha^2\, t^{\frac{2}{2+\alpha}} \\
v(t) &= V_1\, t^{\frac{4+\alpha}{2+\alpha}} + V_2\, t^{\frac{4}{2+\alpha}} \\
\mathbb{E}[Y_t]^2 &= \mu^2\, t^2 + M_1\, t^{\frac{4+\alpha}{2+\alpha}} + M_2\, t^{\frac{4}{2+\alpha}} + M_3\, t^{\frac{2}{2+\alpha}},
\end{aligned}$$

where

$$\begin{aligned}
V_1 &= \mu\frac{(1+\alpha)(2+\alpha)}{4+6\alpha+\alpha^2}\left(\Gamma_\alpha+\frac{2}{(1+\alpha)(2+\alpha)}\right)\Sigma_\alpha^2\\
V_2 &= -\frac{1}{4+5\alpha}\left(\Gamma_\alpha-\frac{\alpha}{2}C_\alpha+\frac{1}{1+\alpha}(1-\frac{\alpha}{2}\Gamma_\alpha)\right)\sigma_S^2\Sigma_\alpha^4\\
M_1 &= -\frac{(2+\alpha)\mu}{(1+\alpha)(4+\alpha)}\left(1-\frac{\alpha}{2}\Gamma_\alpha\right)\sigma_S^2\Sigma_\alpha^2+\frac{\alpha V_1}{(1+\alpha)(4+\alpha)}\sigma_S^2\\
&\quad -\frac{2\mu}{(1+\alpha)(4+\alpha)}\sigma_S^2\Sigma_\alpha^2\\
M_2 &= \frac{1}{4(1+\alpha)^2}\left(1-\frac{\alpha}{2}\Gamma_\alpha\right)\sigma_S^4\Sigma_\alpha^4+\frac{\alpha V_2}{4(1+\alpha)}\sigma_S^2\\
M_3 &= \frac{1}{1+\alpha}\left(1-\frac{\alpha}{2}\Gamma_\alpha\right)\sigma_S^2\Sigma_\alpha^2
\end{aligned}$$

Finally, when $\sigma^2(t)=\Sigma_\alpha^2\,t^{\frac{2}{2-\alpha}}$, we get that

$$\begin{aligned}
&m(t)=\mu\,t-\frac{\sigma_S^2}{2(1+\alpha)}\left(1-\frac{\alpha}{2}\Gamma_\alpha\right)\Sigma_\alpha^2\,t^{\frac{2}{2-\alpha}}\\
&v(t)=V_1\,t^{\frac{4-\alpha}{2-\alpha}}+V_2\,t^{\frac{4}{2-\alpha}}\\
&\mathbb{E}[Y_t]^2=\mu^2\,t^2+M_1\,t^{\frac{4-\alpha}{2-\alpha}}+M_2\,t^{\frac{4}{2-\alpha}}+M_3\,t^{\frac{2}{2-\alpha}},
\end{aligned}$$

where

$$\begin{aligned}
V_1 &= \mu\frac{(1+\alpha)(2-\alpha)}{4+4\alpha-\alpha^2}\left(\Gamma_\alpha+\frac{2}{(1+\alpha)(2-\alpha)}\right)\Sigma_\alpha^2\\
V_2 &= -\frac{1}{4+5\alpha}\left(\Gamma_\alpha-\frac{\alpha}{2}C_\alpha+\frac{1}{1+\alpha}(1-\frac{\alpha}{2}\Gamma_\alpha)\right)\sigma_S^2\Sigma_\alpha^4\\
M_1 &= -\frac{(2-\alpha)\mu}{(1+\alpha)(4-\alpha)}\left(1-\frac{\alpha}{2}\Gamma_\alpha\right)\sigma_S^2\Sigma_\alpha^2+\frac{\alpha V_1}{(1+\alpha)(4-\alpha)}\sigma_S^2\\
&\quad -\frac{2\mu}{(1+\alpha)(4-\alpha)}\sigma_S^2\Sigma_\alpha^2\\
M_2 &= \frac{1}{4(1+\alpha)^2}\left(1-\frac{\alpha}{2}\Gamma_\alpha\right)\sigma_S^4\Sigma_\alpha^4+\frac{\alpha V_2}{4(1+\alpha)}\sigma_S^2\\
M_3 &= \frac{1}{1+\alpha}\left(1-\frac{\alpha}{2}\Gamma_\alpha\right)\sigma_S^2\Sigma_\alpha^2
\end{aligned}$$

