

Analysis-suitable G^1 multi-patch parametrizations for C^1 isogeometric spaces

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Abstract

One key feature of isogeometric analysis is that it allows smooth shape functions. Indeed, when isogeometric spaces are constructed from p -degree splines (and extensions, such as NURBS), they enjoy up to C^{p-1} continuity within each patch. However, global continuity beyond C^0 on so-called multi-patch geometries poses some significant difficulties. In this work, we consider planar multi-patch domains that have a parametrization which is only C^0 at the patch interface. On such domains we study the h -refinement of C^1 -continuous isogeometric spaces. These spaces in general do not have optimal approximation properties. The reason is that the C^1 -continuity condition easily over-constrains the solution which is, in the worst cases, fully *locked* to linears at the patch interface. However, recently [21] has given numerical evidence that optimal convergence occurs for bilinear two-patch geometries and cubic (or higher degree) C^1 splines. This is the starting point of our study. We introduce the class of analysis-suitable G^1 geometry parametrizations, which includes piecewise bilinear parametrizations. We then analyze the structure of C^1 isogeometric spaces over analysis-suitable G^1 parametrizations and, by theoretical results and numerical testing, discuss their approximation properties. We also consider examples of geometry parametrizations that are not analysis-suitable, showing that in this case optimal convergence of C^1 isogeometric spaces is prevented.

1. Introduction

Thanks to the use of smooth B-splines and NURBS, isogeometric methods [12, 17] have revitalized the interest for the use of smooth approximating functions for the numerical solution of partial differential equations. Advantages with respect to C^0 finite element methods are improved accuracy and spectral properties [4, 13, 18, 41] and the possibility to directly discretize differential operators of order higher than 2. In the literature there are indeed many examples of isogeometric methods for 4th order differential problems of relevant interest, such as Kirchhoff-Love plates/shells [3, 7, 23], the Cahn-Hilliard equation [14], and the Navier-Stokes-Korteweg equation [15].

Since higher dimensional spline spaces possess a tensor-product structure, the representation of domains that have a complex geometry is non-trivial. In this paper we focus on multi-patch representations. While the implementation of C^0 -continuity over multi-patch domains is well understood

(see e.g. [5, 24, 39] for strong and [9] for weak imposition of the C^0 conditions), C^1 -continuity is not. Several studies have tackled the problem of constructing function spaces of C^1 or higher order smoothness, such as [10, 19, 43] for spline spaces, [26, 40] for triangulations and [29] comparing both. Nevertheless, the construction of smooth isogeometric spaces with optimal approximation properties on complex geometries is still an open and challenging problem. This is related to the problem of finding parametrizations of smooth surfaces having complex topology, which is a fundamental area of research in the community of Computer Aided Geometric Design (CAGD) over the last decades.

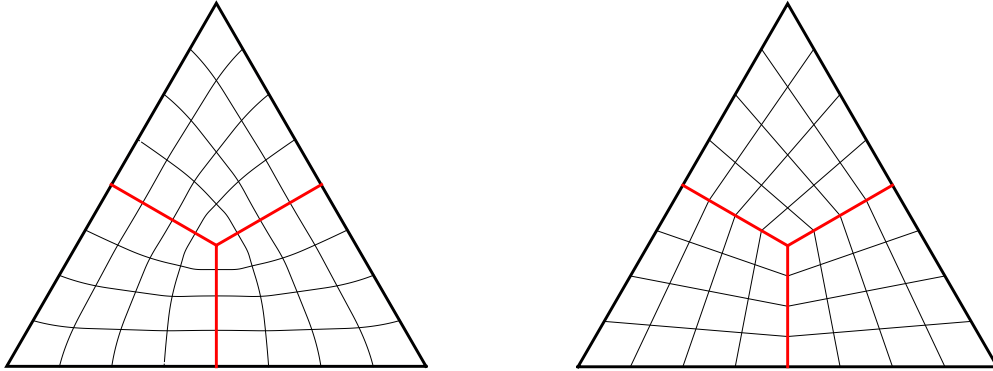


Figure 1: Two possible parametrization schemes: C^1 away from extraordinary points (left) and C^0 everywhere (right).

We review two different strategies for constructing smooth multi-patch geometries and corresponding isogeometric spaces. One possibility is to adopt a geometry parametrization which is globally smooth almost everywhere, with the exception of a neighborhood of the extraordinary points (or edges in 3D), see Figure 1 (left). The other possibility is to use geometry parametrizations that are only C^0 at patch interfaces, see Figure 1 (right). The first option includes subdivision surfaces [11] and the T-spline construction in [38] and, while possessing attractive features, they seem to possess optimal approximation properties for some configurations, see [30], but in general lack accuracy [19, 29]. In our work we consider the second possibility, corresponding to the right part of Figure 1.

The construction of C^1 isogeometric functions over a C^0 parametrization can be interpreted conveniently as geometric continuity G^1 of the graph parametrization. Bilinear multi-patch parametrizations of a planar domain have been analyzed in [8, 27], where it is shown that there exists a minimal determining set with local degrees of freedom for the space of (mapped) piecewise polynomial functions, with global C^1 continuity, if the polynomial degree is high enough (4 if some additional conditions are fulfilled, 5 in general). The recent preprint [28] generalizes the previous results, by using advanced homology techniques, to arbitrary parametrizations, allowing both triangular and quadrilateral patches.

In the work [21], the authors consider splines instead of polynomials within each patch, construct a basis, analyze the space dimensionality of some configurations and, for the first time, perform numerical tests to evaluate the order of convergence when each patch is h -refined. We recall that

within the isogeometric framework, the concept of h -refinement, equivalent to knot insertion, is one of the three constructions to increase accuracy of the spline spaces (see [12]). Remarkably, [21] gives numerical evidence of optimal convergence for C^1 splines of degree 3 (or higher), on a two-patch bilinear geometry. The authors also show an example of *over-constrained* C^1 isogeometric spaces, corresponding to a two-patch non-bilinear geometry parametrization. We refer to the latter situation as C^1 *locking*. Our work develops the underlying theory. While the previous papers give explicit characterizations in the form of minimal determining sets [8, 27] or basis constructions [20, 21], we use an implicit characterization of the continuity conditions and derive from it information on the structure of the isogeometric space. As in [21], our interest is in the impact of h -refinement. We study h -refinement for arbitrary degree and regularity, both theoretically and numerically, and point out its performance depending on the geometry parametrization.

We set up our notation in Section 2. In Section 3 we define the class of analysis-suitable (AS) G^1 geometry parametrizations, which includes the bilinear ones and the extensions presented in Section 3.4 of [21]. Then, in Section 5.1, we study the structure of C^1 isogeometric spaces over AS G^1 two-patch geometries. Here, we give an explanation of the optimal convergence of p -degree isogeometric functions, having C^1 continuity across the patch interface and up to C^{p-2} continuity within the patches. Furthermore, we discuss why C^1 locking occurs for C^{p-1} continuity within the patches. Note that in this paper we do not derive explicit error estimates, which will be the topic of a future paper. In Section 5.2 we analyze C^1 isogeometric spaces constructed over more general geometry parametrizations and conclude that h -convergence is suboptimal beyond AS G^1 geometries. The extensions to surface domains and to NURBS are briefly discussed in Sections 6 and 7, respectively. Numerical tests on two- and multi-patch domains are reported in Section 8. There we present a significant example: the multi-patch parametrization of a smooth simply-connected planar domain. The question of existence and construction methods for AS G^1 multi-patch parametrizations of arbitrary geometries remains to be studied. We summarize our results and draw conclusions in Section 9.

2. Planar multi-patch spline parametrizations and isogeometric spaces

Given an interval or a rectangle ω , we denote by $\mathcal{S}_r^p(\omega)$ the spline space of degree p (in each direction) and continuity C^r at all interior knots. The knot mesh in the parametric domain is assumed to be uniform, with mesh-size h (which is not explicitly indicated in the notation) and interior knot multiplicity $p-r$. We write \mathcal{S}_r^p instead of $\mathcal{S}_r^p(\omega)$ when the domain ω is obvious from the context. We allow $r \geq p$, which stands for C^∞ continuity, that is, the case of global tensor-product polynomials on ω . In this case we also use the notation $\mathcal{P}^p(\omega) = \mathcal{S}_p^p(\omega) = \mathcal{S}_{p+1}^p(\omega) = \dots$

Consider a planar multi-patch domain of interest

$$\Omega = \Omega^{(1)} \cup \dots \cup \Omega^{(N)} \subset \mathbb{R}^2, \quad (1)$$

where the closed sets $\Omega^{(i)}$ form a regular partition (with disjoint interior). For simplicity, we do

not allow hanging nodes. Each $\Omega^{(i)}$ is assumed to be a spline patch, that is

$$\mathbf{F}^{(i)} : [0, 1] \times [0, 1] = \widehat{\Omega} \rightarrow \Omega^{(i)}, \quad (2)$$

where $\mathbf{F}^{(i)} \in \mathcal{S}_r^p(\widehat{\Omega}) \times \mathcal{S}_r^p(\widehat{\Omega})$. We assume

$$r \geq 1 \quad (3)$$

($r \geq p$ means we have Bezier patches $\Omega^{(i)}$) and we assume the parametrizations are not singular, i.e., for all i and for all $(u, v) \in \widehat{\Omega}$,

$$\det \begin{bmatrix} D_u \mathbf{F}^{(i)}(u, v) & D_v \mathbf{F}^{(i)}(u, v) \end{bmatrix} \neq 0. \quad (4)$$

For the sake of simplicity we do not consider more general configurations, e.g., non-uniform knot meshes, different degree or continuity parameters in each parametric direction, different continuity at different knots, or different spline spaces for different patches. Indeed, our simple configuration already presents the key features and difficulties we are interested in.

We assume global continuity of the patch parametrizations. This means the following. Let us fix $\Gamma = \Gamma^{(i,j)} = \Omega^{(i)} \cap \Omega^{(j)}$. When using the superscript as (i, j) in the paper, we assume implicitly that i and j are such that $\Gamma^{(i,j)}$ is not a point or an empty set. Let $\mathbf{F}^{(L)}, \mathbf{F}^{(R)}$ be given such that

$$\begin{aligned} \mathbf{F}^{(L)} : [-1, 0] \times [0, 1] &= \widehat{\Omega}^{(L)} \rightarrow \Omega^{(L)} = \Omega^{(i)}, \\ \mathbf{F}^{(R)} : [0, 1] \times [0, 1] &= \widehat{\Omega}^{(R)} \rightarrow \Omega^{(R)} = \Omega^{(j)}, \end{aligned} \quad (5)$$

where $(\mathbf{F}^{(L)})^{-1} \circ \mathbf{F}^{(i)}$ and $(\mathbf{F}^{(R)})^{-1} \circ \mathbf{F}^{(j)}$ are linear transformations (combinations of a translation, rotation and symmetry). Moreover, the parametrizations agree at $u = 0$, i.e., there is a $\mathbf{F}_0 : [0, 1] \rightarrow \mathbb{R}^2$ with

$$\Gamma = \{\mathbf{F}_0(v) = \mathbf{F}^{(L)}(0, v) = \mathbf{F}^{(R)}(0, v), v \in [0, 1]\}. \quad (6)$$

An example is depicted in Figure 2.

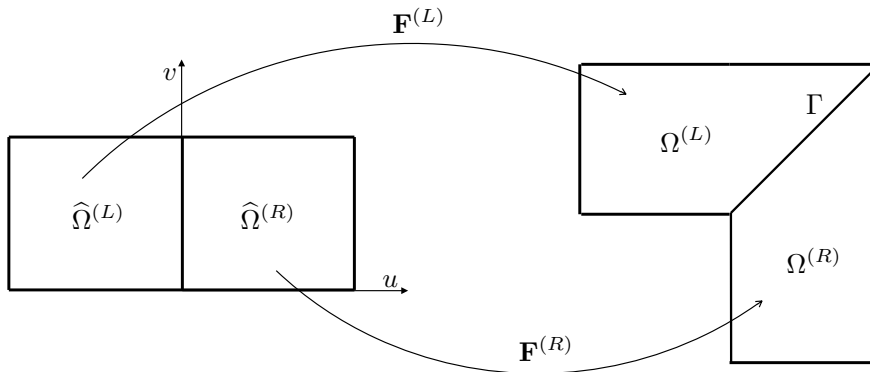


Figure 2: Example of the general setting of (5)–(6).

Remark 1. The domain $\Omega = \Omega^{(1)} \cup \dots \cup \Omega^{(N)}$ can be endowed with a spline manifold structure as defined, e.g., in [16, 28, 36]. In the framework of [36], each pair of adjacent subsets $\Omega^{(i)}$, $\Omega^{(j)}$ is naturally associated with a chart $[-1, 1] \times [0, 1] = \widehat{\Omega}^{(L)} \cup \widehat{\Omega}^{(R)}$ through the maps $\mathbf{F}^{(L)}$ and $\mathbf{F}^{(R)}$.