Appendix B. Proofs

Proof of Proposition 2. The proof essentially follows a standard procedure, but with the use of upper bounds that are strictly related to the functions we are dealing with. We first prove that

$$|g_\alpha(t,x) - g_\alpha(t,y)| \leq C\sqrt{\frac{\dot{\sigma}(t)}{\sigma(t)}}\,|x-y|, \tag{B.1}$$

where, from now on, $C > 0$ denotes different constants, possibly depending on α but not on time and whose specific values are irrelevant. In fact, since

$$|g_\alpha(t,x) - g_\alpha(t,y)| = |\partial_x g_\alpha(t,\theta)\,(x-y)|,$$

for some $\theta \in [\min\{x,y\}, \max\{x,y\}]$, it is sufficient to prove that

$$|\partial_x g_\alpha(t,\theta)\,| \leq C\sqrt{\frac{\dot{\sigma}(t)}{\sigma(t)}}.$$

This inequality is satisfied, since

$$g_\alpha(t,x) = \sqrt{\frac{2\sigma(t)\dot{\sigma}(t)}{1+\alpha}}\sqrt{1-\alpha\frac{x^2}{2\sigma^2(t)}},$$

and

$$|\partial_x g_\alpha(t,x)| = \sqrt{\frac{2\sigma(t)\dot{\sigma}(t)}{1+\alpha}}\,\frac{-\alpha\frac{|x|}{2\sigma^2(t)}}{\sqrt{1-\alpha\frac{x^2}{2\sigma^2(t)}}} \leq C\sqrt{\frac{\dot{\sigma}(t)}{\sigma(t)}}.$$

For $m \in \mathbb{N}$, define the sequence of adapted processes

$$X_t^0 \equiv 0, \qquad X_t^{m+1} = \int_0^t g_\alpha(s, X_s^m)\,dW_s.$$

Then, $\forall\ m \in \mathbb{N}$ and $\forall\ 0 \leq t \leq T$, we get

$$E_t^m = \mathbb{E}\left[\left(X_t^{m+1} - X_t^m\right)^2\right] \leq C\left(\frac{1}{2}\right)^m \sigma^2(t). \tag{B.2}$$

In fact, (B.2) holds for $m = 0$ since

$$E_t^0 = \mathbb{E}\left[\left(X_t^1\right)^2\right] = \mathbb{E}\left[\int_0^t g_\alpha^2(s,0)ds\right] = \frac{\sigma^2(t)}{1+\alpha}.$$

Now, assume that (B.2) holds for $m-1$; we prove that it also holds for m. We get

$$
\begin{aligned}
E_t^m &= \mathbb{E}\left[\left(X_t^{m+1}-X_t^m\right)^2\right] = \mathbb{E}\left[\left(\int_0^t \left(g_\alpha(s,X_s^m)-g_\alpha(s,X_s^{m-1})\right) dW_s\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t \left(g_\alpha(s,X_s^m)-g_\alpha(s,X_s^{m-1})\right)^2\, ds\right] \\
&\leq C\int_0^t \frac{\dot\sigma(s)}{\sigma(s)}\mathbb{E}\left[\left(X_t^m-X_t^{m-1}\right)^2\right]\, ds \\
&\leq C\left(\frac{1}{2}\right)^{m-1}\int_0^t \frac{\dot\sigma(s)}{\sigma(s)}\sigma^2(s)\, ds = C\left(\frac{1}{2}\right)^m \sigma^2(t),
\end{aligned}
$$

where the first inequality follows from (B.1) while the second inequality is due by induction. Thanks to (B.2) and $\sum_{m=0}^{\infty}(1/2)^m < \infty$, the following inequalities hold

$$
\begin{aligned}
\mathbb{E}\left[(X_t^m)^2\right] \;&\leq\; \sum_{k=0}^{m-1}\mathbb{E}\left[\left(X_t^{k+1}-X_t^k\right)^2\right] = \sum_{k=0}^{m-1} E_t^k \leq C\sum_{k=0}^{m-1}\left(\frac{1}{2}\right)^k \sigma^2(t) \\
&\leq\; C\sigma^2(t) < \infty.
\end{aligned} \tag{B.3}
$$

For each $m \in \mathbb{N}$, define the process

$$
M_t^m = X_t^{m+2} - X_t^{m+1} = \int_0^t \left(g_\alpha(s,X_s^{m+1})-g_\alpha(s,X_s^m)\right) dW_s
$$

and denote by $< M^m >_t$ its square variation. Process M_t^m is a continuous true martingale since $\mathbb{E}\left[< M^m >_T\right] = E_T^{m+1} < \infty$. Using Doob and Markov inequalities, we get

$$
\mathbb{E}\left[\max_{t\in[0,T]}(M_t^m)^2\right] \leq 4\mathbb{E}[< M^m >]_T = E_T^{m+1} \leq C\left(\frac{1}{2}\right)^{m+1}\sigma^2(T), \tag{B.4}
$$

$$
\mathbb{P}\left(\max_{t\in[0,T]}(M_t^m)^2 \geq 2^{-\frac{m}{2}}\right) \leq C\,2^{\frac{m}{2}}\left(\frac{1}{2}\right)^{m+1}\sigma^2(T) \leq C\left(\frac{1}{\sqrt{2}}\right)^m \sigma^2(T). \tag{B.5}
$$

Since $\sum_{m=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^m < \infty$, from the Borel-Cantelli lemma we can deduce that for almost every ω there exists $\bar m(\omega) \in \mathbb{N}$ such that for all $m \geq \bar m(\omega)$

$$
\max_{t\in[0,T]} M_t^m(\omega) = \max_{t\in[0,T]}|X_t^{m+2}(\omega) - X_t^{m+1}(\omega)| \leq 2^{-\frac{m}{4}} \tag{B.6}
$$

and therefore, for any $r \in \mathbb{N}$,

$$\max_{t\in[0,T]} |X_t^{m+r}(\omega) - X_t^{m+1}(\omega)| \leq \sum_{j=2}^{r} |X_t^{m+j}(\omega) - X_t^{m+j-1}(\omega)|$$

$$\leq \sum_{j=2}^{r} 2^{-(m+j-2)/4} = 2^{-m/4} \sum_{j=0}^{r} 2^{-j/4} \leq C\, 2^{-m/4}. \tag{B.7}$$

This shows the existence of the limiting process $X_t = \lim_{m\to\infty} X_t^m$ $a.s$ ($X_0 = 0$ $a.s$), which is a continuous adapted process such that $\mathbb{P}\left(\int_0^T g_\alpha^2(t, X_t)\, dt < \infty\right) = 1$ and $X_t = \int_0^t g_\alpha(s, X_s)\, ds$ $a.s.$ The last two assertions follow from (B.1), (B.3) and the dominated convergence theorem. □

Proof of Proposition 3. It suffices to observe that

$$u_t = \frac{\mu - r}{\sigma_S g_\alpha(t, \Omega_t)} \quad = \quad \frac{\mu - r}{\sigma_S \sqrt{2\sigma(t)\dot{\sigma}(t)}} \frac{\sqrt{1+\alpha}}{\sqrt{\phi_\alpha(t, \Omega_t)}},$$

where $\phi_\alpha(t, x) = 1 - \frac{\alpha x^2}{2\sigma^2(t)}$. Since $\phi_\alpha(t, x)$ is greater than 1, we deduce that $u_t \leq \frac{C}{\sqrt{\sigma(t)\dot{\sigma}(t)}}$. □

Proof of Proposition 4. It suffices to show that the local $\mathbb{Q}$-martingale (28) has finite quadratic variation. In fact,

$$\frac{d}{dt}\mathbb{E}^{\mathbb{Q}}\,[\int_o^t g_\alpha^2(s, \Omega_s) ds\,] = \mathbb{E}^{\mathbb{Q}}\,[g_\alpha^2(t, \Omega_t)\,] \sim \sigma(t)\dot{\sigma}(t)\left(1 + \frac{1}{2\sigma^2(t)}\mathbb{E}^{\mathbb{Q}}\,[\Omega_t^2]\right).$$

By Schwarz inequality, we get

$$\mathbb{E}^{\mathbb{Q}}\,[\Omega_t^2] = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Omega_t^2\right] \leq \left(\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2\right]\right)^{1/2} \left(\mathbb{E}\,[\Omega_t^4]\right)^{1/2},$$

where

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2 = \exp\left(-\int_0^T 2u_t dW_t - \frac{1}{2}\int_0^T (2u_t)^2 dt\right) \exp\left(\int_0^T u_t^2 dt\right).$$

As already observed in the proof of Proposition 3, it holds $0 \leq u_t \leq \frac{C}{\sqrt{\sigma(t)\dot{\sigma}(t)}}$. Assuming (27) and using Novikov theorem, we deduce that

$$\mathbb{E}\left[\exp\left(-\int_0^T 2u_t dW_t - \frac{1}{2}\int_0^T (2u_t)^2 dt\right)\right] = 1,$$

and therefore

$$\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2\right] \leq \exp\left(C^2\int_0^T \frac{1}{\sigma(t)\dot{\sigma}(t)}dt\right) \equiv C(T) < +\infty.$$

Since $\left(\mathbb{E}\left[\Omega_t^4\right]\right)^{1/2} = \sqrt{C_\alpha}\sigma^2(t)$, we deduce that $\mathbb{E}^{\mathbb{Q}}\left[g_\alpha^2(t,\Omega_t)\right]$ is dominated (up to a constant, possibly depending on T) by $\sigma(t)\dot{\sigma}(t)$. This concludes the proof. $\square$

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