Definition 1 (Isogeometric spaces). The isogeometric space corresponding to \mathcal{S}_r^p and Ω is given as

$$\mathcal{V} = \left\{ \phi : \Omega \rightarrow \mathbb{R} \text{ such that } \phi \circ \mathbf{F}^{(i)} \in \mathcal{S}_r^p(\widehat{\Omega}), i = 1, \dots, N \right\}. \quad (7)$$

Furthermore we have

$$\mathcal{V}^0 = \mathcal{V} \cap C^0(\Omega), \quad (8)$$

and

$$\mathcal{V}^1 = \mathcal{V} \cap C^1(\Omega). \quad (9)$$

The graph $\Sigma \subset \Omega \times \mathbb{R}$ of an isogeometric function $\phi : \Omega \rightarrow \mathbb{R}$ is naturally split into patches Σ^i having the parametrizations

$$\begin{bmatrix} \mathbf{F}^{(i)} \\ g^{(i)} \end{bmatrix} : [0, 1] \times [0, 1] = \widehat{\Omega} \rightarrow \Sigma^{(i)} \quad (10)$$

where $g^{(i)} = \phi \circ \mathbf{F}^{(i)}$.

In order to analyze the smoothness of an isogeometric function along one interface $\Gamma = \Gamma^{(i,j)} = \Omega^{(i)} \cap \Omega^{(j)}$, we introduce

$$\begin{aligned} \begin{bmatrix} \mathbf{F}^{(L)} \\ g^{(L)} \end{bmatrix} &: [-1, 0] \times [0, 1] = \widehat{\Omega}^{(L)} \rightarrow \Sigma^{(i)} = \Sigma^{(L)}, \\ \begin{bmatrix} \mathbf{F}^{(R)} \\ g^{(R)} \end{bmatrix} &: [0, 1] \times [0, 1] = \widehat{\Omega}^{(R)} \rightarrow \Sigma^{(j)} = \Sigma^{(R)}, \end{aligned} \quad (11)$$

where $g^{(L)}$, $g^{(R)}$ are defined obviously as extensions of (5), see Figure 3. Continuity of ϕ is implied by the continuity of the graph parametrization, which we assume and set to

$$g_0(v) = g^{(L)}(0, v) = g^{(R)}(0, v), \quad (12)$$

for all $v \in [0, 1]$, analogous to (6).

3. C^1 isogeometric spaces

If an isogeometric function is C^1 within each patch (condition (3)), with non-singular parametrization (condition (4)) and is globally continuous (condition (12)), then it is globally C^1 if and only if there exists a well defined tangent plane at each point of the interfaces $\Sigma^{(i)} \cap \Sigma^{(j)}$. Focusing on

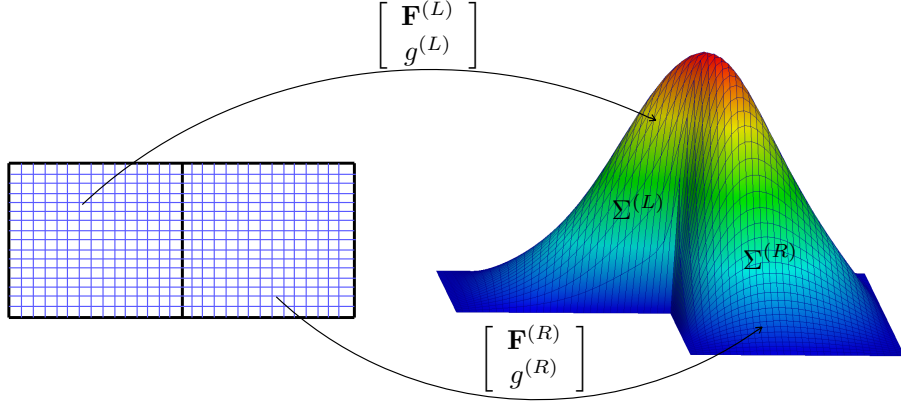


Figure 3: Example of the general setting of (11).

one interface, with notation (11), the tangent planes from the left and right sides are formed by the two pairs of vectors

$$\begin{bmatrix} D_u \mathbf{F}^{(L)}(0, v) \\ D_u g^{(L)}(0, v) \end{bmatrix} \text{ and } \begin{bmatrix} D_v \mathbf{F}_0(v) \\ D_v g_0(v) \end{bmatrix} \text{ as well as } \begin{bmatrix} D_u \mathbf{F}^{(R)}(0, v) \\ D_u g^{(R)}(0, v) \end{bmatrix} \text{ and } \begin{bmatrix} D_v \mathbf{F}_0(v) \\ D_v g_0(v) \end{bmatrix},$$

respectively. These are three different vectors (the vector tangent to $\Sigma^{(i)} \cap \Sigma^{(j)}$ is in common) that form a unique tangent plane, i.e. they are coplanar, if and only if they are linearly dependent. In other words the isogeometric function ϕ is C^1 on $\Omega^{(i)} \cup \Omega^{(j)}$ if and only if

$$\det \begin{bmatrix} D_u \mathbf{F}^{(L)}(0, v) & D_u \mathbf{F}^{(R)}(0, v) & D_v \mathbf{F}_0(v) \\ D_u g^{(L)}(0, v) & D_u g^{(R)}(0, v) & D_v g_0(v) \end{bmatrix} = 0 \quad (13)$$

for all $v \in [0, 1]$. In the context of isogeometric methods, the domain Ω and its parametrization are given at the first stage, then (13) is the condition on the isogeometric function in parametric coordinates (i.e., $g^{(L)}$ and $g^{(R)}$) that gives C^1 continuity of the isogeometric function in physical coordinates.

In CAGD literature, condition (13) is named *geometric continuity of order 1*, in short G^1 , and is commonly stated as in the following Definition (see, e.g., [2, 25, 34]).

Definition 2 (G^1 -continuity at $\Sigma^{(i)} \cap \Sigma^{(j)}$). Given the parametrizations $\mathbf{F}^{(L)}$, $\mathbf{F}^{(R)}$, $g^{(L)}$, $g^{(R)}$ as in (5), (11), fulfilling (3), (4) and (12), we say that the graph parametrization is G^1 at the interface $\Sigma^{(i)} \cap \Sigma^{(j)}$ if there exist $\alpha^{(L)} : [0, 1] \rightarrow \mathbb{R}$, $\alpha^{(R)} : [0, 1] \rightarrow \mathbb{R}$ and $\beta : [0, 1] \rightarrow \mathbb{R}$ such that for all $v \in [0, 1]$,

$$\alpha^{(L)}(v)\alpha^{(R)}(v) > 0 \quad (14)$$

and

$$\alpha^{(R)}(v) \begin{bmatrix} D_u \mathbf{F}^{(L)}(0, v) \\ D_u g^{(L)}(0, v) \end{bmatrix} - \alpha^{(L)}(v) \begin{bmatrix} D_u \mathbf{F}^{(R)}(0, v) \\ D_u g^{(R)}(0, v) \end{bmatrix} + \beta(v) \begin{bmatrix} D_v \mathbf{F}_0(v) \\ D_v g_0(v) \end{bmatrix} = \mathbf{0}. \quad (15)$$

The sign condition (14) on $\alpha^{(L)}$ and $\alpha^{(R)}$ forbids, on general surfaces, the presence of cusps. However, for the graph of a function it is obvious.

For our study it is useful to express G^1 continuity as in Definition 2 since the coefficients $\alpha^{(L)}$, $\alpha^{(R)}$ and β play an important role. Since the first two equations of (15) are linearly independent, $\alpha^{(L)}$, $\alpha^{(R)}$ and β are uniquely determined, up to a common multiplicative factor, by $\mathbf{F}^{(L)}$ and $\mathbf{F}^{(R)}$, i.e. from the equation

$$\alpha^{(R)}(v)D_u\mathbf{F}^{(L)}(0, v) - \alpha^{(L)}(v)D_u\mathbf{F}^{(R)}(0, v) + \beta(v)D_v\mathbf{F}_0(v) = \mathbf{0}. \quad (16)$$

Precisely, we have the following proposition which can also be found in [32, 33, 35].

Proposition 1. *Given any $\mathbf{F}^{(L)}$, $\mathbf{F}^{(R)}$ in the setting of Section 2, then (16) holds if and only if $\alpha^{(S)}(v) = \gamma(v)\bar{\alpha}^{(S)}(v)$, for $S \in \{L, R\}$, and $\beta(v) = \gamma(v)\bar{\beta}(v)$, where*

$$\bar{\alpha}^{(S)}(v) = \det \begin{bmatrix} D_u\mathbf{F}^{(S)}(0, v) & D_v\mathbf{F}_0(v) \end{bmatrix}, \quad (17)$$

$$\bar{\beta}(v) = \det \begin{bmatrix} D_u\mathbf{F}^{(L)}(0, v) & D_u\mathbf{F}^{(R)}(0, v) \end{bmatrix}, \quad (18)$$

and $\gamma : [0, 1] \rightarrow \mathbb{R}$ is any scalar function. In addition, $\gamma(v) \neq 0$ if and only if (14) holds. Moreover, there exist functions $\beta^{(S)}(v)$, for $S \in \{L, R\}$, such that

$$\beta(v) = \alpha^{(L)}(v)\beta^{(R)}(v) - \alpha^{(R)}(v)\beta^{(L)}(v). \quad (19)$$

Proof. Obviously (16) determines $\alpha^{(L)}(v)$, $\alpha^{(R)}(v)$ and $\beta(v)$ up to a common factor $\gamma(v)$ and, since the two vectors $D_u\mathbf{F}^{(L)}(0, v)$ and $D_v\mathbf{F}_0(v)$ are linearly independent because of (4), $\alpha^{(L)}(v)$, $\alpha^{(R)}(v)$ and $\beta(v)$ are uniquely determined up to a common factor $\gamma(v)$ by (16). We have

$$\det \begin{bmatrix} D_u\mathbf{F}^{(L)}(0, v) & D_u\mathbf{F}^{(R)}(0, v) & D_v\mathbf{F}_0(v) \\ D_u\mathbf{F}^{(L)}(0, v) \cdot \mathbf{e}_k & D_u\mathbf{F}^{(R)}(0, v) \cdot \mathbf{e}_k & D_v\mathbf{F}_0(v) \cdot \mathbf{e}_k \end{bmatrix} = 0, \quad (20)$$

for all $k = 1, 2$, \mathbf{e}_k being the canonical base vectors in \mathbb{R}^2 . Using the Laplace expansion of the above determinant (along the third row) we end up with

$$\bar{\alpha}^{(R)}(v)D_u\mathbf{F}^{(L)}(0, v) - \bar{\alpha}^{(L)}(v)D_u\mathbf{F}^{(R)}(0, v) + \bar{\beta}(v)D_v\mathbf{F}_0(v) = 0,$$

where $\bar{\alpha}^{(L)}$, $\bar{\alpha}^{(R)}$ and $\bar{\beta}$ are as in (17) and (18). In a similar way (13) yields

$$\bar{\alpha}^{(R)}(v)D_u g^{(L)}(0, v) - \bar{\alpha}^{(L)}(v)D_u g^{(R)}(0, v) + \bar{\beta}(v)D_v g_0(v) = 0.$$

Setting

$$\boldsymbol{\nu}(v) = \begin{bmatrix} D_v\mathbf{F}_0(v) \cdot \mathbf{e}_2 \\ -D_v\mathbf{F}_0(v) \cdot \mathbf{e}_1 \end{bmatrix}, \quad \boldsymbol{\tau}(v) = \frac{D_v\mathbf{F}_0(v)}{\|D_v\mathbf{F}_0(v)\|^2},$$

we have

$$\det \begin{bmatrix} \boldsymbol{\nu}(v) & \boldsymbol{\tau}(v) \end{bmatrix} = 1 \quad \text{and} \quad \boldsymbol{\nu}(v) \cdot \boldsymbol{\tau}(v) = 0.$$

The existence of $\beta^{(S)}(v)$, $S \in \{L, R\}$, such that (19) holds is obvious, because of (14). Obviously, $\beta^{(S)}(v)$, for $S \in \{L, R\}$, are not unique. A specific choice, which is of interest, is the following

$$\beta^{(S)}(v) = \frac{D_u \mathbf{F}^{(S)}(0, v) \cdot D_v \mathbf{F}_0(v)}{\|D_v \mathbf{F}_0(v)\|^2}. \quad (21)$$

Indeed, using the expansion

$$\forall \mathbf{v} \in \mathbb{R}^2, \quad \mathbf{v} = \det \begin{bmatrix} \mathbf{v} & \boldsymbol{\tau}(v) \end{bmatrix} \boldsymbol{\nu}(v) - \det \begin{bmatrix} \mathbf{v} & \boldsymbol{\nu}(v) \end{bmatrix} \boldsymbol{\tau}(v),$$

and then

$$\det \begin{bmatrix} \mathbf{v}^{(L)} & \mathbf{v}^{(R)} \end{bmatrix} = \det \begin{bmatrix} \mathbf{v}^{(L)} & \boldsymbol{\nu}(v) \end{bmatrix} \det \begin{bmatrix} \mathbf{v}^{(R)} & \boldsymbol{\tau}(v) \end{bmatrix} \\ - \det \begin{bmatrix} \mathbf{v}^{(L)} & \boldsymbol{\tau}(v) \end{bmatrix} \det \begin{bmatrix} \mathbf{v}^{(R)} & \boldsymbol{\nu}(v) \end{bmatrix},$$

gives (19) and (21) by choosing $\mathbf{v}^{(S)} = D_u \mathbf{F}^{(S)}(0, v)$, for $S \in \{L, R\}$. \square

Remark 2. If (15) holds, then there exist coefficients $\alpha^{(S)} \in \mathcal{S}_{r-1}^{2p-1}([0, 1])$, for $S \in \{L, R\}$, and $\beta \in \mathcal{S}_r^{2p}([0, 1])$. Indeed, this follows from Proposition 1 selecting $\gamma = 1$. See also [35].

Summarizing, in the context of isogeometric methods we consider Ω and its parametrization given. Then for each interface $\alpha^{(L)}$, $\alpha^{(R)}$ and β are determined from (16) as stated in Proposition 1. It should be observed that for planar domains, there always exist $\alpha^{(L)}$, $\alpha^{(R)}$ and β fulfilling (16). This is not the case for surfaces, see Section 6. Then, the C^1 continuity of isogeometric functions is equivalent to the last equation in (15), that is

$$\alpha^{(R)}(v) D_u g^{(L)}(0, v) - \alpha^{(L)}(v) D_u g^{(R)}(0, v) + \beta(v) D_v g_0(v) = 0 \quad (22)$$

for all $v \in [0, 1]$.

We end this section by a statement of the equivalence between C^1 continuity of the isogeometric function and G^1 continuity of its graph parametrization. This is a useful and well known property in the isogeometric community, see e.g. [8, 16, 21, 22, 28, 37]. It is formalized and generalized to higher-order continuity and arbitrary dimension in [16]. We give a detailed proof here, in the framework of Proposition 1, since this will serve for the next steps of Section 5.

Proposition 2. An isogeometric function $\phi \in \mathcal{V}$ belongs to \mathcal{V}^1 if and only if its graph Σ is G^1 continuous on each interface $\Sigma^{(i)} \cap \Sigma^{(j)}$.

Proof. Consider a graph interface $\Sigma^{(i)} \cap \Sigma^{(j)}$ and the corresponding $\Gamma = \Gamma^{(i,j)} = \Omega^{(i)} \cap \Omega^{(j)}$. Let $\alpha^{(S)}(v)$, $\beta^{(S)}(v)$, for $S \in \{L, R\}$ such that (16) holds. Define the vector $\mathbf{d}^{(S)}$ on Γ such that

$$\mathbf{d}^{(S)} \circ \mathbf{F}_0(v) = \begin{bmatrix} D_u \mathbf{F}^{(S)}(0, v) & D_v \mathbf{F}_0(v) \end{bmatrix} \begin{bmatrix} 1 \\ -\beta^{(S)}(v) \end{bmatrix} \frac{1}{\alpha^{(S)}(v)}. \quad (23)$$

The vector $\mathbf{d}^{(S)}$ is transversal to Γ , i.e. linear independent to $D_v \mathbf{F}_0(v)$, since

$$\det \begin{bmatrix} \mathbf{d}^{(S)} \circ \mathbf{F}_0(v) & D_v \mathbf{F}_0(v) \end{bmatrix} = \frac{1}{\alpha^{(S)}(v)} \det \begin{bmatrix} D_u \mathbf{F}^{(S)}(0, v) & D_v \mathbf{F}_0(v) \end{bmatrix} = \frac{1}{\gamma(v)} \neq 0.$$

We have $\mathbf{d}^{(L)} = \mathbf{d}^{(R)} = \mathbf{d}$. Indeed, by using (16) and (19), we get

$$\begin{aligned} & \alpha^{(L)}(v)\alpha^{(R)}(v)\mathbf{d}^{(L)} \circ \mathbf{F}_0(v) - \alpha^{(L)}(v)\alpha^{(R)}(v)\mathbf{d}^{(R)} \circ \mathbf{F}_0(v) \\ &= \alpha^{(R)}(v)D_u\mathbf{F}^{(L)}(0, v) - \alpha^{(L)}(v)D_u\mathbf{F}^{(R)}(0, v) + (\alpha^{(L)}(v)\beta^{(R)}(v) - \alpha^{(R)}(v)\beta^{(L)}(v))D_v\mathbf{F}_0(v) \\ &= \mathbf{0}. \end{aligned}$$

Futhrermore, \mathbf{d} is not in general unitary and it is continuous, since \mathbf{F}_0 is at least C^1 from (3). For $S \in \{L, R\}$, let $\phi = g^{(S)} \circ [F^{(S)}]^{-1}$ on $\Omega^{(S)}$, then the derivative from side S in direction \mathbf{d} fulfills

$$\nabla^{(S)}\phi(x, y) \cdot \mathbf{d}(x, y) = [D_u g^{(S)}(0, v) D_v g_0(v)] \left[D_u \mathbf{F}^{(S)}(0, v) D_v \mathbf{F}_0(v) \right]^{-1} \cdot \mathbf{d} \circ \mathbf{F}_0(v), \quad (24)$$

for all $(x, y) \in \Gamma$, where here and in what follows we implicitly assume the relation $\mathbf{F}_0(v) = (x, y)$.

We obtain directly from the definition of $\mathbf{d}^{(S)}$ that

$$\left[D_u \mathbf{F}^{(S)}(0, v) D_v \mathbf{F}_0(v) \right]^{-1} \cdot \mathbf{d} \circ \mathbf{F}_0(v) = \frac{1}{\alpha^{(S)}(v)} \begin{bmatrix} 1 \\ -\beta^{(S)}(v) \end{bmatrix}. \quad (25)$$

Substituting (25) back to (24) we then obtain

$$\nabla^{(S)}\phi(x, y) \cdot \mathbf{d}(x, y) = \frac{D_u g^{(S)}(0, v) - \beta^{(S)}(v)D_v g_0(v)}{\alpha^{(S)}(v)} \quad (26)$$

for $S \in \{R, L\}$. Therefore $\nabla^{(L)}\phi(x, y) \cdot \mathbf{d}(x, y) = \nabla^{(R)}\phi(x, y) \cdot \mathbf{d}(x, y)$ if and only if (by (26))

$$\frac{D_u g^{(L)}(0, v) - \beta^{(L)}(v)D_v g_0(v)}{\alpha^{(L)}(v)} = \frac{D_u g^{(R)}(0, v) - \beta^{(R)}(v)D_v g_0(v)}{\alpha^{(R)}(v)}.$$

That, after multiplying both sides by $\alpha^{(L)}(v)\alpha^{(R)}(v)$ and using (19), is equivalent to (22). \square

Remark 3. As a consequence of Proposition 2, C^1 isogeometric spaces over C^0 planar multi-patch parametrizations fit in the isoparametric framework.

4. Analysis-suitable G^1 parametrizations

At each interface $\Gamma = \Gamma^{(i,j)} = \Omega^{(i)} \cap \Omega^{(j)}$, given any regular and orientation preserving $\mathbf{F}^{(L)}$, $\mathbf{F}^{(R)}$, as in (5), there exist coefficients $\alpha^{(L)}$, $\alpha^{(R)}$ and β , with $\alpha^{(L)}(v)\alpha^{(R)}(v) > 0$, such that (16) holds. Then (22) expresses C^1 continuity of isogeometric functions in terms of $\alpha^{(L)}$, $\alpha^{(R)}$ and β . Optimal approximation properties of the isogeometric space on Ω holds under restrictions on $\alpha^{(L)}$, $\alpha^{(R)}$ and β , i.e. on the geometry parametrization. This leads to the definition below.

Definition 3 (Analysis-suitable G^1 -continuity). $\mathbf{F}^{(L)}$ and $\mathbf{F}^{(R)}$ are *analysis-suitable G^1 -continuous* at the interface Γ (in short, AS $G^1(\Gamma)$ or AS G^1) if there exist $\alpha^{(L)}, \alpha^{(R)}, \beta^{(L)}, \beta^{(R)} \in \mathcal{P}^1([0, 1])$ such that (16) and (19) hold, that is for all $v \in [0, 1]$

$$\alpha^{(R)}(v)D_u\mathbf{F}^{(L)}(0, v) - \alpha^{(L)}(v)D_u\mathbf{F}^{(R)}(0, v) = (\alpha^{(R)}(v)\beta^{(L)}(v) - \alpha^{(L)}(v)\beta^{(R)}(v))D_v\mathbf{F}_0(v).$$

Remark 4. As in Remark 3, we observe that C^1 isogeometric spaces over AS G^1 multi-patch parametrizations fit in the isoparametric framework.

The class of AS G^1 parametrizations contains the bilinear ones and more

Proposition 3. Any $\mathbf{F}^{(L)} \in \mathcal{P}^1(\Omega^{(L)}) \times \mathcal{P}^1(\Omega^{(L)})$ and $\mathbf{F}^{(R)} \in \mathcal{P}^1(\Omega^{(R)}) \times \mathcal{P}^1(\Omega^{(R)})$ are AS G^1 -continuous at Γ . Moreover, for any $\alpha^{(L)}, \alpha^{(R)} \in \mathcal{P}^1([0, 1])$ strictly positive and $\beta^{(L)}, \beta^{(R)} \in \mathcal{P}^1([0, 1])$ there exist $\mathbf{F}^{(L)} \in \mathcal{P}^1(\Omega^{(L)})$ and $\mathbf{F}^{(R)} \in \mathcal{P}^1(\Omega^{(R)})$ fulfilling (16).

Proof. The statement follows directly from Proposition 1 and (21). \square

Remark 5. The class of AS G^1 parametrizations is wider than only bilinear. We will show some examples later in Section 8 where $\mathbf{F}^{(L)}$ and $\mathbf{F}^{(R)}$ are higher order polynomials at $\{0\} \times [0, 1]$.

5. Two-patch geometry

In this section we analyze the two-patch geometry. This is the simplest geometric configuration that allows us to focus on the C^1 -continuity constraints that are associated with each patch interface. For the space of isogeometric C^0 functions, that is \mathcal{V}^0 defined in (7)-(8), optimal convergence under h -refinement is known since the results in [1, 6] apply directly. C^1 -continuity constrains traces and transversal derivatives at Γ of functions $\phi \in \mathcal{V}^1$. Therefore the approximation properties of the space \mathcal{V}^1 follow from the ones of traces of functions and transversal derivatives of functions of \mathcal{V}^1 at Γ . Let $\alpha^{(S)}(v), \beta^{(S)}(v)$, for $S \in \{L, R\}$ such that (16) holds. We define the transversal vector \mathbf{d} to Γ as in (23) and introduce the space of traces and transversal directional derivatives on Γ

$$\mathcal{V}_\Gamma^1 = \{\Gamma \ni (x, y) \mapsto [\phi(x, y), \nabla\phi(x, y) \cdot \mathbf{d}(x, y)], \text{ such that } \phi \in \mathcal{V}^1\} \quad (27)$$

and its pullback

$$\widehat{\mathcal{V}}_\Gamma^1 = \{[\phi, \nabla\phi \cdot \mathbf{d}] \circ \mathbf{F}_0, \text{ such that } \phi \in \mathcal{V}^1\}. \quad (28)$$

With this choice we have that $[\phi, \nabla\phi \cdot \mathbf{d}] \circ \mathbf{F}_0 = [g_0, g_1]$ is equivalent to

$$g^{(S)}(u, v) = g_0(v) + (\beta^{(S)}(v)g'_0(v) + \alpha^{(S)}(v)g_1(v))u + O(u^2), \quad (29)$$

thanks to (26).

Remark 6. The transversal vector \mathbf{d} depends on $\alpha^{(L)}, \alpha^{(R)}, \beta^{(L)}, \beta^{(R)}$, and so do \mathcal{V}_Γ^1 and $\widehat{\mathcal{V}}_\Gamma^1$. The function trace in $\widehat{\mathcal{V}}_\Gamma^1$ fulfills $\phi \circ \mathbf{F}_0 \in \mathcal{S}_r^p$, this is not true for the transversal derivative $\nabla\phi \cdot \mathbf{d}$ which in general is a rational function of some degree higher than p (a similar situation is analyzed in [42]).

5.1. AS G^1 -continuous two-patch geometry

The next result gives the key properties of the space $\widehat{\mathcal{V}}_\Gamma^1$ (defined in (28)) in the case of AS G^1 parametrizations.

Theorem 1. *Let $\Omega = \Omega^{(L)} \cup \Omega^{(R)}$, and let $\mathbf{F}^{(L)}$ and $\mathbf{F}^{(R)}$ be AS G^1 at the interface $\Gamma = \partial\Omega^{(L)} \cup \partial\Omega^{(R)}$. Then*

$$\mathcal{S}_{r+1}^p([0, 1]) \times \mathcal{S}_r^{p-1}([0, 1]) \subset \widehat{\mathcal{V}}_\Gamma^1. \quad (30)$$

Proof. For linear $\alpha^{(S)}$ and $\beta^{(S)}$ ($S \in \{L, R\}$) and $[g_0, g_1] \in \mathcal{S}_{r+1}^p([0, 1]) \times \mathcal{S}_r^{p-1}([0, 1])$, we have

$$g^{(S)}(u, v) = g_0(v) + (\beta^{(S)}(v)g_0'(v) + \alpha^{(S)}(v)g_1(v))u \in \mathcal{S}_r^p(\widehat{\Omega}^{(S)}). \quad (31)$$

Then the statement is a direct consequence of (29). \square

Theorem 1 guarantees that the trace space for the function value $\{\phi \circ \mathbf{F}_0 : \phi \in \mathcal{V}^1\}$ includes all splines of degree p and regularity at least $r+1$, and independently the trace space for the transversal derivatives of the function $\{(\nabla\phi \cdot \mathbf{d}) \circ \mathbf{F}_0 : \phi \in \mathcal{V}^1\}$ includes all splines of degree $p-1$ and regularity at least r . This suggests that \mathcal{V}_Γ^1 enjoys optimal approximation order, and consequently for the whole space \mathcal{V}^1 when $r < p-1$.

However, if $r = p-1$ and the parametrization is not trivial, the space \mathcal{V}_Γ^1 suffers of C^1 locking, that is, h -refinement does not improve the approximation properties for ϕ and $\nabla\phi \cdot \mathbf{d}$ independently. The following theorem gives some understanding of this phenomenon.

Theorem 2. *Let $\Omega = \Omega^{(L)} \cup \Omega^{(R)}$, $\mathbf{F}^{(L)}$, $\mathbf{F}^{(R)}$ be AS G^1 at the interface $\Gamma = \Omega^{(L)} \cap \Omega^{(R)}$, and $r = p-1$. Furthermore, let \mathcal{G}_0 and \mathcal{G}_1 be two spaces such that*

$$\mathcal{G}_0 \times \mathcal{G}_1 \subseteq \widehat{\mathcal{V}}_\Gamma^1. \quad (32)$$

If either $\beta^{(L)} \neq 0$ or $\beta^{(R)} \neq 0$ then the dimension of \mathcal{G}_0 is independent of h . If $\alpha^{(L)}$ and $\alpha^{(R)}$ are linearly independent, then the dimension of \mathcal{G}_1 is independent of h .

Proof. Let $[g_0, 0] = [\phi, \nabla\phi \cdot \mathbf{d}] \circ \mathbf{F}_0$ and assume that $\beta^{(L)}$ is a linear function not identically zero. Furthermore assume that if there exists a $v_0 \in [0, 1]$, such that $\beta^{(L)}(v_0) = 0$, then from the assumption $\beta^{(R)}(v_0) \neq 0$. Using (29), i.e.,

$$g^{(S)}(u, v) = g_0(v) + \beta^{(S)}(v)g_0'(v)u + O(u^2),$$

and using $g^{(S)}(u, v) \in \mathcal{S}_{p-1}^p(\widehat{\Omega}^{(S)})$ then g_0 is a spline in $\mathcal{S}_{p-1}^p([0, 1])$ and also g_0' is C^{p-1} , therefore g_0 is in fact a degree p global polynomial. The same conclusion holds when there exists a $v_0 \in [0, 1]$ such that $\beta^{(L)}(v_0) = \beta^{(R)}(v_0) = 0$ but v_0 is not a knot of the spline space $\mathcal{S}_{p-1}^p([0, 1])$ under consideration. If, instead, $\beta^{(L)}(v_0) = \beta^{(R)}(v_0) = 0$ and v_0 is a knot of the spline space $\mathcal{S}_{p-1}^p([0, 1])$, then g_0 has in general $p-1$ continuous derivatives at v_0 . In conclusion if $g_0 \in \mathcal{G}_0$ then it has to be piecewise polynomial of degree p on at most two intervals, and is in $C^{p-1}([0, 1])$ globally. Hence the dimension of \mathcal{G}_0 is at most $p+2$.

Take now $[0, g_1] = [\phi, \nabla\phi \cdot \mathbf{d}] \circ \mathbf{F}_0$ and assume $\alpha^{(L)}$ and $\alpha^{(R)}$ are linearly independent, that is, their linear combination gives the whole space $\mathcal{P}^1([0, 1])$. Now (29), i.e.,

$$g^{(S)}(u, v) = \alpha^{(S)}(v)g_1(v)u + O(u^2),$$

after taking the derivative in the variable u and evaluating at $u = 0$ gives

$$\frac{\partial}{\partial u}g^{(S)}(0, v) = \alpha^{(S)}(v)g_1(v),$$

and, by suitable linear combination for $S = L$ and $S = R$ yields

$$C_L \frac{\partial}{\partial u}g^{(L)}(0, v) + C_R \frac{\partial}{\partial u}g^{(R)}(0, v) = g_1(v).$$

Since $g^{(S)}(u, v) \in \mathcal{S}_{p-1}^p(\hat{\Omega}^{(S)})$, $g_1 \in \mathcal{S}_{p-1}^p([0, 1])$. Similarly

$$C'_L \frac{\partial}{\partial u}g^{(L)}(0, v) + C'_R \frac{\partial}{\partial u}g^{(R)}(0, v) = vg_1(v),$$

which means $vg_1 \in \mathcal{S}_{p-1}^p([0, 1])$. in conclusion $g_1 \in \mathcal{S}_{p-1}^{p-1}([0, 1])$, therefore the dimension of \mathcal{G}_1 is at most p . \square

When $\beta^{(L)} = \beta^{(R)} = 0$ and for all $v \in [0, 1]$ we have $\alpha^{(L)}(v) = C\alpha^{(R)}(v)$ then

$$D_u\mathbf{F}^{(L)}(0, v) = CD_u\mathbf{F}^{(R)}(0, v),$$

that is, the parametrization is trivially G^1 in the sense that it can be made C^1 by scaling the variable u by a multiplicative factor in (one of) the two patches. If we are in such a case, as for C^1 parametrizations, one can easily conclude $\mathcal{S}_r^p([0, 1]) \times \mathcal{S}_r^p([0, 1]) = \widehat{\mathcal{V}}_\Gamma^1$. If we are not in such a special configuration, then Theorem 2 states that we can *independently* approximate the trace and transversal derivative of a function by isogeometric C^1 functions, but for h -refinement the approximation error does not converge to zero.

5.2. General two-patch geometry

In this section we study the approximation property of \mathcal{V}^1 , for two-patch geometry parametrizations beyond analysis-suitable G^1 -continuity. We consider a specific case where $\mathbf{F}^{(L)}$ is the identity mapping while $\mathbf{F}^{(R)} \in [\mathcal{S}_r^p(\hat{\Omega}^{(R)})]^2$, and select

$$\begin{aligned} \alpha^{(L)}(v) &= 1 \\ \alpha^{(R)}(v) &= D_u\mathbf{F}^{(R)}(0, v) \cdot \mathbf{e}_1 \in \mathcal{S}_r^p([0, 1]), \\ \beta^{(L)}(v) &= 0 \\ \beta^{(R)}(v) &= D_u\mathbf{F}^{(R)}(0, v) \cdot \mathbf{e}_2 \in \mathcal{S}_r^p([0, 1]). \end{aligned} \tag{33}$$

Then $\mathbf{d} = \mathbf{e}_1$ and

$$\widehat{\mathcal{V}}_\Gamma^1 \equiv \mathcal{V}_\Gamma^1 = \left\{ \left[\phi, \frac{\partial}{\partial x}\phi \right], \text{ such that } \phi \in \mathcal{V}^1 \right\}. \tag{34}$$

Equation (29) simplifies to

$$\begin{aligned} & \left[\phi, \frac{\partial \phi}{\partial x} \right] \circ \mathbf{F}_0 = [g_0, g_1] \\ & \Leftrightarrow \begin{cases} g^{(L)}(u, v) = g_0(v) + g_1(v)u + O(u^2), \\ g^{(R)}(u, v) = g_0(v) + (\beta^{(R)}(v)g'_0(v) + \alpha^{(R)}(v)g_1(v))u + O(u^2). \end{cases} \end{aligned} \quad (35)$$

The following statement gives a full characterization of the space \mathcal{V}_Γ^1 . We use $\mathcal{S}_r^{-1}([0, 1])$ to indicate the null space \emptyset , and we recall that $p_\alpha = 0$ is not allowed.

Theorem 3. *Let $\Omega = \Omega^{(L)} \cup \Omega^{(R)}$, $\mathbf{F}^{(L)}$ be the identity and $\alpha^{(R)}$, $\alpha^{(L)}$, $\beta^{(R)}$, $\beta^{(L)}$ as in (33). Assume $\alpha^{(R)} \in \mathcal{S}_r^{p_\alpha}([0, 1])$, with $\alpha^{(R)} \notin \mathcal{S}_r^{p_\alpha-1}([0, 1])$.*

If $\beta^{(R)} = 0$, then

$$\mathcal{V}_\Gamma^1 = \mathcal{S}_r^p([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1]). \quad (36)$$

If instead $\beta^{(R)} \in \mathcal{S}_r^{p_\beta}([0, 1])$ with $\beta^{(R)} \notin \mathcal{S}_r^{p_\beta-1}([0, 1])$, then

$$\mathcal{V}_\Gamma^1 = \mathcal{S}_{r+1}^{\min\{p, p-p_\beta+1\}}([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1]). \quad (37)$$

Proof. Since $\mathbf{F}^{(L)}$ is the identity we have $\mathcal{V}_\Gamma^1 = \widehat{\mathcal{V}}_\Gamma^1 \subset \mathcal{S}_r^p([0, 1]) \times \mathcal{S}_r^p([0, 1])$.

Consider first the case $\beta^{(R)} = 0$. It is easy to see that

$$\widehat{\mathcal{V}}_\Gamma^1 \supset \mathcal{S}_r^p([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1]).$$

Indeed, given any $[g_0, g_1] \in \mathcal{S}_r^p([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1])$ we can find $g^{(S)}(u, v) \in \mathcal{S}_r^p(\widehat{\Omega}^{(S)})$ such that (35) holds. In particular, take

$$g^{(R)}(u, v) = g_0(v) + \alpha^{(R)}(v)g_1(v)u.$$

Moreover, $\mathcal{S}_r^p([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1])$ is equal to the space $\widehat{\mathcal{V}}_\Gamma^1$ due to the second condition in (35), which is

$$g^{(R)}(u, v) = g_0(v) + \alpha^{(R)}(v)g_1(v)u + O(u^2)$$

for $\beta^{(R)} = 0$, since $g^{(R)}(u, v) \in \mathcal{S}_r^p(\widehat{\Omega}^{(R)})$ forbids g_1 to be of a polynomial degree higher than $p - p_\alpha$.

The second case, $\beta^{(R)} \in \mathcal{S}_r^{p_\beta}([0, 1])$ with $\beta^{(R)} \notin \mathcal{S}_r^{p_\beta-1}([0, 1])$, is similar. Again, we have $\widehat{\mathcal{V}}_\Gamma^1 \supset \mathcal{S}_{r+1}^{\min\{p, p-p_\beta+1\}}([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1])$. Indeed, given any

$$[g_0, g_1] \in \mathcal{S}_{r+1}^{\min\{p, p-p_\beta+1\}}([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1])$$

we can find $g^{(S)}(u, v) \in \mathcal{S}_r^p(\widehat{\Omega}^{(S)})$ such that (35) holds. This time, take

$$g^{(R)}(u, v) = g_0(v) + (\beta^{(R)}(v)g'_0(v) + \alpha^{(R)}(v)g_1(v))u.$$

As before $\mathcal{S}_{r+1}^{\min\{p, p-p_\beta+1\}}([0, 1]) \times \mathcal{S}_r^{p-p_\alpha}([0, 1])$ is equal to the space $\widehat{\mathcal{V}}_\Gamma^1$ due to the second condition in (35), that is

$$\mathcal{S}_r^p(\widehat{\Omega}^{(R)}) \ni g^{(R)}(u, v) = g_0(v) + (\beta^{(R)}(v)g'_0(v) + \alpha^{(R)}(v)g_1(v))u + O(u^2).$$

This concludes the proof. \square

Remark 7. In the setting of this section, $\mathbf{F}^{(L)}$ being the identity and $\mathbf{F}^{(R)} \in \mathcal{S}_r^p(\widehat{\Omega}^{(R)}) \times \mathcal{S}_r^p(\widehat{\Omega}^{(R)})$, we always have $p_\alpha \leq p$, $p_\beta \leq p$, which follows from (33). Then Theorem 3 guarantees

$$\mathcal{V}_\Gamma^1 = \widehat{\mathcal{V}}_\Gamma^1 \supset \mathcal{S}_p^1([0, 1]) \times \mathcal{S}_{p-1}^0([0, 1]) = \mathcal{P}^1 \times \mathcal{P}^0.$$

This is in accordance with the well known fact that isoparametric functions reproduce all linear polynomials.

We remark that any restriction on \mathcal{V}_Γ^1 as stated in Theorem 3 affects the approximation properties of $\mathcal{V}^1|_{\Omega^{(L)}} \subset \mathcal{S}_r^p(\Omega^{(L)})$ and consequently of \mathcal{V}^1 itself. For the sake of convenience, this is summarized in the corollary below.

Corollary 1. *Optimal order of convergence for h-refinement can not be achieved if $\deg \alpha^{(R)} > 1$ or $\deg \beta^{(R)} > 1$. In particular C^1 locking, i.e. no convergence, occurs for $\deg \alpha^{(R)} \geq p - r$ or $\deg \beta^{(R)} \geq p - r$.*

Some examples will be considered and further analyzed in Section 8.

6. C^1 isogeometric spaces on surfaces

The result of the previous sections can be extended to a multi-patch surface Ω

$$\Omega = \Omega^{(1)} \cup \dots \cup \Omega^{(N)} \subset \mathbb{R}^3.$$

We can set up an isogeometric function space over the surface

$$\mathcal{V} = \left\{ \phi : \Omega \rightarrow \mathbb{R} \text{ such that } \phi \circ \mathbf{F}^{(i)} \in \mathcal{S}_r^p(\widehat{\Omega}), i = 1, \dots, N \right\}, \quad (38)$$

and again $\mathcal{V}^0 = \mathcal{V} \cap C^0(\Omega)$ and $\mathcal{V}^1 = \mathcal{V} \cap C^1(\Omega)$. As for the planar case, $\mathbf{F}^{(i)} : \widehat{\Omega} \rightarrow \Omega^{(i)}$, though $\Omega^{(i)}$ is now a surface patch. More important, for the function space $C^1(\Omega)$ to be well defined, the surface Ω itself needs to be C^1 , i.e., the surface needs to have a well defined tangent plane in every point. For simplicity, we focus on a single interface, that is a two-patch geometry, where each patch is parameterized via

$$\begin{aligned} \mathbf{F}^{(L)} : [-1, 0] \times [0, 1] &= \widehat{\Omega}^{(L)} \rightarrow \Omega^{(L)} \subset \mathbb{R}^3, \\ \mathbf{F}^{(R)} : [0, 1] \times [0, 1] &= \widehat{\Omega}^{(R)} \rightarrow \Omega^{(R)} \subset \mathbb{R}^3 \end{aligned} \quad (39)$$

with $\mathbf{F}^{(S)} \in (\mathcal{S}_r^p(\widehat{\Omega}^{(S)}))^3$ for $S \in \{L, R\}$. We ask the parameterization to be G^1 , i.e., there exist $\alpha^{(L)} : [0, 1] \rightarrow \mathbb{R}$, $\alpha^{(R)} : [0, 1] \rightarrow \mathbb{R}$ and $\beta : [0, 1] \rightarrow \mathbb{R}$ such that $\forall v \in [0, 1]$, $\alpha^{(L)}(v)\alpha^{(R)}(v) > 0$ and

$$\alpha^{(R)}(v)D_u \mathbf{F}^{(L)}(0, v) - \alpha^{(L)}(v)D_u \mathbf{F}^{(R)}(0, v) + \beta(v)D_v \mathbf{F}_0(v) = \mathbf{0}. \quad (40)$$

Then, the surface gradient of a function $\phi : \Omega \rightarrow \mathbb{R}$ can be computed as follows. First, the surface Ω is extended to $\Omega_\epsilon \subset \mathbb{R}^3$ by defining

$$\mathbf{G}^{(S)} : \widehat{\Omega}^{(S)} \times]-\epsilon, \epsilon[\rightarrow \Omega_\epsilon^{(S)} \quad (41)$$

with

$$\mathbf{G}^{(S)}(u, v, w) = \mathbf{F}^{(S)}(u, v) + w\mathbf{N}^{(S)}(u, v), \quad (42)$$

where $\mathbf{N}^{(S)}$ is the unit normal vector of $\Omega^{(S)}$. Now, we define the extension Φ of the function ϕ via $\hat{\Phi}^{(S)}(u, v, w) = \hat{\phi}^{(S)}(u, v)$ for $S \in \{L, R\}$. The surface gradient of ϕ is then given as the three-dimensional gradient of the extension Φ in Ω_ϵ restricted to the surface Ω , i.e.

$$\nabla_\Omega \phi = \nabla \Phi|_\Omega, \quad (43)$$

where

$$\nabla \Phi \circ \mathbf{G}^{(S)} = \nabla \hat{\Phi}^{(S)} \cdot (\nabla \mathbf{G}^{(S)})^{-1} \quad \text{on } \hat{\Omega}^{(S)}. \quad (44)$$

By construction, the gradient is tangential to the surface.

Proposition 2 can be generalized to surface domains. For the sake of simplicity we consider a simplified case, by assuming that $\mathbf{F}^{(S)}$ can be projected onto the (x, y) -plane without self intersections. Note that this is not a limitation of the concept but facilitates the following propositions. Then we have

$$\mathbf{F}^{(S)} = (F_1^{(S)}, F_2^{(S)}, F_3^{(S)})^T = (\mathbf{P}^{(S)}, f_3^{(S)} \circ \mathbf{P}^{(S)})^T \quad (45)$$

where $\mathbf{P}^{(S)} = (F_1^{(S)}, F_2^{(S)})^T$ is the planar parameterization of the projected surface and f_3 is an (isogeometric) function from the planar projection $\bar{\Omega}^{(S)} = \mathbf{P}^{(S)}(\hat{\Omega}^{(S)})$ to \mathbb{R} . Then the following result is a direct corollary of Proposition 2, see also [16].

Proposition 4. *An isogeometric function $\phi \in \mathcal{V}$ as in (38) belongs to \mathcal{V}^1 if and only if its graph Σ as a 2-dimensional surface in \mathbb{R}^4 is G^1 -continuous at each interface $\Sigma^{(i)} \cap \Sigma^{(j)}$.*

The definition for a G^1 -continuous graph of an isogeometric function ϕ is formally the same as in Definition 2, however now the graph of $\phi : \Omega \rightarrow \mathbb{R}$ is a two-dimensional surface in \mathbb{R}^4 . As for planar domains considered in Section 3, the coefficients for the G^1 condition are still determined by the parameterization $\mathbf{F}^{(S)}$, in fact by two of its three components. In our simplified case, we are in the following situation.

Proposition 5. *Consider surface patches*

$$\mathbf{F}^{(L)} = (\mathbf{P}^{(L)}, F_3^{(L)})^T, \quad \mathbf{F}^{(R)} = (\mathbf{P}^{(R)}, F_3^{(R)})^T$$

and functions $g^{(L)}, g^{(R)}$ fulfilling Definition 2. Then the coefficients $\alpha^{(L)}, \alpha^{(R)}$ and β in (40) are given by $\alpha^{(S)}(v) = \gamma(v)\bar{\alpha}^{(S)}(v)$, for $S \in \{L, R\}$, and $\beta(v) = \gamma(v)\bar{\beta}(v)$, where

$$\bar{\alpha}^{(S)}(v) = \det \begin{bmatrix} D_u \mathbf{P}^{(S)}(0, v) & D_v \mathbf{P}^{(S)}(0, v) \end{bmatrix}, \quad (46)$$

$$\bar{\beta}(v) = \det \begin{bmatrix} D_u \mathbf{P}^{(L)}(0, v) & D_u \mathbf{P}^{(R)}(0, v) \end{bmatrix}, \quad (47)$$

and $\gamma : [0, 1] \rightarrow \mathbb{R}$ is any scalar function. In addition, $\gamma(v) \neq 0$ if and only if $\alpha^{(L)}(v)\alpha^{(R)}(v) > 0$. Moreover, there exist functions $\beta^{(S)}(v)$, for $S \in \{L, R\}$, such that

$$\beta(v) = \alpha^{(L)}(v)\beta^{(R)}(v) - \alpha^{(R)}(v)\beta^{(L)}(v). \quad (48)$$

Again, we omit the details of the proof, as it is a direct consequence of Proposition 1.

In analogy to the planar case, to guarantee optimal approximation properties of the isogeometric space we ask some structure for the trace and derivative trace of isogeometric functions, by restricting the geometry parametrization to the class of analysis-suitable G^1 -continuous functions. The definition is omitted being exactly the same as Definition 3. In the case of (45), the parametrization $\mathbf{P}^{(S)}$ of the planar projection $\bar{\Omega}$ of the geometry Ω needs to be AS G^1 . Furthermore, the third component $F_3^{(S)}$ of $\mathbf{F}^{(S)}$ as well as the function $\hat{\phi}^{(S)}$ have to fulfill the same conditions, for $S \in \{L, R\}$. This means that Remarks 3–4 extend to surfaces: C^1 isogeometric spaces over G^1 and AS G^1 multi-patch parametrizations fit in the isoparametric framework. Eventually, $\phi \in \mathcal{V}^1(\Omega)$ if and only if $\bar{\phi} \in \mathcal{V}^1(\bar{\Omega})$, where $\bar{\phi} \circ \mathbf{P}^{(S)} = \phi \circ \mathbf{F}^{(S)}$. Then, results of Section 5 apply directly to AS G^1 surfaces, at least when the geometry parametrization fulfills (45).

Remark 8. *As a consequence of Proposition 5, C^1 isogeometric spaces over G^1 multi-patch surface parametrizations fit in the isoparametric framework.*

7. NURBS spaces

The results presented in Sections 3–5 are not limited to spline spaces, but, for example, are generalizable to NURBS. One possibility is to generate rational planar geometries from AS G^1 surface patches. A rational patch

$$\mathbf{F}^{(i)} : \hat{\Omega} \rightarrow \Omega^{(i)} \subset \mathbb{R}^2 \quad (49)$$

with

$$\mathbf{F}^{(i)} = \left(\frac{F_1^{(i)}}{F_3^{(i)}}, \frac{F_2^{(i)}}{F_3^{(i)}} \right)^T \quad (50)$$

can be interpreted as a surface patch in \mathbb{R}^3 , with

$$\tilde{\mathbf{F}}^{(i)} = \left(F_1^{(i)}, F_2^{(i)}, F_3^{(i)} \right)^T, \quad (51)$$

when transformed to homogeneous coordinates. Recall, that two points $\tilde{\mathbf{F}}, \tilde{\mathbf{F}}'$ in homogeneous coordinates correspond to the same point in Euclidean space, if there exists a $\lambda \neq 0$ such that $\tilde{\mathbf{F}} = \lambda \tilde{\mathbf{F}}'$. An AS G^1 surface with surface patches (51) generates a rational AS G^1 geometry via (50). Therefore this construction fits into the framework of Section 6. For the sake of brevity, we only present one example (Figure 7) in the next Section 8.2.2 and postpone further studies. We want to point out that not all multi-patch NURBS geometries fit into this framework. The surface representation in (51) may not be G^1 (or not even C^0) but the rational patches representing the graph still join G^1 . Such a configuration is given for the circle presented in Figure 16. However, in that example the graph surface is not analysis-suitable G^1 .

8. Numerical tests

In this section, we present numerical tests which illustrate the previous theoretical results. The simulations have been obtained by the isogeometric analysis library IGATTOOLS, see [31] for an introduction. For related numerical studies see [20, 21, 29, 30].

8.1. Model problem

We consider the following bilaplacian problem on $\Omega \subset \mathbb{R}^2$, where the unknown is denoted by w and the data by f ,

$$\begin{cases} \Delta^2 w = f & \Omega, \\ w = 0 & \partial\Omega, \\ \nabla w \cdot \mathbf{n} = 0 & \partial\Omega. \end{cases} \quad (52)$$

In our tests the data f is selected in order to have an analytic exact solution w . Here \mathbf{n} is the unit normal vector to the boundary $\partial\Omega$. Let

$$\mathcal{U}_0 = \{v \in H^2(\Omega), v = 0 \text{ on } \partial\Omega \text{ and } \nabla v \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

the variational form of (52) is, find $w \in \mathcal{U}_0$, such that

$$\int_{\Omega} \Delta w \Delta v \, dx \, dy = \int_{\Omega} f v \, dx \, dy, \quad \forall v \in \mathcal{U}_0. \quad (53)$$

In what follows, we consider two types of geometries, the ones which are analysis-suitable G^1 and the ones which are not. Let us start with the analysis-suitable G^1 geometries.

8.2. Analysis-suitable G^1 geometries

We consider five different analysis-suitable G^1 geometry mappings with different numbers of patches. For all cases, we select the degrees $p = 3, 4, 5$ and regularities $1 \leq r \leq p - 1$.

8.2.1. Two-patch geometry (L-shape)

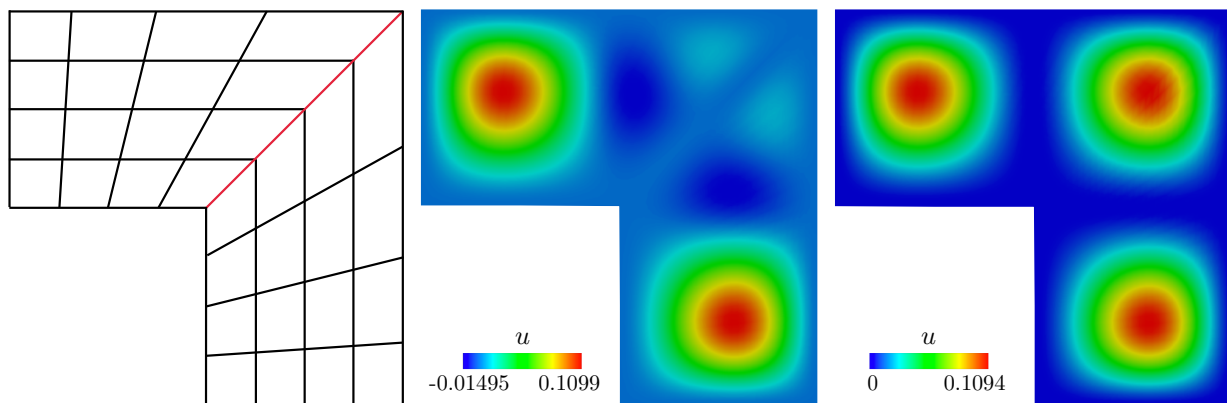


Figure 4: Two-patch L-shaped domain (left), a solution affected by C^1 locking for $p = 3, r = 2$ (center), and a correct numerical solution for $p = 3, r = 1$ (right).

We start with the L-shaped geometry consisting of two patches. The L-shaped domain is given in Figure 4 (left), where the red edge is the common edge of both patches. Here the parametrization of both patches is bilinear. This is obviously an AS G^1 geometry (recall Proposition 3). Using Theorem 1, the optimal convergence is achieved if and only if $r \leq p - 2$, in agreement with the

results of Section 4. If we focus on degree $p = 3$, C^1 locking is evident when the regularity equals to $r = p - 1 = 2$, see Figure 4 (center). In particular, the solution on the interface line equals to 0. In order to circumvent C^1 locking, we decrease the regularity, see Figure 4 (right). Figure 5 gives the convergence curves for degrees $p = 3, 4$ and 5 . We obtain expected results for all degrees, *i.e.* we have convergence (which is optimal) if and only if $r + 2 \leq p$.

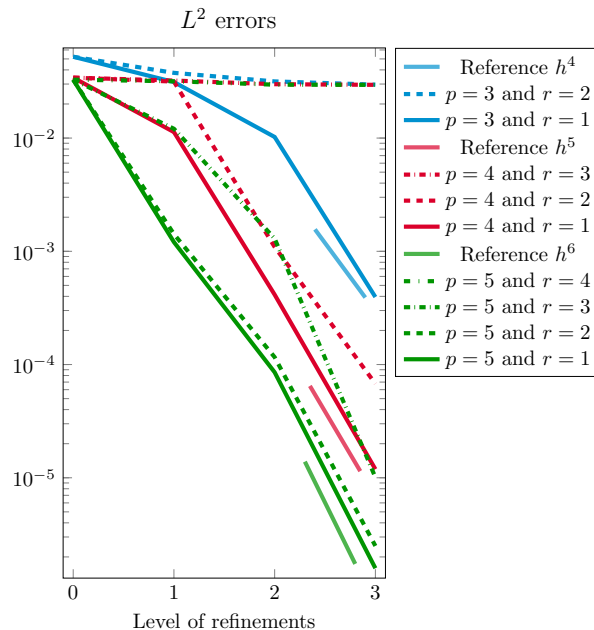


Figure 5: Convergence results for the L-shaped domain.

8.2.2. Multi-patch geometries (triangle, quarter of a circle and rectangle)

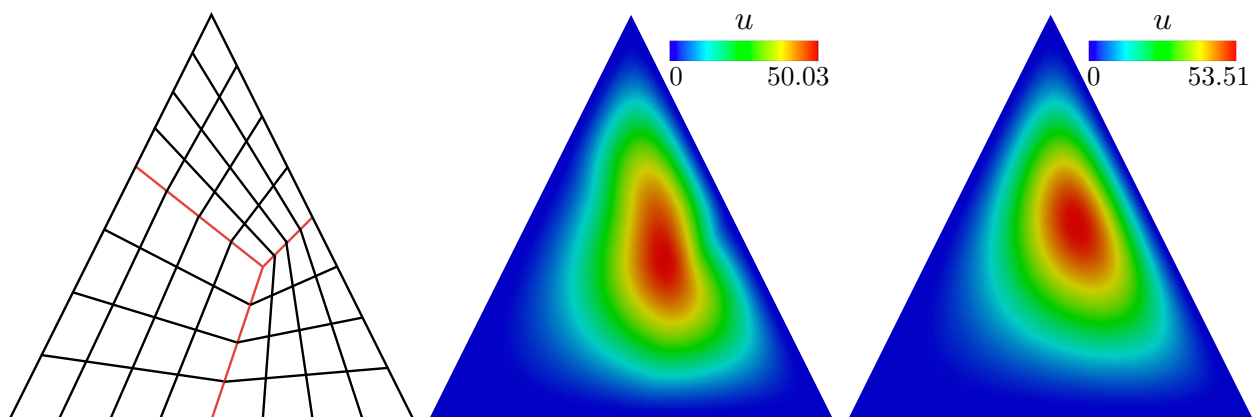


Figure 6: Three-patch triangle (left), a solution affected by C^1 locking for $p = 3$, $r = 2$ (center), and a correct numerical solution for $p = 3$, $r = 1$ (right).

In what follows, we consider cases with three or more patches, starting with an example with three bilinear patches forming a triangle (see Figure 6 (left)) and another with three bi-quadratic

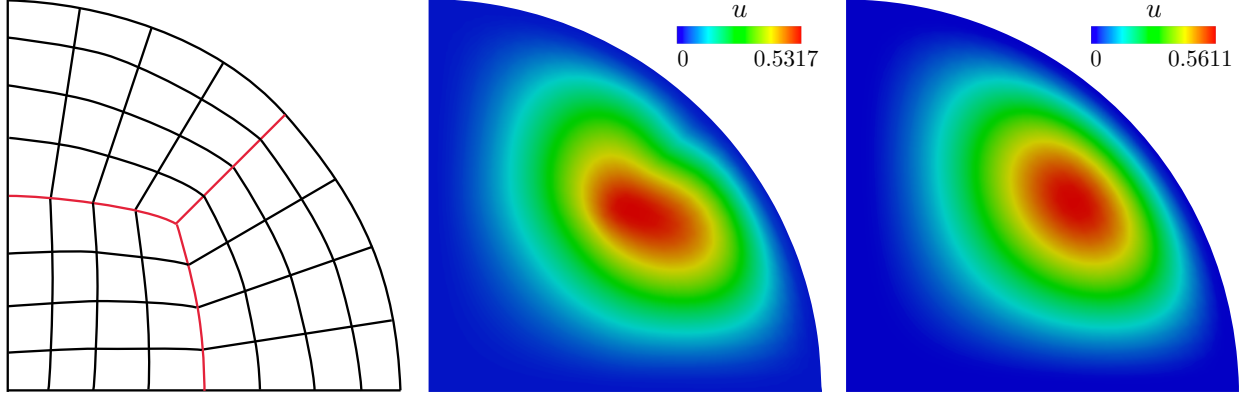


Figure 7: Three-patch quarter of a circle (left), a solution affected by C^1 locking for $p = 3, r = 2$ (center), and a correct numerical solution for $p = 3, r = 1$ (right).

patches forming the quarter of a circle (see Figure 7 (left)). Both are AS G^1 geometries. Note that the quarter of the circle is composed of NURBS patches, and is obtained following Section 7 construction, from a geometry parametrization that is an AS G^1 surface in homogeneous coordinates. The details of this construction are presented in the appendix. Figures 6 (center) and 7 (center) show C^1 locking which appears with $p = 3$ and $r = 2$ that we can circumvent by reducing the regularity, see Figures 6 (right) and 7 (right). The expected convergence orders are given in Figure 8.

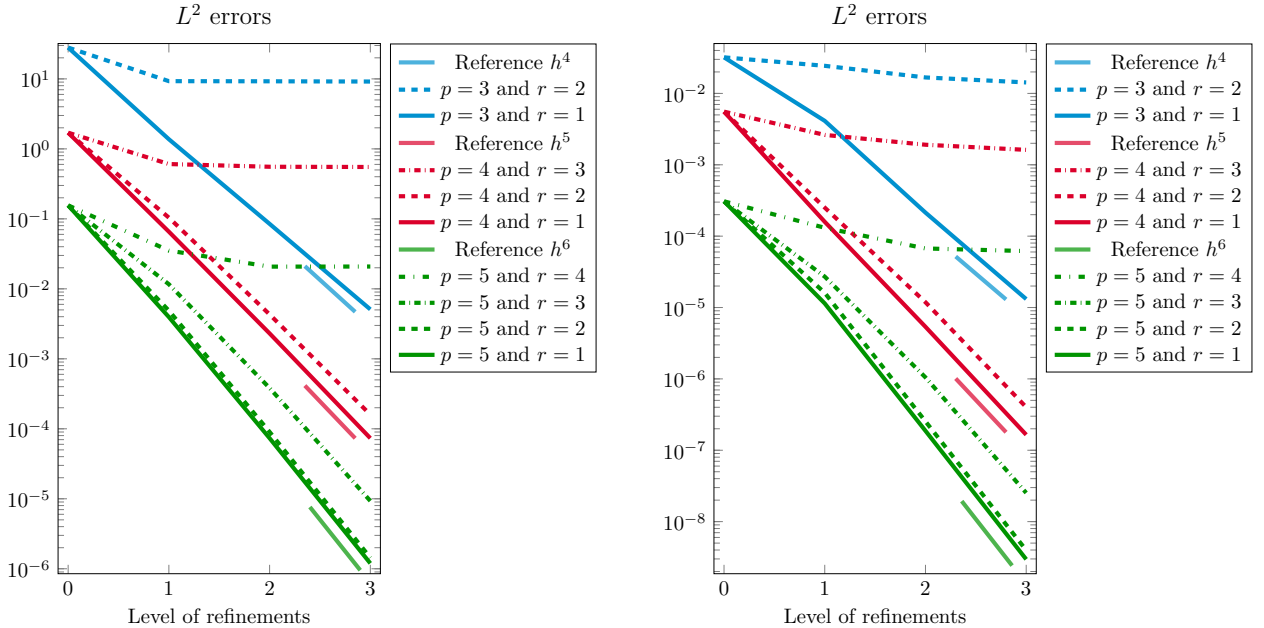


Figure 8: Convergence results for the triangle (left) and the quarter of circle (right).

Next we consider a rectangle composed of four patches, see Figure 9 (left). As shown in this figure, we consider a particular case where two interfaces are collinear. This configuration is among

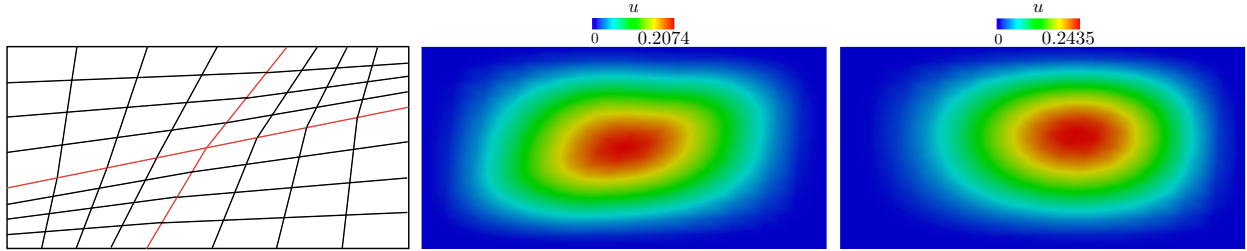


Figure 9: Four-patch rectangle (left), a solution affected by C^1 locking for $p = 3$, $r = 2$ (center), and a correct numerical solution for $p = 3$, $r = 1$ (right).

the ones analyzed in [8, Section 11.2.3]. The results are presented in Figure 9 for degree $p = 3$ and regularity $r = 2$ (center) and $r = 1$ (right). Figure 11 (left) gives the convergence orders, which are as expected.

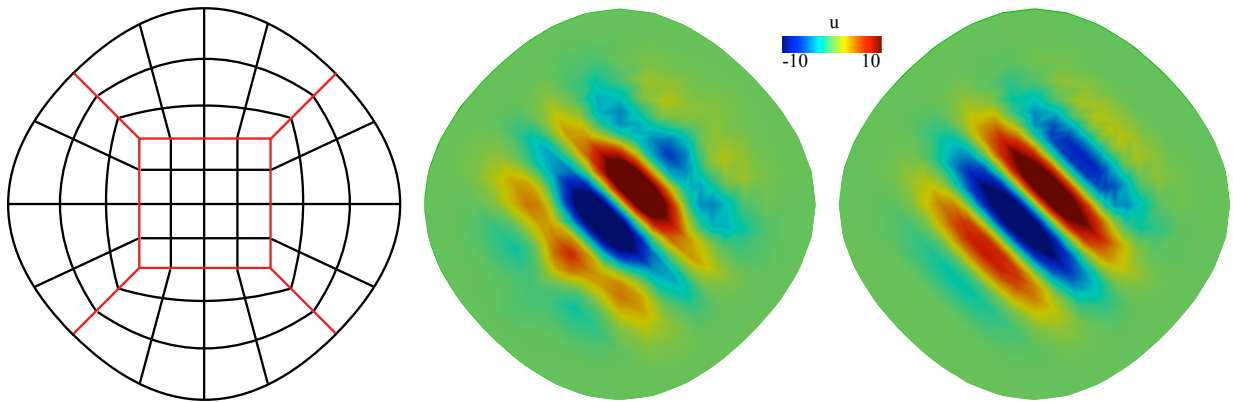


Figure 10: Five-patch simply-connected domain with smooth boundary (left), a solution affected by C^1 locking for $p = 3$, $r = 2$ (center), and a correct numerical solution for $p = 3$, $r = 1$ (right).

The final and most relevant example of AS G^1 geometry is reported in Figure 10 (left). This is a five patch decomposition of a simply-connected domain with smooth boundary. The parametrization of each patch is bi-quadratic, and the domain boundary is C^1 . Given the boundary control points and parametrization, the interior control points have been selected in order to fulfil the AS G^1 conditions. Unlike [32], here not only the mesh curves, i.e., patch interfaces, but also their parametrization need to be chosen properly. We refer to the Appendix for the complete description. The results are presented in Figure 10 for degree $p = 3$ and regularity $r = 2$ (center) and $r = 1$ (right). Figure 11 (right) gives the convergence orders, which are again as expected.

8.3. Non-analysis-suitable G^1 geometries

In this section we study two different examples of non-analysis-suitable G^1 geometries. First we consider a two-patch parametrization of a rectangle, where we compare a distorted parametrization with the undistorted identity mapping, see Figure 13. As a second example we consider a five-patch circle.

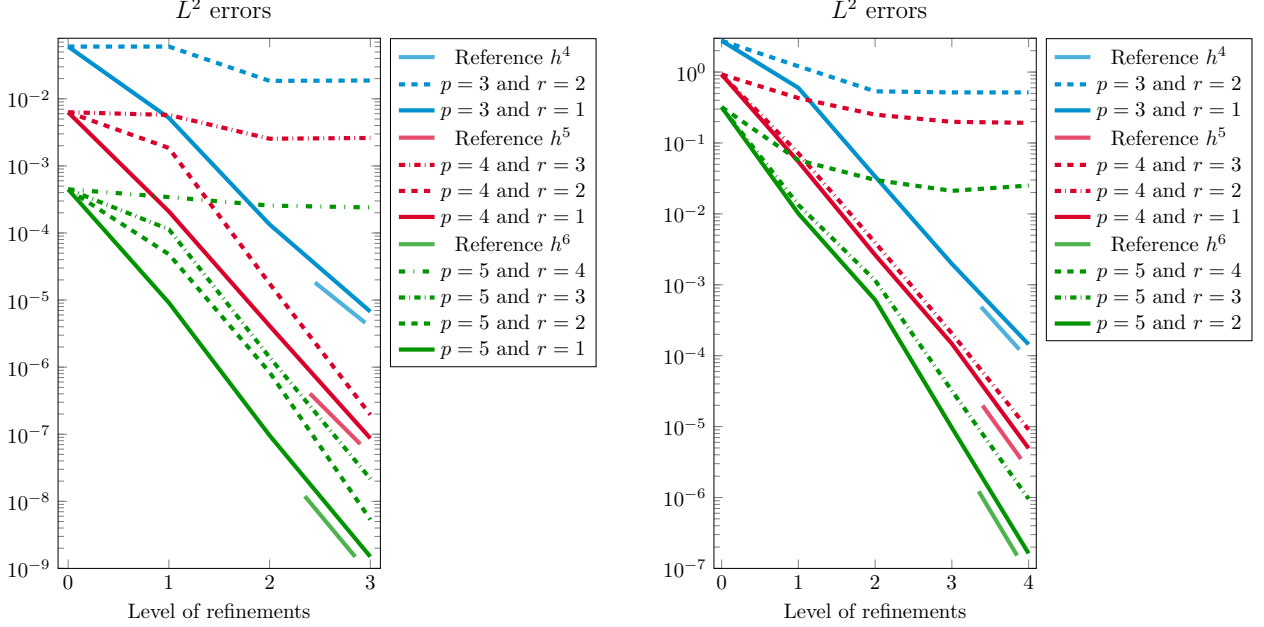


Figure 11: Convergence results for the four-patch rectangle (left) and the five-patch simply-connected domain with smooth boundary (right).

8.3.1. Two-patch geometry (quadratically distorted rectangle)

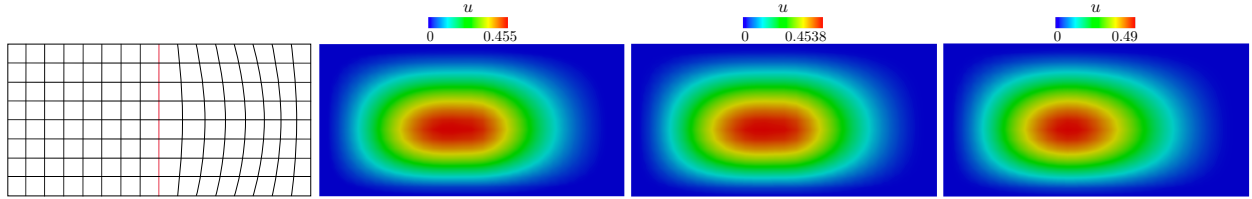


Figure 12: Non-AS G^1 rectangle (left), two solutions affected by C^1 locking for $p = 3$, $r = 2$ (center-left) and $p = 3$, $r = 1$ (center-right), as well as a correct numerical solution for $p = 4$, $r = 1$ (right).

We consider a rectangle domain $[-1, 1] \times [0, 1]$ formed by two patches, where the left one is linear (identity) and right one is quadratic in the horizontal direction and linear in the vertical, see Figure 12 (left). This case illustrates the configuration covered by Theorem 3 and Corollary 1. As shown in Figure 12, with degree $p = 3$, C^1 locking is manifested for both $r = 2$ and $r = 1$, see respectively the second and the third columns of the figure. In order to have convergence, a degree of at least $p = 4$ has to be selected, see Figure 12 (right).

However, as anticipated in Corollary 1, we cannot expect optimal convergence. This is a direct implication of Theorem 3, which for this case ($p_\alpha = 2$ and $\beta^{(R)} = 0$) states that for all $\phi \in \mathcal{V}^1$

$$\left. \frac{\partial \phi}{\partial x} \right|_{\Gamma} \in \mathcal{S}_r^{p-p_\alpha}(\Gamma) = \mathcal{S}_1^2(\Gamma). \quad (54)$$

Assume that the exact solution w is smooth enough and let w_h be the numerical solution. Then, using (54), together with the usual approximation estimates in Sobolev norms and the trace inequality

for Sobolev spaces, gives

$$C_{approx} h^{2.5} \simeq \left\| \frac{\partial w}{\partial x} - \frac{\partial w_h}{\partial x} \right\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_{trace} \|w - w_h\|_{H^2(\Omega)},$$

instead of the optimal order of convergence, that is h^3 when measuring the error in the $H^2(\Omega)$ -norm.

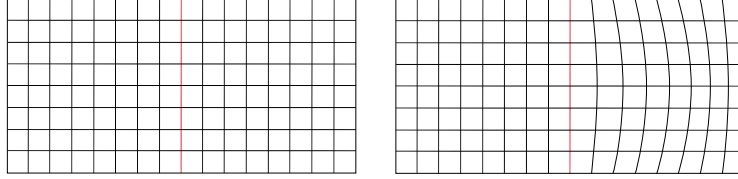


Figure 13: Two-patch rectangle with identity mapping (left), two-patch rectangle with a quadratic distorted patch (right).

In Figure 14 we report the convergence results and, due to the specificity of this case, both L^2 and H^2 norms are plotted. While C^1 locking is easily recognized, from these numerical tests it is difficult to measure the expected sub-optimality of the asymptotic behavior. This is likely a numerical artifact due to the imposition of the C^1 constraint in our implementation, which is discussed in Section 8.4. We further analyze this example in Figure 15, where we compute, for $p = 4$ and $r = 1$, the error only on the left patch, that is $[-1, 0] \times [0, 1]$, and compare the results for this geometry and for a reference geometry formed by the identity mapping on both patches (see Figure 13). Note that on the left patch both parametrizations are the identity mapping.

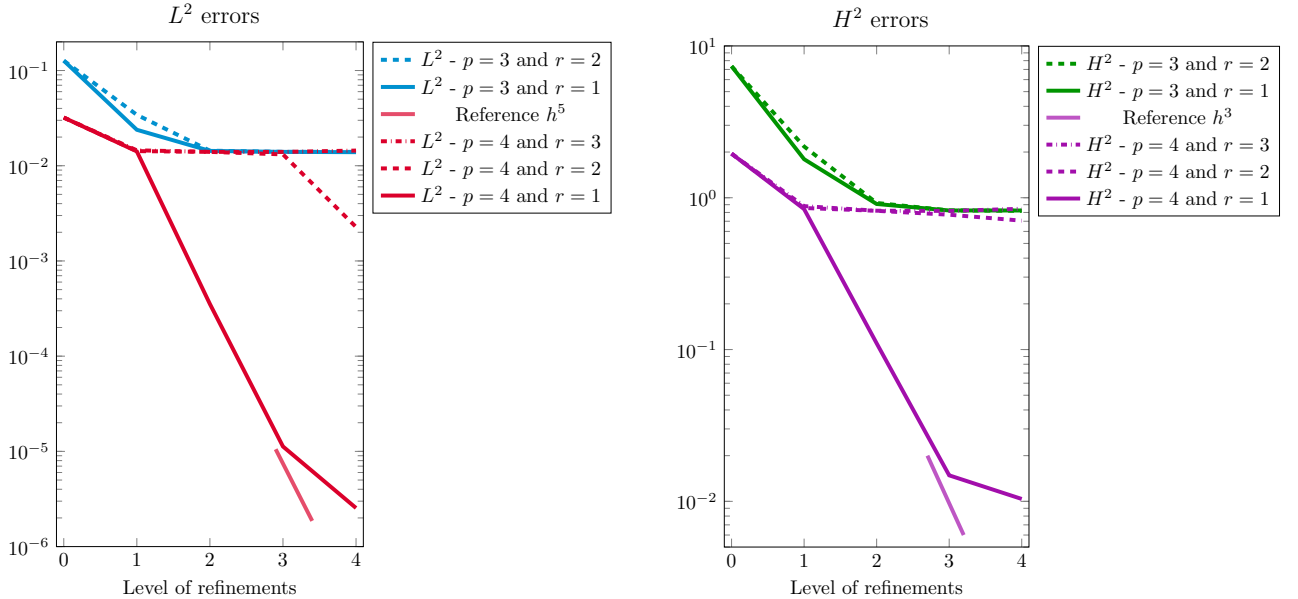


Figure 14: Convergence results for the non-AS G^1 two-patch rectangle, with error in L^2 (left) and H^2 (right).

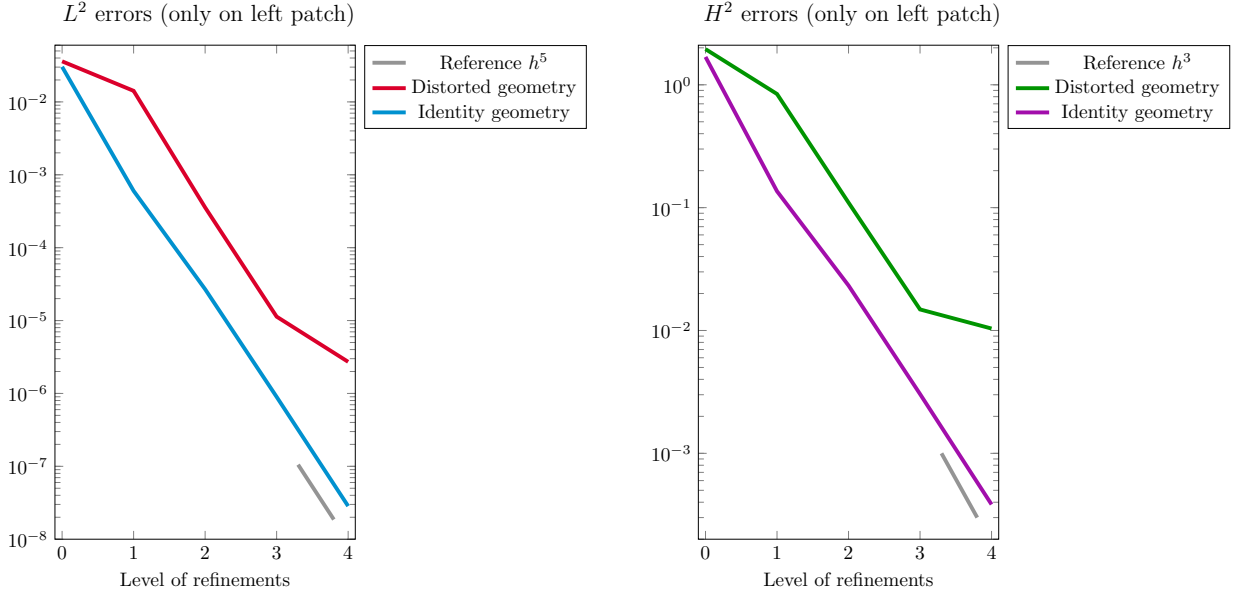


Figure 15: Comparison of convergence between the two geometries given in Figure 13 for $p = 4$ and $r = 1$; the error is computed only in the left subdomain $[-1, 0] \times [0, 1]$ in L^2 (left plot) and H^2 (right plot).

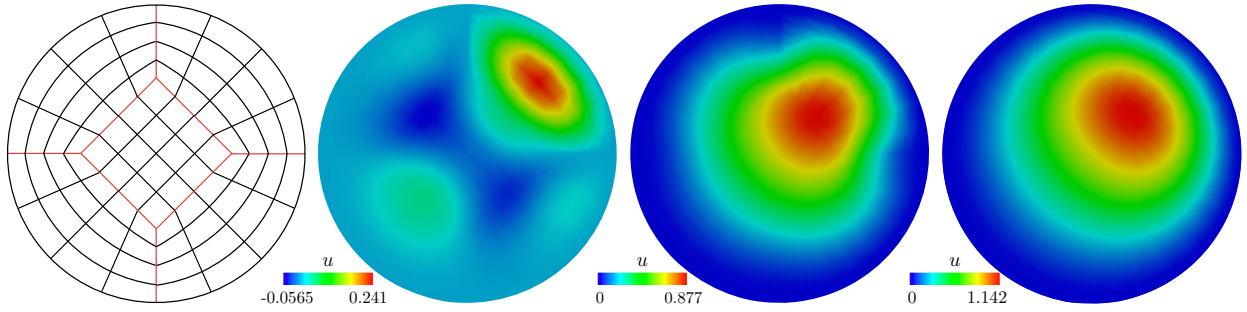


Figure 16: Five-patch circle (left), two solutions affected by C^1 locking for $p = 3$, $r = 2$ (center-left) and for $p = 3$, $r = 1$ (center-right), as well as a correct numerical solution for $p = 4$, $r = 1$ (right).

8.3.2. Multi-patch geometry (circle)

In the last example, we study an exact circle composed of five patches, given in Figure 16 (left). Here, we are interested in testing rational parametrizations that are beyond the framework presented in Section 7, since their homogeneous representation is not an analysis-suitable G^1 surface. Even though our theory does not apply, the numerical results obtained are consistent with our findings. For degree $p = 3$ the numerical solution suffers of C^1 locking, as one can see in Figure 16 for regularity $r = 2$ (center-left) and $r = 1$ (center-right). For degree $p = 4$ and regularity $r = 1$, see Figure 16 (right), we observe convergence to a solution. Figure 17 (right) gives the convergence behavior for degrees $p = 3, 4$ and 5. Sub-optimality in all situations is manifested.

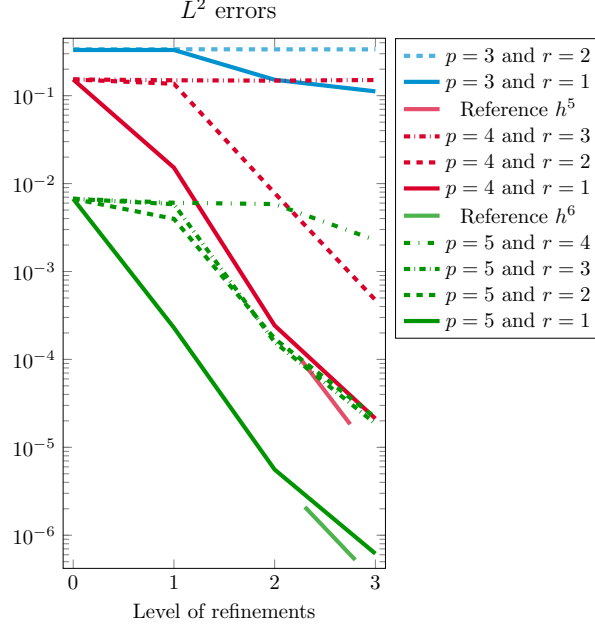


Figure 17: Convergence results for the five-patch circle.

8.4. Numerical implementation of C^1 continuity

In this section, we describe the numerical implementation of C^1 continuity that we have used in order to obtain all the numerical examples given previously. Let A and b , be the matrix and the right and side of the system corresponding to the variational formulation (53), where no boundary condition or continuity condition are included. Furthermore, let C_1 be the C^1 constraint matrix in symmetric form, i.e.

$$(C_1)_{i,j} = \int_{\Gamma} \llbracket \vec{\nabla} \phi_i \cdot \vec{n} \rrbracket \llbracket \vec{\nabla} \phi_j \cdot \vec{n} \rrbracket,$$

where \vec{n} is a normal unitary vector and $\llbracket \cdot \rrbracket$ is the jump at the patch interface Γ . Let N_0 be the change of basis matrix from the fully unconstrained space to the subspace fulfilling boundary condition and C^0 continuity. N_0 can be seen as obtained from the identity $I_{N_{unc}}$ – where N_{unc} is the number of degrees of freedom without constraints – by removing the columns with index corresponding to the degrees of freedom of the boundary conditions and by summing the columns of degrees of freedom that are shared on Γ . Since it is not trivial to compute a C^1 continuous basis analytically, we operate numerically by computing the null-space

$$N_1 = \text{null}(N_0^T C_1 N_0). \quad (55)$$

Then we solve the following problem: find z such that

$$N_1^T N_0^T A N_0 N_1 z = N_1^T N_0^T b, \quad (56)$$

and obtain the solution in the unconstrained initial spline basis as $x = N_0 N_1 z$.

The numerical computation of (55) is a hard task for non AS G^1 geometries. Indeed, the non-zero eigenvalues of $N_0^T C_1 N_0$ are not well separated from the eigenvalues that are numerically zero (close to machine precision), as shown in Figure 18 for the distorted rectangular domain in Figure 12 (left), with $p = 4$, $r = 1$. For the next refinement step, the distinction may not be possible anymore. By comparison, Figure 19 shows the eigenvalues for the L-shaped domain with the same degree and regularity. In this case, it is easier to distinguish between numerical zeros (around 10^{-14}) and non-zero eigenvalues (above 10^{-2}). The two configurations are structurally equivalent, i.e. the topology of the control point grid is the same.

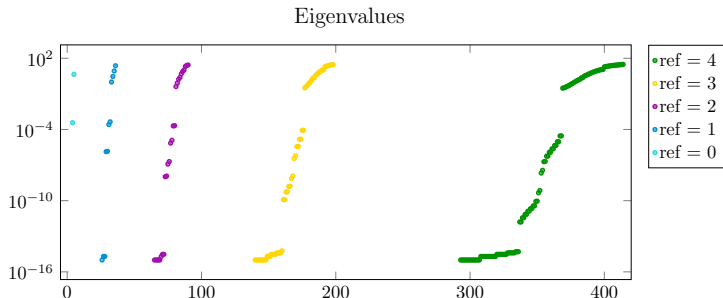


Figure 18: Eigenvalues of the domain given in Figure 12 (left) for $p = 4$ and $r = 1$, and different h -refinement levels.

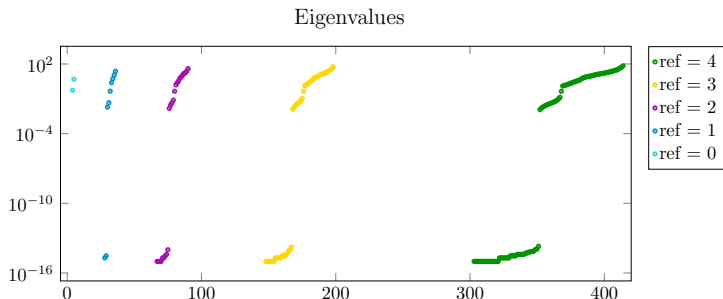


Figure 19: Eigenvalues of the L-shaped domain given in Figure 4 (left) for $p = 4$ and $r = 1$, and different h -refinement levels.

9. Conclusions

In this paper, we have studied C^1 -smooth isogeometric function spaces over multi-patch geometries. We have considered geometry parametrizations composed of multiple patches that meet C^0 at patch interfaces. As it is common for isogeometric methods, the geometry parametrization is given initially and considered fixed. From that, the geometric continuity conditions are derived. The same conditions need to be satisfied for the graph parametrizations of the isogeometric functions in order to obtain a C^1 -smooth function space over the given geometry.

We have studied how these conditions affect the traces and the transversal derivative of the isogeometric function spaces along the patch interfaces. Indeed, if the trace or transversal derivative

at the interface is over-constrained, the approximation order of the isogeometric function space is reduced. We identify a class of configurations that allows for optimal approximation properties of C^1 isogeometric spaces, so called *analysis-suitable G^1* (AS G^1) geometry parametrizations. We show numerically that AS G^1 geometries indeed allow for optimal approximation. We have not developed a complete approximation theory with classical error estimates, but this will be the topic of a future paper. Parametrizations that are not AS G^1 cause suboptimal order of approximation. In the worst case the convergence under h -refinement is prohibited, a behaviour that we have named C^1 locking.

We have addressed mainly the case of planar B-spline geometries, but we have briefly discussed the generalization to surfaces and NURBS patches. All the results are supported and confirmed by numerical tests for various degrees and orders of regularity of the spline space. We have numerically solved a bilaplacian problem over several AS and non AS G^1 geometries, by a standard Galerkin approach. As we pointed out, the numerical implementation of the C^1 conditions poses some non-negligible difficulties for complex configurations of non AS G^1 geometries.

An important question that remains to be studied in more detail is the flexibility of analysis-suitable G^1 geometries. As we have shown, the AS G^1 class contains bilinear patches but extends to more general configurations. We formulate the problem of flexibility in the following way: Given a collection of boundary curves, is it possible to find patches that interpolate the boundary curves and that form an AS G^1 geometry? For piecewise linear boundary curves, the problem can be solved by using bilinear patches. In Figure 9 we have shown the interesting example of a piecewise biquadratic AS G^1 parametrization of a C^1 simply-connected domain. The extension to arbitrary degrees and topology deserves further investigation.

The construction for AS G^1 surfaces could be more difficult. In the planar case, the interior parametrization of the patches may be modified in order to achieve analysis-suitability. This is not feasible for surfaces, as the surface itself changes if the parametrization of the interior is changed. AS G^1 constructions are possible if the surface is given as the image of a planar domain. For the general setting, an explicit construction remains an open problem to be considered in future research.

Acknowledgments

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Appendix A: AS G^1 parametrization of a quarter of circle

In this appendix we discuss in detail the construction of the geometry parametrization of the example shown in Figure 7 in Section 8.2.2. This is based on the NURBS setting of Section 7 and on the ideas in [8] as well as [21, Section 3.4]. Each patch of the three-patch geometry is constructed from a combination of a bilinear mapping

$$\mathbf{B}^{(i)} : [0, 1]^2 \rightarrow Q^{(i)} \subset \Delta = \{(u, v) \in [0, 1]^2 : u + v \leq 1\}$$

onto a quadrilateral $Q^{(i)}$ within a reference triangle Δ and a global mapping

$$\tilde{\mathbf{G}} : \Delta \rightarrow \tilde{\Omega} \subset \mathbb{R}^3$$

from the reference triangle to the surface patch $\tilde{\Omega}$ in homogeneous coordinates. These mappings and corresponding domains are shown in Figure 20.

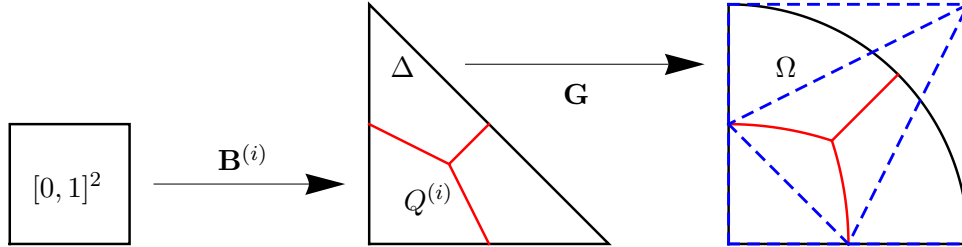


Figure 20: Mappings $\mathbf{B}^{(i)}$ and \mathbf{G} and corresponding domains.

Here the mapping $\tilde{\mathbf{G}}$ is a triangular Bézier patch of total degree $p = 2$ with

$$\tilde{\mathbf{G}}(s, t) = \sum_{i+j+k=2} g_{i,j,k} s^i t^j (1-s-t)^k \frac{i! j! k!}{2},$$

with control points

$$\begin{aligned} g_{0,0,2} &= (0, 0, 1)^T & g_{0,1,1} &= (0, \sqrt{2}, 2\sqrt{2})^T & g_{0,2,0} &= (0, 1, 1)^T \\ g_{1,0,1} &= (\sqrt{2}, 0, 2\sqrt{2})^T & g_{1,1,0} &= (2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2})^T \\ g_{2,0,0} &= (1, 0, 1)^T \end{aligned}$$

in homogeneous coordinates. In Figure 20, the dashed blue lines represent the corresponding control point grids in Cartesian coordinates. Each quadrilateral $Q^{(1)}$, $Q^{(2)}$ and $Q^{(3)}$ is formed by one corner point of the triangle Δ , two adjacent edge midpoints as well as the center of gravity $(1/3, 1/3)^T$. The bilinear mapping $\mathbf{B}^{(i)}$ is (up to rotations of the parameter domain) completely determined by its image $Q^{(i)}$. By construction, the mapping $\mathbf{F}^{(i)} = \mathbf{G} \circ \mathbf{B}^{(i)}$ is a rational bi-quadratic function in each component.

Using this construction, the theory developed in Sections 6 and 7 can be applied to the example in Figure 7.

Appendix B: AS G^1 parametrization of a smooth simply-connected domain

In the following we give a description of the domain depicted in Figure 10. The domain is composed of five bi-quadratic patches and the boundary is C^1 -smooth. The central patch is just the shifted unit square $\Omega^{(c)} = [-\frac{1}{2}, \frac{1}{2}]^2$ parameterized by

$$\mathbf{F}^{(c)}(u, v) = \left(u - \frac{1}{2}, v - \frac{1}{2} \right)^T.$$

The top patch $\Omega^{(t)}$ is parameterized by

$$\mathbf{F}^{(t)}(u, v) = \begin{pmatrix} (u - \frac{1}{2}) \left(1 + \frac{\sqrt{17}-3}{2}v - \frac{\sqrt{17}-5}{2}v^2 \right) \\ 2v^2 (u - u^2) + \frac{1}{2} \left(1 + \frac{\sqrt{17}-3}{2}v - \frac{\sqrt{17}-5}{2}v^2 \right) \end{pmatrix}$$

and the parametrizations $\mathbf{F}^{(l)}$, $\mathbf{F}^{(b)}$ and $\mathbf{F}^{(r)}$ for the left, bottom and right patches are given by rotations of the top patch.

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